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PERSISTENCE AND GLOBAL STABILITY OF A STAGE-STRUCTURED FOOD CHAIN MODEL WITH DISTRIBUTED MATURATION DELAY AND HARVESTING

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Abstract. A three-species food chain model with stage-structure and harvesting is proposed and analyzed, at which the maturation delay is modeled as a distributed for each species, to allow for the possibility that individuals may take different amounts of time to mature. It is assumed in the model that immature predator (immature top predator) do not have the ability to feed on prey (predator), and the mature predator (mature top predator) do not feed on immature prey (immature predator). Mathematical analysis of the model with regard to positivity of solutions, permanence and global stability are analyzed. By using comparison arguments, some sufficient conditions are obtained to guarantee the permanence of the model. Also, by the mean of an iterative technique, we established sufficient conditions for the global attractivity of the boundary equilibria and the positive equilibrium.

Keywords: stage structure; global stability; harvesting; distributed delay; persistence; iteration.

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1. Introduction

Modeling and analysis of the dynamics of predator-prey populations are one of the greatest challenge in the study of ecological systems. The most accepted and extensively studied class

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of models in population dynamics is the Lotka-Volterra which models the interaction among various species. The Lotka-Volterra model is also called the simplest food chain model, being composed of two populations (prey and predator). This model has been extended by introducing a third predator, called the top-predator or superpredator. The resulting system is called a three-level food chain model. There have been some interesting and impressive results on investigating the dynamics of three species predator-prey systems. In [1], a theory that could be central to all of ecology, is reviewed and extended to include the field of ecology, and arguments are offered defending the position that the research program , called Food Chain Model, could be regarded as the central theory of ecology. In literature, many authors have studied food chain models (see, for example[2]-[7]). In the classical predator-prey model it is assumed that each individual predator admits the same ability to attack prey and each individual prey admits the same risk to be attacked by predator. This assumption seems not to be realistic for many animals.

Most population models in the literature assume that all individuals are identical and do not take into account any age structure. However, the growth of species often has its development process, while in each stage of its development, it always shows different characteristic. For instance, the mature species have preying capacity, while the immature predator species are not able to prey, where they are raised by their parents, and the rate they attacking at prey and the reproductive rate can be ignored.

It has been argued that for many biological and ecological reasons, that stage structure with discrete or distributed delay should be taken into consideration in deriving predator-prey models, because it plays an important role in modeling of multi-species population dynamics, profound much simpler ways to simulate the diversity than other models and exhibits real world phenomenon. Aiello and Freedman [8], considered a single species growth model with stage structure which is a generalization of the classical logistic model. Following Aiello and Freedman [8], many authors have studied stage-structured models with discrete delay and some significant work was carried out (see, for example, [9]-[17]). Also, many authors have used distributed delay term to allow for the possibility that individuals may take different amount of time to mature, for more information, (see, for example[18]-[22]).

Motivated by the work of Al-Omari and Gourley [20], we will focus on a stage structure Lotka-Volterra three species food chain model with distributed maturation delay and harvesting for each species. Therefore, we shall study the following system

$$
\begin{aligned}
x_1'(t) &= \alpha_1 x_2(t)) - (\gamma_1 + h_1)x_1(t) - \alpha_1 \int_0^\infty x_2(t-s) f_1(s) e^{-(\gamma_1+h_1)s}\, ds \\
x_2'(t) &= \alpha_1 \int_0^\infty x_2(t-s) f_1(s) e^{-(\gamma_1+h_1)s}\, ds - d_1 x_2^2(t) - b_1 x_2(t) y_2(t) - h_2 x_2(t) \\
y_1'(t) &= \alpha_2 y_2(t) x_2(t) - \alpha_2 \int_0^\infty y_2(t-s) x_2(t-s) f_2(s) e^{-(\gamma_2+h_3)s}\, ds \\
&\quad -(\gamma_2 + h_3) y_1(t) \\
y_2'(t) &= \alpha_2 \int_0^\infty y_2(t-s) x_2(t-s) f_2(s) e^{-(\gamma_2+h_3)s}\, ds - d_2 y_2^2(t) - h_4 y_2(t) \\
&\quad -b_2 y_2(t) z_2(t) \\
z_1'(t) &= \alpha_3 y_2(t) z_2(t) - \alpha_3 \int_0^\infty y_2(t-s) z_2(t-s) f_3(s) e^{-(\gamma_3+h_5)s}\, ds \\
&\quad -(\gamma_3 + h_5) z_1(t), \\
z_2'(t) &= \alpha_3 \int_0^\infty y_2(t-s) z_2(t-s) f_3(s) e^{-(\gamma_3+h_5)s}\, ds - d_3 z_2^2(t) - h_6 z_2(t),
\end{aligned}
$$

where $x_1(t)$, $y_1(t)$ and $z_1(t)$ denote, respectively, the density of immature prey, immature predator and immature top-predator members and $x_2(t)$,$y_2(t)$ and $z_2(t)$ stand for the mature population densities of prey, predator and top-predator members, respectively. The h_1, h_2, h_3, h_4, h_5 and h_6 are the harvesting efforts of the immature prey, mature prey, immature predator, mature predator, immature top-predator and the mature top-predator populations, respectively. d_1, d_2 and d_3 measure the death and intra-specific competition rate of the mature prey, mature predator and mature top predator, respectively; b_1 and b_2 stand for the per capita per unit time predation rate of the predator and top predator, respectively. γ_1, γ_2 and γ_3 denote the death rate of the immature prey, immature predator and immature top predator, respectively. α_1, α_2 and α_3 measure the birth rate of the immature species of prey, predator and top predator, respectively. Most of works in literature, assume that all individuals belong to the same species take the same amount of time to mature. We want to address here the point about the uncertainty in the maturation delay. We propose to introduce a distributed delay term allowing for a distribution of maturation times, weighted by a probability densities, $f_i(s)$, where $\int_0^\infty f_i(s)\,ds = 1$ and $f_i \geq 0$, $i = 1,2,3$. Note that the term $\alpha_1 x_2(t-s)$ is the number of immature prey species born at time $t-s$ per unit time, and is taken as proportional to the number of prey adults then around, and $e^{-(\gamma_1+h_1)s}$ denotes the probability of an individual from the first species, born at time $t-s$ still being alive and not harvested at time t. Then $\alpha_1 \int_0^\infty x_2(t-s) f_1(s) e^{-(\gamma_1+h_1)s}\,ds$ will total up the contributions from all previous times. That is, the term $K_1 = \alpha_1 \int_0^\infty x_2(t-s) f_1(s) e^{-(\gamma_1+h_1)s}\,ds$ represents the transformation of immature prey species to mature prey species. Similarly, the terms $K_2 = \alpha_2 \int_0^\infty y_2(t-s) x_2(t-s) f_2(s) e^{-(\gamma_2+h_3)s}\,ds$ and $K_3 = \alpha_3 \int_0^\infty y_2(t-s) z_2(t-s) f_3(s) e^{-(\gamma_3+h_5)s}\,ds$ represent the transformation of immature predator species to mature predator species and the transformation of immature top predator species to mature top predator species, respectively.

Notice that, the total number of individuals that are immature at time $t = 0$ (i.e. the initial conditions for x_1, y_1 and z_1) such that $x_1(0), y_1(0), z_1(0) \geq 0$, are given by

$$x_1(0) = \alpha_1 \int_{-\infty}^{0} \left(\int_{-s}^{\infty} f_1(\xi)\,d\xi \right) x_2(s) e^{(\gamma_1+h_1)s}\,ds, \tag{1.1}$$

$$y_1(0) = \alpha_2 \int_{-\infty}^{0} \left(\int_{-s}^{\infty} f_2(\xi)\,d\xi \right) y_2(s) x_2(s) e^{(\gamma_2+h_3)s}\,ds, \tag{1.2}$$

$$z_1(0) = \alpha_3 \int_{-\infty}^{0} \left(\int_{-s}^{\infty} f_3(\xi)\,d\xi \right) y_2(s) z_2(s) e^{(\gamma_3+h_5)s}\,ds, \tag{1.3}$$

Note that, in system (1.1), the second, fourth and sixth equations are uncoupled from the first, third and fifth equations. Thus, it is sufficient to consider the second, fourth and sixth equations on their own. But for simplicity, we will assume that the kernels $f_i(s)$, $i = 1,2,3$ has compact support, that is, $f_i(s) = 0$ for all $s \geq \tau$, for some $\tau > 0$ this will apply on K_1, K_2 and K_3 defined above. Of course, we still assume that $\int_0^\tau f_i(s)\,ds = 1$ and $f_i \geq 0$, $i = 1,2,3$. This implies that no individual ever takes longer than τ units of time to mature. Accordingly, we will study the following system

$$x_2'(t) = \alpha_1 \int_0^\tau x_2(t-s) f_1(s) e^{-(\gamma_1+h_1)s}\, ds - d_1 x_2^2(t) - b_1 x_2(t) y_2(t) - h_2 x_2(t)$$

$$y_2'(t) = \alpha_2 \int_0^\tau y_2(t-s) x_2(t-s) f_2(s) e^{-(\gamma_2+h_3)s}\, ds - d_2 y_2^2(t) - h_4 y_2(t)$$

$$-b_2 y_2(t) z_2(t)$$

$$z_2'(t) = \alpha_3 \int_0^\tau y_2(t-s) z_2(t-s) f_3(s) e^{-(\gamma_3+h_5)s}\, ds - d_3 z_2^2(t) - h_6 z_2(t).$$

For initial data, we assume that

$$x_2(t) = \phi_1(t) > 0,\ y_2(t) = \phi_2(t) > 0 \text{ and } z_2(t) = \phi_3(t) > 0 \text{ for } -\tau < t \le 0,$$

(1.4)

$$\text{with } \ x_2(0), y_2(0), z_2(0) > 0.$$

Note that, we can use Theorem 1 of Al-Omari and Gourley[20] to prove that every solution $(x_2(t), y_2(t), z_2(t))$ of system (1.4) is positive for all $t > 0$.

2. Permanence

Recently, permanence concerning the long time survival of species population appears to be a very important concept of stability from the viewpoint of mathematical ecology. Thus, in this section, we are looking for sufficient conditions that guarantee the permanence of system (1.4) with initial condition (1.4). In order to discuss the permanence of system (1.4) we need the following result from [11].

Lemma 1. *Let $u(t)$ be the solution of*

$$u'(t) = \alpha \int_0^\tau f(s) e^{-\gamma s} u(t-s)\, ds - \beta u^2(t) - A u(t), \tag{2.5}$$

where $u(t) > 0$ for $-\tau \le t \le 0$. If $0 \le A < \alpha \int_0^\tau f(s) e^{-\gamma s}\, ds$, then $\lim_{t\to\infty} u(t) = \hat{u}$, where $\hat{u} = \beta^{-1}\left[\alpha \int_0^\tau f(s) e^{-\gamma s}\, ds - A\right]$. But, if $A > \alpha \int_0^\tau f(s) e^{-\gamma s}\, ds$, then $\lim_{t\to\infty} u(t) = 0$.

Theorem 1. *System (1.4) with initial conditions (1.4) is permanent provided that*

$a_i > 0$, $i = 1,2$, *where*

$$(H1): a_1 = d_1d_2d_3 - b_1d_3K_2 - b_2d_1K_3$$
$$a_2 = (K_1K_2K_3 - h_2K_2K_3 - h_4d_1K_3 - h_6d_1d_2)\left(1 - \frac{b_1K_2}{d_1d_2} - \frac{b_2K_3}{d_2d_3}\right) - h_6b_1K_2$$

Proof. Let $(x_2(t), y_2(t), z_2(t))$ is a solution of system (1.4) which satisfies (1.4). The proof will be split in two steps. As a first step, let $\bar{x} = \limsup_{t\to\infty} x_2(t)$, $\bar{y} = \limsup_{t\to\infty} y_2(t)$ and $\bar{z} = \limsup_{t\to\infty} z_2(t)$. Now, according to the first equation of system (1.4), it follows from the positivity of the solution that

$$x_2'(t) \le \alpha_1 \int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}x_2(t-s)\,ds - d_1x_2^2(t) - h_2x_2(t).$$

Consider the equation

$$u'(t) = \alpha_1 \int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}u(t-s)\,ds - d_1u^2(t) - h_2u(t),$$

with, $u(s) \ge x_2(s) > 0$ for all $s \in [-\tau, 0]$. Then by Lemma 1 we derive that

$$\lim_{t\to\infty} u(t) = d_1^{-1}\left[\alpha_1 \int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}\,ds - h_2\right].$$

Clearly, by comparison, $x_2(t) \le u(t)$ and therefore

$$\bar{x} \le d_1^{-1}\left[\alpha_1 \int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}\,ds - h_2\right].$$

Thus, for $\varepsilon > 0$ sufficiently small, there exists $t_1 > 0$ such that if $t > t_1$, then

$$x_2(t) \le d_1^{-1}\left[\alpha_1 \int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}\,ds - h_2\right] + \varepsilon := N_1. \tag{2.6}$$

Note that, as a consequence of the positivity of a_2, then $N_1 > 0$. For $t > t_1 + \tau$, let $v(t)$ be a solution of

$$v'(t) = N_1\alpha_2 \int_0^\tau f_2(s)e^{-(\gamma_2+h_3)s}v(t-s)\,ds - d_2v^2(t) - h_4v(t).$$

Now,

$$\begin{aligned} y_2'(t) &= \alpha_2 \int_0^\tau y_2(t-s)x_2(t-s)f_2(s)e^{-(\gamma_2+h_3)s}\,ds - d_2y_2^2(t) - h_4y_2(t) - b_1z_2(t)y_2(t) \\ &\le N_1\alpha_2 \int_0^\tau f_2(s)e^{-(\gamma_2+h_3)s}y_2(t-s)\,ds - d_2y_2^2(t) - h_4y_2(t). \end{aligned}$$

By comparison, $y_2(t) \leq v(t)$. But, by lemma 1 again, we have

$$\overline{y} \leq \frac{K_2\left(\frac{K_1-h_2}{d_1}+\varepsilon\right)-h_4}{d_2}.$$

Therefore, for $\varepsilon > 0$ sufficiently small there exists $t_2 > t_1+\tau$ such that if $t > t_2$,

$$y_2(t) \leq \frac{K_2\,(K_1-h_2)-h_4 d_1}{d_1 d_2}+\varepsilon := N_2 \tag{2.7}$$

Similarly, from the third equation of system (1.4) and (2.7) we get

$$\overline{z} \leq \frac{K_3\,[K_2(K_1-h_2)-h_4 d_1]-h_6 d_1 d_2}{d_1 d_2 d_3}.$$

Hence, for $\varepsilon > 0$ sufficiently small there is $t_3 > t_2+\tau$ such that if $t > t_3$,

$$z_2(t) \leq \frac{K_3\,[K_2(K_1-h_2)-h_4 d_1]-h_6 d_1 d_2}{d_1 d_2 d_3}+\varepsilon := N_3. \tag{2.8}$$

Now, for the second step, let $\underline{x} = \liminf_{t\to\infty} x_2(t)$, $\underline{y} = \liminf_{t\to\infty} y_2(t)$ and $\underline{z} = \liminf_{t\to\infty} z_2(t)$. Therefore, from the first equation of system (1.4) and (2.7) that for $t > t_3+\tau$, we have

$$x_2'(t) \geq \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1+h_1)s} x_2(t-s)\,ds - d_1 x_2^2(t) - h_2 x_2(t) - b_1 N_2 x_2(t).$$

By comparison, it follows that

$$\underline{x} \geq \frac{(K_1-h_2)-b_1\left[\frac{K_2(K_1-h_2)-h_4 d_1}{d_1 d_2}+\varepsilon\right]}{d_1}.$$

Since $\varepsilon > 0$ is arbitrary and sufficiently small, we conclude that

$$\underline{x} \geq \frac{(K_1-h_2)-b_1\left[\frac{K_2(K_1-h_2)-h_4 d_1}{d_1 d_2}\right]}{d_1}.$$

Therefore, for $\varepsilon > 0$ sufficiently small there is $t_4 > t_3+\tau$ such that if $t > t_4$,

$$x_2(t) > \frac{(K_1-h_2)-b_1\left[\frac{K_2(K_1-h_2)-h_4 d_1}{d_1 d_2}\right]}{d_1}-\varepsilon := n_1. \tag{2.9}$$

From the second equation of system (1.4), (2.8) and (2.9) that for $t > t_4+\tau$,

$$y_2'(t) \geq n_1 \alpha_2 \int_0^\tau y_2(t-s) f_2(s) e^{-(\gamma_2+h_3)s}\,ds - d_2 y_2^2(t) - h_4 y_2(t) - b_2 N_3 y_2(t).$$

By comparison, we obtain from (2.8) and (2.9) that

$$\begin{aligned} \underline{y} \;\geq\; & \frac{1}{d_2}\left[K_2\left(\frac{K_1-h_2-b_1\left(\frac{K_1K_2-h_2K_2-h_4d_1}{d_1d_2}\right)}{d_1}-\varepsilon\right)-h_4\right. \\ & \left.-b_2\left(\frac{K_3(K_1K_2-h_2K_2-h_4d_1)-h_6d_1d_2}{d_1d_2d_3}+\varepsilon\right)\right]. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary small, we can conclude that

$$\underline{y} \geq \frac{1}{d_2}\left[\left(\frac{K_1K_2}{d_1}-\frac{K_2h_2}{d_1}-h_4\right)\left(1-\frac{b_1K_2}{d_1d_2}-\frac{b_2K_3}{d_2d_3}\right)+\frac{h_6b_2}{d_3}\right].$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $t_5 > t_4+\tau$ such that if $t > t_5$,

$$y_2(t) > \frac{1}{d_2}\left[\left(\frac{K_1K_2}{d_1}-\frac{K_2h_2}{d_1}-h_4\right)\left(1-\frac{b_1K_2}{d_1d_2}-\frac{b_2K_3}{d_2d_3}\right)+\frac{h_6b_2}{d_3}\right]-\varepsilon := n_2. \tag{2.10}$$

Similarly, we derive from the third equation of system (1.4) that

$$\underline{z} \geq \frac{a_2}{d_1d_2d_3},$$

where a_2 is defined in $(H1)$. Therefore, for $\varepsilon > 0$ sufficiently small there exists a $t_6 > t_5+\tau$ such that if $t > t_6$,

$$z_2(t) > \frac{a_2}{d_1d_2d_3}-\varepsilon := n_3. \tag{2.11}$$

We note that if $(H1)$ holds and $\varepsilon > 0$ is chosen sufficiently small, $n_i > 0$, $i = 1,2,3$. Therefore, we have completed the proof of the theorem.

3. GLOBAL STABILITY OF EQUILIBRIA

For many biological and ecological systems, the global stability of steady states of the system is an interesting and important issue, because the analysis of global stability is very much useful than the use of only local stability analysis in the biological point of view. In the case of global stability all the individuals co-exist and trajectories are initiated to the equilibrium point. In this section, we shall derive sufficient conditions that guarantee the global stability of all equilibria of system (1.4). It is easy to show that system (1.4) has at least three nonnegative equilibria, $E_0 = (0,0,0)$, $E_1 = (d_1^{-1}(K_1-h_2),0,0)$ and $E_2 = (\hat{x_2},\hat{y_2},0)$.

$$\hat{x_2} = \frac{d_2k_1+b_1h_4-d_2h_2}{b_1k_2+d_1d_2}, \quad \hat{y_2} = \frac{k_1k_2-h_2k_2-d_1h_4}{b_1k_2+d_1d_2} \tag{3.12}$$

It is obvious that the equilibrium E_0 always exists without any restrictions. The equilibrium E_1 exists if $K_1 > h_2$, while the equilibrium E_2 exists if $k_1 > h_2$ and $k_1k_2 - h_2k_2 - d_1h_4 > 0$.

By analysing the corresponding characteristic equation around these equilibria, it is found that E_0 is unstable when $K_1 < h_2$, E_1 is locally stable when $K_1 > h_2$. Also, system (1.4) has a unique positive equilibrium $E_* = (x_2^*, y_2^*, z_2^*)$ where

$$x_2^* = \frac{A_1}{\Delta},\ y_2^* = \frac{A_2}{\Delta},\ z_2^* = \frac{A_3}{\Delta}, \tag{3.13}$$

where,

$$\begin{aligned}
A_1 &= K_1d_2d_3 + K_1K_3b_2 - h_2d_2d_3 - h_2b_2K_3 - b_1b_2h_6 + b_1h_4d_3 \\
A_2 &= K_1K_2d_3 - K_2h_2d_3 + d_1b_2h_6 - h_4d_1d_3 \\
A_3 &= K_1K_2K_3 - K_2K_3h_2 - K_3d_1h_4 - K_2b_1h_6 - d_1d_2h_6 \\
\Delta &= d_1d_2d_3 + b_1d_3K_2 + d_1b_2K_3.
\end{aligned}$$

In this section, we shall prove theorems on the global stability of the equilibria E_0, E_1, E_2 and E_*.

3.1. *Global stability of* E_0.

We shall show that when the equilibria E_1 and E_2 do not exist, that is $K_1 < h_2$, then E_0 is globally asymptotically stable.

Theorem 2. *Assume $K_1 < h_2$, Then $(x_2(t), y_2(t), z_2(t)) \to (0,0,0)$ as $t \to \infty$.*

Proof. Consider the functional

$$\begin{aligned}
V(t) &= \alpha_2\alpha_3x_2(t) + \alpha_1\alpha_2\alpha_3\int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}\int_{t-s}^t x_2(\xi)\,d\xi\,ds \\
&\quad + b_1\alpha_3y_2(t) + b_1\alpha_2\alpha_3\int_0^\tau f_2(s)\int_{t-s}^t e^{-(\gamma_2+h_3)(t-\xi)}y_2(\xi)x_2(\xi))\,d\xi\,ds \\
&\quad + b_1b_2z_2(t) + b_1b_2\alpha_3\int_0^\tau f_3(s)\int_{t-s}^t e^{-(\gamma_3+h_5)(t-\xi)}y_2(\xi)z_2(\xi))\,d\xi\,ds.
\end{aligned}$$

Note that $V(x_2,y_2,z_2) \geq 0$ and $V(x_2,y_2,z_2) = 0$ if and only if $x_2 = y_2 = z_2(t) = 0$. Then for t sufficiently larg,

$$\begin{aligned} V'(t) &= \alpha_1\alpha_2\alpha_3 x_2(t)\int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}\,ds - d_1\alpha_2\alpha_3 x_2^2(t) - h_2\alpha_2\alpha_3 x_2(t) - d_2 b_1\alpha_3 y_2^2(t) \\ &\quad -h_4 b_1\alpha_3 y_2(t) - \alpha_2\alpha_3 b_1(\gamma_2+h_3)\int_0^\tau f_2(s)\int_{t-s}^t e^{-(\gamma_2+h_3)(t-\xi)}y_2(\xi)x_2(\xi))\,d\xi\,ds \\ &\quad -\alpha_3 b_1 b_2(\gamma_3+h_5)\int_0^\tau f_3(s)\int_{t-s}^t e^{-(\gamma_3+h_5)(t-\xi)}y_2(\xi)z_2(\xi))\,d\xi\,ds \\ &\quad -d_3 b_1 b_2 z_2^2 - h_6 b_1 b_2 z_2 \\ &\leq \alpha_1\alpha_2\alpha_3 x_2(t)\int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}\,ds - h_2\alpha_2\alpha_3 x_2(t) \\ &= \alpha_2\alpha_3 x_2(t)(K_1-h_2) < 0. \end{aligned}$$

A direct application of the well known Liapunov-LaSalle theorem [23] (Theorem 2.5.3 of Kuang [24]) shows that $\lim_{t\to\infty} x_2(t) = 0$, $\lim_{t\to\infty} y_2(t) = 0$ and $\lim_{t\to\infty} z_2(t) = 0$. The proof of the theorem is complete.

3.2. ***Global stability of*** E_1. We shall prove a theorem on the global stability of E_1 when E_2 is unstable. That is, when immature mortality of the prey or harvesting effort of the immature or adult prey is low enough, and when the immature mortality of the predator or harvesting effort of the immature or adult predator is high enough and also when the immature mortality or harvesting effort of the immature top predator and the harvesting effort of mature top predator are high enough.

Theorem 3. *Assume* $K_1 > h_2$, $K_2(\frac{K_1-h_2}{d_1}) < h_4$ *and* $K_3 < h_6$ *hold. Then* $(x_2(t),y_2(t),z_2(t)) \to (\bar{x}_2,0,0)$ *as* $t\to\infty$.

Proof. By positivity of y_2 we have from the first equation of (1.4)

$$x_2'(t) \leq \alpha_1\int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}x_2(t-s))\,ds - d_1 x_2^2(t)) - h_2 x_2(t).$$

By comparison, $x_2(t)$ is bounded above by the solution $w(t)$ of

$$\begin{aligned} &w'(t) = \alpha_1\int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}w(t-s)\,ds - d_1 w^2(t) - h_2 w(t), \\ &w(s) = x_2(s), \quad s\in[-\tau,0]. \end{aligned}$$

But by Lemma 1, $\lim_{t\to\infty} w(t) = d^{-1}(K_1 - h_2)$. Therefore,

$$\limsup_{t\to\infty} x_2(t) \le \lim_{t\to\infty} w(t) = \frac{K_1 - h_2}{d_1}. \tag{3.14}$$

Now, since $K_2(\frac{K_1-h_2}{d_1}) < h_4$, let $\varepsilon > 0$ be sufficiently small and satisfying

$$K_2(\frac{K_1 - h_2}{d_1} + \varepsilon) - h_4 < 0,$$

therefore, there exists $t_1 > 0$ such that if $t > t_1$, $x_2(t) < d^{-1}(K_1 - h_2) + \varepsilon$. Applying this into the second equation of (1.4) for $t \ge t_1 + \tau$ we get

$$\begin{aligned} y_2'(t) &\le \alpha_2 \int_0^\tau f_2(s) e^{-(\gamma_2+h_3)s} y_2(t-s) x_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t) \\ &\le \alpha_2 \left(\frac{K_1 - h_2}{d_1} + \varepsilon \right) \int_0^\tau f_2(s) e^{-(\gamma_2+h_3)s} y_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t). \end{aligned}$$

This leads to $y_2(t) \le u(t)$ where $u(t)$ satisfies

$$u'(t) = \alpha_2 \int_0^\tau f_2(s) e^{-(\gamma_2+h_3)s} u(t-s)\, ds - d_2 u^2(t)) - h_4 u(t),$$

then from Lemma 1, $u(t) \to 0$ as $t \to \infty$, and so $y_2(t) \to 0$ as $t \to \infty$. Next we shall show that

$$\liminf_{t\to\infty} x_2(t) \ge \frac{K_1 - h_2}{d_1}. \tag{3.15}$$

Let $\varepsilon > 0$, then since $y_2(t) \to 0$ there exists $t_2 > t_1 + \varepsilon$ such that if $t > t_2$, $0 \le y_2(t) \le \varepsilon$. Then, from the first equation of system (1.4) for $t \ge T_2$, we have

$$x_2'(t) \ge \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1+h_1)s} x_2(t-s)\, ds - d_1 x_2^2(t) - b_1 \varepsilon x_2(t) - h_2 x_2(t).$$

By comparison, it follows that

$$\liminf_{t\to\infty} x_2(t) \ge \frac{K_1 - h_2 - b_1 \varepsilon}{d_1},$$

letting $\varepsilon \to 0$, We conclude that $\liminf_{t\to\infty} x_2(t) \ge \frac{K_1-h_2}{d_1}$. Consequently, $x_2(t) \to d_1^{-1}(K_1 - h_2)$ as $t \to \infty$.

It follows from the third equation of system (1.4) that for $t > t_2 + \tau$,

$$z_2'(t) \le \alpha_3 \varepsilon \int_0^\tau f_3(s) e^{-(\gamma_3+h_5)s} z_2(t-s)\, ds - d_3 z_2^2(t) - h_6 z_2(t).$$

which yields that $\lim_{t\to\infty} z_2(t) = 0$. This completes the proof of the theorem.

3.3. ***Global stability of*** E_2**.** Now, we shall prove a theorem on the global stability of E_2 in the situation when the harvesting effort of the mature top predator is high enough and the the transformation of immature predator species to mature predator species is low enough.

Theorem 4. *The positive equilibrium* $E_2 = (\hat{x}_2, \hat{y}_2, 0)$ *of system (1.4) is globally stable provided that*

$$d_1 d_2 h_6 > K_3(K_1K_2 - h_2K_2 - d_1h_4) > 0. \tag{3.16}$$

and

$$d_1 d_2 > b_1 K_2. \tag{3.17}$$

Proof. Let $\varepsilon > 0$ sufficiently small satisfying

$$K_3\left(\frac{K_1K_2 - h_2K_2 - d_1h_4}{d_1d_2} + \varepsilon\right) - h_6 < 0.$$

Then from the third equation of system (1.4) and (2.7) there exists $t_1 > 0$ such that if $t > t_1$

$$\begin{aligned} z_2'(t) &\leq \alpha_3 N_2 \int_0^\tau f_3(s) e^{-(\gamma_3+h_5)s} z_2(t-s)\,ds - d_3 z_2^2(t) - h_6 z_2(t). \\ &= \alpha_3\left(\frac{K_2(K_1-h_2) - h_4d_1}{d_1d_2}\right)\int_0^\tau f_3(s) e^{-(\gamma_3+h_5)s} z_2(t-s)\,ds - d_3 z_2^2(t) - h_6 z_2(t). \end{aligned}$$

Let $u(t)$ satisfies

$$u'(t) = \alpha_3\left(\frac{K_2(K_1-h_2) - h_4d_1}{d_1d_2}\right)\int_0^\tau f_3(s) e^{-(\gamma_3+h_5)s} u(t-s)\,ds - d_3 u^2(t) - h_6 u(t).$$

By comparison $z_2(t) \leq u(t)$, and therefore by Lemma 1 and (3.16) we conclude that $\lim_{t\to\infty} u(t) = 0$. Thus $\lim_{t\to\infty} z_2(t) = 0$, for all $t > 0$. Therefore, for $\varepsilon > 0$, there is $t_2 > t_1$ such that $0 \leq z_2(t) \leq \varepsilon$. It follows from the first equation of system (1.4) that for $t > t_2$,

$$x_2'(t) \leq \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1+h_1)s} x_2(t-s))\,ds - d_1 x_2^2(t)) - h_2 x_2(t).$$

Then by comparison, it follows

$$U_1 = \limsup_{t\to\infty} x_2(t) \leq \frac{K_1 - h_2}{d_1} := N_1^{x_2},$$

hence, for $\varepsilon > 0$ there is $t_3 > 0$ such that if $t > t_3$, $x_2(t) \leq N_1^{x_2} + \varepsilon$. We derive from the second equation of system (1.4) for $t > t_3 + \tau$,

$$y_2'(t) \leq \alpha_2(N_1^{x_2} + \varepsilon) \int_0^\tau f_2(s) e^{-(\gamma_2 + h_3)s} y_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t).$$

A standard comparison argument shows that

$$U_2 = \limsup_{t \to \infty} y_2(t) \leq \frac{K_2(N_1^{x_2} + \varepsilon) - h_2}{d_2}.$$

Since this is true for arbitrary $\varepsilon > 0$, we conclude that

$$U_2 = \limsup_{t \to \infty} y_2(t) \leq \frac{K_2 N_1^{x_2} - h_4}{d_2} := N_1^{y_2}.$$

Thus, for $\varepsilon > 0$ there is $t_4 > t_3 + \tau$ such that if $t > t_4$, $y_2(t) \leq N_1^{y_2} + \varepsilon$. Now, from the first equation of system (1.4) for $t > t_4$,

$$x_2'(t) \geq \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1 + h_1)s} x_2(t-s))\, ds - d_1 x_2^2(t)) - h_2 x_2(t) - b_1(N_1^{y_2} + \varepsilon) x_2(t).$$

Hence,

$$V_1 = \liminf_{t \to \infty} x_2(t) \geq \frac{K_1 - h_2 - b_1(N_1^{y_2} + \varepsilon)}{d_1}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$V_1 = \liminf_{t \to \infty} x_2(t) \geq \frac{K_1 - h_2 - b_1 N_1^{y_2}}{d_1} := M_1^{x_2}.$$

Thus, for any $\varepsilon > 0$ sufficiently small, there exists $t_5 > t_4 + \tau$ such that if $t > t_5$, $x_2(t) \geq M_1^{x_2} - \varepsilon$. Similarly, from the second equation of system (1.4) for $t > t_5 + \tau$,

$$y_2'(t) \geq \alpha_2(M_1^{x_2} - \varepsilon) \int_0^\tau f_2(s) e^{-(\gamma_2 + h_3)s} y_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t) - b_2 \varepsilon y_2(t).$$

By comparison, we obtain that

$$V_2 = \liminf_{t \to \infty} x_2(t) \geq \frac{K_2(M_1^{x_2} - \varepsilon) - h_4 - b_2 \varepsilon}{d_2},$$

where we conclude that for arbitrary $\varepsilon > 0$,

$$V_2 = \liminf_{t \to \infty} x_2(t) \geq \frac{K_2 M_1^{x_2} - h_4}{d_2} := M_1^{y_2}.$$

Thus, for $\varepsilon > 0$ there is $t_6 > t_5 + \tau$ such that if $t > t_6$, $y_2 \geq M_1^{y_2} - \varepsilon$. From the first equation of system (1.4) we have for $t > t_6$,

$$x_2'(t) \leq \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1 + h_1)s} x_2(t-s))\, ds - d_1 x_2^2(t)) - h_2 x_2(t) - b_1 (M_1^{y_2} - \varepsilon) x_2(t).$$

We then have,

$$U_1 = \limsup_{t\to\infty} x_2(t) \leq \frac{K_1 - h_2 - b_1 M_1^{y_2}}{d_1} := N_2^{x_2}.$$

Hence, for $\varepsilon > 0$, there exists $t_7 > t_6 + \tau$ such that if $t > t_7$, $x_2 \leq N_2^{x_2} + \varepsilon$. This implies from the second equation of system (1.4) for $t > t_7 + \tau$,

$$y_2'(t) \leq \alpha_2 (N_2^{x_2} + \varepsilon) \int_0^\tau f_2(s) e^{-(\gamma_2 + h_3)s} y_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t).$$

By comparison, and for $\varepsilon > 0$ sufficiently small we have

$$U_2 = \limsup_{t\to\infty} y_2(t) \leq \frac{K_2 N_2^{x_2} - h_4}{d_2} := N_2^{y_2}.$$

Continuing this process, we obtain four sequences $N_n^{x_2}$, $N_n^{y_2}$, $M_n^{x_2}$, $M_n^{y_2}$, $n = 1, 2, 3, \cdots$ such that, for $n \geq 2$,

$$\begin{aligned} N_n^{x_2} &= \frac{K_1 - h_2 - b_1 M_{n-1}^{y_2}}{d_1} \\ N_n^{y_2} &= \frac{K_2 N_n^{x_2} - h_4}{d_2} \\ M_n^{x_2} &= \frac{K_1 - h_2 - b_1 N_n^{y_2}}{d_1} \\ M_n^{y_2} &= \frac{K_2 M_n^{x_2} - h_4}{d_2}. \end{aligned} \tag{3.18}$$

By combining these, we get

$$N_n^{y_2} = \frac{(d_1 d_2 - b_1 K_2)(K_2(K_1 - h_2) - d_1 h_4)}{(d_1 d_2)^2} + \frac{b_1^2 K_2^2}{d_1^2 d_2^2} N_{n-1}^{y_2}.$$

or

$$N_n^{y_2} = \frac{(d_1 d_2 + b_1 K_2)(d_1 d_2 - b_1 K_2)}{(d_1 d_2)^2} \hat{y}_2 + \frac{b_1^2 K_2^2}{d_1^2 d_2^2} N_{n-1}^{y_2}. \tag{3.19}$$

Note that (3.17) implies that

$$\frac{b_1^2 K_2^2}{d_1^2 d_2^2} < 1.$$

We claim that $N_n^{y_2}$ is monotonically decreasing sequence that is bounded below by $\hat{y}_2$. The boundedness below by $\hat{y}_2$ follows immediately from (3.19) by induction. Then, by (3.19)

$$\begin{aligned}
\frac{N_n^{y_2}}{N_{n-1}^{y_2}} &= \frac{(d_1d_2-b_1K_2)(K_2(K_1-h_2)-d_1h_4)}{(d_1d_2)^2N_{n-1}^{y_2}}+\frac{b_1^2K_2^2}{d_1^2d_2^2}\\
&\leq \frac{(d_1d_2-b_1K_2)(K_2(K_1-h_2)-d_1h_4)}{(d_1d_2)^2\hat{y}_2}+\frac{b_1^2K_2^2}{d_1^2d_2^2}\\
&= \frac{(b_1K_2+d_1d_2)(d_1d_2-b_1K_2)}{d_1^2d_2^2}+\frac{b_1^2K_2^2}{d_1^2d_2^2}=1,
\end{aligned}$$

so that $N_n^{y_2}$ is monotonically decreasing. Hence $N_n^{y_2}$ converge to a limit which, by (3.19) equals $\hat{y}_2$.

Of course, convergence of $N_n^{y_2}$ implies convergence of the remaining three sequences, that is

$$\lim_{t\to\infty} N_n^{x_2}=\hat{x}_2,\ \lim_{t\to\infty} N_n^{y_2}=\hat{y}_2,\ \lim_{t\to\infty} M_n^{x_2}=\hat{x}_2,\ \lim_{t\to\infty} M_n^{y_2}=\hat{y}_2.$$

Therefore,

$$U_1=V_1=\hat{x}_2,\quad U_2=V_2=\hat{y}_2.$$

This completes the proof of the theorem.

3.4. ***Global stability of*** E_***.** Finally, we shall prove the global stability of E_* when all other equilibria are unstable.

Theorem 5. *Let the initial data satisfy (1.4), and assume x_2^*, y_2^* and z_2^* satisfy (3.13) such that $A_i>0$, $i=1,2,3$. Then the positive equilibrium $E_*=(x_2^*,y_2^*,z_2^*)$ of system (1.4) is globally stable provided that*

$$d_1d_2d_3>d_3b_1K_2+d_1b_2K_3. \tag{3.20}$$

Proof. Denote

$$U_1=\limsup_{t\to\infty} x_2(t),\ U_2=\limsup_{t\to\infty} y_2(t),\ U_3=\limsup_{t\to\infty} z_2(t),$$

$$V_1=\liminf_{t\to\infty} x_2(t),\ V_2=\liminf_{t\to\infty} y_2(t),\ V_3=\liminf_{t\to\infty} z_2(t).$$

It follows from the first equation of system (1.4), that

$$x_2'(t)\leq \alpha_1\int_0^\tau f_1(s)e^{-(\gamma_1+h_1)s}x_2(t-s)\,ds-d_1x_2^2(t)-h_2x_2(t).$$

By comparison we have

$$U_1 = \limsup_{t\to\infty} x_2(t) \le \frac{K_1 - h_2}{d_1} := N_1^{x_2}.$$

Then, for $\varepsilon > 0$ sufficiently small there is $t_1 > 0$ such that if $t > t_1$, $x_2(t) \le N_1^{x_2} + \varepsilon$. Thus, from the second equation of system (1.4) we have

$$y_2'(t) \le \alpha_2 (N_1^{x_2} + \varepsilon) \int_0^\tau f_2(s) e^{-(\gamma_2 + h_3)s} y_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t).$$

A standard comparison argument shows that

$$U_2 = \limsup_{t\to\infty} y_2(t) \le \frac{K_2(N_1^{x_2} + \varepsilon) - h_4}{d_2},$$

since $\varepsilon > 0$ is arbitrary, we conclude that

$$U_2 \le \frac{K_2 N_1^{x_2} - h_4}{d_2} := N_1^{y_2}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $t_2 > t_1 + \tau$ such that if $t > t_2$, $y_2(t) \le N_1^{y_2} + \varepsilon$. We derive from the third equation of system (1.4) for $t > t_2 + \tau$,

$$z_2'(t) \le \alpha_3 (N_1^{y_2} + \varepsilon) \int_0^\tau f_3(s) e^{-(\gamma_3 + h_5)s} z_2(t-s)\, ds - d_3 z_2^2(t) - h_6 z_2(t).$$

By comparison, it follows that

$$U_3 = \limsup_{t\to\infty} z_2(t) \le \frac{K_3(N_1^{y_2} + \varepsilon) - h_6}{d_3},$$

since this is true for $\varepsilon > 0$ sufficiently small, we conclude that

$$U_3 \le \frac{K_3 N_1^{y_2} - h_6}{d_3} := N_1^{z_2}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is $t_3 > t_2 + \tau$ such that if $t > t_3$, $z_2(t) \le N_1^{z_2} + \varepsilon$. Again, from the first equation of system (1.4) we have

$$x_2'(t) \ge \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1 + h_1)s} x_2(t-s)\, ds - d_1 x_2^2(t) - h_2 x_2(t) - b_1 (N_1^{y_2} + \varepsilon) x_2(t).$$

Thus if $t > t_3$, $x_2(t) \ge v(t)$ with suitable initial condition, where $v(t)$ is the solution of

$$v'(t) = \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1 + h_1)s} v(t-s)\, ds - d_1 v^2(t) - h_2 v(t) - b_1 (N_1^{y_2} + \varepsilon) v(t).$$

Hence,

$$V_1 = \liminf_{t\to\infty} x_2(t) \ge \lim_{t\to\infty} v(t) = \frac{K_1 - h_2 - b_1(N_1^{y_2} + \varepsilon)}{d_1},$$

since $\varepsilon > 0$ is arbitrary, we have

$$V_1 \geq \frac{K_1 - h_2 - b_1 N_1^{y_2}}{d_1} := M_1^{x_2}.$$

Therefore, for any $\varepsilon > 0$ sufficiently small, there is $t_4 > t_3 + \tau$ such that if $t > t_4$, $x_2(t) \geq M_1^{x_2} - \varepsilon$. It follows from the second equation of system (1.4) that for $t > t_4 + \tau$,

$$y_2'(t) \leq \alpha_2 (M_1^{x_2} - \varepsilon) \int_0^\tau f_2(s) e^{-(\gamma_2 + h_3)s} y_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t) - b_2 (N_1^{z_2} + \varepsilon) y_2(t).$$

By comparison, and for $\varepsilon > 0$ sufficiently small we have

$$V_2 = \liminf_{t \to \infty} y_2(t) \geq \frac{K_2 M_1^{x_2} - h_4 - b_2 N_1^{z_2}}{d_2} := M_1^{y_2}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $t_5 > t_4 + \tau$ such that if $t > t_5$, $y_2(t) \geq M_1^{y_2} - \varepsilon$. Similarly, it follows from the third equation of system (1.4) for $t > t_5 + \tau$,

$$z_2'(t) \leq \alpha_3 (M_1^{y_2} - \varepsilon) \int_0^\tau f_3(s) e^{-(\gamma_3 + h_5)s} z_2(t-s)\, ds - d_3 z_2^2(t) - h_6 z_2(t),$$

which gives

$$V_3 = \liminf_{t \to \infty} z_2(t) \geq \frac{K_3 M_1^{y_2} - h_6}{d_3} := M_1^{z_2}.$$

Thus, for $\varepsilon > 0$ sufficiently small, there is $t_6 > t_5 + \tau$ such that if $t > t_6$, $z_2(t) \geq M_1^{z_2} - \varepsilon$. We derive from the first equation of system (1.4) that for $t > t_6$

$$x_2'(t) \leq \alpha_1 \int_0^\tau f_1(s) e^{-(\gamma_1 + h_1)s} x_2(t-s)\, ds - d_1 x_2^2(t) - h_2 x_2(t) - b_1 (M_1^{y_2} - \varepsilon) x_2(t).$$

By comparison we have

$$U_1 = \limsup_{t \to \infty} x_2(t) \leq \frac{K_1 - h_2 - b_1 M_1^{y_2}}{d_1} := N_2^{x_2}.$$

Then, for $\varepsilon > 0$ sufficiently small there is $t_7 > 0$ such that if $t > t_7$, $x_2(t) \leq N_2^{x_2} + \varepsilon$. Thus, from the second equation of system (1.4), for $t > t_7 + \tau$ we have

$$y_2'(t) \leq \alpha_2 (N_2^{x_2} + \varepsilon) \int_0^\tau f_2(s) e^{-(\gamma_2 + h_3)s} y_2(t-s)\, ds - d_2 y_2^2(t) - h_4 y_2(t) - b_2 (M_1^{z_2} - \varepsilon) y_2(t).$$

A standard comparison argument shows that

$$U_2 = \limsup_{t \to \infty} y_2(t) \leq \frac{K_2 (N_2^{x_2} + \varepsilon) - h_4 - b_2 (M_1^{z_2} - \varepsilon)}{d_2},$$

since $\varepsilon > 0$ is arbitrary, we conclude that

$$U_2 \leq \frac{K_2 N_1^{x_2} - h_4 - b_2 M_1^{z_2}}{d_2} := N_2^{y_2}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $t_8 > t_7 + \tau$ such that if $t > t_8$, $y_2(t) \leq N_2^{y_2} + \varepsilon$. We derive from the third equation of system (1.4) for $t > t_8 + \tau$,

$$z_2'(t) \leq \alpha_3 (N_2^{y_2} + \varepsilon) \int_0^\tau f_3(s) e^{-(\gamma_3 + h_5)s} z_2(t-s)\, ds - d_3 z_2^2(t) - h_6 z_2(t).$$

By comparison, it follows that

$$U_3 = \limsup_{t \to \infty} z_2(t) \leq \frac{K_3 (N_2^{y_2} + \varepsilon) - h_6}{d_3},$$

since this is true for $\varepsilon > 0$ sufficiently small, we conclude that

$$U_3 \leq \frac{K_3 N_2^{y_2} - h_6}{d_3} := N_2^{z_2}.$$

One now sees that the transition from the $(n-1)$th to the nth step in this iterative process, for $n \geq 2$, is given by

$$\begin{aligned}
N_n^{x_2} &= \frac{K_1 - h_2 - b_1 M_{n-1}^{y_2}}{d_1}, \\
N_n^{y_2} &= \frac{K_2 N_n^{x_2} - h_4 - b_2 M_{n-1}^{z_2}}{d_2}, \\
N_n^{z_2} &= \frac{K_3 N_n^{y_2} - h_6}{d_3}, \\
M_n^{x_2} &= \frac{K_1 - h_2 - b_1 N_n^{y_2}}{d_1}, \\
M_n^{y_2} &= \frac{K_2 M_n^{x_2} - h_4 - b_2 N_n^{z_2}}{d_2}, \\
M_n^{z_2} &= \frac{K_3 M_n^{y_2} - h_6}{d_3},
\end{aligned}$$

and, of course,

$$M_n^{x_2} \leq V_1 \leq U_1 \leq N_n^{x_2}, \;\; M_n^{y_2} \leq V_2 \leq U_2 \leq N_n^{y_2} \;\text{ and }\; M_n^{z_2} \leq V_3 \leq U_3 \leq N_n^{z_2}.$$

We see at once that

$$N_{n+1}^{z_2} = \frac{A_3 (d_1 d_2 d_3 - d_1 b_2 K_3 - b_1 d_3 K_2)}{(d_1 d_2 d_3)^2} + \frac{(d_1 b_2 K_3 + b_1 d_3 K_2)^2}{(d_1 d_2 d_3)^2} N_n^{z_2}. \tag{3.21}$$

We can rewrite (3.21) into

$$N_{n+1}^{z_2} = \frac{(d_1d_2d_3)^2 - (d_1b_2K_3 + b_1d_3K_2)^2}{(d_1d_2d_3)^2} z_2^* + \frac{(d_1b_2K_3 + b_1d_3K_2)^2}{(d_1d_2d_3)^2} N_n^{z_2}. \tag{3.22}$$

Note that (3.20) implies that

$$\frac{(d_1b_2K_3 + b_1d_3K_2)^2}{(d_1d_2d_3)^2} < 1.$$

We claim that $N_n^{z_2}$ is monotonically decreasing sequence that is bounded below by z_2^*. The boundedness below by z_2^* follows immediately from (3.22) by induction. Then by (3.21)

$$\begin{aligned}
\frac{N_{n+1}^{z_2}}{N_n^{z_2}} &= \frac{A_3\,(d_1d_2d_3 - d_1b_2K_3 - b_1d_3K_2)}{(d_1d_2d_3)^2 N_n^{z_2}} + \frac{(d_1b_2K_3 + b_1d_3K_2)^2}{(d_1d_2d_3)^2} \\
&\le \frac{A_3\,(d_1d_2d_3 - d_1b_2K_3 - b_1d_3K_2)}{(d_1d_2d_3)^2 z_2^*} + \frac{(d_1b_2K_3 + b_1d_3K_2)^2}{(d_1d_2d_3)^2} \\
&= \frac{\Delta(d_1d_2d_3 - d_1b_2K_3 - b_1d_3K_2)}{(d_1d_2d_3)^2} + \frac{(d_1b_2K_3 + b_1d_3K_2)^2}{(d_1d_2d_3)^2} \\
&= \frac{(d_1d_2d_3 + b_1d_3K_2 + d_1b_2K_3)\,(d_1d_2d_3 - d_1b_2K_3 - b_1d_3K_2) + (d_1b_2K_3 + b_1d_3K_2)^2}{(d_1d_2d_3)^2} \\
&= 1.
\end{aligned}$$

This means that $N_n^{z_2}$ is monotonically decreasing sequence. Hence, $\lim_{t\to\infty} N_n^{z_2}$ exists. Taking $n \to \infty$, then from (3.22) we have

$$\lim_{n\to\infty} N_n^{z_2} = z_2^*. \tag{3.23}$$

The analysis of the remaining five sequences is similar. That is, convergence of $N_n^{z_2}$ implies convergence of the other five sequences. Therefore, from (3.23) we have

$$\lim_{n\to\infty} N_n^{x_2} = x_2^*,\ \lim_{n\to\infty} M_n^{x_2} = x_2^*,\ \lim_{n\to\infty} N_n^{y_2} = y_2^*,\ \lim_{n\to\infty} M_n^{y_2} = y_2^*,\ \lim_{n\to\infty} M_n^{z_2} = z_2^*.$$

Thus,

$$U_1 = V_1 = x_2^*, \quad U_2 = V_2 = y_2^*, \quad U_3 = V_3 = z_2^*.$$

The proof of the theorem is complete.

4. Discussion

In this paper, based on the work of Al-Omari and Gourley [20], we have proposed and discussed a stage structure three-species food chain model with harvesting and in which the maturation time for each species is not always the same for all individuals. Our results Theorem 1 shows that the system (1.4) is permanent provided $(H1)$ holds true. By using an iterative technique, we discussed the global stability of all equilibria of system (1.4). In Theorem 2 we construct suitable Lyapunov functional, sufficient conditions are established for the global stability of the trivial equilibrium E_0, we get that the trivial equilibrium is globally stable if the death rate of the immature prey species, γ_1, the harvesting effort of immature prey species, h_1, are large enough and significant harvesting effort among mature prey species, h_2. By Theorem 3, we see that if K_2 and K_3 are low enough, and the harvesting rate of the mature predator, h_4, the intra-specific competition rate of the mature prey, d_1, and the harvesting effort of the mature top predator, h_6, are high enough, the prey population will be persistent, but the predator and top predator will go to extinction, that is E_1 is globally stable. By Theorem 4 we can see that the top predator will go to extinction, but the prey and predator populations will be permanent if the harvesting effort of the mature top predator is high enough satisfying (3.16) and the transformation of immature predator species to mature predator species satisfying (3.17). Finally, the result of global stability of the equilibrium E^* in Theorem 5 implies that the three species model system coexists, is permanent, and the trivial and all other semitrivial solutions are unstable. Our results show that the behavior of harvesting on the three species affect the dynamical behavior of system (1.4). That is, it can prevent them from dying out. Also, it show the dynamics of our model depends on the maturation delay of the predator population as represented by the probability density function $f_i(s)$, $i = 1,2,3$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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