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## DYNAMICS OF A STOCHASTIC THREE-SPECIES NUTRIENT-PHYTOPLANKTON-ZOOPLANKTON MODEL

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**Abstract.** We investigate a stochastic three-species nutrient-phytoplankton-zooplankton model in this paper. We verify that the system admits a unique positive global solution starting from any positive initial value firstly. Then the sufficient conditions that guarantee the population extinction and persistence in the mean are derived. The results show that weaker white noise will strengthen the stability of the system, while stronger white noise will result in extinction of population. We also show that around the positive equilibrium point of deterministic system, the stochastic system is weakly persistent in the mean under some conditions.

**Keywords:** stochastic nutrient-phytoplankton-zooplankton model; extinction and persistence in the mean; asymptotic behavior.

**2010 AMS Subject Classification:** 60H10.

## 1. Introduction

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In marine ecology, almost all aquatic life is based upon plankton, which are the most abundant form of life floating freely and weakly swimming organisms near the surfaces of all aquatic environments, namely, lakes, rivers, estuaries and oceans [1]. Plankton are the first link in the marine food chain, which is eaten by many organisms, including mussels, fishes, birds and mammals (for example, baleen whales). Plankton are made up of phytoplankton (the primary producers) and zooplankton (the microscopic animals that eat other plankton). Phytoplankton are capable of photo-synthesis in presence of sunlight and serve as the basic food source and occupy the first trophic level for all aquatic food chains. Besides, phytoplankton also control the global carbon cycle which has a significant impact on the climate regulation. Phytoplankton are consumed by zooplankton, their animal counterparts, which considered to be most favorable food sources for fish and other aquatic animals.

A remarkable common feature of phytoplankton population is rapid increase of biomass due to rapid cell proliferation, these type of rapid changes in phytoplankton population density are known as blooms. Phytoplankton blooms are a widespread phenomenon present in lakes, rivers and oceans spanning the globe, and it may be categorized into spring blooms and red tides. Spring blooms occur seasonally due to changes in temperature or nutrient availability. Red tides are the result of localized outbreaks associated with water temperature.

In recent years, many authors attempted to explain the bloom phenomenon by different approaches, and find that toxic phytoplankton certainly play an important role in the bloom phenomenon, there is also some evidences show that zooplankton grazing plays a crucial role in red-tide outbreak[2]. Recently, there has been a considerable scientific attention towards the study of toxin-producing phytoplankton-zooplankton systems. J. Chattopadhyay et al.[3] formulated a toxin-producing phytoplankton-zooplankton model with deterministic system, and the authors have analyzed the role of toxin production with various combinations of grazing rates and toxin liberation rates. T. Saha et al. [4] studied the dynamical analysis of toxin producing phytoplankton-zooplankton interactions, they consider the elementary dynamical properties of the toxin-phytoplankton-zooplankton interacting model in absence of time delay, and establish the existence of local Hopf-bifurcation as the time delay crosses a threshold value and prove the existence of stability switching phenomena as well as the global existence of periodic orbits.

S. Jang et al. [5] proposed periodic systems of phytoplankton-zooplankton interactions with toxin producing phytoplankton to study the effects of TPP upon extinction and persistence of the populations. More related literature can be seen in [6, 7, 8, 9, 10].

However, unlike the above models, in this paper, we model the harmful effect of phytoplankton by choosing the zooplankton grazing function as a type-IV function due to prey toxicity. This type of functional response implies that for large phytoplankton density, the predation rate decreased, which resembles a group defence mechanism for phytoplankton against zooplankton and hence a type-IV function would be an appropriate choice to model zooplankton predation. Mukhopadhyay et al.[2] consider the following nutrient-phytoplankton-zooplankton model with holling type-IV :

$$\begin{cases} \frac{dN}{dt} = D(N^0 - N) - \frac{aPN}{k+N} + \gamma_1 P + \varepsilon_1 Z, \\ \frac{dP}{dt} = \frac{a_1 PN}{k+N} - \frac{cPZ}{(P^2/i)+P+b} - (\gamma + D_1)P, \\ \frac{dZ}{dt} = Z[\frac{c_1 P}{(P^2/i)+P+b} - \varepsilon - D_2], \end{cases} \quad (1.1)$$

where  $N, P$  and  $Z$  denote the instantaneous concentrations of nutrient, phytoplankton and zooplankton at time  $t$ , respectively, subject to the non-negative initial condition  $N(0) = N_0 \geq 0, P(0) = P_0 \geq 0$  and  $Z(0) = Z_0 \geq 0$ . The system parameters are all non-negative and are interpreted as follows:

$a$ : maximal nutrient uptake rate of phytoplankton;

$a_1$ : nutrient-phytoplankton conversion rate ( $a_1 < a$ );

$c$ : maximal phytoplankton uptake rate of zooplankton;

$c_1$ : phytoplankton-zooplankton conversion rate ( $c_1 < c$ );

$N^0$ : constant input concentrations of nutrient;

$D, D_1, D_2$ : washout rates for nutrient, phytoplankton and zooplankton respectively;

$i$ : the zooplankton's immunity from, or tolerance of, the phytoplankton;

$\gamma, \varepsilon$ : death rates for phytoplankton and zooplankton;

$\gamma_1, \varepsilon_1$ : nutrient recycling rates from dead phytoplankton and zooplankton, respectively with

$\gamma_1 < \gamma, \varepsilon_1 < \varepsilon$ .

In [2], the authors have analyzed stability and bifurcation behavior of different equilibrium points of system (1.1), and considered diffusion-driven instability as well as stability. The result

showed that, when  $\varepsilon + D_2 < c_1$ ,  $\gamma + D_1 < \frac{a_1 N^*}{k + N^*}$ , the model (1.1) has a positive equilibrium point  $E^* = (N^*, P^*, Z^*)$ . In fact, since aquatic ecosystems are generally complex, open systems, the natural growth and death rate of populations are inevitably affected by environmental stochasticity. Therefore, it is meaningful to further incorporate the environmental stochasticity into the underlying system (1.1), which could provide us a deeper understanding for the real ecosystems. For the convenience of calculation, we assume  $a_1 = a$ ,  $c_1 = c$ . So far as our knowledge is concerned, no results related to nutrient-phytoplankton-zooplankton model with stochastic perturbation have been reported. Therefore, we assume the fluctuation of uptake rates  $a$  and  $c$  are subjected to the Gaussian white noise, that is,  $a \rightarrow a + \sigma_1 \dot{B}_1(t)$ ,  $c \rightarrow c + \sigma_2 \dot{B}_1(t)$ , then we obtain the following stochastic model:

$$\begin{cases} dN = [D(N^0 - N) - \frac{aPN}{k+N} + \gamma_1 P + \varepsilon_1 Z]dt - \frac{\sigma_1 PN}{k+N} dB_1(t), \\ dP = [\frac{aPN}{k+N} - \frac{cPZ}{(P^2/i)+P+b} - (\gamma + D_1)P]dt + \frac{\sigma_1 PN}{k+N} dB_1(t) - \frac{\sigma_2 PZ}{(P^2/i)+P+b} dB_2(t), \\ dZ = Z[\frac{cP}{(P^2/i)+P+b} - \varepsilon - D_2]dt + \frac{\sigma_2 PZ}{(P^2/i)+P+b} dB_2(t), \end{cases} \quad (1.2)$$

where  $B_1(t), B_2(t)$  are independent Brownian motions defined on the complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual normal conditions (i.e. it is right continuous and increasing while  $\mathcal{F}_0$  contains all P-null sets).  $\sigma_1^2, \sigma_2^2$  represent the intensity of the white noise.

Scholars have pointed out that random fluctuations may change the dynamics of population models greatly and studied the effect of the noise on the dynamic behavior of the population models, see [11, 12, 14, 13, 15, 16] and the references therein. For example, Zhao and Yuan [11] analyzed the stationary distribution and ergodicity of a stochastic phytoplankton allelopathy model under regime switching, which takes both white and colored noises into account. Liu and Wang [12] investigate a double delayed (maturation delay for toxin producing phytoplankton and gestation delay for zooplankton) bioeconomic phytoplankton zooplankton system with commercial harvesting on zooplankton and environmental stochasticity, they studied the existence and uniqueness of the global positive solution as well as the stochastic stability and existence of stochastic Hopf bifurcation.

This paper is organized as follows. In Section 2, we show the existence and uniqueness of the global positive solution. In Section 3, we establish the sufficient conditions of population

extinction and persistence in the mean. We obtain the asymptotic behavior of stochastic system near the positive equilibrium point of deterministic system in Section 4. The paper ends with some conclusions in Section 5.

## 2. Existence and uniqueness of the global positive solution

As we know, in order for a stochastic differential equation to have a unique global solution (i.e. no explosion in a finite time) for any given initial value, the function involved with stochastic system are generally required to satisfy the linear growth condition and local Lipschitz condition [17, 18]. However, the function of system (1.2) do not satisfy the linear growth condition, though they are local Lipschitz condition, so for any given initial value  $(N(0), P(0), Z(0)) \in \mathbb{R}_+^3$ , there only a unique positive local solution  $(N(t), P(t), Z(t))$  on  $t \in [0, \tau_e)$  a.s., where  $\tau_e$  is the explosion time. To show this solution is global, we only to prove  $\tau_e \rightarrow \infty$  a.s..

To obtain the global positive solution of system (1.2), we consider an auxiliary equation:

$$W(t) = N(t) + P(t) + Z(t),$$

clearly, we have

$$dW(t) = dN(t) + dP(t) + dZ(t),$$

using system (1.2) in the above expression, we obtain

$$\begin{aligned} \frac{dW(t)}{dt} &= DN^0 - DN + (\gamma_1 - \gamma)P + (\varepsilon_1 - \varepsilon)Z - D_1P - D_2Z, \\ &\leq DN^0 - DN - D_1P - D_2Z, \\ &\leq DN^0 - \hat{D}W(t), \end{aligned} \tag{2.1}$$

where  $\hat{D} = \min\{D, D_1, D_2\}$ . Thus

$$\frac{dW(t)}{dt} + \hat{D}W(t) \leq DN^0.$$

Now, applying the theorem of differential inequalities, we can get

$$0 < W(t) \leq W(0)e^{-\hat{D}t} + \frac{DN^0}{\hat{D}},$$

as  $t \rightarrow \infty$ , we obtain

$$0 < W(t) = N(t) + P(t) + Z(t) \leq \frac{DN^0}{\hat{D}} \triangleq C_1, \quad (2.2)$$

**Theorem 2.1.** *For any initial value  $(N(0), P(0), Z(0)) \in R_+^3$ , system (1.2) has a unique global positive solution  $(N(t), P(t), Z(t))$  for all  $t > 0$ , and the solution will remain in  $R_+^3$  with probability one.*

**Proof.** Let  $k_0 > 0$  be sufficiently large such that  $N_0 > \frac{1}{k_0}, P_0 > \frac{1}{k_0}, Z_0 > \frac{1}{k_0}$ . For each integer  $k \geq k_0$ , define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : \min\{N(t), P(t), Z(t)\} \leq \frac{1}{k}\}.$$

We set  $\inf \Phi = \infty$  ( $\Phi$  denotes the empty set). Obviously,  $\tau_k$  is increasing on  $k$  and  $\tau_k < \tau_e$ . Thus,  $\tau_\infty = \lim_{k \rightarrow +\infty} \tau_k$  make senses, and  $\tau_\infty \leq \tau_e$  a.s.. Hence, we only need to prove  $\tau_\infty \rightarrow \infty$  a.s.. If the statement is false, then for any constant  $T > 0$ , there is an  $\varepsilon \in (0, 1)$  and an integer  $k_1 \geq k_0$  such that

$$P\{\tau_k \leq T\} \geq \varepsilon, \quad \forall k \geq k_1. \quad (2.3)$$

According to (2.2), we define a  $C^2$ -function  $V(N, P, Z)$  by

$$V(N, P, Z) = -\ln \frac{N}{C_1} - \ln \frac{P}{C_1} - \ln \frac{Z}{C_1}.$$

By Ito's formula, we have

$$dV(N, P, Z) = LV(N, P, Z)dt + \frac{\sigma_1(P - N)}{k + N}dB_1(t) + \frac{\sigma_2(Z - P)}{(P^2/i) + P + b}dB_2(t),$$

where

$$\begin{aligned} LV(N, P, Z) &= -\frac{DN^0}{N} + D + \frac{aP}{k+N} - \frac{\gamma P}{N} - \frac{\varepsilon_1 Z}{N} - \frac{aN}{k+N} + \frac{cZ}{P^2/i + P + b} \\ &\quad + \gamma + D_1 - \frac{cP}{P^2/i + P + b} + \varepsilon + D_2 + \frac{\sigma_1^2(P^2 + N^2)}{2(k+N)^2} + \frac{\sigma_2^2 Z^2}{2(P^2/i + P + b)^2}. \end{aligned}$$

Noting that  $N(t) \leq C_1, P(t) \leq C_1, Z(t) \leq C_1$ , we can obtain

$$\begin{aligned} LV(N, P, Z) &\leq D + \frac{aP}{k} + \frac{cZ}{b} + \gamma + D_1 + \varepsilon + D_2 + \frac{\sigma_1^2(P^2 + N^2)}{2k^2} + \frac{\sigma_2^2 Z^2}{2b^2} \\ &\leq D + \frac{aC_1}{k} + \frac{cC_1}{b} + \gamma + D_1 + \varepsilon + D_2 + \frac{\sigma_1^2(2C_1^2)}{2k^2} + \frac{\sigma_2^2 C_1^2}{2b^2} \\ &= D + \gamma + D_1 + \varepsilon + D_2 + \left(\frac{a}{k} + \frac{c}{b}\right)C_1 + \left(\frac{\sigma_1^2}{k^2} + \frac{\sigma_2^2}{2b^2}\right)C_1^2 \triangleq C_2. \end{aligned}$$

Therefore

$$dV(N, P, Z) \leq C_2 dt + \frac{\sigma_1(P - N)}{k + N} dB_1(t) + \frac{\sigma_2(Z - P)}{(P^2/i) + P + b} dB_2(t).$$

Integrating and take the expectation of both sides, we have

$$EV(N(\tau_\varepsilon \wedge T), P(\tau_\varepsilon \wedge T), Z(\tau_\varepsilon \wedge T)) \leq V(N_0, P_0, Z_0) + C_2 T.$$

And set  $\Omega_k = \{\tau_k \leq T\}$  for  $k \geq k_1$ , then in the view of (2.3), we obtain  $P(\Omega_k) \geq \varepsilon$ . Noting that for every  $\omega \in \Omega_k$ , there is at least one of  $P(\tau_k, \omega), N(\tau_k, \omega), Z(\tau_k, \omega)$  equals  $\frac{1}{k}$ , then

$$V(P(\tau_k), N(\tau_k), Z(\tau_k))(k - 1 - \ln k) \geq \ln(kC_1),$$

that is

$$\begin{aligned} V(N_0, P_0, Z_0) + C_2 T &\geq E(1_{\Omega_k} V(N(\tau_k \wedge T), P(\tau_k \wedge T), Z(\tau_k \wedge T))) \\ &\geq P(\Omega_k) V(N(\tau_k), P(\tau_k), Z(\tau_k)) \\ &\geq \varepsilon \ln(kC_1), \end{aligned}$$

here  $1_{\Omega_k}$  is the indicator function of  $\Omega_k$ . Letting  $k \rightarrow \infty$  yields the contradiction

$$\infty > V(N_0, P_0, Z_0) + \lambda T = +\infty.$$

Therefore we obtain that  $\tau_\infty = \infty$  a.s.. This completes the proof.

### 3. Extinction and persistence in mean

In this section, we investigate persistence and extinction of system (1.2). For obtaining the main results, we need the following Definition and Lemma.

**Definition 3.1.**([19]) (1)  $P(t)$  is said to be extinct if there exist a constant  $l < 0$  such that

$$\limsup_{t \rightarrow \infty} \frac{\ln P(t)}{t} \leq l < 0 \text{ a.s.};$$

(2)  $P(t)$  is said to be weak persistent in mean if there exist a constant  $l > 0$  such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(s) ds \leq l < +\infty \text{ a.s.};$$

(3)  $P(t)$  is said to be strong persistent in mean if there exist a constant  $l > 0$  such that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(s) ds \geq l > 0 \text{ a.s..}$$

**Lemma 3.1.**([20]) Suppose that  $x(t) \in C(\Omega \times [0, \infty), R_+)$ , if there are positive constant  $T, \mu_0, \mu > 0$  such that

- (1) If  $\ln x(t) \leq \mu t - \mu_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t)$ , then  $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s) ds \leq \frac{\mu}{\mu_0}$  a.s.;
- (2) If  $\ln x(t) \geq \mu t - \mu_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t)$ , then  $\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s) ds \geq \frac{\mu}{\mu_0}$  a.s.,

for  $t \geq T$ , where  $\beta_i (1 \leq i \leq n)$  is a constant.

**Lemma 3.2.**([21]). Let  $(N(t), P(t), Z(t))$  be the solution of system (1.2) with initial value  $(N(0), P(0), Z(0)) \in R_+^3$ . Then

$$\lim_{t \rightarrow +\infty} \frac{\int_0^t \frac{\sigma_1 N}{k+N} dB_1(s)}{t} = 0, \lim_{t \rightarrow +\infty} \frac{\int_0^t \frac{\sigma_2 Z}{P^2/i+P+b} dB_2(s)}{t} = 0, \lim_{t \rightarrow +\infty} \frac{\int_0^t \frac{\sigma_2 P}{(P^2/i)+P+b} dB_2(s)}{t} = 0. \text{ a.s.}$$

**Theorem 3.1.** Let  $(N(t), P(t), Z(t))$  be the solution of system (1.2), then the following statements hold:

- (i) If  $\sigma_1 > \frac{a}{\sqrt{2(\gamma+D_1)}}$ , then  $P(t)$  and  $Z(t)$  are all extinct;
- (ii) If  $\sigma_1, \sigma_2$  are small enough such that  $\frac{DN^0}{D} > \max\{\frac{\Theta(k+C_1)}{a}, \frac{b(\epsilon+D_2)}{c}\}$ , then  $P(t)$  is strong persistent in mean a.s., and  $Z(t)$  is weak persistent in mean a.s., where  $\Theta = (\gamma + D_1 + \frac{\sigma_1^2}{2} + \frac{cC_1}{b} + \frac{\sigma_2^2 C_1^2}{2b^2})$ .

**Proof.** (i) By the positiveness of the solution  $(N(t), P(t), Z(t))$  of system (1.2).

An application of Itô's formula to  $\ln P(t)$  yields

$$\begin{aligned} d \ln P(t) &= \left[ \frac{aN}{k+N} - \frac{cZ}{P^2/i+P+b} - (\gamma + D_1) - \frac{\sigma_1^2 N^2}{2(k+N)^2} - \frac{\sigma_2^2 Z^2}{2(P^2/i+P+b)^2} \right] dt \\ &\quad + \frac{\sigma_1 N}{k+N} dB_1(t) - \frac{\sigma_2 Z}{P^2/i+P+b} dB_2(t) \\ &\leq \left[ \frac{aN}{k+N} - (\gamma + D_1) - \frac{\sigma_1^2 N^2}{2(k+N)^2} \right] dt + \frac{\sigma_1 N}{k+N} dB_1(t) - \frac{\sigma_2 Z}{P^2/i+P+b} dB_2(t). \end{aligned} \quad (3.1)$$

Integrating both sides of (3.1) from 0 to  $t$  yields

$$\begin{aligned} \ln P(t) &= -\frac{\sigma_1^2}{2} \int_0^t \left( \frac{N}{k+N} - \frac{a}{\sigma_1^2} \right)^2 ds - (\gamma + D_1)t + \frac{a^2}{2\sigma_1^2} t + M_1(t) - M_2(t) + \ln P(0) \\ &\leq -(\gamma + D_1)t + \frac{a^2}{2\sigma_1^2} t + M_1(t) - M_2(t) + \ln P(0) \end{aligned} \quad (3.2)$$

where  $M_1(t) = \int_0^t \frac{\sigma_1 N}{k+N} dB_1(s)$ ,  $M_2(t) = \int_0^t \frac{\sigma_2 Z}{P^2/i+P+b} dB_2(s)$ . Dividing by  $t$  on both sides of (3.2), we obtain

$$\frac{\ln P(t)}{t} \leq -(\gamma + D_1 - \frac{a^2}{2\sigma_1^2}) + \frac{M_1(t)}{t} - \frac{M_2(t)}{t} + \frac{\ln P(0)}{t}. \quad (3.3)$$



Obviously,  $\limsup_{t \rightarrow \infty} \frac{\ln P(0)}{t} = 0$  *a.s.*, combining with Lemma 3.2, since  $\sigma_1 > \frac{a}{\sqrt{2(\gamma+D_1)}}$ , taking the limit superior of both sides of (3.3) leads to

$$\limsup_{t \rightarrow \infty} \frac{\ln P(t)}{t} \leq -(\gamma + D_1 - \frac{a^2}{2\sigma_1^2}) < 0.$$

which implies  $\lim_{t \rightarrow \infty} P(t) = 0$  *a.s.*. Then for any  $\varepsilon > 0$ , there exist  $T > 0$  and a set  $\Omega_\varepsilon$  such that  $P(\Omega_\varepsilon) > 1 - \varepsilon$  and  $P(t) < \varepsilon$  for  $t > T$ ,  $\omega \in \Omega_\varepsilon$ .

An application of *Itô's* formula to the third equation of system (1.2), we have

$$\begin{aligned} d \ln Z(t) &\leq (\frac{c}{b}P - \varepsilon - D_2 - \frac{\sigma_2^2 P^2}{2(P^2/i + P + b)^2})dt + \frac{\sigma_2 P}{(P^2/i + P + b)}dB_2(t) \\ &\leq (\frac{c}{b}\varepsilon - \varepsilon - D_2)dt + \frac{\sigma_2 P}{(P^2/i + P + b)}dB_2(t). \end{aligned} \quad (3.4)$$

Integrating both sides of (3.4) from 0 to  $t$ , and dividing by  $t$  on both sides, yields

$$\frac{\ln Z(t)}{t} \leq (\frac{c}{b}\varepsilon - \varepsilon - D_2) + \frac{M_3(t)}{t} + \frac{\ln Z(0)}{t}. \quad (3.5)$$

where  $M_3(t) = \int_0^t \frac{\sigma_2 P}{(P^2/i + P + b)}dB_2(s)$ , analogously, by using of the Lemma 3.2, and combining with  $\limsup_{t \rightarrow \infty} \frac{\ln Z(0)}{t} = 0$  *a.s.*, and a random small constant  $\varepsilon$  such that  $\frac{c}{b}\varepsilon - \varepsilon - D_2 < 0$ , we can get

$$\limsup_{t \rightarrow \infty} \frac{\ln Z(t)}{t} \leq (\frac{c}{b}\varepsilon - \varepsilon - D_2) < 0.$$

which implies  $\lim_{t \rightarrow \infty} Z(t) = 0$  *a.s.*.

(ii) Combining with (2.2), and application of *Itô's* formula to  $\ln P(t)$  yields

$$d \ln P(t) \geq [\frac{aN}{k + C_1} - \gamma - D_1 - \frac{\sigma_1^2}{2} - \frac{cC_1}{b} - \frac{\sigma_2^2 C_1^2}{2b^2}]dt + \frac{\sigma_1 N}{k + N}dB_1(t) - \frac{\sigma_2 Z}{P^2/i + P + b}dB_2(t). \quad (3.6)$$

Integrating both sides of (3.6) from 0 to  $t$ , and dividing by  $t$  on both sides, we have

$$\frac{\ln P(t)}{t} \geq \frac{a}{k + C_1} \frac{1}{t} \int_0^t N dt - \Theta + \Psi. \quad (3.7)$$

where  $\Theta = (\gamma + D_1 + \frac{\sigma_1^2}{2} + \frac{cC_1}{b} + \frac{\sigma_2^2 C_1^2}{2b^2})$ ,  $\Psi = \frac{M_1(t)}{t} - \frac{M_2(t)}{t} + \frac{\ln P(0)}{t}$ .

Besides, by (2.1), we have

$$dW(t) \leq (DN^0 - \hat{D}W)dt.$$

Integrating both side of above inequation from 0 to  $t$ , we obtain

$$N(t) + P(t) + Z(t) - (N(0) + P(0) + Z(0)) \leq DN^0 t - \hat{D} \int_0^t N dt - \hat{D} \int_0^t P dt - \hat{D} \int_0^t Z dt. \quad (3.8)$$

According to system (1.2), (2.1)and (2.2), taking the limit superior of both side of (3.8), we have, for all  $t \geq 0$ ,

$$\frac{\hat{D}}{t} \int_0^t N dt \leq DN^0 - \frac{\hat{D}}{t} \int_0^t P dt. \quad (3.9)$$

By (3.7) and (3.9), we can get

$$\frac{\ln P(t)}{t} \geq \frac{a}{k+C_1} \frac{DN^0}{\hat{D}} - \Theta - \frac{a}{k+C_1} \frac{1}{t} \int_0^t P(s) ds + \Psi$$

Notice that  $\limsup_{t \rightarrow \infty} \frac{\ln Z(0)}{t} = 0$  a.s., and by Lemma 3.1, we can have

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t P(s) ds \geq \frac{DN^0}{\hat{D}} - \frac{\Theta(k+C_1)}{a} > 0 \text{ a.s..}$$

By Definition 3.1,  $P(t)$  is strong persistent in mean a.s..

Similarly, an application of  $It\delta'$ 's formula to  $\ln Z(t)$  yields

$$d \ln Z(t) \leq \left( \frac{c}{b} P - \varepsilon - D_2 \right) dt + \frac{\sigma_2 P Z}{(P^2/i) + P + b} dB_2(t). \quad (3.10)$$

Integrating both side of (3.10) from 0 to  $t$ , then dividing by  $t$  on both sides, we can obtain

$$\frac{\ln Z(t) - \ln Z(0)}{t} \leq \frac{c}{b} \int_0^t P dt - (\varepsilon + D_2) + \frac{M_3(t)}{t}. \quad (3.11)$$

Similarly, by (3.9), we can obtain

$$\frac{\hat{D}}{t} \int_0^t P dt \leq DN^0 - \frac{\hat{D}}{t} \int_0^t Z dt. \quad (3.12)$$

By (3.11) and (3.12), we can get

$$\frac{\ln Z(t)}{t} \leq \frac{\ln Z(0)}{t} + \frac{cDN^0}{b\hat{D}} - (\varepsilon + D_2) - \frac{c}{bt} \int_0^t Z dt + \frac{M_3(t)}{t}, \quad (3.13)$$

by the strong law of large numbers, we obtain

$$\lim_{t \rightarrow \infty} \frac{M_3(t)}{t} = 0 \text{ a.s..}$$

and obviously  $\lim_{t \rightarrow \infty} \frac{\ln Z(0)}{t} = 0$  a.s.. Using Lemma 3.2, it follows from (3.10) that

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t Z dt \leq \frac{b \left( \frac{cDN^0}{b\hat{D}} - \varepsilon - D_2 \right)}{c} \text{ a.s..}$$

Since  $\frac{DN^0}{D} > \frac{b(\varepsilon+D_2)}{c}$ , so  $Z(t)$  is weak persistent in mean a.s., this completes the proof.

#### 4. The asymptotic behavior of the positive equilibrium point

If  $\varepsilon + D_2 < c, \gamma + D_1 < \frac{aN^*}{k+N^*}$ , then the corresponding deterministic system of (1.2) exist the interior equilibrium point  $E^* = (N^*, P^*, Z^*)$ , however, the point  $E^* = (N^*, P^*, Z^*)$  is not the equilibrium point of system (1.2), therefore, it is necessary to consider the asymptotic behavior of stochastic system near the positive equilibrium point of deterministic system.

**Theorem 4.1.** *Suppose that  $(N(t), P(t), Z(t))$  is any solution of system (1.2) with initial value  $(N(0), P(0), Z(0))$ , if the coefficients of system (1.2) satisfy  $\varepsilon + D_2 < c, \gamma + D_1 < \frac{aN^*}{k+N^*}$  and  $\eta_1 > 0, \eta_2 > 0, \eta_3 > 0$ , then the system will be weakly persistent in the mean under the expectation around the positive equilibrium point  $E^* = (N^*, P^*, Z^*)$  in the long run, mathematically, it says that*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t [\eta_1(N(s) - N^*)^2 + \eta_2(P(s) - P^*)^2 + \eta_3(Z(s) - Z^*)^2] ds \leq \mathbb{k},$$

where

$$\begin{aligned} \eta_1 &= D - \frac{(\gamma+2D-\gamma_1+\frac{a}{k+N^*})\varepsilon}{2} - \frac{(\varepsilon+D+D_2-\varepsilon_1)}{2\varepsilon}, \\ \eta_2 &= \gamma - \gamma_1 + D_1 - 2\sigma_1^2 + \frac{cZ^*}{b^2} - \frac{cP^*Z^*}{i} - \frac{(\gamma+2D-\gamma_1+\frac{a}{k+N^*})}{2\varepsilon} - \frac{(\varepsilon-\varepsilon_1+\gamma-\gamma_1+D+D_2+\frac{c(P^*)^2}{b^2i}+\frac{cP^*}{b^2}+\frac{2c}{b})}{2\varepsilon}, \\ \eta_3 &= (\varepsilon - \varepsilon_1 + D_2 - 2\sigma_2^2) - \frac{(\varepsilon+D+D_2-\varepsilon_1)\varepsilon}{2} - \frac{(\varepsilon-\varepsilon_1+\gamma-\gamma_1+D+D_2+\frac{c(P^*)^2}{b^2i}+\frac{cP^*}{b^2}+\frac{2c}{b})\varepsilon}{2}, \\ \mathbb{k} &= 2\sigma_1^2(P^*)^2 + 2\sigma_2^2(Z^*)^2 + \frac{\sigma_1^2P^*}{2} + \frac{\sigma_2^2P^*C_1^2}{2b^2} + \frac{\sigma_2^2Z^*}{2}. \end{aligned}$$

**Proof.** Because of the condition of  $\varepsilon + D_2 < c, \gamma + D_1 < \frac{aN^*}{k+N^*}$ , then the corresponding deterministic system of (1.2) exist the interior equilibrium point  $E^* = (N^*, P^*, Z^*)$ , and satisfy the following conditions

$$\begin{cases} D(N^0 - N^*) - \frac{aPN^*}{k+N^*} + \gamma_1P^* + \varepsilon_1Z^* = 0, \\ \frac{aN^*}{k+N^*} - \frac{cZ^*}{((P^*)^2/i)+P^*+b} - (\gamma + D_1) = 0, \\ \frac{cP^*}{((P^*)^2/i)+P^*+b} - \varepsilon - D_2 = 0, \end{cases} \quad (4.1)$$

Define a  $C^2$ -function  $V : R^3 \rightarrow R_+$  by

$$\begin{aligned} V(N, P, Z) &= \frac{1}{2}(N + P + Z - N^* - P^* - Z^*)^2 + (P - P^* - P^* \ln \frac{P}{P^*}) + (Z - Z^* - Z^* \ln \frac{Z}{Z^*}) \\ &= V_1 + V_2 + V_3, \end{aligned}$$

where  $V_1 = \frac{1}{2}(N + P + Z - N^* - P^* - Z^*)^2$ ,  $V_2 = P - P^* - P^* \ln \frac{P}{P^*}$ ,  $V_3 = Z - Z^* - Z^* \ln \frac{Z}{Z^*}$ .

Applying *Itô's* formula, we can obtain

$$dV(N, P, Z) = dV_1 + dV_2 + dV_3,$$

in detail

$$\begin{aligned} dV_1 &= (N + P + Z - N^* - P^* - Z^*)(dN + dP + dZ) + \frac{1}{2}(dN + dP + dZ)^2 \\ &= (N + P + Z - N^* - P^* - Z^*)(DN^0 - DN + \gamma_1 P + \varepsilon_1 Z - \gamma P - D_1 P \\ &\quad - \varepsilon Z - D_2 Z + \frac{2\sigma_1^2 P^2 N^2}{2(k+N)^2} + \frac{2\sigma_2^2 P^2 Z^2}{2(P^2/i + P + b)^2})dt, \end{aligned}$$

making use of (4.1), and the elementary inequalities  $(a + b)^2 \leq 2a^2 + 2b^2$ ,  $2ab \leq a^2 + b^2$ , we have

$$\begin{aligned} dV_1 &\leq [-D(N - N^*)^2 - (\gamma - \gamma_1 + D_1 - 2\sigma_1^2)(P - P^*)^2 - (\varepsilon - \varepsilon_1 + D_2 - 2\sigma_2^2)(Z - Z^*)^2 \\ &\quad - (\gamma + 2D - \gamma_1)(P - P^*)(N - N^*) - (\varepsilon + D + D_2 - \varepsilon_1)(N - N^*)(Z - Z^*) \\ &\quad - (\varepsilon - \varepsilon_1 + \gamma - \gamma_1 + D + D_2)(P - P^*)(Z - Z^*) + 2\sigma_1^2(P^*)^2 + 2\sigma_2^2(Z^*)^2]dt, \end{aligned}$$

noting that

$$\begin{aligned} &- (\gamma + 2D - \gamma_1)(P - P^*)(N - N^*) - (\varepsilon + D + D_2 - \varepsilon_1)(N - N^*)(Z - Z^*) \\ &- (\varepsilon - \varepsilon_1 + \gamma - \gamma_1 + D + D_2)(P - P^*)(Z - Z^*) \\ &\leq (\gamma + 2D - \gamma_1)|P - P^*||N - N^*| + (\varepsilon + D + D_2 - \varepsilon_1)|N - N^*||Z - Z^*| \\ &+ (\varepsilon - \varepsilon_1 + \gamma - \gamma_1 + D + D_2)|P - P^*||Z - Z^*|, \end{aligned}$$

we have

$$\begin{aligned} dV_1 &\leq [-D(N - N^*)^2 - (\gamma - \gamma_1 + D_1 - 2\sigma_1^2)(P - P^*)^2 - (\varepsilon - \varepsilon_1 + D_2 - 2\sigma_2^2)(Z - Z^*)^2 \\ &\quad (\gamma + 2D - \gamma_1)|P - P^*||N - N^*| + (\varepsilon + D + D_2 - \varepsilon_1)|N - N^*||Z - Z^*| \\ &\quad + (\varepsilon - \varepsilon_1 + \gamma - \gamma_1 + D + D_2)|P - P^*||Z - Z^*| + 2\sigma_1^2(P^*)^2 + 2\sigma_2^2(Z^*)^2]dt. \end{aligned}$$

Similarly,

$$\begin{aligned}
dV_2 &= \left(1 - \frac{P^*}{P}\right)dP + \frac{P^*}{2P^2}(dP)^2 \\
&\leq (P - P^*)\left(\frac{aN}{k+N} - \frac{aN^*}{k+N^*} + \frac{cZ^*}{(P^*)^2/i + P^* + b} - \frac{cZ}{P^2/i + P + b}\right)dt \\
&\quad + \frac{P^*}{2}\left(\frac{\sigma_1^2 N^2}{(k+N)^2} + \frac{\sigma_2^2 Z^2}{(P^2/i + P + b)^2}\right)dt + \frac{\sigma_1(P - P^*)N}{k+N}dB_1(t) - \frac{\sigma_2((P - P^*))Z}{P^2/i + P + b}dB_2(t) \\
&\leq \left[\frac{a}{k+N^*}|P - P^*||N - N^*| + \frac{c}{b^2}\left(\frac{(P^*)^2}{i} + P^* + b\right)|P - P^*||Z - Z^*| - \frac{c}{b^2}Z^*(P - P^*)^2\right. \\
&\quad \left. + \frac{\sigma_1^2 P^*}{2} + \frac{\sigma_2^2 P^* C_1^2}{2b^2}\right]dt + \frac{\sigma_1(P - P^*)N}{k+N}dB_1(t) - \frac{\sigma_2((P - P^*))Z}{P^2/i + P + b}dB_2(t),
\end{aligned}$$

and

$$\begin{aligned}
dV_3 &= \left(1 - \frac{Z^*}{Z}\right)dZ + \frac{Z^*}{2Z^2}(dZ)^2 \\
&\leq (Z - Z^*)\left(\frac{cP}{P^2/i + P + b} - \frac{cP^*}{(P^*)^2/i + P^* + b}\right)dt + \frac{\sigma_2^2 Z^*}{2}dt + \frac{\sigma_2(Z - Z^*)P}{P^2/i + P + b}dB_2(t) \\
&\leq \left[\frac{c}{b}|P - P^*||Z - Z^*| + \frac{cP^*Z^*}{i}(P - P^*)^2 + \frac{\sigma_2^2 Z^*}{2}\right]dt + \frac{\sigma_2(Z - Z^*)P}{P^2/i + P + b}dB_2(t).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
dV &= dV_1 + dV_2 + dV_3 \\
&\leq [-D(N - N^*)^2 - (\gamma - \gamma_1 + D_1 - 2\sigma_1^2 + \frac{cZ^*}{b^2} - \frac{cP^*Z^*}{i})(P - P^*)^2 - (\varepsilon - \varepsilon_1 + D_2 - 2\sigma_2^2)(Z - Z^*)^2 \\
&\quad + (\gamma + 2D - \gamma_1 + \frac{a}{k+N^*})|P - P^*||N - N^*| + (\varepsilon + D + D_2 - \varepsilon_1)|N - N^*||Z - Z^*| \\
&\quad + (\varepsilon - \varepsilon_1 + \gamma - \gamma_1 + D + D_2 + \frac{c(P^*)^2}{b^2 i} + \frac{c}{b^2}P^* + \frac{2c}{b})|P - P^*||Z - Z^*| + 2\sigma_1^2(P^*)^2 + 2\sigma_2^2(Z^*)^2 \\
&\quad + \frac{\sigma_1^2 P^*}{2} + \frac{\sigma_2^2 P^* C_1^2}{2b^2} + \frac{\sigma_2^2 Z^*}{2}]dt + \frac{\sigma_1(P - P^*)N}{k+N}dB_1(t) + \frac{\sigma_2((P^*Z - PZ^*))}{P^2/i + P + b}dB_2(t),
\end{aligned}$$

by Young's inequality, we have

$$\begin{aligned}
|P - P^*||N - N^*| &\leq \left[\frac{(P - P^*)^2}{2\varepsilon} + \frac{\varepsilon(N - N^*)^2}{2}\right], \\
|N - N^*||Z - Z^*| &\leq \left[\frac{(N - N^*)^2}{2\varepsilon} + \frac{\varepsilon(Z - Z^*)^2}{2}\right], \\
|P - P^*||Z - Z^*| &\leq \left[\frac{(P - P^*)^2}{2\varepsilon} + \frac{\varepsilon(Z - Z^*)^2}{2}\right],
\end{aligned}$$

then

$$dV \leq [-\eta_1(N - N^*)^2 - \eta_2(P - P^*)^2 - \eta_3(Z - Z^*)^2 + \mathbb{k}]dt + \frac{\sigma_1(P - P^*)N}{k + N}dB_1(t) + \frac{\sigma_2((P^*Z - PZ^*))}{P^2/i + P + b}dB_2(t), \quad (4.2)$$

$$\text{where } \eta_1 = D - \frac{(\gamma + 2D - \gamma_1 + \frac{a}{k + N^*})\varepsilon}{2} - \frac{(\varepsilon + D + D_2 - \varepsilon_1)}{2\varepsilon},$$

$$\eta_2 = \gamma - \gamma_1 + D_1 - 2\sigma_1^2 + \frac{cZ^*}{b^2} - \frac{cP^*Z^*}{i} - \frac{(\gamma + 2D - \gamma_1 + \frac{a}{k + N^*})}{2\varepsilon} - \frac{(\varepsilon - \varepsilon_1 + \gamma - \gamma_1 + D + D_2 + \frac{c(P^*)^2}{b^2i} + \frac{cP^*}{b^2} + \frac{2c}{b})}{2\varepsilon},$$

$$\eta_3 = (\varepsilon - \varepsilon_1 + D_2 - 2\sigma_2^2) - \frac{(\varepsilon + D + D_2 - \varepsilon_1)\varepsilon}{2} - \frac{(\varepsilon - \varepsilon_1 + \gamma - \gamma_1 + D + D_2 + \frac{c(P^*)^2}{b^2i} + \frac{cP^*}{b^2} + \frac{2c}{b})\varepsilon}{2},$$

$$\mathbb{k} = 2\sigma_1^2(P^*)^2 + 2\sigma_2^2(Z^*)^2 + \frac{\sigma_1^2P^*}{2} + \frac{\sigma_2^2P^*C_1^2}{2b^2} + \frac{\sigma_2^2Z^*}{2}.$$

Integrating (4.2) from 0 to  $t$  on both sides results in

$$V(N(t), P(t), Z(t)) - V(N(0), P(0), Z(0)) \leq \int_0^t [-\eta_1(N(s) - N^*)^2 - \eta_2(P(s) - P^*)^2 - \eta_3(Z(s) - Z^*)^2]ds + \mathbb{k}t + M(t), \quad (4.3)$$

where  $M(t) = \int_0^t \frac{\sigma_1(P - P^*)N}{k + N}dB_1(s) + \frac{\sigma_2((P^*Z - PZ^*))}{P^2/i + P + b}dB_2(s)$ . Taking expectation on both sides of (4.3) leads to

$$EV(N(t), P(t), Z(t)) - EV(N(0), P(0), Z(0)) \leq E \int_0^t [-\eta_1(N(s) - N^*)^2 - \eta_2(P(s) - P^*)^2 - \eta_3(Z(s) - Z^*)^2]ds + \mathbb{k}t.$$

Therefore, we have

$$\limsup \frac{1}{t} E \int_0^t [\eta_1(N(s) - N^*)^2 + \eta_2(P(s) - P^*)^2 + \eta_3(Z(s) - Z^*)^2]ds \leq \mathbb{k}.$$

This completes the proof.

## 5. Conclusion

In this paper, we consider a stochastic three-species nutrient-phytoplankton-zooplankton model. Firstly, We verify that the system admits a unique positive global solution starting from any positive initial value. Then we obtain the sufficient conditions of population extinction and persistence in the mean, the results show that if  $\sigma_1 > \frac{a}{\sqrt{2(\gamma + D_1)}}$ , then  $P(t)$  and  $Z(t)$  are all extinct; if  $\sigma_1, \sigma_2$  are small enough such that  $\frac{DN^0}{D} > \max\{\frac{\Theta(k + C_1)}{a}, \frac{b(\varepsilon + D_2)}{c}\}$ , then  $P(t)$  is strong persistent in mean and  $Z(t)$  is weak persistent in mean. Our results also imply that weaker white noise

will strengthen stability of the system, while stronger white noise will result in extinction of population. Furthermore, we obtain the asymptotic behavior of stochastic system near the positive equilibrium point of deterministic system, that is, the stochastic system is weakly persistent in the mean under some conditions.

### **Conflict of Interests**

The authors declare that there is no conflict of interests.

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### **REFERENCES**

- [1] A. EL. Abdllaoui, J. Chattopadhyay, O. Arino, Comparisons, by models, of some basic mechanisms acting on the dynamics of the zooplankton-toxic phytoplankton system, *Math. Models Methods Appl. Sci.* 12 (10) (2002) 1421-1251.
- [2] B. Mukhopadhyay, R. Bhattacharyya, Modelling phytoplankton allelopathy in a nutrient-plankton model with spatial heterogeneity, *Ecol. Model.* 198 (2006) 163-173.
- [3] J. Chattopadhyay, R. Sarkar, S. Mandal, Toxin producing plankton may act as a biological control for planktonic blooms-Field study and mathematical modelling, *J. Theor. Biol.* 215 (3) (2002) 333-344.
- [4] T. Saha, M. Bandyopadhyay, Dynamical analysis of toxin producing phytoplankton-zooplankton interactions, *Nonlinear Anal., Real World Appl.* 10 (1) (2009) 314-332.
- [5] S. Jang, J. Baglama, L. Wu, Dynamics of phytoplankton-zooplankton systems with toxin producing phytoplankton, *Appl. Math. Comput.* 227 (2014) 717-740.
- [6] Y. Wang, H. Wang, W. Jiang, Hopf-transcritical bifurcation in toxic phytoplankton-zooplankton model with delay, *J. Math. Anal. Appl.* 415 (2) (2014) 574-594.
- [7] S. R. J. Janga, J. Baglama, J. Rick, Nutrient-phytoplankton-zooplankton-models with a toxin, *Math. Comput. Model.* 43 (1-2) (2006) 105-118.
- [8] R. R. Sarkar, S. Pal, J. Chattopadhyay, Role of two toxin-producing plankton and their effect on phytoplankton-zooplankton system-a mathematical study supported by experimental findings, *Biosystem*, 80 (2005) 11-23.
- [9] S. Khare, O.P. Misra, J. Dhar, Role of toxin producing phytoplankton on a plankton ecosystem, *Nonlinear Anal., Hybrid Syst.* 4 (2010) 496-502.
- [10] A. Chatterjee, S. Pal, S. Chatterjee, Bottom up and top down effect on toxin producing phytoplankton and its consequence on the formation of plankton bloom, *Appl. Math. Comput.* 218 (2011) 3387-3398.

- [11] Y. Zhao, S. Yuan, T. Zhang, The stationary distribution and ergodicity of a stochastic phytoplankton allelopathy model under regime switching, *Commun. Nonlinear Sci. Numer. Simulat.* 37 (2016) 131-142.
- [12] C. Liu, L. Wang, Q. Zhang, Y. Yan, Dynamical analysis in a bioeconomic phytoplankton zooplankton system with double time delays and environmental stochasticity, *Physica A*, 482 (2017) 682-698.
- [13] C. Liu, L. Yu, Q. Zhang, Y. Li, Dynamic analysis of a hybrid bioeconomic plankton system with double time delays and stochastic fluctuations, *Appl. Math. Comput.* 316 (2018) 115-137.
- [14] Y. Zhao, S. Yuan, T. Zhang, Stochastic periodic solution of a non-autonomous toxic-producing phytoplankton allelopathy model with environmental fluctuation, *Commun. Nonlinear Sci. Numer. Simulat.* 44 (2017) 266-276.
- [15] S. Jang, E. Allen, Deterministic and stochastic nutrient-phytoplankton-zooplankton models with periodic toxin producing phytoplankton, *Appl. Math. Comput.* 271 (2015) 52-67.
- [16] P. Mandal, S. Abbas, M. Banerjee, A comparative study of deterministic and stochastic dynamics for a non-autonomous allelopathic phytoplankton model, *Appl. Math. Comput.* 238 (2014) 300-318.
- [17] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, New York: Wiley, 1972.
- [18] A. Friedman, *Stochastic Differential Equations and their Applications*, New York: Academic Press, 1976.
- [19] W. Zuo, D. Jiang, Stationary distribution and periodic solution for stochastic predator-prey systems with nonlinear predator harvesting, *Commun. Nonlinear Sci. Numer. Simulat.* 36 (2016) 65-68.
- [20] M. Liu, K. Wang, Survival analysis of a stochastic cooperation system in a polluted environment, *J. Biol. Syst.* 19 (2011) 183-204.
- [21] Z. Chang, X. Meng, X. Lu, Analysis of a novel stochastic SIRS epidemic model with two different saturated incidence rates, *Physica A*, 472 (2017) 103-116.