Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2019, 2019:24
https://doi.org/10.28919/cmbn/4075
ISSN: 2052-2541

OPTIMAL HARVESTING AND STABILITY ANALYSIS IN A LESLIE-GOWER DELAYED PREDATOR-PREY MODEL

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Abstract. A delayed Leslie-Gower predator-prey model with continuous threshold prey harvesting is studied. Existence and local stability of the positive equilibrium of the system with or without delay are completely determined in the parameter plane. Considering delay as parameter, we investigate the effect of delay on stability of the coexisting equilibrium. It is observed that there are stability switches and a Hopf bifurcation occurs when the delay crosses some critical values. Employing the normal form theory, the direction and stability of the Hopf bifurcations are explicitly determined by the parameters of the system. Optimal harvesting is also investigated and some numerical simulations are given to support and extend our theoretical results.

Keywords: harvesting; Hopf bifurcation; retarded optimal control; stability analysis.

2010 AMS Subject Classification: 37N25.

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Received March 27, 2019

1. INTRODUCTION

Leslie has introduced a predator-prey model [1], including support capability that the environment provides predators is proportional to the number of prey. Leslie advances that the growth rate of predators and preys admits an upper limit which can be approached under certain conditions: for the predator when the number of prey is high, for the prey when predator numbers (can be also the number of prey) is low [1,2]. The Leslie-Gower term means in absence of preys, the predators have an oscillatory behavior.

There are many predator-prey models in the literature with Leslie-Gower term or a modified Leslie-Gower term and Holling type II functional response [3, 5–8, 11–13]. Some of them analyze bifurcations [3,4,14], persistence [9] or seasonally varying parameters [10]. The Leslie-Gower predator-prey model has not yet been analyzed as in this paper, considering optimal harvest and response function of type III.

Profit, over-exploitation and extinction of a species being harvested are primary concerns in ecology and commercial harvesting industries. Thus, current research incorporates a harvesting component in mathematical models to study the effects it has on one or multiple species. This has attracted interest from the commercial harvesting industry and from many scientific communities including biology, ecology, and economics.

Most predator-prey models in the literature consider either constant or linear harvesting functions [15, 16, 19, 20]. Recently, Tchinda et al., Tankam et al. [21, 23] considered a system of delay differential equations modeling the predator-prey dynamics with continuous threshold prey harvesting and Holling response function of type III. In [21], the model system was given by

$$
\left\{
\begin{array}{rcl}
\dot{x}(t) & = & \varphi(x(t)) - my(t)p(x(t)) - H(x(t)), \\
\dot{y}(t) & = & [-d + cmp(x(t-\tau))]y(t),
\end{array}
\right.
\tag{1}
$$

where $x(t)$ and $y(t)$ represent the population of preys and predators at time t respectively. The parameter d is the natural mortality rate of predators. Parameters c and m are positive constants. The function

$$
\varphi(x) = rx\left(1 - \frac{x}{K}\right) \tag{2}
$$

models the behavior of preys in absence of predators, where r denote the growth rate of preys when x is small, and K is the capacity of the environment to support the preys. The functions $H(x)$ and $p(x)$ which are the harvesting function of the preys and the response function of predators to preys respectively, are defined by

$$H(x) = \begin{cases} 0 \text{ if } x < T, \\ \\ \dfrac{h(x-T)}{h+x-T} \text{ if } x \geq T, \end{cases} \tag{3}$$

and

$$p(x) = \frac{x^2}{ax^2+bx+1}, \tag{4}$$

where a is a positive constant and b is a nonnegative constant. This function is one of potential response function of predators to preys, modeling the consumption of preys by predators. It reflects very small predation when the number of preys is small($p'(0) = 0$), and a group of advantage for the preys when the number of prey is hight ($p(x)$ tends to $\frac{1}{a}$ when x tends to infinity). For the harvesting function, T is the threshold value. In this way, once the prey population reaches the size $x = T$, then harvesting starts and increases smoothly to a limit value h. Here, a time delay τ is in the predator response term $p(x(t))$ in the predator equation. This delay can be regarded as a gestation period or reaction time of the predators.

In [23], System (1) has been investigated, but with a piecewise linear threshold policy harvesting given by

$$H(x) = \begin{cases} 0 \text{ if } x < T_1, \\ \\ \dfrac{h(x-T_1)}{T_2-T_1} \text{ if } T_1 \leq x \leq T_2, \\ \\ h \text{ if } x \geq T_2. \end{cases} \tag{5}$$

This piecewise linear threshold policy harvesting has been previously introduced in [22] in a predator prey model without delay where a Holling response function of type II was considered.

In these models, global qualitative and bifurcation analysis are combined to determine the global dynamics of the model. But, note that, all these models do not take into account the fact that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. This assumption leads the Leslie- Gower formulation.

On the other hand, time delay plays an important role in many biological dynamical systems, being particularly relevant in ecology, where time delays have been recognized to contribute critically to the stable or unstable outcome of prey densities due to predation. The introduction of time delay into the population model is more realistic to model the interaction between the predator and prey populations and the population models with time delay are of current research interest in mathematical biology [29, 31]. There is extensive literature about the effects of delay on the dynamics of predator-prey models.

In this paper, we consider a delayed Leslie-Grower predator-prey model both with refuge and the piecewise linear threshold policy harvesting given by Eq. (5). The Leslie-Gower formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. Indeed, Leslie introduced a predator-prey model where the carrying capacity of the predator environment is proportional to the number of prey [1, 2]. He stresses the fact that there are upper limits to the rates of increase of both prey x and predator y, which are not recognized in the Lotka-Volterra model.

This paper is organized as follows. In the Section 2, we give a description of the model. In Section 3, some preliminary results on the boundedness of solutions for System (6) when Eq.(5) are given. Existence and unicity of equilibria are investigated. Section 4 deals with the linear stability analysis of the model system with and without time delay. In Section 5, direction and stability of Hopf bifurcation are presented. In Section 6, optimal harvest policy of population model is derived. Numerical results to illustrate the analytical findings are presented in Section 7 and, finally, a summary is presented in Section 8.

2. THE MODEL

It is well known that time delay can play an important role in biological dynamical systems, where it has been recognized to contribute critically to the stable or unstable outcome of prey densities due to predation. Therefore, let us analyze the following delayed predator-prey model:

$$
\left\{
\begin{array}{rcl}
\dot{x}(t) & = & \Big(r_1 - b_1 x(t)\Big)x(t) - a_1(1-m)x(t)y(t) \\
 & - & H\big(x(t)\big), \\
\dot{y}(t) & = & \Big[r_2 - \dfrac{a_2 y(t-\tau)}{(1-m)x(t-\tau)}\Big]y(t),
\end{array}
\right.
\tag{6}
$$

where $x(t)$ denotes the Prey population at time t and $y(t)$ the Predator population at time t. All parameters are positive and m is such that $0 \leq m < 1$. This parameter is the rate of refuge of prey population. This means that when $m = 0$, all preys are available for predation. $mx(t)$ models the capacity of a refuge at time t and so refuge protecting $mx(t)$ of the prey population. It therefore remains $(1-m)x(t)$ of the preys available for predation. Parameters r_1 and r_2 are the intrinsic growth rate of the preys and predators respectively, a_1 denotes the predation rate per unit of time, $\dfrac{r_1}{b_1}$ is the carrying capacity of the prey's environment and $\dfrac{r_2}{a_2}x(t)$ is the carrying capacity of the predator's environment which is proportional to the number of prey. Here, we incorporate a single discrete delay $\tau > 0$ in the negative feedback of the predator's density.

Let us denote by $\mathbb{R}_2^+$ the nonnegative quadrant and by $\text{int}(\mathbb{R}_2^+)$ the positive quadrant. For $\theta \in [-\tau, 0]$, we use the following conventional notation:

$$x_t(\theta) = x(t+\theta).$$

Then the initial conditions for this system take the form

$$
\left\{
\begin{array}{rcl}
\dot{x}_0(\theta) & = & \phi_1(\theta), \\
\dot{y}_0(\theta) & = & \phi_2(\theta),
\end{array}
\right.
\tag{7}
$$

for all $\theta \in [-\tau, 0]$, where $(\phi_1, \phi_2) \in C([-\tau, 0], \mathbb{R}_+^2), x(0) = \phi_1(0) > 0$ and $y(0) = \phi_2(0) > 0$.

For ecological reason, as in [23], we make the following assumption. We assume that:

: (i) $0 < x(0) \leq \dfrac{r_1}{b_1} \equiv K$;

: (ii) $T_1 < T_2 < K$.

In fact, the first assumption comes from the fact that it is not plausible to have an initial value of the preys $x(0)$ at time $t = 0$ which is greater than the carrying capacity K of the preys. Moreover if $T_1 = T_2$, then the harvesting function becomes a discrete harvesting. In other hand, if we assume $T_2 \geq K$, then we will not have some harvest after T_2 since the first assumption leads to $0 \leq x(t) \leq K$.

3. Preliminary results

3.1. Boundedness of solutions. We start by showing that solutions of System (6) and System (5) that start in $\mathbb{R}_+^2$ will remain there and are uniformly bounded. Indeed, we have the following theorem.

Theorem 1. *: Let Assumption 2-(i) holds. Then, every solution of System (6) that starts in $\mathbb{R}_+^2$ will remain there and is uniformly bounded.*

Proof. : Let $(x_0, y_0) \in \mathbb{R}_+^2$ be given and let us denote for each $t \geq 0$, $(x(t), y(t))$ the orbit of System (6) passing through (x_0, y_0) at $t = 0$. Then, we can find that $(x(t), y(t)) \in \mathbb{R}_+^2$ for all $t \geq 0$. Thus, every solution of System (1) that starts in $\mathbb{R}_+^2$ will remain there. From the $\dot{x}$-equation of System (6), we have

$$\dot{x}(t) \leq \Big(r_1 - b_1 x(t)\Big) x(t).$$

Applying a differential inequality [28] gives

$$x(t) \leq \frac{1}{\dfrac{b_1}{r_1} + \left(\dfrac{1}{x(0)} - \dfrac{b_1}{r_1}\right) e^{-r_1 t}}$$

for all $t \geq 0$. Since $0 < x(0) \leq \dfrac{r_1}{b_1}$ from Assumption 2-(i), it follows that $x(t) \leq \dfrac{r_1}{b_1}$ for all $t \geq 0$. Now, let us check for the boundedness of $y(t)$.

From the predator equation, we have $\dot{y}(t) \leq r_2 y(t)$. Hence, for $t > \tau$, $y(t) \leq y(t-\tau)e^{r_2\tau}$. This equation is equivalent for $t > \tau$, to

$$y(t-\tau) \geq y(t)e^{-r_2 t}. \tag{8}$$

Moreover, for any $\delta > 1$, there exists a positive T_δ such that for $t > T_\delta$, $x(t) < \delta \dfrac{r_1}{b_1}$. Eq.(8) gives for $t > T_\delta + \tau$,

$$\dot{y}(t) < y(t) \left(r_2 - \frac{a_2 e^{-r_2\tau}}{\delta(1-m)\dfrac{r_1}{b_1}} y(t) \right),$$

which implies, by the same arguments use for x, that $\limsup\limits_{t\to+\infty} y(t) < \dfrac{r_2}{a_2 \delta(1-m)\frac{r_1}{b_1} e^{r_2\tau}}$. The conclusion of this lemma holds for $\delta \to 1$. $\square$

3.2. Equilibria of the model. In this section we analyze some equilibria properties of System (6)-(5). These steady states, which are determined analytically by setting $\dot{x} = \dot{y} = 0$, are independent of the delay τ. The following results holds:

Proposition 1. *: Let $K = \dfrac{r_1}{b_1}$, $b_0 = b_1 + \dfrac{r_2 a_1 (1-m)^2}{a_2}$, $K_0 = \dfrac{r_1}{b_0}$, $\varphi : x \longmapsto (r_1 - b_1 x)x$ and $\varphi_0 : x \longmapsto (r_1 - b_0 x)x$.*

(1) System (6)-(5) has one or more equilibria with $y = 0$ (without predators).
- *One equilibrium in $\mathbb{R}_+^* \times \{0\}$ under some conditions. More precisely,*
  - *if $\varphi(T_2) > h$, then $E_1(x_1, 0)$ is the unique equilibrium of the model with $x_1 \in \left[\dfrac{K}{2}, K\right]$ if $T_2 \leq \dfrac{K}{2}$, or $x_1 \in [T_2, K]$ if $T_2 \geq \dfrac{K}{2}$.*
  - *if $\varphi(T_2) \leq h$ and $T_2 \geq \dfrac{K}{2}$, then $\tilde{F}(\tilde{x}, 0)$ is the unique equilibrium of the model with $\tilde{x} \in [T_1, T_2]$.*
- *Two equilibria $\tilde{F}(\tilde{x}, 0)$ and $\tilde{E}\left(\dfrac{K}{2}, 0\right)$ in $\mathbb{R}_+^* \times \{0\}$, where $\tilde{x} \in [T_1, T_2]$ under the conditions $T_2 \leq \dfrac{K}{2}$, $\varphi(T_2) \leq h$ and $\varphi\left(\dfrac{K}{2}\right) = h$.*
- *Three equilibria $\tilde{F}(\tilde{x}, 0)$, $E_1(x_1, 0)$ and $E_2(x_2, 0)$ in $\mathbb{R}_+^* \times \{0\}$, where $\tilde{x} \in [T_1, T_2]$, $x_1 \in \left]\dfrac{K}{2}, K\right]$, $x_2 \in \left[T_2, \dfrac{K}{2}\right]$ under the conditions $T_2 \leq \dfrac{K}{2}$, $\varphi(T_2) \leq h$ and $\varphi\left(\dfrac{K}{2}\right) > h$.*

(2) Under some conditions, System (6)-(5) has one or more coexistence equilibria.
- *A unique equilibrium in these different cases:*
  - *if $K_0 < T_1$, then $G_0(K_0, y_0)$ is the equilibrium of the model.*
  - *if $K_0 \in [T_1, T_2]$, then $G(x^*, y^*)$ is the equilibrium of the model, with $x^* \in [T_1, T_2]$ and $y^* = \dfrac{r_2(1-m)x^*}{a_2}$.*
  - *if $K_0 \geq T_2$ and $\varphi(T_2) > h$, then $G_1(x_1^*, y_1^*)$ is the equilibrium of the model with $x_1^* \in [T_2, K_0]$ and $y_1^* = \dfrac{r_2(1-m)x_1^*}{a_2}$.*
- *Two equilibria $G(x^*, y^*) \in [T_1, T_2] \times \mathbb{R}_+^*$ and $\tilde{G}_0\left(\dfrac{K_0}{2}, \tilde{y}_0\right)$ when $\dfrac{K_0}{2} > T_2$, $\varphi_0(T_2) \leq h$ and $\varphi_0\left(\dfrac{K_0}{2}\right) = h$.*
- *Tree equilibria $G(x^*, y^*) \in [T_1, T_2] \times \mathbb{R}_+^*$, $G_1(x_1^*, y_1^*) \in \left]\dfrac{K_0}{2}, K_0\right] \times \mathbb{R}_+^*$ and $G_2(x_2^*, y_2^*) \in \left[T_2, \dfrac{K_0}{2}\right[\times \mathbb{R}_+^*$ when $\dfrac{K_0}{2} > T_2$, $\varphi_0(T_2) \leq h$ and $\varphi_0\left(\dfrac{K_0}{2}\right) > h$.*

Remark 1. *: Concerning parameters K and K_0, we always have $K_0 \leq K$.*

Proof. : An equilibrium $S(x,y)$ of the model is solution of Eq.(9) when $x < T_1$, Eq. (10) when $T_1 \leq x \leq T_2$ and Eq. (11) when $x \geq T_2$, where

$$\begin{cases} (r_1 - b_1 x)x - a_1(1-m)xy = 0, \\ \\ \left[r_2 - \dfrac{a_2 y}{(1-m)x}\right]y = 0, \end{cases} \tag{9}$$

$$\begin{cases} (r_1 - b_1 x)x - a_1(1-m)xy - \dfrac{h(x-T_1)}{T_2 - T_1} = 0, \\ \\ \left[r_2 - \dfrac{a_2 y}{(1-m)x}\right]y = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} (r_1 - b_1 x)x - a_1(1-m)xy - h = 0, \\ \\ \left[r_2 - \dfrac{a_2 y}{(1-m)x}\right]y = 0. \end{cases} \tag{11}$$

From the second equation of System (9), System (10) or System (11), we have $y = 0$ or $y = \dfrac{r_2(1-m)x}{a_2}$.

When $y = 0$, the equilibria $(0,0)$ and $\left(\dfrac{r_1}{b_1}, 0\right)$ exist on $[0, T_1[$. This is impossible since $\dfrac{r_1}{b_1} = K > T_1$. Moreover, we have the following equations,

$$\begin{aligned} &(r_1 - b_1 x)x - a_1(1-m)xy - \frac{h(x-T_1)}{T_2 - T_1} = 0 \text{ on } [T_1, T_2], \\ &\textit{and} \\ &(r_1 - b_1 x)x - a_1(1-m)xy - h = 0 \text{ on } [T_2, K]. \end{aligned}$$

- On $[T_1, T_2]$, the identity at the equilibrium gives equation $-b_1 x^2 + \left(r_1 - \dfrac{h}{T_2 - T_1}\right)x + \dfrac{hT_1}{T_2 - T_1} = 0$ which admits a unique positive solution.

  Let us consider $f(x) = -b_1 x^2 + \left(r_1 - \dfrac{h}{T_2 - T_1}\right)x + \dfrac{hT_1}{T_2 - T_1}$. Then $f(T_1) > 0$ and $f(T_2) = \varphi(T_2) - h$. Hence, if $\varphi(T_2) \leq h$, a unique solution exists on $[T_1, T_2]$.
- On $[T_2, K]$, the identity at the equilibrium gives equation $-b_1 x^2 + r_1 x - h = 0$. Its discriminant is

$$\Delta = r_1^2 - 4b_1 h = 4b_1 \left(\varphi\left(\frac{K}{2}\right) - h\right).$$

Hence, if $\frac{K}{2} > h$, there are two positive solutions, which are both on $[T_2, K]$, when $T_2 < \frac{K}{2}$ and $\varphi(T_2) \leq h$. Besides, when $\varphi(T_2) > h$, just one of the solutions is on $[T_2, K]$.

Still according to the sign of the discriminant Δ, if $\varphi(\frac{K}{2}) = h$, $x = \frac{K}{2}$ is the unique solution on $[T_2, K]$ when $\frac{K}{2} \geq T_2$. There is no solution when $\frac{K}{2} < T_2$.

When $y \neq 0$, from the second equation of System (9), System (10) and System (11), we have $y = \frac{r_2(1-m)}{a_2}x$. Replacing it in the first equation gives $(r_1 - b_0 x)x - H(x) = 0$. On $[0, T_1]$, the unique solution of this equation is $x = K_0$, which exists if and only if $K_0 \leq T_1$. Moreover, we have the following equations,

$$
\begin{aligned}
&(r_1 - b_0 x)x - \frac{h(x - T_1)}{T_2 - T_1} = 0 \text{ on } [T_1, T_2] \\
&\textit{and} \\
&(r_1 - b_0 x)x - h = 0 \text{ on } [T_2, K].
\end{aligned}
$$

- On $[T_1, T_2]$, if $K_0 < T_1$, there is no equilibrium on $[T_1, T_2]$. Else, the identity at the equilibrium gives equation $-b_0 x^2 + \big(r_1 - \frac{h}{T_2 - T_1}\big)x + \frac{hT_1}{T_2 - T_1} = 0$ which admits a unique positive solution.

  Let us consider $f_0(x) = -b_0 x^2 + \big(r_1 - \frac{h}{T_2 - T_1}\big)x + \frac{hT_1}{T_2 - T_1}$. Then $f_0(T_1) = b_0 T_1 (K_0 - T_1) > 0$ and $f_0(T_2) = \varphi_0(T_2) - h$. Hence, if $\varphi_0(T_2) \leq h$, a unique solution exists on $[T_1, T_2]$.
- On $[T_2, K]$, the identity at the equilibrium gives $-b_0 x^2 + r_1 x - h = 0$. Its discriminant is $\Delta_0 = r_1^2 - 4b_0 h = 4b_0 \left(\varphi_0(\frac{K_0}{2}) - h\right)$. Hence, when $\frac{K_0}{2} > h$, there are two positive solutions, which are both on $[T_2, K_0]$, when $T_2 < \frac{K_0}{2}$ and $\varphi_0(T_2) \leq h$. Besides, when $\varphi_0(T_2) > h$, just one of the solutions is on $[T_2, K]$ (particularly on $[T_2, K_0]$).

  Still according to the sign of the discriminant, when $\varphi_0\left(\frac{K_0}{2}\right) = h$, $x = \frac{K_0}{2}$ is the unique solution on $[T_2, K]$ when $\frac{K_0}{2} \geq T_2$. There is no solution when $\frac{K_0}{2} < T_2$.

$\square$

Remark 2. *: We summarize the results about equilibria in Fig. 1 and Fig. 2.*

4. Stability analysis

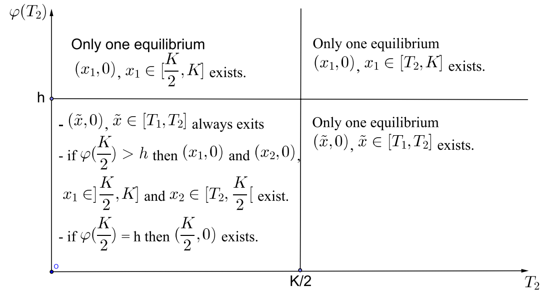


FIGURE 1. Existence and number of equilibria when $y = 0$.

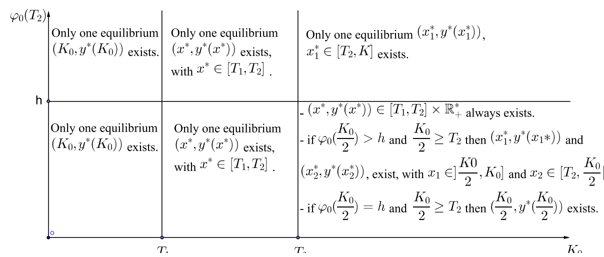


FIGURE 2. Existence and number of equilibria when $y \neq 0$ and $y^*(x^*) = \frac{r_2(1-m)x^*}{a_2}$.

4.1. Stability of equilibria when $\tau = 0$. The Jacobian matrix $J(x,y)$ of System (6) at the equilibrium (x,y) when $T_1 \leq x \leq T_2$, is given by

$$\begin{pmatrix} \varphi'(x) - \dfrac{h}{T_2 - T_1} - a_1(1-m)y & -a_1(1-m)x, \\ \\ \dfrac{a_2 y^2}{(1-m)x^2} & r_2 - \dfrac{2a_2 y}{(1-m)x} \end{pmatrix}.$$

We notice that $r_2 \geq 0$ is always an eigenvalue of any equilibrium $E(x,0)$, which is therefore unstable.

Concerning stability of any equilibrium $G(x^*,y^*)$ with $y^* \neq 0$, the following theorem holds.

Theorem 2. *: Let consider*

$$\Delta_1 = \Big[\varphi'(x^*) - a_1(1-m)y^* - r_2\Big]^2 - 4\Big[2a_1(1-m)r_2 y^* - r_2\varphi'(x^*)\Big],$$
$$\Delta_2 = \Big[\varphi'(x^*) - a_1(1-m)y^* - \frac{h}{T_2 - T_1} - r_2\Big]^2 - 4\Big[2a_1(1-m)r_2 y^* - r_2[\varphi'(x^*) - \frac{h}{T_2 - T_1}]\Big].$$

(1) Let consider an equilibrium $G(x^*,y^*)$ with $x^* \in [0, T_1[\cup]T_2, K]$.

- *If $\Delta_1 > 0$, then the equilibrium is a stable node when $-\varphi'(x^*) + a_1(1-m)y^* + r_2 > 0$ and $2a_1(1-m)r_2y^* - r_2\varphi'(x^*) > 0$.*

- *If* $\Delta_1 = 0$*, then the equilibrium is a stable node when* $-\varphi'(x^*) + a_1(1-m)y^* + r_2 > 0$.
- *If* $\Delta_1 < 0$*, then the equilibrium is a stable focus when* $-\varphi'(x^*) + a_1(1-m)y^* + r_2 > 0$.
- *If* $-\varphi'(x^*) + a_1(1-m)y^* + r_2 = 0$ *and* $2a_1(1-m)r_2y^* - r_2\varphi'(x^*) > 0$*, then the equilibrium is a center.*

(2) Let consider an equilibrium $G(x^*, y^*)$ *with* $x^* \in [T_1, T_2]$.

- *If* $\Delta_2 > 0$*, then the equilibrium is a stable node when* $-\varphi'(x^*) + a_1(1-m)y^* + \dfrac{h}{T_2 - T_1} + r_2 > 0$ *and* $2a_1(1-m)r_2y^* - r_2(\varphi'(x^*) - \dfrac{h}{T_2 - T_1}) > 0$.
- *If* $\Delta_2 = 0$*, then the equilibrium is a stable node when* $-\varphi'(x^*) + a_1(1-m)y^* + \dfrac{h}{T_2 - T_1} + r_2 > 0$.
- *If* $\Delta_2 < 0$*, then the equilibrium is a stable focus when* $-\varphi'(x^*) + a_1(1-m)y^* + \dfrac{h}{T_2 - T_1} + r_2 > 0$.
- *If* $-\varphi'(x^*) + a_1(1-m)y^* + \dfrac{h}{T_2 - T_1} + r_2 = 0$ *and* $2a_1(1-m)r_2y^* - r_2(\varphi'(x^*) - \dfrac{h}{T_2 - T_1}) > 0$*, then the equilibrium is a center.*

Proof. : The Jacobian matrix $J(x^*, y^*)$ of System (6) at the equilibrium (x^*, y^*) becomes

$$\begin{pmatrix} \varphi'(x^*) - a_1(1-m)y^* - H'(x^*) & -a_1(1-m)x^* \\ & \\ r_2\dfrac{y^*}{x^*} & -r_2, \end{pmatrix},$$

where $H'(x) = 0$ for $x \in [0, T_1[\cup]T_2, K]$ and $H'(x) = \dfrac{h}{T_2 - T_1}$ for $x \in [T_1, T_2]$.

Therefore, the eigenvalues are given by the following equation:

$$\begin{aligned} &\lambda^2 + \lambda\Big[-\varphi'(x^*)) + a_1(1-m)y^* + H'(x^*) + r_2\Big] \\ &+2a_1(1-m)r_2y^* - r_2(\varphi'(x^*) - H'(x^*)) = 0. \end{aligned} \tag{12}$$

The discriminant of this equation is given by

$$\begin{aligned} \Delta_= &\Big[\varphi'(x^*) - a_1(1-m)y^* - H'(x) - r_2\Big]^2 \\ -4&\Big[2a_1(1-m)r_2y^* - r_2[\varphi'(x^*) - H'(x)]\Big], \end{aligned}$$

which is equal to Δ_1 on $[0, T_1[\cup]T_2, K]$ and Δ_2 on $[T_1, T_2]$.

- When $\Delta > 0$, $J(x^*, y^*)$ has two positive eigenvalues which are both negatives if $-\varphi'(x^*) + a_1(1-m)y^* + H'(x^*) + r_2 > 0$ and $2a_1(1-m)r_2y^* - r_2(\varphi'(x^*) - H'(x^*)) > 0$.

- When $\Delta = 0$, $J(x^*, y^*)$ has one positive eigenvalue which is negative if $-\varphi'(x^*) + a_1(1-m)y^* + H'(x^*) + r_2 > 0$.
- When $\Delta < 0$, $J(x^*, y^*)$ has two conjugated complex eigenvalues with a positive real part equal to $\varphi'(x^*) - a_1(1-m)y^* - H'(x) - r_2$.
- When $-\varphi'(x^*) + a_1(1-m)y^* + H'(x^*) + r_2 = 0$ and $2a_1(1-m)r_2y^* - r_2(\varphi'(x^*) - H'(x^*)) > 0$, $J(x^*, y^*)$ has pure imaginary eigenvalues.

Hence, the conclusions follow. □

Remark 3. *: The importance of this section is due to the fact that, if an equilibrium of System (6)-(5) is unstable for $\tau = 0$, it remains unstable for $\tau > 0$ [24, 25]. Then, any equilibrium of System (6) in the form $E(x, 0)$ is unstable when $\tau > 0$. Concerning stability of equilibria when $\tau > 0$, we only consider the coexistence equilibria.*

4.2. Stability of coexistence Equilibria for $\tau > 0$ and Hopf Bifurcation.

In order to analyze the stability of coexistence equilibria $G(x^*, y^*)$, let us define new variables $u(t) = x(t) - x^*$ and $v(t) = y(t) - y^*$. Then the linearization of System (6) at G gives

$$\left\{ \begin{array}{rcl} \dot{u}(t) & = & \Big[r_1 - 2b_1x^* - a_1(1-m)y^* - H'(x^*)\Big]u(t) \\ & - & a_1(1-m)x^*v(t), \\ \dot{v}(t) & = & -\Psi'(x^*)y^{*2}u(t-\tau) - r_2v(t-\tau), \end{array} \right. \tag{13}$$

where $H'(x^*) = 0$ for $x^* \in [0, T_1[\cup[T_2, K]$, $H'(x^*) = \dfrac{h}{T_2 - T_1}$ for $x^* \in [T_1, T_2]$ and $\Psi(x^*) = \dfrac{a_2}{(1-m)x^*}$.

The characteristic equation of System (13) at $G(x^*, y^*)$ is given by

$$\lambda^2 - \alpha\lambda + r_2\lambda e^{-\lambda\tau} - r_2\Big(\alpha_1 - a_1(1-m)y^*\Big)e^{-\lambda\tau} = 0, \tag{14}$$

where $\alpha = r_1 - 2b_1x^* - a_1(1-m)y^* - H'(x^*)$.

Note that for $\tau = 0$, the characteristic equations (14) becomes

$$\lambda^2 + (r_2 - \alpha)\lambda - r_2\Big(\alpha - a_1(1-m)y^*\Big) = 0. \tag{15}$$

Since the sum and product of roots are $-(r_2 - \alpha)$ and $-r_2\Big(\alpha - a_1(1-m)y^*\Big)$ respectively, the two roots of (15) are real and negative or complex conjugate with negative real parts if and only if

$$r_2 - \alpha > 0 \quad \text{and} \quad \alpha - a_1(1-m)y^* < 0. \tag{16}$$

Hence, in the absence of time delay, the system is locally asymptotically stable if and only if $r_2 - \alpha > 0$ and $\alpha - a_1(1-m)y^* < 0$.

Now, for $\tau > 0$, if $\lambda = i\omega$ is a root of equation (14), then we have

$$-\omega^2 + \alpha\omega + r_2 i\omega\big(\cos\omega\tau - i\sin\omega\tau\big) - c\big(\cos w\tau - i\sin w\tau\big) = 0,$$

where $c = r_2\Big(\alpha - a_1(1-m)y^*\Big)$.

Separating real and imaginary parts gives

$$r_2\omega\sin\omega\tau - c\cos\omega\tau = \omega^2 \text{ and } r_2\omega\cos\omega\tau + c\sin\omega\tau = \alpha\omega. \tag{17}$$

Eliminating τ by squaring and adding equations of (17), we get the algebraic equation

$$r_2^2\omega^6 + \Big[c^2 + r_2^2(\alpha^2 - r_2^2)\Big]\omega^4 + c^2(\alpha^2 - 2r_2^2)\omega^2 - c^4 = 0. \tag{18}$$

Substituting $\omega^2 = \eta$ in the above equation gives a cubic equation in η of the form

$$r_2^2\eta^3 + \Big[c^2 + r_2^2(\alpha^2 - r_2^2)\Big]\eta^2 + c^2(\alpha^2 - 2r_2^2)\eta - c^4 = 0. \tag{19}$$

Observe that conditions (16) implies $\alpha < r_2$. Since $r_2^2 > 0$ and $-c^4 < 0$, if $c^2 + r_2^2(\alpha^2 - r_2^2) > 0$ or $\alpha^2 - 2r_2^2 < 0$, then by Descartes' rule of sign, Eq.(19) has at least one positive root.

If $\alpha \in]-r_2, r_2[$, then $\alpha^2 - 2r_2^2 < 0$ and Eq.(19) has only one positive root. If $\alpha < -r_2$, then $c^2 + r_2^2(\alpha^2 - r_2^2) > 0$ and Eq.(19) has at least one positive root. So, for any cases, Eq.(19) has at least one positive root.

The following theorem gives a criterion for the switching in the stability behavior of $G^*(x^*, y^*)$ in terms of the delay parameter τ.

Theorem 3. *: Suppose that $G(x^*, y^*)$ exists and is locally asymptotically stable for System (6) with $\tau = 0$. Also let $\eta_0 = \omega_0^2$ be a positive root of Eq.(19). Then there exists a value $\tau = \tau_0$ such that G is locally asymptotically stable for $\tau \in (0, \tau_0]$ and unstable for $\tau > \tau_0$. Furthermore, the system undergoes a Hopf bifurcation at G when $\tau = \tau_0$.*

Proof. : Since ω_0 is a solution of Eq.(18), the characteristic Eq.(14) has the pair of purely imaginary roots $\pm i\omega_0$. From Eq.(17), τ_n^0 for $n = 0, 1, \ldots$ as a function of ω_0 is given by

$$\tau_n^0 = \frac{1}{w_0} \arccos\left\{ \frac{w_0^2(-c+\alpha r_2)}{c^2 + r_2^2 w_0^2} \right\} + \frac{2\pi n}{w_0}. \tag{20}$$

For $\tau = 0$, theorem 2 ensures that G is locally asymptotically stable. Hence, by Butler's lemma [27], G remains stable up to the minimum value of τ_n^0, obtained here for $n = 0$, i.e. for $\tau < \tau_0^0$, so that $\tau^0 = \min_{n \geq 0} \tau_n^0 \equiv \tau_0^0$. The theorem can be completely proved if we can show that

$$sign\left\{ \frac{d(R_e\lambda(\tau))}{d\tau}) \right\}\Big|_{\lambda = i\omega_0} > 0.$$

Differentiating equation (14) with respect to τ yields

$$\Big[2\lambda - \alpha + \Big(r_2 - r_2\tau\lambda + c\tau\Big)e^{-\lambda\tau}\Big]\frac{d\lambda}{d\tau} = \big(r_2\lambda^2 - c\lambda\big)e^{-\lambda\tau}, \tag{21}$$

which gives

$$\begin{aligned}
\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} &= \frac{2\lambda - \alpha + \Big(r_2 - r_2\tau\lambda + c\tau\Big)e^{-\lambda\tau}}{\big(r_2\lambda^2 - c\lambda\big)e^{-\lambda\tau}}, \\
&= -\frac{2\lambda^2 - \alpha\lambda}{\lambda^2(\lambda^2 - \alpha\lambda)} - \frac{r_2}{\lambda(c - r_2\lambda)} - \frac{\tau}{\lambda}, \\
&= -\frac{1}{\lambda^2 - \alpha\lambda} - \frac{1}{\lambda^2} - \frac{r_2}{\lambda(c - r_2\lambda)} - \frac{\tau}{\lambda}.
\end{aligned}$$

Thus, $\mu_0 = sign\left\{ \frac{d(R_e\lambda(\tau))}{d\tau}) \right\}\Big|_{\lambda = iw_0}$ is given by

$$\begin{aligned}
\mu_0 &= sign\left\{ R_e\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} \right\}\Big|_{\lambda = iw_0}, \\
&= sign\left\{ R_e\left[-\frac{1}{\lambda^2 - \alpha\lambda} - \frac{1}{\lambda^2} - \frac{r_2}{\lambda(c - r_2\lambda)} \right] \right\}\Big|_{\lambda = iw_0}, \\
&= sign\left\{ \frac{w_0^2}{w_0^4 + \alpha^2 w_0^2} + \frac{1}{w_0^2} - \frac{r_2^2 w_0^2}{r_2^2 w_0^4 + w_0^2 c^2} \right\} \\
&= sign\left\{ \frac{r_2^2 w_0^4 + 2c^2 w_0^2 + \alpha^2 c^2}{(w_0^4 + \alpha^2 w_0^2)(r_2 w_0^2 + c^2)} \right\} > 0.
\end{aligned}$$

Hence, $sign\left\{ \frac{d(R_e\lambda(\tau))}{d\tau}) \right\}\Big|_{\tau = \tau^0, \lambda = i\omega_0} > 0$. The transversality condition is satisfied and a Hopf bifurcation occurs at $\tau = \tau^0$. This achieves the proof. □

5. PROPERTIES OF HOPF BIFURCATION

In this section, we give some properties of the Hopf bifurcation presented in Theorem 3 and analyse the stability of bifurcated periodic solutions occurring through Hopf bifurcations by using the normal form theory and the center manifold reduction for retarded functional differential equations (RFDEs) due to Hassard, Kazarinoff and Wan [17]. We assume that System (6) undergoes Hopf bifurcation at the positive equilibrium $G(x^*,y^*)$ for $\tau=\tau_j^0$, $(j=0,1,2,...)$ and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation.

Let $x_1(t)=x(t)-x^\star$ and $x_2(t)=y(t)-y^\star$. Then, system (6)-(5) is equivalent to :

$$
(22) \qquad \begin{cases} \dot{x}_1(t) &= \Big[r_1-2b_1x^*-a_1(1-m)y^*\Big]x_1(t) \\ & -H'(x^*)x_1(t)-a_1(1-m)x^*x_2(t) \\ & +f_1\big(x_1(t),x_2(t)\big), \\ \dot{x}_2(t) &= -\Psi'(x^*)y^{*2}x_1(t-\tau)-r_2x_2(t-\tau) \\ & +f_2\big(x_2(t),x_1(t-\tau),x_2(t-\tau)\big), \end{cases}
$$

where

$$
\begin{aligned} f_1\big(x_1(t),x_2(t)\big) &= -a_1(1-m)x_1(t)x_2(t) \\ & -b_1x_1^2(t), \end{aligned}
$$

and

$$
\begin{aligned} & f_2\big(x_2(t),x_1(t-\tau),x_2(t-\tau)\big)=r_2(x_2(t)+y^\star) \\ & -\Big[\Psi(x_1(t-\tau)+x^*)(x_2(t-\tau)+y^*)\Big]\big(x_2(t)+y^\star\big) \\ & +\Psi'(x^*)y^{*2}x_1(t-\tau)+r_2x_2(t-\tau). \end{aligned}
$$

Let $\tau=\tau_j^0+\mu$. Then, $\mu=0$ corresponds to Hopf bifurcation value of System (6) at the positive equilibrium $G(x^\star,y^\star)$. Since System (6) is equivalent to System (22), in the following discussion we use System (22).

In System (22), let $\bar{x}_k(t) = x_k(\tau t)$ and drop the bars for simplicity of notation. Then, System (22) can be rewritten as a system of RFDEs in $\mathscr{C}\big([-1,0],\mathbb{R}^2\big)$ of the form:

$$
\left\{
\begin{array}{rcl}
\dot{x}_1(t) & = & (\tau_j^0+\mu)\Big[r_1 - 2b_1x^* - a_1(1-m)y^*\Big]x_1(t) \\
 & - & (\tau_j^0+\mu)H'(x^*)x_1(t) \\
 & - & (\tau_j^0+\mu)a_1(1-m)x^*x_2(t) \\
 & + & (\tau_j^0+\mu)f_1\big(x_1(t),x_2(t)\big), \\
\dot{x}_2(t) & = & -(\tau_j^0+\mu)\Psi'(x^*)y^{*2}x_1(t-\tau) \\
 & - & (\tau_j^0+\mu)r_2x_2(t-\tau) \\
 & + & (\tau_j^0+\mu)f_2\big(x_2(t),x_1(t-\tau),x_2(t-\tau)\big).
\end{array}
\right. \tag{23}
$$

Define the linear operator $L(\mu):\mathscr{C}\to\mathbb{R}^2$ and the nonlinear operator $f(\cdot,\mu):\mathscr{C}\to\mathbb{R}^2$ by:

$$
\begin{aligned}
L_\mu(\phi) = {}& (\tau_j^0+\mu)\begin{pmatrix} J_0 & J_1 \\ \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \\ \phi_2(0) \end{pmatrix} \\
&+(\tau_j^0+\mu)\begin{pmatrix} 0 & 0 \\ \\ -\Psi'(x^*)y^{*2} & -r_2 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \\ \phi_2(-1) \end{pmatrix}
\end{aligned} \tag{24}
$$

and

$$
f(\phi,\mu) = (\tau_j^0+\mu)\begin{pmatrix} f_1\big(\phi_1(0),\phi_2(0)\big) \\ \\ f_2\big(\phi_2(0),\phi_1(-1),\phi_2(-1)\big) \end{pmatrix} \tag{25}
$$

respectively, where $\phi = (\phi_1,\phi_2)^T \in \mathscr{C}$, $J_0 = r_1 - 2b_1x^* - a_1(1-m)y^* - H'(x^*)$, $J_1 = -a_1(1-m)x^*$.

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta,\mu)$, $-1\le\theta\le 0$ whose elements are of bounded variation such that

$$
L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) \quad \text{for} \quad \phi\in\mathscr{C}\big([-1,0],\mathbb{R}^2\big). \tag{26}
$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_j^0+\mu)\begin{pmatrix} J_0 & J_1 \\ & \\ 0 & 0 \end{pmatrix}\delta(\theta) + (\tau_j^0+\mu)\begin{pmatrix} 0 & 0 \\ & \\ -\Psi'(x^*)y^{*2} & -r_2 \end{pmatrix}\delta(\theta+1), \tag{27}$$

where δ is the Dirac delta function defined by

$$\delta(\theta) = \begin{cases} 0 \text{ if } \theta \neq 0, \\ \\ 1 \text{ if } \theta = 0. \end{cases} \tag{28}$$

For $\phi \in \mathscr{C}\big([-1,0],\mathbb{R}^2\big)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta} \text{ if } \theta \in [-1,0), \\ \\ \displaystyle\int_1^0 d\eta(\mu,s)\phi(s) \text{ if } \theta = 0, \end{cases} \tag{29}$$

and

$$R(\mu)\phi = \begin{cases} 0 \text{ if } \theta \in [-1,0), \\ \\ f(\mu,\phi) \text{ if } \theta = 0. \end{cases} \tag{30}$$

Then, System (23) is equivalent to

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t, \tag{31}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \in [-1,0]$.

For $\psi \in \mathscr{C}^1\left([0,1],\mathbb{R}^2\right)$, define

$$A^*\psi = \begin{cases} -\dfrac{d\psi(s)}{ds} \text{ if } s \in (0,1], \\ \\ \displaystyle\int_1^0 d\eta(t,0)\phi(-t) \text{ if } s = 0, \end{cases} \tag{32}$$

and a bilinear inner product

$$\begin{aligned} \langle \psi(s), \phi(\theta) \rangle &= \bar{\psi}(0)\phi(0) \\ &\quad - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta) d\eta(\theta)\phi(\xi) d\xi, \end{aligned} \tag{33}$$

where $\eta(\theta) = \eta(\theta,0)$. In addition, by Theorem 3 we know that $\pm i\omega_0\tau_j^0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. Let us assume that $q(\theta)$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_j^0$ and $q^*(s)$ is the eigenvector of A^* corresponding to $-i\omega_0\tau_j^0$.

Let $q(\theta) = \begin{pmatrix} 1, & v_1 \end{pmatrix}^T e^{i\omega_0\tau_j^0\theta}$ and $q^*(s) = D\begin{pmatrix} 1, & v_1^* \end{pmatrix}^T e^{i\omega_0\tau_j^0 s}$. From the above discussion, it is easy to know that $A(0)q(0) = i\omega_0\tau_j^0 q(0)$ and $A^*(0)q^*(0) = -i\omega_0\tau_j^0 q^*(0)$. That is

$$\tau_j^0 \begin{pmatrix} J_0 & J_1 \\ \\ 0 & 0 \end{pmatrix} q(0)$$

$$+\tau_j^0 \begin{pmatrix} 0 & 0 \\ \\ -\Psi'(x^*)y^{*2} & -r_2 \end{pmatrix} q(-1) = iw_0\tau_j^0 q(0)$$

and

$$\tau_j^0 \begin{pmatrix} J_0 & 0 \\ \\ J_1 & -r_2 \end{pmatrix} q^*(0)$$

$$+\tau_j^0 \begin{pmatrix} 0 & -\Psi'(x^*)y^{*2} \\ \\ 0 & -r_2 \end{pmatrix} q^*(-1) = -iw_0\tau_j^0 q^*(0).$$

Thus, we can easily obtain

$$q(\theta)=\left(1\,,\ \frac{J_0-i\omega_0}{a_1(1-m)x^*}\right)^T e^{i\omega_0\tau_j^0\theta}, \tag{34}$$

$$q^*(s)=D\left(1\,,\ \frac{J_0+i\omega_0}{\Psi'(x^*)y^{*2}e^{-i\omega_0\tau_j^0}}\right)^T e^{i\omega_0\tau_j^0 s}. \tag{35}$$

In order to assure $\langle\bar{q}^*(s),q(\theta)\rangle=1$, we need to determine the value of D. From (33), we have

$$\begin{aligned}
&\langle q^*(s),q(\theta)\rangle=\bar{q}^*(0)q(0)\\
&-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi\\
&=\bar{q}^*(0)q(0)\\
&-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{D}\Big(1,\ \bar{\nu}_1^*\Big)e^{-i\omega_0\tau_j^0(\xi-\theta)}d\eta(\theta)\Big(1,\ \nu_1\Big)^T e^{i\omega_0\tau_j^0\xi}d\xi\\
&=\bar{q}^*(0)q(0)\\
&-\bar{q}^*(0)\int_{-1}^{0}\theta e^{i\omega_0\tau_j^0\theta}d\eta(\theta)q(0)\\
&=\bar{q}^*(0)q(0)\\
&-\bar{q}^*(0)\tau_j^0\begin{pmatrix}0 & 0\\ & \\ -\Psi'(x^*)y^{*2} & -r_2\end{pmatrix}\Big(-e^{-i\omega_0\tau_j^0}\Big)q(0)\\
&=\bar{D}\Big[1+\nu_1\bar{\nu}_1^*-\tau_j^0e^{-i\omega_0\tau_j^0}\bar{\nu}_1^*(\Psi'(x^*)y^{\star 2}+r_2)\Big].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\bar{D}&=\frac{1}{1+\nu_1\bar{\nu}_1^*-\tau_j^0e^{-i\omega_0\tau_j^0}\bar{\nu}_1^*(\Psi'(x^*)y^{\star 2}+r_2)},\\
D&=\frac{1}{1+\bar{\nu}_1\nu_1^*-\tau_j^0e^{i\omega_0\tau_j^0}\nu_1^*(\Psi'(x^*)y^{\star 2}+r_2)}.
\end{aligned} \tag{36}$$

Using the same notations as in [17], we first compute the coordinates to describe the center manifold $\mathscr{C}_0$ at $\mu=0$. Let x_t be the solution of Eq. (22) when $\mu=0$. Define

$$\begin{aligned}
&z(t)=\langle q^*\ ,\ x_t\rangle,\\
&W(t,\theta)=x_t(\theta)-2\mathscr{R}_e\Big(z(t)q(\theta)\Big)\\
&=x_t(\theta)-\Big(z(t)q(\theta)+\bar{z}(t)\bar{q}(\theta)\Big).
\end{aligned} \tag{37}$$

On the center manifold $\mathscr{C}_0$, we have

$$W(t,\theta)=W(z,\bar{z},\theta), \tag{38}$$

where

$$\begin{aligned} W(z,\bar{z},\theta) &= W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}\frac{\bar{z}^2}{2} \\ &+W_{30}(\theta)\frac{z^3}{6}+\cdots, \end{aligned} \tag{39}$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. We only consider real solutions. For solution $x_t \in \mathscr{C}_0$ of Eq.(22), since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= i\omega_0\tau_j^0 z \\ &+ \bar{q}^*(0)f\Big(0,W(z,\bar{z},0)+2\mathscr{R}_e\big(z(t)q(\theta)\big)\Big) \\ &\equiv i\omega_0\tau_j^0 z+\bar{q}^*(0)f_0(z,\bar{z}). \end{aligned} \tag{40}$$

We rewrite this equation as

$$\dot{z}(t)=i\omega_0\tau_j^0 z+g\big(z,\bar{z}\big), \tag{41}$$

where

$$\begin{aligned} g\big(z,\bar{z}\big) &= g_{20}(\theta)\frac{z^2}{2}+g_{11}(\theta)z\bar{z}+g_{02}\frac{\bar{z}^2}{2} \\ &+g_{21}(\theta)\frac{z^2\bar{z}}{2}+\cdots \end{aligned} \tag{42}$$

Then, $x_t(\theta)=\big(x_{1t}(\theta),x_{2t}(\theta)\big)$ and $q(\theta)=\big(1,\nu_1\big)^T e^{i\omega_0\tau_j^0\theta}$. So, from Eq.(37) and Eq.(39), it follows that

$$\begin{aligned} x_t(\theta) &= W(t,\theta)+2\mathscr{R}_e\Big(z(t)q(\theta)\Big) \\ &= W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}\frac{\bar{z}^2}{2} \\ &+ \big(1,\nu_1\big)^T e^{i\omega_0\tau_j^0\theta}z(t)+\big(1,\bar{\nu}_1\big)^T e^{-i\omega_0\tau_j^0\theta}\bar{z}(t)+\cdots \end{aligned} \tag{43}$$

Then, we have

$$
(44)\quad
\begin{aligned}
x_{1t}(0) &= z+\bar{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}\\
&+ W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}+\cdots\\
x_{2t}(0) &= \nu_1 z+\bar{\nu}_1\bar{z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}\\
&+ W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+\cdots\\
x_{1t}(-1) &= ze^{-i\omega_0\tau_j^0}+\bar{z}e^{i\omega_0\tau_j^0}+W_{20}^{(1)}(-1)\frac{z^2}{2}\\
&+ W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}+\cdots\\
x_{2t}(-1) &= \nu_1 ze^{-i\omega_0\tau_j^0}+\bar{\nu}_1\bar{z}e^{i\omega_0\tau_j^0}+W_{20}^{(2)}(-1)\frac{z^2}{2}\\
&+ W_{11}^{(2)}(-1)z\bar{z}+W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2}+\cdots
\end{aligned}
$$

It follows together with Eq.(25) that

$$
(45)\quad
\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z})\\
&= \bar{q}^*(0)f(0,x_t)=\tau_j^0\bar{D}\begin{pmatrix} 1, & \bar{\nu}_1^* \end{pmatrix}\times\\
&\quad\begin{pmatrix}
-b_1x_{1t}^2(0)-a_1(1-m)x_{1t}(0)x_{2t}(0);\\
\\
\\
-\Psi'(x^*)x_{2t}(-1)x_{2t}(0)\\
-\Psi'(x^*)x_{1t}(-1)x_{2t}(-1)x_{2t}(0);\\
-\Psi'(x^*)y^*x_{1t}(-1)x_{2t}(-1)\\
-\Psi'(x^*)y^*x_{1t}(-1)x_{2t}(0);\\
-\dfrac{\Psi''(x^*)}{2}y^*x_{1t}^2(-1)x_{2t}(-1)\\
-\dfrac{\Psi''(x^*)}{2}y^*x_{1t}^2(-1)x_{2t}(0);\\
-\dfrac{\Psi''(x^*)}{2}y^{*2}x_{1t}^2(-1)x_{2t}(0)\\
-\dfrac{\Psi^{(3)}(x^*)}{6}y^{*2}x_{1t}^3(-1)-\cdots
\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{z^2}{2}\{2\tau_j^0\bar{D}[-b_1-\nu_1 a_1(1-m)\\
&- \bar{\nu}_1^*(\Psi(x^*)\nu_1^2 e^{-i\omega_0\tau_j^0}-\Psi'(x^*)y^*\nu_1 e^{-2i\omega_0\tau_j^0}\\
&- y^*\nu_1 e^{-i\omega_0\tau_j^0}-\frac{\Psi''(x^*)}{2}y^{*2}e^{-2i\omega_0\tau_j^0}\Big)\Big]\Big\}\\
&+ \frac{\bar{z}^2}{2}\{2\tau_j^0\bar{D}[-b_1-\bar{\nu}_1 a_1(1-m)\\
&- \bar{\nu}_1^*(\Psi(x^*)\bar{\nu}_1^2 e^{i\omega_0\tau_j^0}-\Psi'(x^*)y^*\bar{\nu}_1 e^{2i\omega_0\tau_j^0}\\
&- y^*\bar{\nu}_1 e^{i\omega_0\tau_j^0}-\frac{\Psi''(x^*)}{2}y^{*2}e^{2i\omega_0\tau_j^0}\Big)\Big]\Big\}\\
&+ z\bar{z}\{2\tau_j^0\bar{D}[-b_1-\Re_e(\nu_1)a_1(1-m)\\
&- \bar{\nu}_1^*(\Re_e(\nu_1\bar{\nu}_1 e^{-i\omega_0\tau_j^0})\Psi(x^*)-\Re_e(\bar{\nu}_1)y^*\Psi'(x^*)\\
&- y^*\Re_e(\nu_1 e^{i\omega_0\tau_j^0})-y^{*2}\Psi''(x^*)\Big)\Big]\Big\}\\
&+ \frac{z^2\bar{z}}{2}\Big\{\tau_j^0\bar{D}\Big[-b_1\Big((2W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\Big)\\
&- a_1(1-m)(2W_{11}^{(2)}(0)+W_{20}^{(2)}(0)+\bar{\nu}_1 W_{20}^{(1)}(0))\\
&+ 2\nu_1 a_1(1-m)W_{11}^{(1)}(0))\\
&- \bar{\nu}_1^*\Psi(x^*)(2\nu_1 W_{11}^{(2)}(0)+\bar{\nu}_1 W_{20}^{(2)}(0)e^{i\omega_0\tau_j^0})\\
&+ \bar{\nu}_1^*\Psi(x^*)(\bar{\nu}_1 W_{20}^{(2)}(-1)+2\nu_1 W_{11}^{(2)}(-1))\\
&- \bar{\nu}_1^*\Psi'(x^*)(2\nu_1\bar{\nu}_1 e^{-2i\omega_0\tau_j^0})\\
&+ \bar{\nu}_1^*\Psi'(x^*)(2\nu_1\bar{\nu}_1+2\nu_1^2 e^{-2i\omega_0\tau_j^0})\\
&- \bar{\nu}_1^*\Psi'(x^*)y^*(2W_{11}^{(2)}(-1)e^{-i\omega_0\tau_j^0}(1+\nu_1)\\
&+ \bar{\nu}_1^*\Psi'(x^*)y^*W_{20}^{(1)}(-1)e^{i\omega_0\tau_j^0}(1+\bar{\nu}_1)\\
&- 2W_{11}^{(2)}(0)e^{-i\omega_0\tau_j^0}+W_{20}^{(2)}(0)e^{i\omega_0\tau_j^0}\\
&+ \bar{\nu}_1 W_{20}^{(1)}(-1)+2\nu_1 W_{11}^{(1)}(-1))\\
&- \bar{\nu}_1^*\frac{\Psi''(x^*)y^*}{2}(6\nu_1 e^{-i\omega_0\tau_j^0}+4\nu_1+2\bar{\nu}_1 e^{-2i\omega_0\tau_j^0})\\
&+ 4y^*\bar{\nu}_1^*\frac{\Psi''(x^*)y^*}{2}W_{11}^{(1)}(-1)e^{-i\omega_0\tau_j^0}\\
&+ 2y^*e^{i\omega_0\tau_j^0}W_{20}^{(1)}(-1)\\
&- \bar{\nu}_1^*\frac{\Psi'''(x^*)y^*}{3}\Big(2e^{-i\omega_0\tau_j^0}+e^{i\omega_0\tau_j^0}\Big).
\end{aligned}
$$

Comparing the coefficient with Eq.(42) gives

$$
\begin{aligned}
g_{20} &= 2\tau_j^0\bar{D}(-b_1-\nu_1 a_1(1-m))\\
&- 2\tau_j^0\bar{D}\bar{\nu}_1^*\Psi(x^*)\nu_1^2 e^{-i\omega_0\tau_j^0}\\
&- 2\tau_j^0\bar{D}\Psi'(x^*)y^*\nu_1 e^{-2i\omega_0\tau_j^0}\\
&- 2\tau_j^0\bar{D}y^*\nu_1 e^{-i\omega_0\tau_j^0}\\
&- 2\tau_j^0\bar{D}\frac{\Psi''(x^*)}{2}y^{*2}e^{-2i\omega_0\tau_j^0},
\end{aligned}
$$

$$
\begin{aligned}
g_{02} &= 2\tau_j^0\bar{D}(-b_1-\nu_1 a_1(1-m)) \\
&- 2\tau_j^0\bar{D}\bar{\nu}_1^*\Psi(x^*)\nu_1^2 e^{-i\omega_0\tau_j^0} \\
&- 2\tau_j^0\bar{D}\Psi'(x^*)y^*\nu_1 e^{-2i\omega_0\tau_j^0} \\
&- 2\tau_j^0\bar{D}y^*\nu_1 e^{-i\omega_0\tau_j^0} \\
&- 2\tau_j^0\bar{D}\frac{\Psi''(x^*)}{2}y^{*2}e^{-2i\omega_0\tau_j^0},
\end{aligned}
$$

$$
\begin{aligned}
g_{11} &= 2\tau_j^0\bar{D}(-b_1-\Re_e(\nu_1)a_1(1-m)) \\
&- 2\tau_j^0\bar{D}\bar{\nu}_1^*\Re_e(\nu_1\bar{\nu}_1 e^{-i\omega_0\tau_j^0})\Psi(x^*) \\
&- 2\tau_j^0\bar{D}\bar{\nu}_1^*\Re_e(\bar{\nu}_1)y^*\Psi'(x^*) \\
&- 2\tau_j^0\bar{D}\bar{\nu}_1^*y^*\Re_e(\nu_1 e^{i\omega_0\tau_j^0}) \\
&- 2\tau_j^0\bar{D}\bar{\nu}_1^*y^{*2}\Psi''(x^*),
\end{aligned}
$$

$$
\begin{aligned}
g_{21} &= -\tau_j^0\bar{D}b_1(2W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)) \\
&- \tau_j^0\bar{D}a_1(1-m)(2W_{11}^{(2)}(0)+W_{20}^{(2)}(0)) \\
&+ \tau_j^0\bar{D}\bar{\nu}_1 a_1(1-m)W_{20}^{(1)}(0)+2\nu_1 W_{11}^{(1)}(0)) \\
&- \bar{\nu}_1^*\Psi(x^*)(2\nu_1 W_{11}^{(2)}(0)+\bar{\nu}_1 W_{20}^{(2)}(0)e^{i\omega_0\tau_j^0}) \\
&+ \bar{\nu}_1^*\Psi(x^*)(\bar{\nu}_1 W_{20}^{(2)}(-1)+2\nu_1 W_{11}^{(2)}(-1)) \\
&- \bar{\nu}_1^*\Psi'(x^*)(2\nu_1\bar{\nu}_1 e^{-2i\omega_0\tau_j^0}+2\nu_1\bar{\nu}_1 \\
&- 2\bar{\nu}_1^*\Psi'(x^*)\nu_1^2 e^{-2i\omega_0\tau_j^0}) \\
&- \bar{\nu}_1^*\Psi'(x^*)y^*2W_{11}^{(2)}(-1)e^{-i\omega_0\tau_j^0}(1+\nu_1) \\
&+ \bar{\nu}_1^*\Psi'(x^*)y^*W_{20}^{(1)}(-1)e^{i\omega_0\tau_j^0}(1+\bar{\nu}_1) \\
&- 2\bar{\nu}_1^*\Psi'(x^*)y^*W_{11}^{(2)}(0)e^{-i\omega_0\tau_j^0} \\
&+ \bar{\nu}_1^*\Psi'(x^*)y^*W_{20}^{(2)}(0)e^{i\omega_0\tau_j^0}+\bar{\nu}_1 W_{20}^{(1)}(-1) \\
&+ 2\bar{\nu}_1^*\Psi'(x^*)y^*\nu_1 W_{11}^{(1)}(-1) \\
&- \bar{\nu}_1^*\frac{\Psi''(x^*)y^*}{2}(6\nu_1 e^{-i\omega_0\tau_j^0}+4\nu_1) \\
&+ 2\bar{\nu}_1^*\frac{\Psi''(x^*)y^*}{2}\bar{\nu}_1 e^{-2i\omega_0\tau_j^0} \\
&+ 4\bar{\nu}_1^*\frac{\Psi''(x^*)y^*}{2}y^*W_{11}^{(1)}(-1)e^{-i\omega_0\tau_j^0} \\
&+ 2y^*e^{i\omega_0\tau_j^0}W_{20}^{(1)}(-1) \\
&- \bar{\nu}_1^*\frac{\Psi'''(x^*)y^*}{3}\Big(2e^{-i\omega_0\tau_j^0}+e^{i\omega_0\tau_j^0}\Big)
\end{aligned} \tag{46}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From Eq.(31) and Eq.(37), we have

$$\dot{W} = \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q}$$

$$(47)\qquad = \begin{cases} AW - 2\mathscr{R}_e\Big\{\bar{q}^*(0)f_0q(\theta)\Big\} \text{ if } \theta \in [-1;0), \\ \\ AW - 2\mathscr{R}_e\Big\{\bar{q}^*(0)f_0q(\theta)\Big\} + f_0 \text{ if } \theta = 0, \end{cases}$$

$$\equiv^{\text{def}} \; AW + \mathscr{H}\big(z,\bar{z},\theta\big),$$

where

$$(48)\qquad \begin{aligned} \mathscr{H}\big(z,\bar{z},\theta\big) &= \mathscr{H}_{20}(\theta)\frac{z^2}{2} + \mathscr{H}_{11}(\theta)z\bar{z} \\ &+ \mathscr{H}_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \end{aligned}$$

Substituting the corresponding series into Eq.(47) and comparing the coefficients give

$$(49)\qquad \begin{aligned} (A - 2i\omega_0\tau_j^0)W_{20}(\theta) &= -\mathscr{H}_{20}(\theta), \\ \\ AW_{11}(\theta) &= -\mathscr{H}_{11}(\theta). \end{aligned}$$

From Eq.(47), we know that for $\theta \in [-1,0)$,

$$(50)\qquad \begin{aligned} \mathscr{H}\big(z,\bar{z},\theta\big) &= -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) \\ &= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \end{aligned}$$

Comparing the coefficient with Eq.(48) gives

$$(51)\qquad -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) = H_{20}(\theta),$$

$$(52)\qquad -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) = H_{11}(\theta).$$

From Eq.(49) and Eq.(51) and the definition of A, it follows that

$$(53)\qquad \dot{W}(\theta) = 2iw_0\tau_j^0W_{20} + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Notice that $q(\theta) = \Big(1, \nu_1\Big)^T e^{iw_0\tau_j^0\theta}$. Hence,

$$W_{20}(\theta) \quad = \frac{ig_{20}}{w_0\tau_j^0}q(0)e^{i\omega_0\tau_j^0\theta} + \frac{i\bar{g}_{02}}{3w_0\tau_j^0}\bar{q}(0)e^{-i\omega_0\tau_j^0\theta} \\ +E_1 e^{2i\omega_0\tau_j^0\theta}, \tag{54}$$

where $E_1 = \left(E_1^{(1)}, E_1^{(2)}\right) \in \mathbb{R}^2$ is a constant vector. Similarly, from Eq.(49) and Eq.(52), we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{w_0\tau_j^0}q(0)e^{i\omega_0\tau_j^0\theta} + \frac{i\bar{g}_{11}}{w_0\tau_j^0}\bar{q}(0)e^{-i\omega_0\tau_j^0\theta} + E_2, \tag{55}$$

where $E_2 = \left(E_2^{(1)}, E_2^{(2)}\right) \in \mathbb{R}^2$ is also a constant vector.

In what follows, we will seek appropriate E_1 and E_2. From the definition of A and Eq.(49), we obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_j W_{20}(0) - H_{20}(0), \tag{56}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{57}$$

where $\eta(\theta) = \eta(0,\theta)$. By Eq.(47), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_j^0 \times \\ \begin{pmatrix} -b_1 - \nu_1 a_1(1-m) \\ \\ -\Psi(x^*)\nu_1^2 e^{-i\omega_0\tau_j^0} - \Psi'(x^*)y^*\nu_1 e^{-2i\omega_0\tau_j^0} \\ -y^*\nu_1 e^{-i\omega_0\tau_j^0} - \dfrac{\Psi''(x^*)}{2}y^{*2}e^{-2i\omega_0\tau_j^0} \end{pmatrix}, \tag{58}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_j^0 \times \\ \begin{pmatrix} -b_1 - \Re_e(\nu_1)a_1(1-m) \\ \\ -\Re_e(\nu_1\bar{\nu}_1 e^{-i\omega_0\tau_j^0})\Psi(x^*) - \Re_e(\bar{\nu}_1)y^*\Psi'(x^*) \\ -y^*\Re_e(\nu_1 e^{i\omega_0\tau_j^0}) - y^{*2}\Psi''(x^*) \end{pmatrix}. \tag{59}$$

Substituting Eq.(54) and Eq.(58) into Eq.(56) and using the fact that

$$\begin{aligned}
\left(i\omega_0\tau_j^0 I - \int_{-1}^{0} e^{i\omega_0\tau_j^0\theta} d\eta(\theta)\right) q(0) &= 0, \\
\left(-i\omega_0\tau_j^0 I - \int_{-1}^{0} e^{-i\omega_0\tau_j^0\theta} d\eta(\theta)\right) \bar{q}(0) &= 0,
\end{aligned} \tag{60}$$

we obtain

$$\left(2i\omega_0\tau_j^0 I - \int_{-1}^{0} e^{2i\omega_0\tau_j^0\theta} d\eta(\theta)\right) E_1 = 2\tau_j^0 \times \begin{pmatrix} -b_1 - \nu_1 a_1(1-m) \\ \\ -\Psi(x^*)\nu_1^2 e^{-i\omega_0\tau_j^0} - \Psi'(x^*) y^* \nu_1 e^{-2i\omega_0\tau_j^0} \\ -y^*\nu_1 e^{-i\omega_0\tau_j^0} - \dfrac{\Psi''(x^*)}{2} y^{*2} e^{-2i\omega_0\tau_j^0} \end{pmatrix}.$$

This leads to

$$\begin{pmatrix} 2i\omega_0 - J_0 & -J_1 \\ \\ \Psi'(x^*) y^{*2} e^{-2i\omega_0\tau_j^0} & 2i\omega_0 + r_2 e^{-2i\omega_0\tau_j^0} \end{pmatrix} E_1 =$$

$$2\begin{pmatrix} -b_1 - \nu_1 a_1(1-m) \\ \\ -\Psi(x^*)\nu_1^2 e^{-i\omega_0\tau_j^0} - \Psi'(x^*) y^* \nu_1 e^{-2i\omega_0\tau_j^0} \\ -y^*\nu_1 e^{-i\omega_0\tau_j^0} - \dfrac{\Psi''(x^*)}{2} y^{*2} e^{-2i\omega_0\tau_j^0} \end{pmatrix}.$$

Solving this system for E_1 gives

$$E_1^{(1)} = \frac{2}{\sigma} \begin{vmatrix} -b_1 - \nu_1 a_1(1-m) & a_1(1-m)x^* \\ \\ e_0 & 2i\omega_0 + r_2 e^{-2i\omega_0\tau_j^0} \end{vmatrix},$$

where

$$\begin{aligned}
e_0 &= -\Psi(x^*)\nu_1^2 e^{-i\omega_0\tau_j^0} - \Psi'(x^*) y^* \nu_1 e^{-2i\omega_0\tau_j^0} \\
&\quad -y^*\nu_1 e^{-i\omega_0\tau_j^0} - \frac{\Psi''(x^*)}{2} y^{*2} e^{-2i\omega_0\tau_j^0},
\end{aligned}$$

$$E_1^{(2)} = \frac{2}{\sigma}\begin{vmatrix} 2i\omega_0 - J_0 & -b_1 - \nu_1 a_1(1-m) \\ \Psi'(x^*)y^{*2}e^{-2i\omega_0\tau_j^0} & e_0 \end{vmatrix},$$

where

$$\sigma = \begin{vmatrix} 2i\omega_0 - J_0 & a_1(1-m)x^* \\ \Psi'(x^*)y^{*2}e^{-2i\omega_0\tau_j^0} & 2i\omega_0 + r_2 e^{-2i\omega_0\tau_j^0} \end{vmatrix}.$$

Similarly, substituting Eq.(55) and Eq.(59) into (57) gives

$$\begin{pmatrix} -J_0 & -J_1 \\ \Psi'(x^*)y^{*2} & r_2 \end{pmatrix} E_2 =$$

$$2\begin{pmatrix} -b_1 - \Re_e(\nu_1)a_1(1-m) \\ -\Re_e(\nu_1\bar{\nu}_1 e^{-i\omega_0\tau_j^0})\Psi(x^*) - \Re_e(\bar{\nu}_1)y^*\Psi'(x^*) \\ -y^*\Re_e(\nu_1 e^{i\omega_0\tau_j^0}) - y^{*2}\Psi''(x^*) \end{pmatrix}.$$

Therefore,

$$E_2^{(1)} = \frac{2}{\rho}\begin{vmatrix} -b_1 - \Re_e(\nu_1)a_1(1-m) & -J_1 \\ e_1 & r_2 \end{vmatrix},$$

where

$$e_1 = -\Re_e(\nu_1\bar{\nu}_1 e^{-i\omega_0\tau_j^0})\Psi(x^*) - \Re_e(\bar{\nu}_1)y^*\Psi'(x^*)$$
$$-y^*\Re_e(\nu_1 e^{i\omega_0\tau_j^0}) - y^{*2}\Psi''(x^*)$$

$$E_2^{(2)} = \frac{2}{\rho}\begin{vmatrix} -J_0 & -b_1 - \Re_e(\nu_1)a_1(1-m) \\ \Psi'(x^*)y^{*2} & e_1 \end{vmatrix},$$

where

$$\rho = \begin{vmatrix} -J_0 & a_1(1-m)x^* \\ & \\ \Psi'(x^*)y^{*2} & r_2 \end{vmatrix}.$$

Thus, we can determine W_{20} and W_{11} from Eq.(54) and Eq.(55). Furthermore, g_{21} in Eq.(46) can be expressed in terms of parameters and delay. Thus, we can compute the following values

$$\begin{aligned} C_1(0) &= \frac{i}{2w_0\tau_j^0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \nu_2 &= -\frac{\mathscr{R}_e\{C_1(0)\}}{\mathscr{R}_e\left\{\lambda'(\tau_j^0)\right\}}, \\ \beta_2 &= 2\mathscr{R}_e\{C_1(0)\}, \\ T_2 &= -\frac{\mathscr{I}_m\{C_1(0)\} + \nu_2\mathscr{I}_m\left\{\lambda'(\tau_j^0)\right\}}{w_0\tau_j^0}, \end{aligned} \tag{61}$$

which determine the qualities of bifurcating periodic solution in the center manifold at the critical value τ_j^0.

Theorem 4. *[17]: In Eq. (61), the sign of ν_2 determines the direction of the Hopf bifurcation. Thus, if $\nu_2 > 0$, then the Hopf bifurcation is supercritical and the bifurcating periodic solution exists for $\tau_1 > \tau_1^0$. If $\nu_2 < 0$, then the Hopf bifurcation is subcritical and the bifurcating periodic solution exists for $\tau_1 < \tau_1^0$. β_2 determines the stability of the bifurcating periodic solution: The bifurcating periodic solutions are stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. T_2 determines the period of the bifurcating periodic solutions: the period increase if $T_2 > 0$ and decrease if $T_2 < 0$.*

6. Bionomic equilibrium and Optimal Harvest Policy

The first part of this section deals with the *bionomic equilibrium* of System (6). The term bionomic equilibrium is an amalgamation of the concepts of biological equilibrium and economic equilibrium. As we already saw, a biological equilibrium is given by $\dot{x} = 0 = \dot{y}$. The economic equilibrium is said to be achieved when TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for the effort devoted to harvesting).

To discuss the bionomic equilibrium of the prey-predator model, we consider the parameters such as $c =$ cost per unit effort for prey; $p =$ price per unit biomass for the prey.

The net economic rent or net revenue (R) at any time is given by

$$R(x,h,t) = \Big(p\frac{x-T_1}{T_2-T_1} - c\Big)h \text{ if } T_1 \le x \le T_2, \tag{62}$$

and

$$R(x,h,t) = \Big(p - c\Big)h \text{ if } x \ge T_2. \tag{63}$$

The bionomic equilibrium is $P_\infty(x_\infty, y_\infty, h_\infty)$, where x_∞, y_∞ and h_∞ are the positive solutions of the following simultaneous equations

$$\begin{cases} (r_1 - b_1 x)x - a_1(1-m)xy - \dfrac{h(x-T_1)}{T_2-T_1} = 0, \\ \big[r_2 - \dfrac{a_2 y}{(1-m)x}\big]y = 0, \quad \text{if} \quad T_1 \le x \le T_2 \\ \Big(p\dfrac{x-T_1}{T_2-T_1} - c\Big)h = 0, \end{cases} \tag{64}$$

and

$$\begin{cases} (r_1 - b_1 x)x - a_1(1-m)xy - h = 0, \\ \big[r_2 - \dfrac{a_2 y}{(1-m)x}\big]y = 0, \quad \text{if} \quad x \ge T_2 \\ \Big(p - c\Big)h = 0, \end{cases} \tag{65}$$

It may be noted here that if $c > p\dfrac{x-T_1}{T_2-T_1}$ when $T_1 \le x \le T_2$ or if $c > p$ when $x \ge T_2$, i.e. if the prey cost exceeds the revenue obtained from it, then the economic rent obtained from the prey becomes negative. Hence the prey will be closed and no bionomic equilibrium exists. Therefore, for the existence of bionomic equilibrium, it is natural to assume $c < p\dfrac{x-T_1}{T_2-T_1}$ when

$T_1 \leq x \leq T_2$ and $c < p$ when $x \geq T_2$. Then, for $T_1 \leq x \leq T_2$,

$$x_\infty = T_1 + \frac{c}{p}(T_2 - T_1), \tag{66}$$

$$y_\infty = \frac{r_2(1-m)x_\infty}{a_2}, \tag{67}$$

$$h_\infty = \frac{p\Big(r_1 - b_1 x_\infty - a_1(1-m)y_\infty\Big)x_\infty}{c}. \tag{68}$$

It is clear that $h_\infty > 0$ if

$$r_1 - b_1 x_\infty - a_1(1-m)y_\infty > 0. \tag{69}$$

Thus, the bionomic equilibrium $P_\infty(x_\infty, y_\infty, h_\infty)$ exists if $x_\infty \leq T_2$ and inequality (69) holds.

In what follows, our objective is to maximize the total discounted net revenues from the fishery. In commercial exploitation of renewable resources, the fundamental problem from the economic point of view, is to determine the optimal trade-off between present and future harvests. If we look at the problem, it is observed that the marine fishery sectors become more important not only for domestic demand but also from the imperatives of exports.

Symbolically our strategy is to maximize the present value J given by

$$J(h) = \int_0^{t_f} R\big(x(t), h(t), t\big)e^{-\delta t}dt, \tag{70}$$

where $R(x,h,t) = \Big(p\dfrac{x - T_1}{T_2 - T_1} - c\Big)h$ if $T_1 \leq x \leq T_2$, $R(x,h,t) = \Big(p - c\Big)h$ if $x \geq T_2$ and δ denotes the instantaneous annual rate of discount. Our problem is to maximize J subject to the state System (6) by invoking Pontryagin's Maximum principle for retarded optimal control problem [26]. The control variable $h(t)$ is subjected to the constraints $0 \leq h(t) \leq K$. So, in other words, our problem now is to find h^* such that

$$J(h^*) = \max_{h \in \Omega} J(h), \tag{71}$$

where $\Omega = \{h \in L^1(0, t_f); 0 \leq h \leq K\}$.

The existence of an optimal harvesting is due to the concavity of integrand of J with respect to h, a boundedness of the state solutions $(x(t), y(t))$, and the Lipschitz property of the state system (6) with respect to the state variables (see [32]).

Using the Pontryagin's maximum principle for delayed control problem [26, 30], problem (71) is reduced to maximize the Hamiltonian $\mathscr{H}$ defined by:

$$\begin{aligned}
&\mathscr{H}(x(t),y(t),x(t-\tau),y(t-\tau),h(t),\lambda(t)) = \\
&e^{-\delta t}R(x(t),h(t),t)+\lambda_1(r_1-b_1x(t))x(t) \\
&+\lambda_1\Big[-a_1(1-m)x(t)y(t)-H(x(t))\Big] \\
&+\lambda_2\Big[r_2y(t)-\frac{a_2y(t-\tau)y(t)}{(1-m)x(t-\tau)}\Big],
\end{aligned}$$

where $\lambda=(\lambda_1,\lambda_2)$. By the maximal principle, there exists adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ for all $t\geq 0$ such that

$$\left\{\begin{array}{rcl}
\dfrac{d\lambda_1(t)}{dt} &=& -\chi_{[0,t_f-\tau]}(t)\frac{\partial\mathscr{H}}{\partial x(t-\tau)}(t+\tau), \\
 & & -\dfrac{\partial\mathscr{H}}{\partial x(t)}(t) \\
\dfrac{d\lambda_2(t)}{dt} &=& -\chi_{[0,t_f-\tau]}(t)\frac{\partial\mathscr{H}}{\partial y(t-\tau)}(t+\tau), \\
 & & -\dfrac{\partial\mathscr{H}}{\partial y(t)}(t)
\end{array}\right. \tag{72}$$

and

$$\frac{\partial\mathscr{H}}{\partial h(t)}(x(t),y(t),x(t-\tau),y(t-\tau),h(t),\lambda(t))=0, \tag{73}$$

where $\chi_{[0,t_f-\tau]}(t)$ is the indicatrice function on $[0,t_f-\tau]$.

Therefore, we obtain the adjoint system:

$$\left\{\begin{array}{rcl}
\dot\lambda_1(t) &=& -\dfrac{ph}{T_2-T_1}e^{-\delta t}+\lambda_1(t)(-r_1+2b_1x(t)) \\
 &+& \lambda_1(t)\left(a_1(1-m)y(t)+\dfrac{h}{T_2-T_1}\right) \\
 &-& \chi_{[0,t_f-\tau]}(t)\dfrac{a_2y(t+\tau)\lambda_2(t+\tau)y(t)}{(1-m)x^2(t)}, \\
 & & \\
\dot\lambda_2(t) &=& -a_1(1-m)x(t)\lambda_1(t)-r_2\lambda_2(t) \\
 &+& \chi_{[0,t_f-\tau]}(t)\dfrac{a_2y(t+\tau)\lambda_2(t+\tau)}{(1-m)x(t)}.
\end{array}\right. \tag{74}$$

The transversality conditions of System (74) are

$$\lambda_1(t_f)=\lambda_2(t_f)=0.$$

Since $\mathscr{H}$ is linear in the control variable h, the optimal control will be a combination of bang-bang control and singular control. Let

$$\sigma(t) = e^{-\delta t}\left(\frac{p(x-T_1)}{T_2-T_1} - c\right) - \lambda_1(t)\frac{(x-T_1)}{T_2-T_1}.$$

The optimal control $h(t)$ which maximizes $\mathscr{H}$ must satisfy the following conditions:

$$h(t) = K \quad \text{if} \quad \sigma(t) > 0 \tag{75}$$

$$i.e \quad e^{\delta t}\lambda_1(t) < p - \frac{c}{\frac{x-T_1}{T_2-T_1}}, \tag{76}$$

$$h(t) = 0 \quad \text{if} \quad \sigma(t) < 0 \tag{77}$$

$$i.e \quad e^{\delta t}\lambda_1(t) > p - \frac{c}{\frac{x-T_1}{T_2-T_1}}, \tag{78}$$

where $e^{\delta t}\lambda_1(t)$ is the usual shadow price [18] and $p - \frac{c}{\frac{x-T_1}{T_2-T_1}}$ is the net economic revenue on a unit harvest. This shows that $h = K$ or zero according to the shadow price is less than or greater than the net economic revenue on a unit harvest. Economically, condition (76) implies that if the profit after paying all the expenses is positive, then it is beneficial to harvest up to the limit of available effort. Condition (78) implies that when the shadow price exceeds the fisherman's net economic revenue on a unit harvest, then the fisherman will not exert any effort.

When $\sigma(t) = 0$, i.e. when the shadow price equals the net economic revenue on a unit harvest, then the Hamiltonian $\mathscr{H}$ becomes independent of the control variable $h(t)$, i.e. $\partial\mathscr{H}/\partial h = 0$. This is the necessary condition for the singular control $h(t)$ to be optimal over the control set $0 \leq h \leq K$. Thus, the optimal harvesting policy is

$$h(t) = \begin{cases} 0 & if \quad \sigma(t) < 0, \\ h^* & if \quad \sigma(t) = 0, \\ K & if \quad \sigma(t) > 0. \end{cases}$$

Solving $\sigma(t) = 0$, we get

$$\lambda_1(t) = e^{-\delta t}\left(p - \frac{c}{\frac{x-T_1}{T_2-T_1}}\right). \tag{79}$$

Substituting Eq(79) into System (74) gives

$$
\left\{
\begin{array}{l}
\dot{\lambda}_1(t) = -\dfrac{ph}{T_2 - T_1} e^{-\delta t} + e^{-\delta t} \left(p - \dfrac{c}{\dfrac{x - T_1}{T_2 - T_1}} \right) \times \\
\left(-r_1 + 2b_1 x(t) + a_1(1-m)y(t) + \dfrac{h}{T_2 - T_1} \right) \\
-\chi_{[0,t_f-\tau]}(t) \dfrac{a_2 y(t+\tau)\lambda_2(t+\tau)y(t)}{(1-m)x^2(t)}, \\
\dot{\lambda}_2(t) = -a_1(1-m)x(t)e^{-\delta t} \left(p - \dfrac{c}{\dfrac{x - T_1}{T_2 - T_1}} \right) \\
-r_2\lambda_2(t) + \chi_{[0,t_f-\tau]}(t) \dfrac{a_2 y(t+\tau)\lambda_2(t+\tau)}{(1-m)x(t)}.
\end{array}
\right. \tag{80}
$$

Using equilibrium conditions and integrating System (80), we obtain $\lambda_1(t)$ and $\lambda_2(t)$. Solving equation

$$
\lambda_1(t) = p - \frac{c}{\dfrac{x - T_1}{T_2 - T_1}},
$$

we obtain the optimal harvesting efforts h^*.

7. Numerical Simulations

In this section, we give some numerical simulations for a special case of System (6) with harvesting function (5) to support our analytical results in this paper. As an example, we consider systems (6) and (5) with the coefficients $r_1 = 1.1$, $b_1 = 1.1/300$, which gives $K = 300$, $m = 0.1$, $a_1 = 0.11$, $r_2 = 0.2$, $a_2 = 1$, $h = 0.2 * K$, $T_1 = 60$, $T_2 = 90$ and $\tau = 20$. When there is no delay, we choose $x(0) = 40$ and $y(0) = 25$. That is,

$$
\left\{
\begin{array}{rcl}
\dot{x}(t) & = & (1.1 - \dfrac{1.1}{300} * x) * x \\
 & - & 0.11 * (1 - 0.1) * x * y - H(x(t)), \\
\dot{y}(t) & = & \left(0.2 - \dfrac{1.2 * y(t-\tau)}{(1-0.1) * x(t-\tau)} \right) * y(t).
\end{array}
\right. \tag{81}
$$

In Figure 3, we have $\Delta_1 = 117.5377 > 0$, $-\varphi'(K_0) + a_1(1-m)y_0 + r_2 = 11.2751 > 0$ and $2a_1(1-m)r_2y_0 - r_2\varphi'(K_0) = 2.3975 > 0$. So, the conditions of stability of equilibrium $G_0(K_0, y_0)$ are satisfied and G_0 is locally asymptotically stable.

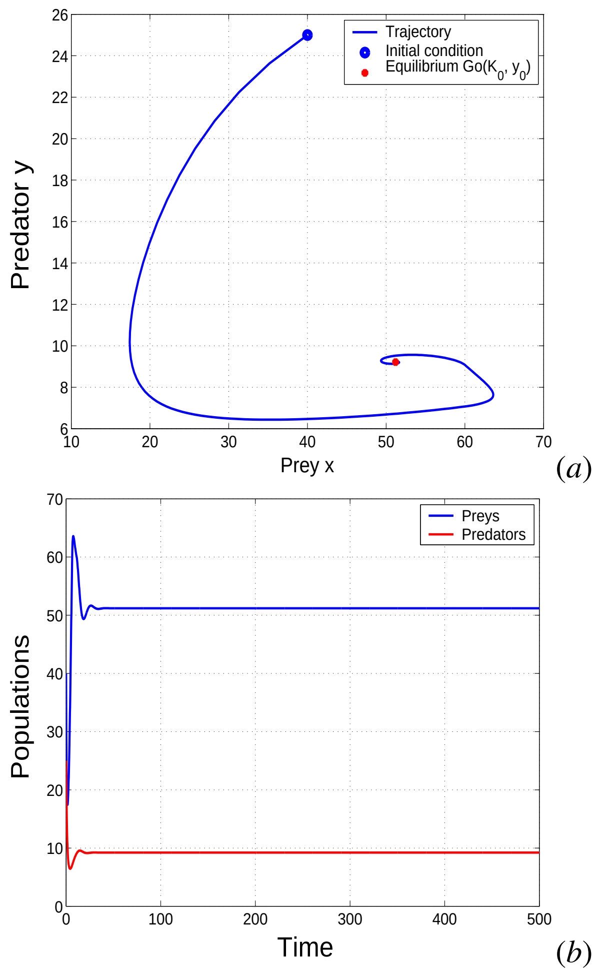
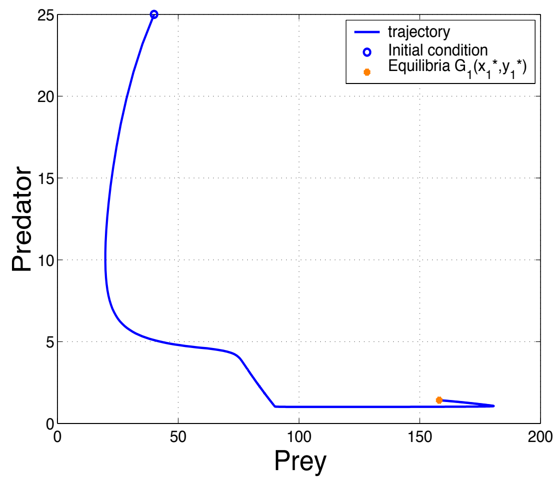


FIGURE 3. The numerical approximations of system (6) when $\tau = 0$ and $K_0 = 51.1945 < T_1$. The positive equilibrium $G_0(51.1945, 9.2150)$ is a asymptotically stable.

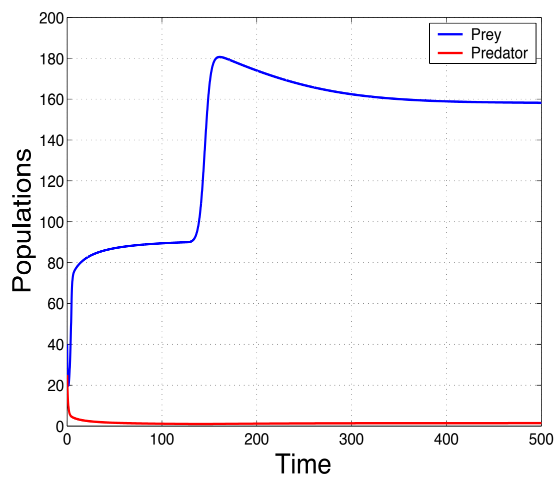
Fig. 4 shows that under some conditions, equilibrium $G_1(x_1^*, y_1^*)$ is the only equilibrium of the model system (6) and is locally asymptotically stable. More precisely, we have $x_1^* - T_2 = 68.065$, $K_0 - x_1^* = 83.28$, $\varphi(T_2) - h = 9.3$, $\Delta_1 = 1.14 \times 10^3$,

$$-\varphi'(x_1^*) + a_1(1-m)y_1^* + r_2 = 33.8253 > 0$$

and $2a_1(1-m)r_2y_1^* - r_2\varphi'(x_1^*) = 0.3396 > 0$. So, we have $T_2 < x_1^* < K_0$, $\varphi(T_2) > h$ and all conditions which give the stability of G_1.



(a)



(b)

FIGURE 4. The numerical approximations of system (6) when $\tau = 0$, $r_2 = 0.01$ and $K_0 = 241.35 > T_2$. The positive equilibrium $G_1(158.0658, 1.4226)$ is a asymptotically stable.

We now present some numerical results of the system for different values of τ. From the above discussion, we may determine the direction of Hopf-bifurcation and the direction of bifurcating periodic solution. We consider the system when the parameter values are given as in Figure (3). So, the model has a positive equilibrium $G_0(51.1945, 9.2150)$ which is locally asymptotically stable for $\tau = 0$. When τ passes through the critical value $\tau = \tau_1^0 = 95.2311$ and $\frac{d(R_e\lambda(\tau))}{d\tau})\Big|_{\lambda=iw_0,\tau=\tau_1^0} = 7.6799 > 0$, the equilibrium G_0 losses its stability and the system (6) experiences Hopf-bifurcation. From Sect. 5 we can determine the nature of the stability and direction of the periodic solution bifurcating from the interior equilibrium at the critical point τ_0^0.

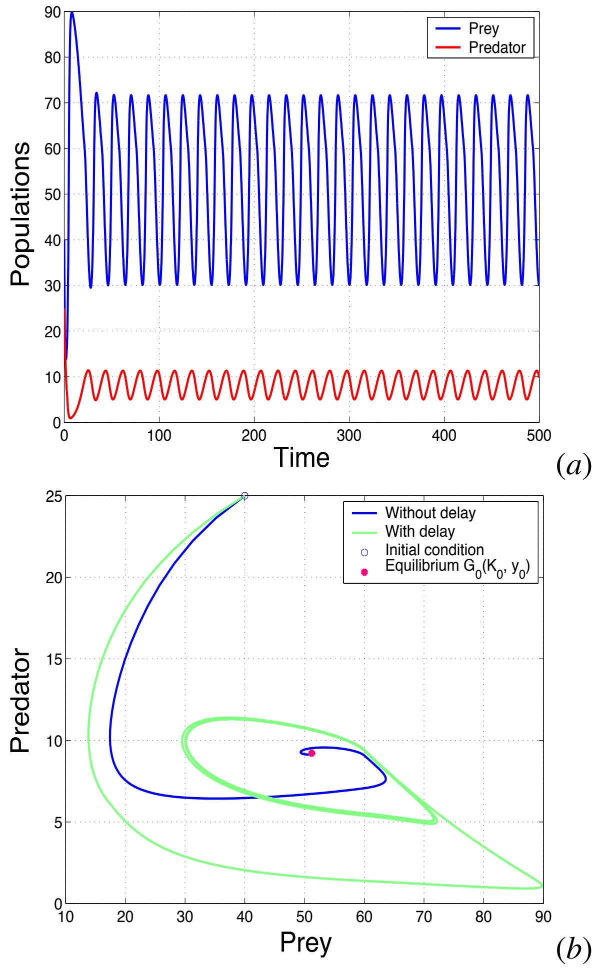


FIGURE 5. Hopf bifurcation behavior of the system (6) around the interior equilibrium $G_0(51.1945, 9.2150)$ when $\tau = \tau_0^0 = 95.2311$. The other parameter are the same as in Fig. (3). We obtain the existence of unstable supercritical bifurcating periodic solution around the interior equilibrium G_0 with the same parameter values as in Fig. (3).

Using (61), we can compute $C_1(0) = 69.7625 - 28.9307\,i$, $\nu_2 = 968.6446 > 0$, $\beta_2 = 139.5250 > 0$ and $T_2 = -120.1525$. Hence, the bifurcating periodic solution exists when τ crosses τ_1^0 from left to right and the corresponding periodic solution is supercritical and unstable (as $\beta_2 > 0$) as evident from Fig. 5 (a)-(b). The negative sign of T_2 indicates the decreasing period of the periodic solution of the system. Moreover, this system is locally asymptotically stable around the interior equilibrium G_0, which is clearly depicted from Fig. 6(a).(b) for $\tau = 16 < \tau_0^0$.

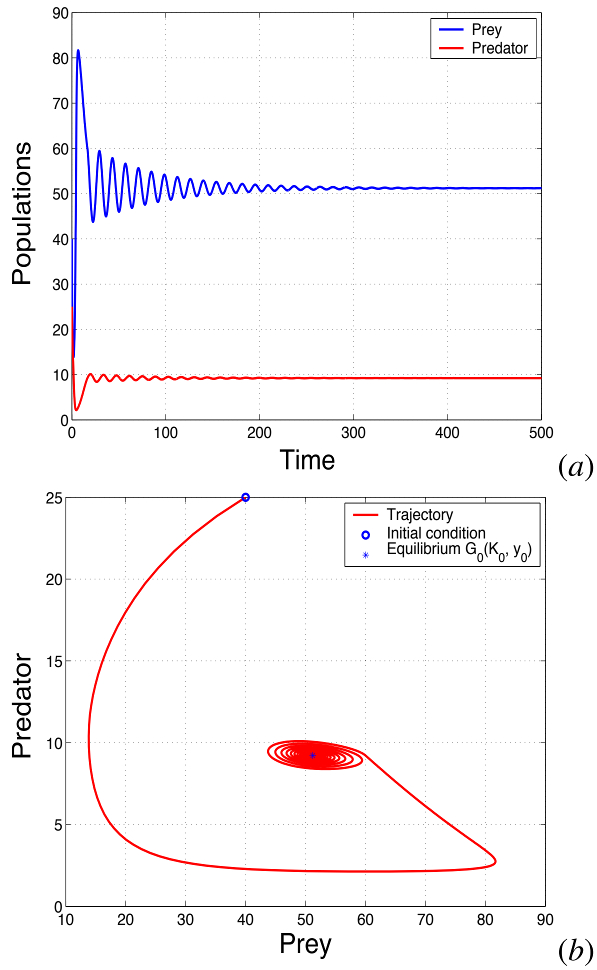


FIGURE 6. The system (6) is globally asymptotically stable around the interior equilibrium G_0 at $\tau = 16 < \tau_0 = 95.2311$. The other parameter values are given in the previous figure.

Figure (7) gives the optimal harvesting of prey in the presence of the two population. We observe that the control increase the period of limit cycle (see Figure (7 a)) and also increase the predator and prey population (see Figure (7 b and c)). In order to obtain this result, the harvesting will be made periodically (see Figure (7 d)). From this figure, it is clear that as the time progresses the prey and predator populations fluctuate in different period depending on the values of the optimal harvesting. We observe that when we harvest, the predator population decrease quickly and the prey population starts to rise rapidly. On the other hand as the predator population rises, the prey population descends speedily. This figure is obtained when $p = 30$, $c = 35$ and $\delta = 0.1$.

8. Conclusions

In this paper, the properties of Hopf bifurcations in a Leslie-Gower Predator-Prey model with delay in predator's equation have been studied. We have also investigated optimal harvesting when the harvesting is given by a continuous function in this model. Although bifurcations in a population dynamics without delay have been investigated by many researchers, there are few papers on the bifurcations of a population dynamics with delay, which have shown direction of global Hopf-bifurcation and optimal harvesting simultaneously. We have obtained sufficient conditions on the parameters for which the delay-induced system is asymptotically stable around the positive equilibrium for all values of the delay parameter and if the conditions are not satisfied, then there exists a critical value of the delay parameter below which the system is stable and above which the system is unstable. By applying the normal form theory and the center manifold theorem, the explicit formulae which determine the stability and direction of the bifurcating periodic solutions have been determined. Our analytical and simulation results show that when the delay τ passes through the critical value τ_0^0, the coexisting equilibrium G_0 losses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from G_0. Also, the amplitude of oscillations increases with increasing τ. For the considered parameter values, it is observed that the Hopf bifurcation is supercritical and the bifurcating periodic solution is unstable. The problem of optimal harvesting policy has been solved by using the new result of retarded optimal control which is an extension of Pontryagin's Maximal principle theory. We hope that the theoretical investigations which have been carried out in this paper will certainly help the experimental ecologists to do some experimental studies and as a consequence the theoretical ecology may be developed to some extent.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

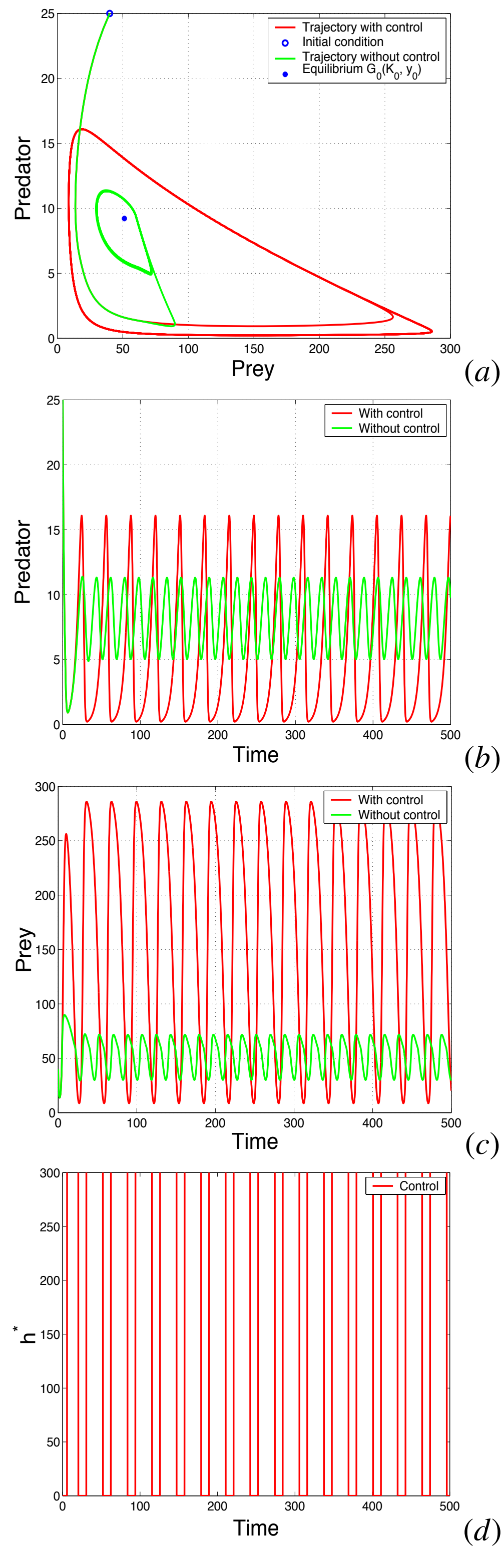


FIGURE 7. Trajectory of the model system (6) with and without the control.

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