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EXTINCTION AND PERSISTENCE OF A STOCHASTIC SIS EPIDEMIC MODEL WITH VERTICAL TRANSMISSION, SPECIFIC FUNCTIONAL RESPONSE AND LÉVY JUMPS

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Abstract. In this paper, we study the dynamics of a stochastic SIS epidemic model with vertical transmission and specific functional response. The environment variability in this work is characterized by Gaussian white noise and Lévy jump noise. We establish the existence and uniqueness of a global positive solution starting from any positive initial value. We also investigate extinction and persistence in mean of the disease. Numerical examples are presented to illustrate the theoretical results.

Keywords: SIS epidemic model; vertical transmission; Gaussian white noise; Lévy jump noise; extinction; persistence in mean.

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1. Introduction

Epidemiology is the study of the spread of diseases in human populations and the factors that are responsible for or contribute to their occurrence. Consequently, it has been investigated by several researchers through study of the dynamical behavior of infectious diseases by mathematical models (see, e.g., [1, 2, 3, 4, 5, 6]). Particularly, the SIS (susceptible-infected-susceptible) epidemic model is often used to model the dynamics of diseases such as bacterial diseases and some sexually transmitted diseases where individuals start to be susceptible, at some stage catch the disease, and after a short infectious period become susceptible again [7].

In recent years, many mathematical models have been formulated to describe the impact of environmental fluctuation on the dynamics of infectious disease, see, e.g., [8, 9, 10, 11, 12, 13]. Gray et al. [14] constructed a stochastic SIS epidemic model with constant population size where the authors not only established threshold value conditions, i.e., the disease dies out or persists but also they showed the existence of a stationary distribution. Zhao and Jiang studied the threshold of a stochastic SIS epidemic model with vaccination in [15]. Teng and Wang [16] discussed a stochastic SIS epidemic model with nonlinear incidence rate. Miao et al. [17] were interested in a stochastic SIS epidemic model with a saturated incidence rate and double epidemic hypothesis. They presented a threshold value on the extinction and persistence for the model.

Vertical transmission of diseases is the transmission of an infection from parent to child during the perinatal period like Rubella, Varicella, Measles, AIDS (HIV infection), Zika fever (see, e.g., [18]). Many authors presented the mathematical analysis of vertically transmitted diseases models, see, e.g., [19, 20, 21]. Most recently, Zhang et al. [22] proposed the following stochastic SIS model with vertical transmission and random perturbations

$$\left\{\begin{aligned} dS(t) = &\quad (\Lambda + bS - \beta SI - dS - BS + \gamma I + bqI)dt + \sigma_1 S dB_1(t), \\ dI(t) = &\quad (bpI + \beta SI - dI - aI - BI - \gamma I)dt + \sigma_2 I dB_2(t), \end{aligned}\right. \tag{1}$$

where $S(t)$ is the number of susceptible individuals, $I(t)$ is the number of infected individuals, Λ is the recruitment rate of susceptibles corresponding to immigration, β is the contact transmission coefficient, B is output rate of susceptibles and infected population corresponding to emigration, a is the disease related death rate, γ is the recovery rate, b and d are the birth rate and natural death rate, respectively. p is the vertical transmission coefficient, with $0 < p < 1$ and $q = 1 - p$. It is assumed that $d + B - b > 0$ [22]. $B_i(t)$ $(i = 1,2)$ are independent standard Brownian motions, and $\sigma_i > 0$ $(i = 1,2)$ represents the intensities of $B_i(t)$, respectively. According to the theory in [22], the basic reproduction number of model (1) is $R_0^S = \beta\Lambda \big/ \left[(d + B - b)(d + B - bp + a + \gamma + \sigma_2^2 \big/ 2)\right]$. Moreover, if $R_0^S < 1$, then the disease will die out, while if $R_0^S > 1$, then the disease will prevail. Also the authors in [22] found that random perturbations can suppress the outbreaks of the disease.

System (1) is a stochastic model driven by white noise only (its solution is continuous). However, the population system may suffer sudden environmental perturbations such as earthquakes, hurricanes, floods, toxic pollutants, etc. These disturbances can not be described by the continuous stochastic model. Thus it is important to model these phenomena by jump processes. Initially, Bao et al. in [23] studied Lotka-Volterra population dynamics with jumps, and they gave some results which revealed that jump processes can influence the properties of the population systems. From then on, many results on epidemic models with jumps have been reported (see, e.g., [24, 25, 26, 27, 28]).

Our aim in this present work is to extend the model presented in [22] to a model with Lévy noise perturbation and specific functional response. In this way, model (1) will be changed into following form

$$(2)\qquad \begin{cases} dS(t) = & (\Lambda + bS - f(S,I)I - dS - BS + \gamma I + bqI)dt + \sigma_1 S dB_1(t) \\ & + \int_{\mathscr{Y}} q_1(u) S(t^-)\tilde{N}(dt,du), \\ dI(t) = & (bpI + f(S,I)I - dI - aI - BI - \gamma I)dt + \sigma_2 I dB_2(t) \\ & + \int_{\mathscr{Y}} q_2(u) I(t^-)\tilde{N}(dt,du), \end{cases}$$

where $S(t^-)$ is the left limit of $S(t)$, $I(t^-)$ is the left limits of $I(t)$, $\tilde{N}(dt,du) = N(dt,du)$ - $\mu(du)dt$, N is a Poisson counting measure on $[0,+\infty)\times\mathscr{Y}$, μ is the characteristic measure of N on a measurable subset $\mathscr{Y}$ of $(0,+\infty)$ with $\mu(\mathscr{Y}) < \infty$, $q_i(u)$ is the jump diffusion coefficient, $i = 1,2$. The incidence rate of the disease is modeled by the following specific functional response

$$f(S,I) = \frac{\beta S}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI},$$

where $\alpha_1, \alpha_2, \alpha_3$ are saturation factors measuring the psychological or inhibitory effect. This specific functional response was introduced by Hattaf et al. [29].

Our study in this paper is as follows. In the next section, we prove the existence of a unique global positive solution with any positive initial value. In Section 3, we present the basic reproduction number R_{jump}, and we show that when $R_{\text{jump}} < 1$ the disease will die out. In Section 4, we will prove that when $R_{\text{jump}} > 1$, the disease is persistent in mean.

Throughout this paper, for the sake of convenience, let $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Set $\mathbb{R}^2_+ = \{(x,y)\in\mathbb{R}^2 : x > 0,\ y > 0\}$. Denote by $\mathbb{E}(X)$ the mathematical expectation of a random variable X.

We consider the following stochastic differential equation with Lévy jumps

$$\begin{cases} dx(t) = F(x(t),t)dt + G(x(t),t)dW(t) + \int_{\mathscr{Y}} H(x(t),t,u)\tilde{N}(dt,du), \ \ t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

where $x(t)\in\mathbb{R}^n$, x_0 represents the initial value and $W(t)$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ independent of the Poisson counting measure. The functional $F(x,t)$ and $H(x,t,u)$ are defined respectively on $\mathbb{R}^n\times[0,+\infty)$ and $\mathbb{R}^n\times[0,+\infty)\times\mathscr{Y}$. $G(x,t)$ is an $n\times m$ matrix. Denote by $\mathscr{C}^{2,1}(\mathbb{R}^n\times[0,+\infty);\mathbb{R}_+)$ the family of all nonnegative functions $U(x,t)$ defined on $\mathbb{R}^n\times[0,+\infty)$ such that they are continuously twice differentiable in x and once in t. If L acts on a function $U \in \mathscr{C}^{2,1}(\mathbb{R}^n \times [0,+\infty);\mathbb{R}_+)$, then

$$\begin{aligned} LU(x,t) &= U_t(x,t) + U_x(x,t)F(x,t) + \frac{1}{2}\text{trace}[G^T(x,t)U_{xx}(x,t)G(x,t)] \\ &\quad + \int_{\mathscr{Y}} [U(x+H(x,t,u)) - U(x,t) - U_x(x,t)H(x,t,u)]\,\mu(du), \end{aligned}$$

where $U_t = \frac{\partial U}{\partial t}$, $U_x = (\frac{\partial U}{\partial x_1}, \ldots.., \frac{\partial U}{\partial x_n})$, $U_{xx} = (\frac{\partial^2 U}{\partial x_i \partial x_j})_{n\times n}$. Then the generalized Itô formula with jumps is given (refer to [30, 31, 32] for more details) by

$$dU(x,t) = LU(x,t)dt + U_x(x,t)G(x,t)dW(t) + \int_{\mathscr{Y}} [U(x+H(x,t,u)) - U(x,t)]\,\tilde{N}(dt,du).$$

In all the sequel, we assume that the following assumptions hold.

$(H1)$: $\int_{\mathscr{Y}} q_i^2(u)\mu(du) < \infty,\ i = 1,2.$

$(H2)$: $1 + q_i(u) > 0,\ u \in \mathscr{Y},\ i = 1,2.$

$(H3)$: $|\ln(1+q_i(u))| \leq M_i$, where M_i is a positive constant, $i = 1,2$. This assumption means that the intensities of Lévy noises are not infinite.

2. Existence and uniqueness of a positive solution

In this section we will establish the existence of a unique global positive solution for our stochastic epidemic model (2).

Theorem 2.1. *For any given initial value $(S(0),I(0)) \in \mathbb{R}_+^2$, there is a unique positive solution $(S(t),I(t))$ of model (2) on $t \geq 0$, and the solution will remain in $\mathbb{R}_+^2$ with probability one, that is to say, $(S(t),I(t)) \in \mathbb{R}_+^2$ for all $t \geq 0$ almost surely (briefly a.s).*

Proof. *Under assumption $(H1)$ and since the coefficients of model (2) are locally Lipschitz continuous, for any given initial value $(S(0),I(0)) \in \mathbb{R}_+^2$, there is a unique local solution $(S(t),I(t))$ on $[0,\tau_e)$, where τ_e is the explosion time* [32, 33]. *To prove that this solution is global, we show that $\tau_e = \infty$ a.s. For this we consider the following stopping time*

$$\tau^+ = \inf\{t \in [0,\tau_e) : S(t) \leq 0 \text{ or } I(t) \leq 0\},$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, $\tau^+ \leq \tau_e$, so if we can show that $\tau^+ = \infty$ a.s, then $\tau_e = \infty$ and $(S(t),I(t)) \in \mathbb{R}_+^2$ for all $t \geq 0$ a.s. Assume $\mathbb{P}(\tau^+ < T) > 0$ for some $T > 0$. Then we consider the function $U(S(t),I(t))$ defined for $(S(t),I(t)) \in \mathbb{R}_+^2$ by

$$U(S(t),I(t)) = \ln S(t) + \ln I(t).$$

Calculating the differential of U along the solution trajectories of system (2), using Itô's formula with jumps, we get, for $\eta \in \{\tau^+ < T\}$, and for all $t \in [0,\tau^+)$,

$$dU = LU dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t) + \int_{\mathscr{Y}} [\ln(1+q_1(u)) + \ln(1+q_2(u))] \tilde{N}(dt,du), \tag{3}$$

where

$$\begin{aligned} LU &= \frac{\Lambda}{S} - (d+B-b) - \frac{f(S,I)I}{S} + \frac{(\gamma+bq)I}{S} + f(S,I) - (d+B-bp+a+\gamma) \\ &\quad -\frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} - \int_{\mathscr{Y}} [q_1(u) - \ln(1+q_1(u))]\,\mu(du) - \int_{\mathscr{Y}} [q_2(u) - \ln(1+q_2(u))]\,\mu(du). \end{aligned}$$

Integrating both side of (3) we obtain

$$\begin{aligned} U(S(t),I(t)) - U(S(0),I(0)) &= \int_0^t \left[\frac{\Lambda}{S} - (d+B-b) - \frac{f(S,I)I}{S} + \frac{(\gamma+bq)I}{S} - \frac{\sigma_1^2}{2}\right] ds \\ &\quad + \int_0^t \left[f(S,I) - (d+B-bp+a+\gamma) - \frac{\sigma_2^2}{2}\right] ds \\ &\quad - \int_0^t \left\{\int_{\mathscr{Y}} [q_1(u) - \ln(1+q_1(u))]\,\mu(du)\right\} ds \\ &\quad - \int_0^t \left\{\int_{\mathscr{Y}} [q_2(u) - \ln(1+q_2(u))]\,\mu(du)\right\} ds \\ &\quad + \sigma_1 B_1(t) + \sigma_2 B_2(t) + \int_0^t \int_{\mathscr{Y}} [\ln(1+q_1(u)) + \ln(1+q_2(u))] \tilde{N}(ds,du). \end{aligned}$$

Hence

$$\begin{aligned} U(S(t),I(t)) &\geq U(S(0),I(0)) + \int_0^t \left[-2(d+B-b) - (a+\gamma+bq) - \beta I - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}\right] ds \\ &\quad - \int_0^t \left\{\int_{\mathscr{Y}} [q_1(u) - \ln(1+q_1(u))]\,\mu(du)\right\} ds \\ &\quad - \int_0^t \left\{\int_{\mathscr{Y}} [q_2(u) - \ln(1+q_2(u))]\,\mu(du)\right\} ds \\ &\quad + \sigma_1 B_1(t) + \sigma_2 B_2(t) + \int_0^t \int_{\mathscr{Y}} [\ln(1+q_1(u)) + \ln(1+q_2(u))] \tilde{N}(ds,du). \end{aligned} \tag{4}$$

Note that $S(\tau^+) = 0$ or $I(\tau^+) = 0$. Thereby,

$$\lim_{t \to \tau^+} U(S(\tau^+), I(\tau^+)) = -\infty.$$

Letting $t \longrightarrow \tau^+$ in (4), we get the contradiction

$$\begin{aligned}
-\infty &\geq U(S(0),I(0)) + \left[-2(d+B-b)-(a+\gamma+bq)-\frac{\sigma_1^2}{2}-\frac{\sigma_2^2}{2}\right]\tau^+ - \beta\int_0^{\tau^+} I(s)ds \\
&\quad -\int_0^{\tau^+}\left\{\int_{\mathscr{Y}}[q_1(u)-\ln(1+q_1(u))]\,\mu(du)\right\}ds \\
&\quad -\int_0^{\tau^+}\left\{\int_{\mathscr{Y}}[q_2(u)-\ln(1+q_2(u))]\,\mu(du)\right\}ds \\
&\quad +\sigma_1 B_1(\tau^+)+\sigma_2 B_2(\tau^+)+\int_0^{\tau^+}\int_{\mathscr{Y}}[\ln(1+q_1(u))+\ln(1+q_2(u))]\tilde{N}(ds,du) \\
&> -\infty.
\end{aligned}$$

Thus, $\tau^+ = \tau_e = \infty$ a.s. Which completes the proof of the theorem.

Denote

$$\Delta = \left\{(S,I)\in\mathbb{R}_+^2 : S+I \leq \frac{\Lambda}{d+B-b}\right\}.$$

The following theorem shows that the set Δ is positive invariant set of the stochastic SIS model with jumps (2), i.e., if $(S(0),I(0))\in\Delta$, then $(S(t),I(t))\in\Delta$ for all $t\geq 0$ a.s.

Theorem 2.2. *The set Δ is almost surely positively invariant of stochastic model (2).*

Proof. *Let $(S(0),I(0))\in\Delta$ and $k_0 \geq 0$ be sufficiently large such that $(S(0),I(0))\in\left(\frac{1}{k_0},\frac{\Lambda}{d+B-b}\right]^2$. For any integer $k\geq k_0$ we define the following stopping times*

$$\tau_k = \inf\left\{t>0 : (S(t),I(t))\in\Delta \text{ and } (S(t),I(t))\notin\left(\frac{1}{k},\frac{\Lambda}{d+B-b}\right]^2\right\},$$

$$\tau = \inf\{t>0 : (S(t),I(t))\notin\Delta\}.$$

We need to prove that $\mathbb{P}(\tau<t)=0$ for any $t>0$. $(\tau<t)\subset(\tau_k<t)$, hence $\mathbb{P}(\tau<t)\leq\mathbb{P}(\tau_k<t)$. So, it suffices to show that $\limsup_{k\to\infty}\mathbb{P}(\tau_k<t)=0$.

We consider the function $V:\mathbb{R}_+^2\to\mathbb{R}_+$ by

$$V(S,I)=\frac{1}{S}+\frac{1}{I}.$$

For all $T\geq 0$ and $0\leq s\leq T\wedge\tau_k$, using Itô's formula, we have

$$dV(S(s),I(s)) = LVds - \frac{\sigma_1}{S(s)}dB_1(s) - \frac{\sigma_2}{I(s)}dB_2(s) - \int_{\mathscr{Y}}\left[\frac{q_1(u)}{S(1+q_1(u))}+\frac{q_2(u)}{I(1+q_1(u))}\right]\tilde{N}(ds,du),$$

where

$$\begin{aligned}
LV &= -\frac{\Lambda}{S^2}+\frac{d+B-b}{S}+\frac{f(S,I)I}{S^2}+\frac{\sigma_1^2}{S}+\int_{\mathscr{Y}}\frac{q_1^2(u)}{S(1+q_1(u))}\mu(du) \\
&\quad +\frac{d+B-bp+a+\gamma}{I}-\frac{f(S,I)}{I}+\frac{\sigma_2^2}{I}+\int_{\mathscr{Y}}\frac{q_2^2(u)}{I(1+q_2(u))}\mu(du).
\end{aligned}$$

Then

$$\begin{aligned} dV(S(s),I(s)) \quad \leq \quad & \left[d+B-b+\frac{f(S,I)I}{S}+\sigma_1^2+\int_{\mathscr{Y}}\frac{q_1^2(u)}{(1+q_1(u))}\mu(du)\right]\frac{ds}{S} \\ & +\left[d+B-bp+a+\gamma+\sigma_2^2+\int_{\mathscr{Y}}\frac{q_2^2(u)}{(1+q_2(u))}\mu(du)\right]\frac{ds}{I} \\ & -\frac{\sigma_1}{S(s)}dB_1(s)-\frac{\sigma_2}{I(s)}dB_2(s)-\int_{\mathscr{Y}}\left[\frac{q_1(u)}{S(1+q_1(u))}+\frac{q_2(u)}{I(1+q_1(u))}\right]\tilde{N}(ds,du). \end{aligned}$$

Hence

$$dV \leq \theta V - \frac{\sigma_1}{S}dB_1(s) - \frac{\sigma_2}{I}dB_2(s) - \int_{\mathscr{Y}}\left[\frac{q_1(u)}{S(1+q_1(u))}+\frac{q_2(u)}{I(1+q_1(u))}\right]\tilde{N}(ds,du), \tag{5}$$

where

$$\begin{aligned} \theta \quad = \quad & \max\left\{d+B-b+\frac{\beta\Lambda}{d+B-b}+\sigma_1^2+\int_{\mathscr{Y}}\frac{q_1^2(u)}{(1+q_1(u))}\mu(du)\,,\right. \\ & \left. d+B-bp+a+\gamma+\sigma_2^2+\int_{\mathscr{Y}}\frac{q_2^2(u)}{(1+q_2(u))}\mu(du)\right\}. \end{aligned}$$

By integrating, taking the expectation on both sides of (5) and applying Fubini's theorem, we get

$$\mathbb{E}\left[V(S(s),I(s))\right] \leq V(S(0),I(0)) + \theta\int_0^s \mathbb{E}\left[V(S(v),I(v))\right]dv.$$

From Gronwall Lemma we have for all $0 \leq s \leq T \wedge \tau_k$*,*

$$\mathbb{E}\left[V(S(s),I(s))\right] \leq V(S(0),I(0))e^{\theta s}.$$

Then

$$\mathbb{E}\left[V(S(T\wedge\tau_k),I(T\wedge\tau_k))\right] \leq V(S(0),I(0))e^{\theta(T\wedge\tau_k)} \leq V(S(0),I(0))e^{\theta T} \ \text{ for any } T \geq 0. \tag{6}$$

Since $V(S(T\wedge\tau_k),I(T\wedge\tau_k)) > 0$ *and some component of* $(S(\tau_k),I(\tau_k))$ *is less than or equal to* $\frac{1}{k}$*, then* $V(S(\tau_k),I(\tau_k)) \geq k$*, which implies that*

$$\mathbb{E}\left[V(S(T\wedge\tau_k),I(T\wedge\tau_k))\right] \geq \mathbb{E}\left[V(S(\tau_k),I(\tau_k))\chi_{\{\tau_k<T\}}\right] \geq k\mathbb{P}(\tau_k < T), \tag{7}$$

where $\chi_{\{\tau_k<T\}}$ *is the indicator function of* $\{\tau_k < T\}$*.*

By (6) and (7), we get that for all $T \geq 0$

$$\mathbb{P}(\tau_k < T) \leq \frac{V(S(0),I(0))e^{\theta T}}{k},$$

then $\limsup_{k\to+\infty}\mathbb{P}(\tau_k < T) = 0$*. This completes the proof.*

3. Extinction of the disease

In this section, we are concerned with the conditions of disappearance of the disease in the system (2). The basic reproduction number of stochastic SIS epidemic model (2) without jumps, that is, $q_i = 0$ $(i = 1,2)$, is as follows

$$R_{\text{noise}} = \frac{\beta\Lambda}{(d+B-b+\alpha_1\Lambda)(d+B-bp+a+\gamma+\frac{\sigma_2^2}{2})}.$$

Now, we define the threshold of our stochastic SIS epidemic model (2) as follows

$$R_{\text{jump}} = \frac{\beta\Lambda}{(d+B-b+\alpha_1\Lambda)(d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)},$$

with

$$\varpi = \int_{\mathscr{Y}} [q_2(u) - \ln(1+q_2(u))]\,\mu(du).$$

Remark 3.1. *We have*

$$\begin{aligned}\frac{\beta S}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI} &= \frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda}\\ &-\frac{\beta(d+B-b)}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 SI)(d+B-b+\alpha_1\Lambda)}(\frac{\Lambda}{d+B-b}-S)\\ &-\frac{\beta\alpha_2\Lambda}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 SI)(d+B-b+\alpha_1\Lambda)}I\\ &-\frac{\beta\alpha_3\Lambda}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 SI)(d+B-b+\alpha_1\Lambda)}SI.\end{aligned}$$

For simplicity we denote $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(s)ds$ if f is an integrable function on $[0,+\infty)$.

Definition 3.1. *System (2) is said to be extinct if* $\lim_{t\to\infty}\langle I(t)\rangle = 0$ *a.s.*

On the extinction of the disease in stochastic system (2) we have the following result.

Theorem 3.1. *Let* $(S(t),I(t))$ *be any solution of system (2) with initial value* $(S(0),I(0)) \in \Delta$. *Then*

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \le (d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)(R_{jump}-1).$$

Moreover, if $R_{jump} < 1$, *then* $\lim_{t\to\infty}\langle I(t)\rangle = 0$ *a.s, and* $\lim_{t\to\infty}\langle S(t)\rangle = \frac{\Lambda}{d+B-b}$ *a.s. That is to say, the disease dies out with probability one.*

Proof. *Let* $(S(0),I(0)) \in \Delta$. *Applying Itô's formula to the second equation of system (2) leads to*

$$\begin{aligned}d\ln I &= \left[f(S,I)-(d+B-bp+a+\gamma)-\frac{\sigma_2^2}{2}-\int_{\mathscr{Y}}[q_2(u)-\ln(1+q_2(u))]\,\mu(du)\right]dt\\ &+\sigma_2 dB_2(t)+\int_{\mathscr{Y}}\ln(1+q_2(u))\tilde{N}(dt,du).\end{aligned}$$

Since $(S(t),I(t)) \in \Delta$, *then by Remark 3.1, we have*

$$f(S,I) \le \frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda}.$$

Hence

$$d\ln I \leq \left[\frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda} - (d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)\right]dt + \sigma_2 dB_2(t) + \int_{\mathscr{Y}} \ln(1+q_2(u))\tilde{N}(dt,du), \tag{8}$$

Then, by integrating inequality (8) we obtain

$$\ln I(t) \leq (d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)(R_{jump}-1)t + G(t) + H(t) + \ln I(0), \tag{9}$$

where $G(t)$ and $H(t)$ are defined by

$$G(t) = \int_0^t \sigma_2 dB_2(s), \;\; H(t) = \int_0^t \int_{\mathscr{Y}} \ln(1+q_2(u))\tilde{N}(ds,du).$$

Thus

$$\langle G,G\rangle_t = \int_0^t \sigma_2^2 ds = t\sigma_2^2, \;\; \langle H,H\rangle_t = t\int_{\mathscr{Y}} [\ln(1+q_2(u))]^2 \mu(du) < tC.$$

By the strong law of large numbers for martingales (see, e.g., [34]*), we have*

$$\lim_{t\to\infty}\frac{G(t)}{t} = \lim_{t\to\infty}\frac{H(t)}{t} = 0 \quad a.s. \tag{10}$$

Dividing by t on the both sides of (9) and letting $t\to\infty$, we get

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq (d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)(R_{jump}-1),$$

which ensures that if, $R_{jump} < 1$ holds, then

$$\lim_{t\to\infty}\langle I(t)\rangle = 0 \quad a.s. \tag{11}$$

On the other hand, we have

$$\begin{aligned} d(S+I) &= [\Lambda - (d+B-b)S - (d+B-b+a)I]dt + \sigma_1 S dB_1(t) + \sigma_2 I dB_2(t) \\ &\quad + \int_{\mathscr{Y}} (q_1(u)S + q_2(u)I)\tilde{N}(ds,du) \\ &= \left[(d+B-b)(\frac{\Lambda}{d+B-b} - S) - (d+B-b+a)I\right]dt \\ &\quad + \sigma_1 S dB_1(t) + \sigma_2 I dB_2(t) \\ &\quad + \int_{\mathscr{Y}} (q_1(u)S + q_2(u)I)\tilde{N}(ds,du). \end{aligned}$$

Then

$$\begin{aligned} \frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} &= (d+B-b)\left\langle \frac{\Lambda}{d+B-b} - S(t)\right\rangle - (d+B-b+a)\langle I(t)\rangle \\ &\quad + \frac{\sigma_1}{t}\int_0^t S(s)dB_1(s) + \frac{\sigma_2}{t}\int_0^t I(s)dB_2(s) \\ &\quad + \frac{1}{t}\int_0^t \int_{\mathscr{Y}} (q_1(u)S + q_2(u)I)\tilde{N}(ds,du). \end{aligned}$$

This yields

$$\left\langle \frac{\Lambda}{d+B-b} - S(t) \right\rangle = \frac{d+B-b+a}{d+B-b} \langle I(t) \rangle + \Phi(t), \tag{12}$$

where

$$\begin{aligned} \Phi(t) &= \frac{1}{d+B-b} \left\{ \frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} - \frac{\sigma_1}{t} \int_0^t S(s) dB_1(s) \right. \\ &\left. - \frac{\sigma_2}{t} \int_0^t I(s) dB_2(s) - \frac{1}{t} \int_0^t \int_{\mathscr{Y}} (q_1(u) S + q_2(u) I) \tilde{N}(ds, du) \right\}. \end{aligned}$$

By the fact that $(S(t), I(t)) \in \Delta$ *and the large number theorem for martingales, we have*

$$\lim_{t\to\infty} \Phi(t) = 0 \quad a.s. \tag{13}$$

Hence from (11), (12) and (13) we get that

$$\lim_{t\to\infty} \left\langle \frac{\Lambda}{d+B-b} - S(t) \right\rangle = 0 \quad a.s,$$

i.e., $\lim_{t\to\infty} \langle S(t) \rangle = \frac{\Lambda}{d+B-b}$ *a.s. The theorem is proved.*

Remark 3.2. *By the basic inequality* $x - 1 - \ln x \geq 0$ *for any* $x > 0$*, we have*

$$\begin{aligned} \varpi &= \int_{\mathscr{Y}} [q_2(u) - \ln(1+q_2(u))] \, \mu(du) \\ &= \int_{\mathscr{Y}} [(1+q_2(u)) - 1 - \ln(1+q_2(u))] \, \mu(du) \geq 0. \end{aligned}$$

Then $R_{jump} \leq R_{noise}$*, thus it is possible that* $R_{jump} < 1 < R_{noise}$*. Which means that the disease in stochastic model (2) with jumps will go extinct with probability one but the disease in model (2) without jumps is persistent in the mean (see Example 5.1).*

4. Persistence in mean of the disease

In this section, we will focus on the conditions which guarantee the persistence in mean of the disease.

Definition 4.1. *System (2) is said to be persistent in mean if* $\liminf_{t\to\infty} \langle I(t) \rangle > 0$ *a.s.*

Next, we give a lemma which will be used to prove persistence in mean of the disease (see Lemma 17 in [35]).

Lemma 4.1. *Let* $f \in \mathscr{C}([0,+\infty) \times \Omega, (0,+\infty))$ *and* $F \in \mathscr{C}([0,+\infty) \times \Omega, \mathbb{R})$ *such that* $\lim_{t\to\infty} \frac{F(t)}{t} = 0$ *a.s. If for all* $t \geq 0$,

(i) $\ln f(t) \geq \delta_0 t - \delta \langle f(t) \rangle t + F(t)$ *a.s, then* $\liminf_{t\to\infty} \langle f(t) \rangle \geq \frac{\delta_0}{\delta}$ *a.s.*
(ii) $\ln f(t) \leq \delta_0 t - \delta \langle f(t) \rangle t + F(t)$ *a.s, then* $\limsup_{t\to\infty} \langle f(t) \rangle \leq \frac{\delta_0}{\delta}$ *a.s,*

where $\delta_0 \geq 0$ *and* $\delta > 0$ *are two real numbers.*

Theorem 4.1. *If* $R_{jump} > 1$*, then the solution* $(S(t), I(t))$ *of system (2) with initial value* $(S(0), I(0)) \in \Delta$ *is persistent in the mean. In addition, we have*

(i) $0 < I^* \leq \liminf_{t\to\infty} \langle I(t) \rangle \leq \limsup_{t\to\infty} \langle I(t) \rangle \leq J^*$,

(ii) $0 < \frac{d+B-b+a}{d+B-b} I^* \leq \liminf_{t\to\infty} \left\langle \frac{\Lambda}{d+B-b} - S(t) \right\rangle \leq \limsup_{t\to\infty} \left\langle \frac{\Lambda}{d+B-b} - S(t) \right\rangle \leq \frac{d+B-b+a}{d+B-b} J^*$,

where

$$
\begin{aligned}
I^* &= \frac{\Lambda(1-\frac{1}{R_{jump}})}{d+B-b+a+\Lambda(\alpha_2+\alpha_3\frac{\Lambda}{d+B-b})},\\
J^* &= \frac{(d+B-b+\alpha_1\Lambda+\alpha_2\Lambda)(1-\frac{1}{R_{jump}})}{\alpha_2(d+B-b)}.
\end{aligned}
$$

Proof. *(i). According to Remark 3.1, we have*

$$
\begin{aligned}
f(S,I) &\geq \frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda} - \frac{\beta(d+B-b)}{d+B-b+\alpha_1\Lambda}(\frac{\Lambda}{d+B-b} - S)\\
&\quad - \frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda}(\alpha_2+\alpha_3\frac{\Lambda}{d+B-b})I.
\end{aligned}
$$

Then

$$
\begin{aligned}
d\ln I &\geq \left[\frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda} - (d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)\right]dt\\
&\quad - \frac{\beta(d+B-b)}{d+B-b+\alpha_1\Lambda}(\frac{\Lambda}{d+B-b} - S)dt\\
&\quad - \frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda}(\alpha_2+\alpha_3\frac{\Lambda}{d+B-b})Idt + \sigma_2 dB_2(t)\\
&\quad + \int_{\mathscr{Y}} \ln(1+q_2(u))\tilde{N}(dt,du).
\end{aligned}
$$

From (12), and integrating the last inequality we have

$$
\begin{aligned}
\ln I(t) &\geq \frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda}(1-\frac{1}{R_{jump}})t\\
&\quad - \frac{\beta}{d+B-b+\alpha_1\Lambda}\left[d+B-b+a+\Lambda(\alpha_2+\alpha_3\frac{\Lambda}{d+B-b})\right]\langle I(t)\rangle t + \Psi(t),
\end{aligned}
$$

where

$$
\Psi(t) = -\frac{\beta(d+B-b)}{d+B-b+\alpha_1\Lambda}\Phi(t)t + G(t) + H(t) + \ln I(0).
$$

Then from (10) and (13) we have

$$
\lim_{t\to\infty}\frac{\Psi(t)}{t} = 0 \;\; a.s.
$$

So, applying Lemma 4.1 we obtain

$$
\liminf_{t\to\infty} \langle I(t)\rangle \geq \frac{\Lambda(1-\frac{1}{R_{jump}})}{d+B-b+a+\Lambda(\alpha_2+\alpha_3\frac{\Lambda}{d+B-b})} = I^*.
$$

On the other hand, we have

$$\begin{aligned} d\ln I &= \left[f(S,I) - (d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)\right]dt + \sigma_2 dB_2(t) \\ &\quad + \int_{\mathscr{Y}} \ln(1+q_2(u))\tilde{N}(dt,du) \\ &\leq \left[\frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda} - (d+B-bp+a+\gamma+\frac{\sigma_2^2}{2}+\varpi)\right. \\ &\quad \left. -\frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda} + \frac{\beta S}{1+\alpha_1 S+\alpha_2 I}\right]dt + \sigma_2 dB_2(t) \\ &\quad + \int_{\mathscr{Y}} \ln(1+q_2(u))\tilde{N}(dt,du). \end{aligned} \tag{14}$$

Note that

$$\begin{aligned} -\frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda} + \frac{\beta S}{1+\alpha_1 S+\alpha_2 I} &= \frac{\beta\left[(d+B-b)S-\Lambda\right] - \beta\Lambda\alpha_2 I}{(d+B-b+\alpha_1\Lambda)(1+\alpha_1 S+\alpha_2 I)} \\ &\leq -\frac{\beta\Lambda\alpha_2}{(d+B-b+\alpha_1\Lambda)(1+\alpha_1 S+\alpha_2 I)} I \\ &\leq -\frac{\beta\Lambda\alpha_2(d+B-b)}{(d+B-b+\alpha_1\Lambda)(d+B-b+\alpha_1\Lambda+\alpha_2\Lambda)} I. \end{aligned}$$

Then, by integrating (14), we obtain

$$\begin{aligned} \ln I(t) &\leq \frac{\beta\Lambda}{d+B-b+\alpha_1\Lambda}(1-\frac{1}{R_{jump}})t - \frac{\beta\Lambda\alpha_2(d+B-b)}{(d+B-b+\alpha_1\Lambda)(d+B-b+\alpha_1\Lambda+\alpha_2\Lambda)}\langle I(t)\rangle t \\ &\quad + G(t) + H(t) + \ln I(0). \end{aligned}$$

By (10) and Lemma 4.1, we get

$$\limsup_{t\to\infty} \langle I(t)\rangle \leq \frac{(d+B-b+\alpha_1\Lambda+\alpha_2\Lambda)(1-\frac{1}{R_{jump}})}{\alpha_2(d+B-b)} = J^*.$$

(ii). From (12) and (13) we have

$$\liminf_{t\to\infty} \left\langle \frac{\Lambda}{d+B-b} - S(t)\right\rangle = \frac{d+B-b+a}{d+B-b} \liminf_{t\to\infty} \langle I(t)\rangle,$$

and

$$\limsup_{t\to\infty} \left\langle \frac{\Lambda}{d+B-b} - S(t)\right\rangle = \frac{d+B-b+a}{d+B-b} \limsup_{t\to\infty} \langle I(t)\rangle.$$

Hence (ii) holds, which finishes the proof.

Remark 4.1. *If $R_{jump} = 1$, then $I^* = J^* = 0$, and consequently*

$$\lim_{t\to\infty} \langle I(t)\rangle = \lim_{t\to\infty} \left\langle \frac{\Lambda}{d+B-b} - S(t)\right\rangle = 0 \quad a.s.$$

Then the stochastic model system (2) is nonpersistent in the mean.

5. Numerical examples

Example 5.1. *We consider the following parameters* $\Lambda = 10$, $\beta = 0.01$, $b = 0.1$, $B = 0.05$, $d = 0.1$, $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $\alpha_3 = 0.05$, $p = 0.7$, $a = 0.01$, $\gamma = 0.2$, $\sigma_1 = 0.01$, $\sigma_2 = 0.2$, $q_1(u) = 0.03$, $q_2(u) = 0.7$, $\mathscr{Y} = (0, +\infty)$, $\mu(\mathscr{Y}) = 1$. *By calculation, we obtain* $R_{jump} = 0.83 < 1$*, then, by Theorem 3.1, we deduce that the disease dies out, and the solution* $(S(t), I(t))$ *of model (2) obeys*

$$\lim_{t\to\infty} \langle S(t)\rangle = 200 \quad a.s,$$

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq -0.08 < 0 \quad a.s.$$

On the other hand, we have $R_{noise} = 1.25 > 1$*, which means that the disease will prevail. This implies that Lévy jumps suppress the disease outbreak.*

Example 5.2. *We keep all the system (2) parameters the same as in Example 5.1 except that* β *is increased to* 0.03 *from* 0.01. *Then* $R_{jump} = 2.5 > 1$*, and we can conclude, by Theorem 4.1, that the disease persists in the population.*

6. Conclusion

In this paper, we consider the dynamical behavior of a stochastic SIS epidemic model with vertical transmission and specific functional response which is perturbed by both Gaussian white noise and Lévy jump noise. The functional response used in this work covers the most functional responses used by several authors such as the saturated incidence rate, the Beddington-DeAngelis functional response, and the Crowley-Martin functional response. First of all, we established the existence and uniqueness of a global positive solution to the stochastic model with jumps. Then we obtain sufficient conditions for extinction of the disease. Also we establish sufficient conditions for persistence in the mean of the disease. We have shown that when R_{jump} is less than one the disease will go to extinction (Theorem 3.1). In the case where it is greater than one, the disease will be persistent in mean (Theorem 4.1). Moreover, we find that R_{noise} is less than R_{jump}. This implies that Lévy jumps can further suppress the disease outbreak (Remark 3.2). In addition, when R_{jump} is equal one the system (2) is nonpersistent in the mean (Remark 4.1). Numerical examples were given to illustrate the results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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