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THE PERIODIC SOLUTIONS OF THE IMPULSIVE STATE FEEDBACK DYNAMICAL SYSTEM

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Abstract. This work reviews some recent advances on the periodic solution of the semi-continuous dynamical system, which consists of two parts: the stability of periodic solution, the homoclinic and heteroclinic bifurcations. In the first part, the order-1 periodic solution is classified into three types at first. Then for type 1 periodic solution, by means of square approximation and a series of switched systems, the periodic solution is approximated by a series of continuous hybrid limit cycles. Hence, a general stability criteria are obtained by the method of successor function similar to the analysis in the ordinary differential equation. In the second part, the homoclinic and heteroclinic cycles are found for some specific parameter value in the prey-predator system. When the parameter varies, the cycles disappear and the system bifurcates an unique order-1 periodic solution. The geometry theory and the successor function are applied to obtain these bifurcations. Finally, we discuss some possible future trends in the periodic solution of the semi-continuous dynamical systems.

Keywords: semi-continuous dynamical systems; impulsive state feedback dynamical systems; periodic solution; successor function; stability; homoclinic and heteroclinic bifurcation.

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1. Introduction

Impulsive control methods have many important applications in various fields such as biology, engineering, medicine etc[1]. There are different kinds of impulses controls pointed out in [2, 3, 1]. Most studies focus on the systems with impulse at fixed times. Recently, the systems with impulses depending on the state (not on the time) have been more attractive and received more attention, which can be formulated as semi-continuous dynamical systems[4]. For convenience, we call the impulsive state feedback system as semi-continuous dynamical system in the following.

In this study, we aim to review the advances of the periodic solutions of of semi-continuous dynamical systems since 2010, particularly the existence and stability of periodic solution and the homoclinic and heteroclinic bifurcations [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39].

An earlier existence result of the periodic solution was obtained by constructing a Bendixson region proposed by Zeng and Chen on 2006. The method of successor function is a more convenient and popular method[4], which is applied to prove the existence in various fields[14, 16, 19, 24, 25, 35, 34]. For the stability of the periodic solutions, the famous Analogue of Poincaré Criterion[2, 40] is applied widely, yet it is not convenient to calculate due to the dependence on initial conditions. E. M. Bonotto et al. investigated the Lyapunov stability and Poisson stability of closed set in semi dynamical systems[41, 42], and extended the Poincaré-Bendixson and LaSalle's theorems to the semi-continuous dynamical systems[36, 43]. The more popular and convenient method, the method of successor function, is also used to study the stability of periodic solution[4]. In the references [26, 25, 35, 34], the authors applied this method to obtain some stability results for the particular semi-continuous dynamical systems. Based on these results, the authors in [44] classified the order-1 periodic solution into three types at first, and then presented a convenient and general stability criteria of the convex periodic solution by square approximation and a series of switched systems.

Unlike the rich results in the bifurcation theory of the ordinary differential equation, there is litter results concerning the impulsive differential equations, especially about the semi-continuous dynamical systems[15, 22, 21, 24, 28]. In this paper, we mainly review the homoclinic bifurcation [24, 28] and heteroclinic bifurcation[21] of a predator-prey system investigated by the geometry theory of semi-continuous dynamical systems.

This paper is organized as follows. Section 2 introduces some preliminary knowledge about the semi-continuous dynamical systems. Section 3 presents the recent results about the existence of periodic solution. The stability of order-1 periodic solution, especially the stability criterion established by a series of approximation hybrid systems is presented in Section 4. The existence of homoclinic and heterclicnic cycles and bifurcations are provided in Section 5. Finally, a brief discussion concludes this paper.

2. Some preliminary knowledge about the semi-continuous dynamical systems

In this section, we introduce some notations and definitions of the semi-continuous dynamical systems, which will be used in the following discussion.

Definition 2.1 ([4, 44, 25]) Consider a two dimensional state dependent impulsive differential equation

$$\left\{\begin{array}{l}\left.\begin{array}{l}\dfrac{dx}{dt}=P(x,y),\\ \dfrac{dy}{dt}=Q(x,y),\end{array}\right\}(x,y)\notin M\{x,y\},\\ \left.\begin{array}{l}\Delta x=\alpha(x,y),\\ \Delta y=\beta(x,y),\end{array}\right\}(x,y)\in M\{x,y\},\end{array}\right. \tag{1}$$

The solution mapping of system (1) is called as the semi-continuous dynamical system denoted by (Ω, f, φ, M), where $(x,y)\in\Omega\subset R^2_+$, $f=f(p,t)$ is the semi-continuous dynamical system mapping with initial point $p=(x_0,y_0)\notin M$, the sets M and N are called the impulse set and phase set, which are lines or curves on R^2_+. The continuous function $\varphi: M\to N$ is called impulse mapping.

Definition 2.2 ([4, 44, 25]) Let $f(p,t):\Omega\to\Omega$ be the semi-continuous dynamical system mapping described by system (1). If there exist points $A\in N$ and $B\in M$, and a time $T>0$ such that

$$B = f(A,T), \text{ and } A = \varphi(B),$$

then, the solution $f(p,T)$ is said to an order-1 periodic solution denoted by $\widehat{AB}$. The orbit $\Gamma = \widehat{AB} \cup \overline{AB}$ is said to an order-1 cycle.

Definition 2.3 ([4, 44, 25]) Suppose the impulse set M and the phase set N in system (1) be straight lines, and the intersection point of phase set N and y axis be E as shown in Fig.1. Then for any point $A \in N$, the distance between the point A and E is denoted by a as the coordinate of point A. The trajectory initiating from A reaches impulse set M at point B, then the impulse function φ maps B to C in phase set N. Point C is called the subsequent point of A, and the coordinate of C is denoted as c. The successor function of A is defined as $F(A) = c - a$.

By a similar way, we can define the order-2 periodic solution and the corresponding successor function.

Definition 2.4 ([4]) Let $f(p,t) : \Omega \to \Omega$ be the semi-continuous dynamical system mapping described by system (1). If there exist points $A \in N$, A_1, B, B_1 and C, time $T_1 > 0$ and $T_2 > 0$ such that

$$A_1 = f(A,T_1) \in M, \; B = \varphi(A_1) \in N, \; B_1 = f(B,T_2) \in M, \text{ and } C = \varphi(B_1) \in N,$$

then, the order-2 successor function of A is defined as $F(A) = c - a$ as shown in Fig. 2, where a and c are denoted as the coordinate of A and C, respectively. If $C = A$, the solution $f(A, T_1 + T_2)$ is said to an order-2 periodic solution with period $T_1 + T_2$.

Definition 2.5 ([44]) The order-1 periodic solution $\Gamma_1 = f(p,t)$ is said to be orbitally stable if there exists $\delta > 0$ and $t_1 > 0$ such that $\rho(f(p_1,t),\Gamma_1) < \varepsilon$, for $t > t_1$ and any $\varepsilon > 0$, where $p_1 \in U(p,\delta) \cap N$.

3. **The existence of periodic solution**

In this section, we present two existence criteria of order-1 periodic solution. The first one is established in [45] for a general planar autonomous impulsive system which is similar with Poincaré-Bendixson theorem of ordinary differential equation. The second one is a general existence criterion by means of successor function[4].

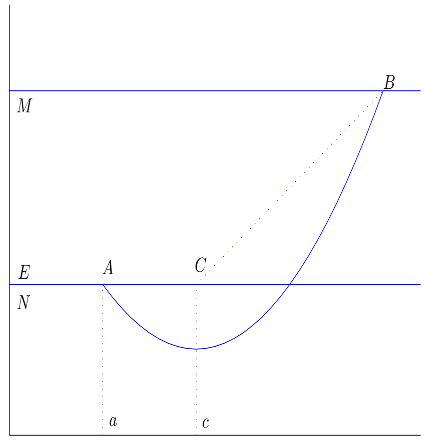

FIGURE 1. The successor function.

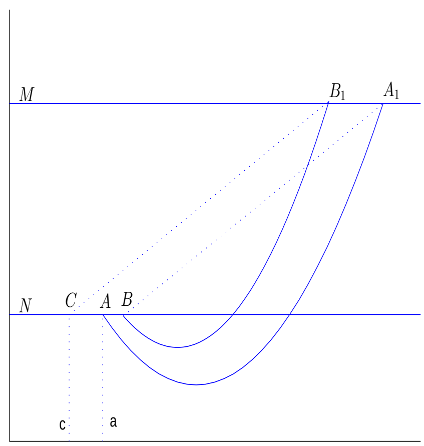

FIGURE 2. The order-2 successor function.

Theorem 3.1([45]) Assume that there exists a bounded closed simple connected region G with boundary $\partial G = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, which has the following properties:

(1) there is no singularity in it;

(2) the boundary $\Gamma_1 = G \cup M$ are non-tangent arc of semi-continuous dynamical system (1);

(3) the boundary $\Gamma_2 \subset \varphi(M)$ is a line segment and satisfies $\varphi(\Gamma_1) \subset \Gamma_2$;

(4) the orbits of system (1) with initial values in $\Gamma_2 \cup \Gamma_1$ will come into the interior of G,

then there must exists an order-1 periodic solution in region G(see Fig. 3).

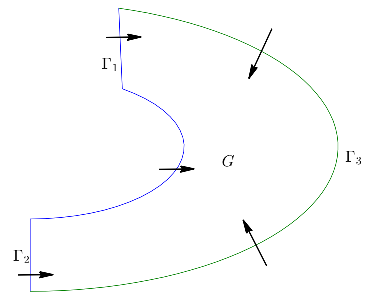

FIGURE 3. The Bendixon Region.

Lemma 3.1([4, 25]) The successor function $F(A)$ is continuous.

Theorem 3.2([4, 25]) In a semi-continuous dynamical system (Ω, f, φ, M), if there are two points A and B in the phase set N such that $F(A) > 0$ and $F(B) < 0$, then there must exist a point $C \in N$ between A and B such that $F(C) = 0$. That is, there is an order-1 periodic solution passing through point C.

The order-2 and order-1 periodic solutions have the following relationship.

Theorem 3.3([4]) If system (1) has an order-2 periodic solution, there must exist an order-1 periodic solution.

4. The stability of order-1 periodic solution

Let $\Gamma = \widehat{AB} \cup \overline{BA}$ denote an order-1 cycle, and assume the trajectory $\widehat{AB}$ with $A \in N$ is not tangent to the impulse set M. The successor point of a point $C \in N$ is E where C is near point A. According to the position between the points A, C and E, the order-1 periodic solution is classified into three types:

(1) Type 1: the order-1 cycle Γ is convex, and the points C and E are at the same side of A as shown in Fig.4.

(2) Type 2: the order-1 cycle Γ is not convex, yet the points C and E are at the same side of A as shown in Fig.5.

(3) Type 3: the points C and E are at different sides of A as shown in Fig.6.

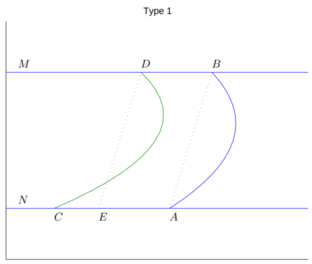

FIGURE 4. Type 1

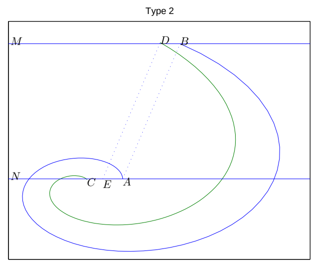

FIGURE 5. Type 2

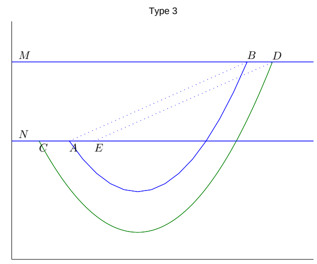

FIGURE 6. Type 3

The type 1 periodic solution is said to be a convex order-1 periodic solution of unilateral asymptotic type. In the following we focus on the general stability criterion of type 1 periodic solution generated by a state dependent impulsive system with linear impulse:

$$
\begin{cases}
\left.\begin{aligned}
\frac{dx}{dt} &= P(x,y),\\
\frac{dy}{dt} &= Q(x,y),
\end{aligned}\right\} x < h,\\
\left.\begin{aligned}
\Delta x &= -\alpha x,\\
\Delta y &= \beta y,
\end{aligned}\right\} x = h.
\end{cases}
\tag{2}
$$

Theorem 4.1([44]) Let $\widehat{AB}$ be a type 1 order-1 periodic solution. For any point $C \in N$ in the neighbourhood of point A, there exist a trajectory through C intersects the phase set N at point E. If for any point C above point A, its successor function satisfies $F(C) < 0$, then the order-1 periodic solution $\widehat{AB}$ is unidirectional stable.

The difficulty in the stability analysis of the semi-continuous dynamical systems is the introduction of impulse function. Hence, some methods of ordinary differential equation cannot be used. The impulse function can be described as

$$
x = f_1(t) = \begin{cases} x_1, & t < t_1,\\ x_2, & t \geq t_1. \end{cases}
\tag{3}
$$

At time t_1, there is an impulse $\Delta x = x_2 - x_1$ as shown in Fig. 7. In order to overcome the difficulty, a piecewise continuous function is introduced to approximate equation (3)

$$x = f_2(t) = \begin{cases} x_1, & t \le t_1, \\ \dfrac{x_2 - x_1}{t_2 - t_1} t + x_1 - \dfrac{x_2 - x_1}{t_2 - t_1} t_1, & t_1 < t < t_2, \\ x_2, & t_2 \le t. \end{cases} \tag{4}$$

Equation (4) is called as a square approximation function of equation (3), as shown in Fig. 8. These two figures show that $f_2(t) \to f_1(t)$ as $t_1 \to t_2$. Based on the similar idea, we will construct the square approximation of system (2).

For an order-1 periodic solution $\widehat{AB}$ with period T, the end points A and B are denoted by $A(x_a, y_a)$ and $B(x_b, y_b)$. The time spend on the line $\overline{AB}$ is zero since B is mapped to A impulsively. In order to use the square approximation of the impulsive map, we assume point B spends time T/n reaching A defined by the following system

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{\alpha n h}{T} \triangleq P_1(x,y), \\ \dfrac{dy}{dt} = \dfrac{n(y_a - y_b)}{T} \triangleq Q_1(x,y), \quad n = 1, 2, \ldots \end{cases} \tag{5}$$

Then we formulate a hybrid system to approximate system (2)

$$\begin{cases} \left.\begin{aligned} \dfrac{dx}{dt} &= P(x,y), \\ \dfrac{dy}{dt} &= Q(x,y), \end{aligned}\right\} \text{initial values in the phase set } x = (1-\alpha)h, \\ \left.\begin{aligned} \dfrac{dx}{dt} &= -\dfrac{\alpha n h}{T} \triangleq P_1(x,y), \\ \dfrac{dy}{dt} &= \dfrac{n(y_a - y_b)}{T} \triangleq Q_1(x,y), \end{aligned}\right\} \text{initial values in the pulse set } x = h. \end{cases} \tag{6}$$

Now, the discontinuous solution of impulse system (2) is approximated by a piecewise continuous solution of system (6). The discontinuous periodic solution is approximated by a a continuous closed periodic cycle.

For simplicity some denotations are introduced

$$Z(x,y),\ X^1(P(x,y), Q(x,y)),\ X^2(P_1(x,y), Q_1(x,y)),$$

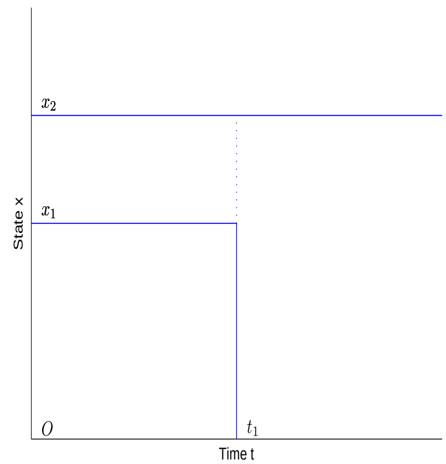

FIGURE 7. The impulse function.

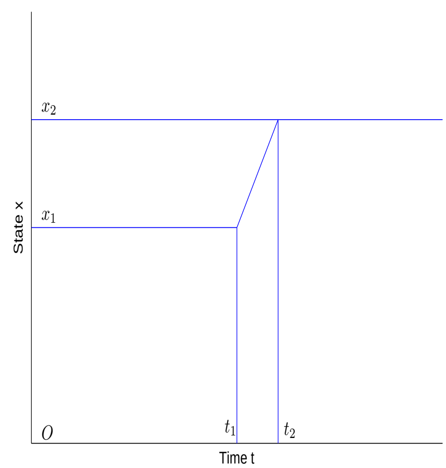

FIGURE 8. The square approximation function.

to rewritten system (6) as

$$\frac{d}{dt}[Z(x,y)] = c_1 X^1 + c_2 X^2, \tag{7}$$

or

$$\begin{cases} \dfrac{dx}{dt} = Z_1(x,y) = c_1 P(x,y) + c_2 P_1(x,y), \\ \dfrac{dy}{dt} = Z_2(x,y) = c_1 Q(x,y) + c_2 Q_1(x,y), \end{cases} \tag{8}$$

where

$$
\begin{cases} c_1 = 1,\ c_2 = 0, \text{ if initial values are in the phase set } x = (1-\alpha)h \\ c_1 = 0,\ c_2 = 1, \text{ if initial values are in the phase set } x = h. \end{cases} \tag{9}
$$

For a periodic cycle $\Gamma = \widehat{AB} \cup \overline{BA}$ of system (2), choosing any point $S_0 \in N$ near point A, there are a series of points $\{S_1, S_2, \ldots, S_k, \ldots\}$, where S_{i+1} is the subsequent point of S_i. Now we construct a coordinate system at the phase set N such that the coordinate of A is zero. Let s_i be the coordinates of the points S_i, $i = 0, 1, \ldots$.

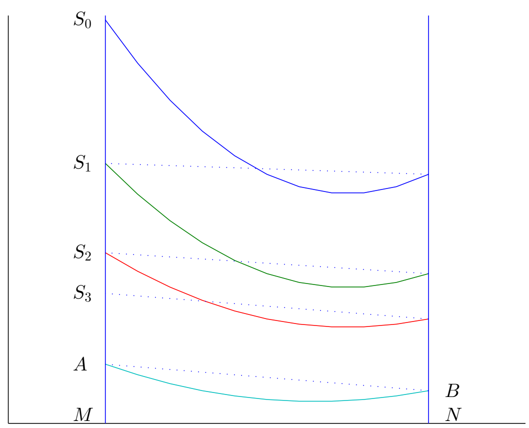

FIGURE 9. The stable order-1 periodic solution.

Lemma 4.1([25, 44]) For any initial point $S_0 \in N$ in the neighborhood of A, if there are a series of points $\{S_0, S_1, \ldots, S_k, \ldots\}$ approach to A when $k \to \infty$, i.e., $\lim_{k\to\infty} s_k = 0$, then the order-1 periodic solution is stable (unidirectional).

Lemma 4.2 (Königs) Suppose that $\bar{s} = f(s)$ is a continuous transform from line segment L to itself, and it has a fixed point $s = 0$. Then the fixed point $s = 0$ is stable(unstable), if the part of curve $\bar{s} = f(s)$ near the origin is in the domain

$$
\left|\frac{\bar{s}}{s}\right| \le 1 - \varepsilon (\ge 1 + \varepsilon), \quad \varepsilon > 0.
$$

Lemma 4.3([44]) Suppose the function $H(x(t), y(t))$ has continuous partial derivatives with respect to x and y, where $x(t)$ and $y(t)$ are continuous functions. The integration of $H(x,y)$

along a closed curve S satisfies

$$\oint_S \frac{dH(x(t),y(t))}{dt}dt = \int_0^T \frac{dH(x(t),y(t))}{dt} = 0,$$

where T is the period of S.

The periodic solution Γ_n generated by hybrid system (6) has period $T+\frac{T}{n}$. For any continuous differential function $D(x(t),y(t))$, we have

Lemma 4.4 Assume that a continuous periodic solution Γ_n square approximates the order-1 periodic solution Γ of unilateral asymptotic type, then

$$\int_\Gamma D(x(t),y(t))dt = \lim_{n\to\infty} \oint_{\Gamma_n} D(x(t),y(t))dt = 0.$$

According to Theorem 4.1, the stability of order-1 periodic solution is if and only if

$$F(S_k) = y_c - y_{S_k} < 0 \tag{10}$$

for any point S_k above point A, where c is the successor point of S_k, y_{S_k} and y_c are the coordinates of points S_k and c, which are shown in Fig. 10. Hence, it is necessary to find a method to calculate the value of $F(S_k)$.

Along the direction of the trajectory $\widehat{AB}$, we establish the curvilinear coordinate (s,n) on point A, where s is the arc length starting from A, n is the length of the normal line, which are shown in Fig. 10. The trajectory passing through S_k intersect the n axis and impulse set M at point a and b, respectively. The trajectory through c intersect n axis at d, where $c=\varphi(b)\in N$. Then we define the successor function of S_k in the curvilinear coordinate system by

$$F^{\oplus}(S_k) = n_d - n_c < 0.$$

Hence, we have the stability condition

$$F(S_k) = y_c - y_{s_k} < 0 \Longleftrightarrow F^{\oplus}(S_k) = n_d - n_c < 0. \tag{11}$$

Taking arc length s as a parameter, the equation of $\widehat{AB}$ and $\overline{AB}$ can be rewritten in the curvilinear coordinate system (s,n)

$$x=\varphi(s),\ y=\psi(s)$$

and

$$x=\varphi_1(s),\ y=\psi_1(s).$$

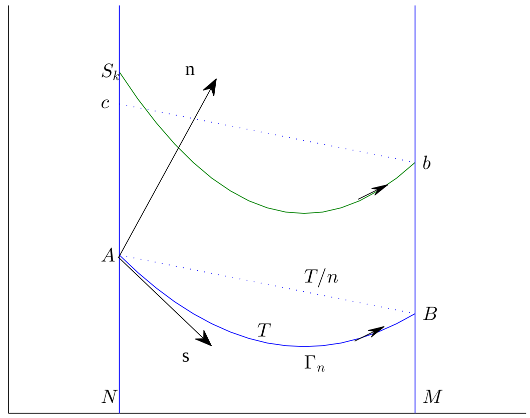

FIGURE 10. The curvilinear coordinate.

Hence the equation of periodic solution Γ_n is

$$\left\{\begin{array}{l} x = \Phi(s) = c_1\varphi(s) + c_2\varphi_1(s), \\ y = \Psi(s) = c_1\psi(s) + c_2\psi_1(s), \end{array}\right. \tag{12}$$

where c_1 and c_2 are defined in equation (9).

For point A, there is a relationship between its rectangular coordinate (x,y) and curvilinear coordinate (s,n)

$$x = \Phi(s) - n\Psi'(s),\ y = \Psi(s) + n\Phi'(s).$$

Let $Z_{10}(x,y)$ and $Z_{20}(x,y)$ be the value of $Z_1(x,y)$ and $Z_2(x,y)$ of periodic solution Γ_n, i.e.,

$$Z_{10}(x,y) = Z_1(\Phi(s),\Psi(s)),\ \ Z_{20}(x,y) = Z_2(\Phi(s),\Psi(s)).$$

From equation (8), it is easy to obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{\Psi'(s) + \Phi'(s)\frac{dn}{ds} + n\Phi''(s)}{\Phi'(s) - \Psi'(s)\frac{dn}{ds} - n\Psi''(s)} \\ &= \frac{Z_2(\Phi(s) - n\Psi'(s), \Psi(s) + n\Phi(s))}{Z_1(\Phi(s) - n\Psi'(s), \Psi(s) + n\Phi(s))} \end{aligned} \tag{13}$$

and

$$\frac{dn}{ds} = \frac{Z_2\Phi'(s) - Z_1\Psi'(s) - n(Z_1\Phi''(s) + Z_2\Psi''(s))}{Z_1\Phi'(s) + Z_2\Psi'(s)} = F(s,n). \tag{14}$$

Suppose the functions Z_1 and Z_2 have continuous partial derivatives, we have

$$\frac{dn}{ds} = F'_n(s,n)\big|_{n=0} n + o(n), \tag{15}$$

where

$$F'_n(s,n)\big|_{n=0} = \frac{Z_{10}^2 Z_{2y0} - Z_{10}Z_{20}(Z_{1y0} + Z_{2x0}) + Z_{20}^2 Z_{1x0}}{(Z_{10}^2 + Z_{20}^2)^{\frac{3}{2}}} = H(s), \tag{16}$$

where Z_{1x0}, Z_{1y0}, Z_{2x0} and Z_{2y0} are the partial derivatives of Z_1 and Z_2 as $n = 0$, respectively.

Hence the first order approximation of equation (14) is

$$\frac{dn}{ds} = H(s)n,$$

and we obtain

$$n = n_0 \exp\Big(\int_0^s H(\tau)d\tau\Big),\ n_0 = n(0). \tag{17}$$

Obviously, if $\int_0^\gamma H(s)ds < 0$, it has $|n(\gamma)| < |n_0|$. Then by Lemma 4.1 and Lemma 4.2, we have the following theorem.

Theorem 4.2 Let $\Gamma_n = \widehat{AB} \cup \overline{BA}$ be the periodic orbit of system (8), γ be the length of Γ_n. The periodic solution Γ_n is stable if

$$\int_0^\gamma H(s)ds < 0. \tag{18}$$

Let $ds = \sqrt{Z_{10}^2 + Z_{20}^2}dt$, the stability condition (18) is rewritten as

(19)

$$\begin{aligned}
\int_0^\gamma H(s)ds &= \int_0^{T+\frac{T}{n}} \frac{Z_{10}^2 Z_{2y0} - Z_{10}Z_{20}(Z_{1y0} + Z_{2x0}) + Z_{20}^2 Z_{1x0}}{Z_{10}^2 + Z_{20}^2} dt \\
&= \int_0^{T+\frac{T}{n}} \Big[Z_{1x0} + Z_{2y0} - \frac{Z_{10}^2 Z_{1y0} + Z_{10}Z_{20}(Z_{1y0} + Z_{2x0}) + Z_{20}^2 Z_{1x0}}{Z_{10}^2 + Z_{20}^2}\Big] dt \\
&= \int_0^{T+\frac{T}{n}} (Z_{1x0} + Z_{2y0})dt - \frac{1}{2}\oint_{\Gamma_n} \frac{d(Z_{10}^2 + Z_{20}^2)}{Z_{10}^2 + Z_{20}^2} dt = \int_0^{T+\frac{T}{n}} (Z_{1x0} + Z_{2y0})dt.
\end{aligned}$$

Theorem 4.3 The periodic solution Γ_n of equation (8) is orbital asymptotical stable if the integral along Γ_n satisfies

$$\int_0^{T+\frac{T}{n}} (Z_{1x0} + Z_{2y0})dt < 0.$$

In addition, according to

$$Z_{1x0} = \frac{\partial Z_1}{\partial x} = \frac{\partial P}{\partial x},\ Z_{2y0} = \frac{\partial Z_2}{\partial y} = \frac{\partial Q}{\partial y},$$

it is easy to get

Theorem 4.4 The periodic solution Γ_n of equation (8) is orbital asymptotical stable if the integral along Γ_n satisfies

$$\int_0^{T+\frac{T}{n}} \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big) dt < 0.$$

Obviously, system (8) approaches (2), i.e.,

$$\Gamma_n \to \Gamma,\ \frac{T}{n} \to 0,\ T + \frac{T}{n} \to T,\ \text{ when } n \to \infty,$$

by Lemma 4.4, we have

Theorem 4.5 If the semi-continuous dynamical system (2) has a type 1 order-1 periodic solution Γ with period T, and the integral along Γ satisfies

$$\int_0^{T} \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big) dt < 0,$$

then the order-1 periodic solution Γ is orbital stable (but not necessarily orbital asymptotical stable).

Corollary 1 If the semi-continuous dynamical system (2) has a type 1 order-1 periodic solution Γ with period T, and the region which contains Γ satisfies

$$\Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big) < 0,$$

then the order-1 periodic solution Γ is orbital stable.

5. The homoclinic and heteroclinic bifurcations in predator-prey models

Unlike the rich results about the bifurcation theory in ordinary differential equations, there is little results about that in semi-continuous dynamical system. In this section, we review some results about the homoclinic and heterclinic bifurcation in the specific predator-prey models[21, 24, 28].

5.1. **The heteroclinic cycle and heteroclinic bifurcation.** A predator prey model with Allee effect is described by

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= r(x-\theta)(1-\frac{x}{K}) - \frac{qxy}{1+qhx}, \\ \frac{dy}{dt} &= \frac{aqy}{1+qhx}(x-b), \end{aligned}\right\} y<\tau, \\ \left.\begin{aligned} \Delta x &= -\alpha x, \\ \Delta y &= -\beta y, \end{aligned}\right\} y=\tau. \end{cases} \tag{20}$$

where $0<\alpha<1$, $0<\beta<1$, $\theta>0$ implies that the prey suffering strong Allee effect. The states $x(t)$ and $y(t)$ represent the population density of prey and predator at time t, respectively. The other parameters and biology background can be found in [24] in detail.

For convenience, in the following we assume $\tau<y^*$ and

$$\text{condition H: } \theta<b<K, \text{ and } \frac{qy^*}{(1+qhb)^2} > r-\frac{2rb}{K}+\frac{r\theta}{K}, \tag{21}$$

where $y^*=(b-\theta)(K-b)(1+qhb)r/(kqb)$.

Lemma 5.1 System (20) is uniformly bounded.

Theorem 5.1 If condition (H) holds, system (20) have two boundary saddle equilibria $N_1(\theta,0)$ and $N_2(K,0)$, and a positive equilibrium N_3. Furthermore, N_3 is locally asymptotically stable.

Let L_1^+ and L_2^- denote the stable manifold of N_1 and unstable manifold of N_2, respectively. Then L_2^- intersects the impulse set $y=\tau$ and phase set $y=(1-\beta)\tau$ at points A and B, and L_1^+ intersects the impulse set $y=\tau$ and phase set $y=(1-\beta)\tau$ at points A_1 and B_1. Both the curves and points are shown in Fig. 11. Since the impulse function $\varphi(x,\alpha)=(1-\alpha)x$ is monotonically increasing with respect to x and monotonically decreasing with respect to α, there must exist a $\alpha^*\in(0,1)$ such that the phase point of A is B_1, i.e., $\varphi(x_A,\alpha^*)=(1-\alpha)x_A=x_{B_1}$.

Hence, there is a closed curve $\Gamma=\widehat{B_1N_1}\cup\overline{N_1N_2}\cup\widehat{N_2A}\cup\overline{AB_1}$ passing through two saddles N_1 and N_2. Hence, system (20) has an order-1 heteroclinic cycle. When $\alpha\in(\alpha^*,1)$, it has $0<(1-\alpha)x_A=x_{A^+}<(1-\alpha^*)x_A=x_{B_1}$. Thus the trajectory from A^+ will cross the $x=0$ axis and has no impulse effect, which implies that system (20) has no periodic solution. When $0<\alpha^0<\alpha^*<1$, system (20) has an unique order-1 periodic solution by the method of successor function.

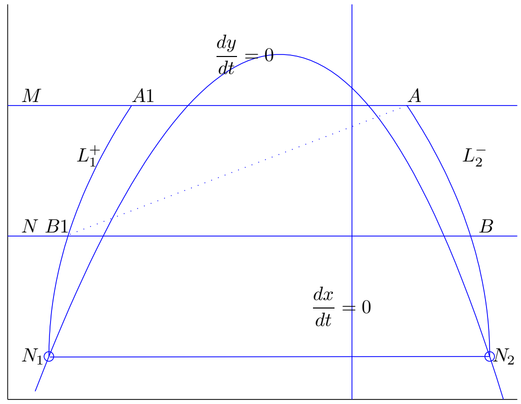

FIGURE 11. The heteroclinic cycle.

Theorem 5.2 If condition (H) holds, there exist α^* and α^0 with $0<\alpha^0<\alpha^*<1$ such that system (20) has an unique order-1 periodic solution for any $\alpha\in(\alpha^0,\alpha^*)$. When $\alpha=\alpha^*$, system (20) has an order-1 heteroclinic cycle. If $\alpha\in(\alpha^*,1)$, system (20) has no order-1 periodic solution.

5.2. **The homoclinic cycle and homoclinic bifurcation.** A predator prey model is described by

$$\left\{\begin{array}{l}\left.\begin{array}{l}\frac{dx}{dt}=x(a-x)(x+k)-xy,\\ \frac{dy}{dt}=y(\theta-d)(x-\lambda)-u(x+k)+\delta(a-x)(x+k)-y,\end{array}\right\}x>h,\\ \left.\begin{array}{l}\Delta x=\tau,\\ \Delta y=-qy,\end{array}\right\}x=h.\end{array}\right. \tag{22}$$

where τ is the constant stocking rate, $0<q<1$ is the harvesting rate, $x(t)$ and $y(t)$ represent the population density of prey and predator at time t, respectively. The other parameters and biology background can be found in [28] in detail. When $\delta=0$ and $4u<(\theta-d)(a-\lambda)^2$, system (22) without impulsive effects has two positive equilibria $E(x_1,y_1)$ and $Q(x_2,y_2)$, where E is a non-saddle singular points and Q is a saddle point.

Lemma 5.2 System (22) is uniformly upper bounded.

Let Γ_A and Γ_B denote the unstable and stable manifold of saddle point Q, L_1 and L_2 denote the isolines $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$. According to Lemma 5.2, Γ_A intersect impulse set M at point $A(x_A, y_A)$. The vertical isolines L_1 passing through Q intersect the set M and N at point $C(x_C, y_C)$ and $D(x_D, y_D)$, respectively. These curves and points are shown in Fig. 12. Since the impulse map $\phi(y,q) = (1-q)y$ is monotonically increasing with respect to y and monotonically decreasing with respect to q, there must exist a $q^* \in (0,1)$ such that the phase point of A is B, i.e., $\phi(y_A, q^*) = (1-q)y_A = y_B$. Hence, there is a cycle $\Gamma = \widehat{BQ} \cup \widehat{QA} \cup \overline{AB}$ passing through Q. That is, system (22) has a homoclinic cycle.

If $q < q^*$ and $y_B \leq \phi(y_c, q)$, $y_D \geq \phi(y_A, q)$, a Bendixion region is constructed to obtain an unique order-1 periodic solution by Theorem 3.1. Hence we have the following theorem.

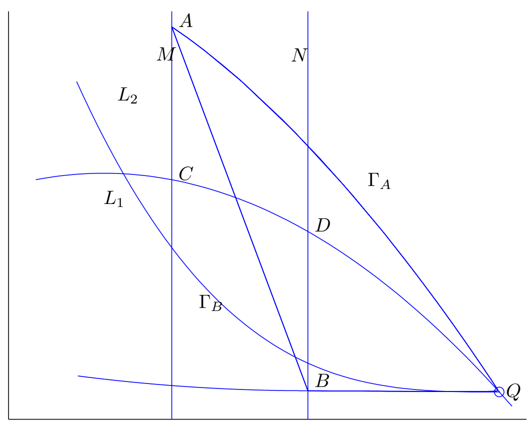

FIGURE 12. The homoclinic cycle.

Theorem 5.3 If $4u < (\theta - d)(a - \lambda)^2$, there is a $q^* \in (0,1)$ such that system (22) has an order-1 homoclinic cycle. If $q < q^*$ and $y_B \leq \phi(y_c, q)$, $y_D \geq \phi(y_A, q)$, then system (22) has no homoclinic cycle and bifurcates an unique order-1 periodic solution.

6. Conclusion

This paper reviews some advances on the stability and bifurcations of the semi-continuous dynamical systems since 2010. For the stability results, we focus on the closed convex order-1 periodic solution, one of three type periodic solutions. A sequences of switched systems are constructed to generate hybrid limit cycles, which are square approximations of order-1 periodic solution. Then a general and simple stability criterion is obtained by the successor function which is similar to the stability analysis in ordinary differential equation. For the bifurcation theory, we mainly consider the homoclinic and heteroclinic bifurcations of prey predator models with state dependent impulsive harvesting. By the successor function and the geometry theory of the semi-continuous systems, there are the homoclinic or heteroclinic cycles for the specific parameter value. When the parameter varies, the cycles disappear and the system bifurcates an unique order-1 periodic solution.

It is worth mentioning that the geometry theory of the semi-continuous dynamical systems is still in the early stage of study, and has many interesting topics to be explored, especially in the following topics.

(1) The current method is only applied to type 1 periodic solution, i.e., the closed convex one of unilateral asymptotical type. It should develop new methods to study the other two types and other orders of periodic solutions.
(2) Comparing with the rich bifurcation theory in the ordinary differential equation, it should make more efforts to the bifurcation theory in the semi-continuous dynamical systems, such as Hopf bifurcation and backward bifurcation.
(3) Most of the current works investigate the two dimensional systems. The more powerful analytical techniques should be introduced to explore the three dimensional or more higher dimensional systems.

Conflict of Interests

The authors declare that there is no conflict of interests.

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