Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2019, 2019:29
https://doi.org/10.28919/cmbn/4203
ISSN: 2052-2541

HARVESTING OF A PREY-PREDATOR MODEL FISHERY IN THE PRESENCE OF COMPETITION AND TOXICITY WITH TWO EFFORT FUNCTIONS

M. R. LEMNAOUAR[1], M. KHALFAOUI[1], Y. LOUARTASSI[1,2,*], I. TOLAIMATE[1]

[1]Mohammed V University in Rabat, Superior School of Technology Salé, LASTIMI, Salé, Morocco

[2]Faculty of Sciences, LabMIA-SI, Rabat, Morocco

Abstract. In this paper, we propose a predator-prey model with harvesting and reserved area for prey with the presence of competition and toxicity with two effort functions. First, we prove the boundedness of the solutions. Then, the existence is studied, as well as the local and global stability of the equilibria. Lyapunov proved this last with certain conditions. The optimal harvesting policy is discussed using the Maximum Principle of Pantryagin. Finally, we ensure our results by numerical simulations.

Keywords: predator–prey system; toxicity; equilibria; stability; competition; optimal harvesting policy.

2010 AMS Subject Classification: 34K18, 34K20, 92B20.

1. Introduction

There is a powerful relationship between prey species and predator species in theoretical ecology and applied mathematics. In recent decades, researchers have proposed many powerful prey-predator models to describe the dynamic behavior between these two types of populations [2, 5, 17], by taking into account optimal harvesting policy [14, 16, 18], toxicity [2, 10, 17], and the competition [6, 15]. As well, the dynamics of biological species has been analyzed.

*Corresponding author
E-mail address: ylouartassi@gmail.com

Received July 4, 2019

Studies have shown that they know a huge growth that is why the zones are divided into fishing and no-fishing areas. It can be introduced as a protective measure hoping that the migration of juveniles will allow rebuilding the depleted fishing grounds. As a result, fisheries need to be managed in an efficient and detailed manner to protect over-exploited stocks. Therefore, it is necessary to control the fishing effort in the different areas taking in consideration the growth of prey and predators. The aim of this paper is to study the competition and toxicity effects [4, 10] on the dynamics of the predator-prey model. The fishing efforts E_1 and E_2 are considered time-dependent, in order to protect certain fish stocks by limiting fishing activities. From [7], we suppose that E_1 and E_2 are expressed by differential equations. In order to preserve fish populations, the regulator imposes taxes τ_1 and τ_2 per unit of biomass of landed fish (with τ_1 and $\tau_2 > 0$). Now, the basic model is governed by the following ordinary equations:

$$
(1) \qquad \begin{cases}
\dfrac{dx}{dt} &= r_1 x\left(1-\frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - u x^2 - \frac{axz}{b+x} - q_1 E_1 x - n_1 xy, \\
\dfrac{dy}{dt} &= (r_2-\sigma_2)y + \sigma_1 x - v y^2 - n_2 xy, \\
\dfrac{dz}{dt} &= \frac{\beta axz}{b+x} - dz - wz - q_2 E_2 z, \\
\dfrac{dE_1}{dt} &= \lambda_1\left(q_1 x(m_1-\tau_1) - c_1\right)E_1, \\
\dfrac{dE_2}{dt} &= \lambda_2\left(q_2 z(m_2-\tau_2) - c_2\right)E_2.
\end{cases}
$$

The explanations of the parameters are presented on this table:

Parameters	Explanation
x	biomass densities of the unreserved areas
y	biomass densities of the reserved areas
z	biomass density of predator species
E_1	the effort applied for harvesting in the unreserved area
E_2	the effort applied for harvesting in the predator populations
r_1, r_2	the intrinsic growth rates of fish population inside reserved and the unserved areas
q_1, q_2	the catchability coefficient in the unreserved area and the predator species
σ_1, σ_2	migration rate from unreserved area to reserved area and reserved area to unreserved area
n_1, n_2	the competition coefficients
m_1, m_2	the fixed selling price per unit biomass of unreserved and predator species
c_1, c_2	the fixed cost of harvesting per unit of effort of unreserved and predator species fish
τ_1, τ_2	the imposed taxes per unit harvested of unreserved zone and predator species
λ_1, λ_2	constants which converts savings into capital
ux^2, vy^2	the reduction terms, in the unreserved area and reserved area respectively, where u and v the coefficients of toxicity
wz	the reduction term for the predator species
d	the death rate of the predator species
β	the conversion rate of predator due to prey
$\frac{axz}{b+x}$	holling type II functional response

Taking into account the biological constraints, and so that there is no decrease of the functions compared to the times of these functions it is necessary that: if there is no migration of fish population from reserved area to unreserved area ($\sigma_2 = 0$) and ($r_1 - \sigma_1 < 0$), we find that $\frac{dx}{dt} < 0$. Similarly, if there is no migration of fish population from unreserved area to reserved area ($\sigma_1 = 0$) and $r_2 - \sigma_2 < 0$, then $\frac{dy}{dt} < 0$.
If $\beta a - d - w < 0$, then $\frac{dz}{dt} < 0$.
If $m_i - \tau_i < 0$, then $\frac{dE_i}{dt} < 0$, for $i = 1, 2$.
Therefore, we assume that:

$$\beta a - d - w > 0,\ r_i - \sigma_i > 0 \text{ and } m_i - \tau_i > 0 \text{ for } i = 1, 2. \tag{2}$$

In this paper, we propose in our model the modification of the effort function of the model Y. Louartassi et al. [9]. Our article is organized as follows. In the following section, we show the bournitude of the solutions of the system (1). In section 3, we study the existence and stability of all the equilibria of our model. Then, we discuss the optimal harvesting policy of model (1) in the section 4. Finally, we present the numerical simulations for to study the stability of equilibria.

2. Positivity and Boundedness of the Solution

In this section, we describe the uniform boundedness of the solutions of the system (1).

Lemma 2.1. *The set* $\Omega = \left\{(x, y, z, E_1, E_2) \in \mathbb{R}^5_+ : x + y + \frac{1}{\beta}z + \frac{E_1}{m_1-\tau_1} + \frac{E_2}{m_2-\tau_2} \leq \frac{G}{d+w}\right\}$ *is a region of attraction for all solutions initiating in the interior of the positive octant, where*
$G = \dfrac{K(r_1 + d + w)^2}{4(r_1 + Ku)} + \dfrac{(r_2 + d + w)^2}{4v}$.

Proof. We pose $Y = x + y + \frac{1}{\beta}z + \frac{E_1}{m_1-\tau_1} + \frac{E_2}{m_2-\tau_2}$. Then,

$$\begin{aligned} \frac{dY}{dt} + (d+w)Y &= (r_1 + d + w)x - (\tfrac{r_1}{K} + u)x^2 - vy^2 + (r_2 + d + w)y - (n_1 + n_2)xy \\ &+ \frac{E_1}{m_1-\tau_1}(\frac{d+w}{\lambda_1} - c_1) + \frac{E_2}{m_2-\tau_2}(\frac{d+w}{\lambda_2} - c_2), \end{aligned}$$

$$\leq \frac{K(r_1+d+w)^2}{4(r_1+Ku)} + \frac{(r_2+d+w)^2}{4v} = G.$$

Applying the theory of differential inequality [1, 9], we get

$$Y < \frac{G}{d+w} - \left(\frac{G}{d+w} - Y(0)\right)\exp\left(-(d+w)t\right)$$

and when $t \to \infty$, $0 < Y \leq \frac{G}{d+w}$, proving the Lemma. □

3. Stability of Equilibria

In this section, we find the positive equilibria, then we study their local stability. We denote the function on the right hand side of the system (1) by $f_i(x,y,z,E_1,E_2)$, for $i = 1,\ldots,5$. Equilibria of model (1) is obtained by solving $f_i(x,y,z,E_1,E_2) = 0$, for $i = 1,\ldots,5$. It can be checked that model (1) has six positive equilibria:

1- $P_1(0,0,0,0,0)$ there is a trivial equilibrium.

2- $P_2(x_2,y_2,0,0,0)$, where (x_2,y_2) is the positive solution of the following equations:

$$\begin{aligned}(r_1-\sigma_1)x - \left(\frac{r_1+Ku}{K}\right)x^2 + \sigma_2 y - n_1 xy &= 0,\\ (r_2-\sigma_2)y + \sigma_1 x - vy^2 - n_2 xy &= 0.\end{aligned} \tag{3}$$

After the calculations, x is satisfied by the following cubic equation:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{4}$$

where:

$$\begin{aligned}
a_3 &= \left(u+\frac{r_1}{K}\right)\left(n_1n_2 - v\left(u+\frac{r_1}{K}\right)\right),\\
a_2 &= \frac{2v(r_1+Ku)(r_1-\sigma_1)}{K} - n_2\sigma_2\left(\frac{r_1}{K}+u\right) - n_1n_2(r_1-\sigma_1) - n_1(r_2-\sigma_2)\left(u+\frac{r_1}{K}\right) + \sigma_1 n_1^2,\\
a_1 &= -v(r_1-\sigma_1)^2 + (r_1-\sigma_1)\left(n_2\sigma_2 + n_1(r_2-\sigma_2)\right) + (r_2-\sigma_2)\sigma_2\left(u+\frac{r_1}{K}\right) - 2\sigma_1\sigma_2 n_1,\\
a_0 &= -\sigma_2(r_2-\sigma_2)(r_1-\sigma_1) + \sigma_1\sigma_2^2.
\end{aligned}$$

Using the result of [10], the above equation (4) had a unique positive solution if the following inequalities hold. According to the criteria of Descartes it is necessary to impose that:

$$\begin{aligned}
a_0 > 0 \quad &\text{if} \quad (r_2-\sigma_2)(r_1-\sigma_1) < \sigma_1\sigma_2,\\
a_1 > 0 \quad &\text{if} \quad (r_1-\sigma_1)\left(n_2\sigma_2 + n_1(r_2-\sigma_2)\right)\\
&+ \quad (r_2-\sigma_2)\sigma_2\left(u+\frac{r_1}{K}\right) > 2\sigma_1\sigma_2 n_1 + v(r_1-\sigma_1)^2,
\end{aligned}$$

$$a_3 < 0 \quad \text{if} \quad n_1 n_2 < v\left(u + \frac{r_1}{K}\right), \tag{5}$$

$$a_2 > 0 \quad \text{if} \quad v(r_1 - \sigma_1) > n_2\sigma_2 + n_1(r_2 - \sigma_2). \tag{6}$$

Then,

$$y_2 = \frac{x_2}{\sigma_2 - n_1 x_2}\left(\left(\frac{r_1 + Ku}{K}\right)x_2 - (r_1 - \sigma_1)\right) > 0,$$

if

$$\frac{(r_1 - \sigma_1)K}{r_1 + Ku} < x_2 < \frac{\sigma_2}{n_1} \quad or \quad \frac{\sigma_2}{n_1} < x_2 < \frac{(r_1 - \sigma_1)K}{r_1 + Ku}. \tag{7}$$

3- In the interior of the equilibrium $P_3(x_3, y_3, z_3, 0, 0)$, i.e. $f_i(x_3, y_3, z_3, 0, 0) = 0$, $i = 1, 2, 3$, we get a positive solution:

$$\begin{aligned} x_3 &= \tfrac{b(d+w)}{\beta a - d - w}, \\ y_3 &= \tfrac{r_2 - \sigma_2 - n_2 x_3 + \sqrt{(r_2 - \sigma_2 - n_2 x_3)^2 + 4\sigma_1 x_3 v}}{2v}, \\ z_3 &= \tfrac{b + x_3}{a x_3}\left((r_1 - \sigma_1 - n_1 y_3)x_3 - (\tfrac{r_1}{K} + u)x_3^2 + \sigma_2 y_3\right) > 0. \end{aligned} \tag{8}$$

if

$$0 < x_3 < \frac{r_1 - \sigma_1 - n_1 y_3 + \sqrt{(r_1 - \sigma_1 - n_1 y_3)^2 + 4(u + \frac{r_1}{K})y_3\sigma_2}}{2(\frac{r_1}{K} + u)}. \tag{9}$$

4- For the equilibrium $P_4(x_4, y_4, 0, (E_1)_4, 0)$, i.e. $f_i(x_4, y_4, 0, (E_1)_4, 0) = 0$, $i = 1, 2, 4$, we get a positive solution:

$$\begin{aligned} x_4 &= \frac{c_1}{q_1(m_1 - \tau_1)}, \\ y_4 &= \frac{(r_2 - \sigma_2 - n_2 x_4) + \sqrt{(r_2 - \sigma_2 - n_2 x_4)^2 + 4v\sigma_1 x_4}}{2v}, \\ (E_1)_4 &= \frac{(r_1 - \sigma_1)x_4 - (\frac{r_1}{K} + u)x_4^2 + \sigma_2 y_4 - n_1 x_4 y_4}{q_1 x_4}. \end{aligned} \tag{10}$$

which is positive if

$$(r_1 - \sigma_1)x_4 + \sigma_2 y_4 > (\frac{r_1}{K} + u)x_4^2 + n_1 x_4 y_4, \tag{11}$$

then,

$$y_4\left(\frac{n_1 c_1}{q_1(m_1 - \tau_1)} - \sigma_2\right) < \frac{c_1}{q_1(m_1 - \tau_1)}\left(r_1 - \sigma_1 - (\frac{r_1}{K} + u)\frac{c_1}{q_1(m_1 - \tau_1)}\right). \tag{12}$$

5- For the equilibrium $P_5(x_5, y_5, z_5, 0, (E_2)_5)$, i.e. $f_5(x_5, y_5, z_5, 0, (E_2)_5) = 0$, we get a positive solution:

$$z_5 = \frac{c_2}{q_2(m_2 - \tau_2)}, \tag{13}$$

and (x_5, y_5) is satisfying the following system of equations:

$$\begin{array}{lcl} (r_1 - \sigma_1)x - (\frac{r_1 + Ku}{K})x^2 + \sigma_2 y - n_1 xy - \frac{axz}{b+x} & = & 0, \\ (r_2 - \sigma_2)y + \sigma_1 x - vy^2 - n_2 xy & = & 0. \end{array} \tag{14}$$

After the calculations, x is satisfied by the following equation:

$$b_5 x^5 + b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0 = 0, \tag{15}$$

where:

$$\begin{array}{rcl} b_5 & = & dn_1 n_2 - d^2 v, \\ b_4 & = & 2bdn_1 n_2 + 2cdv + n_1^2 \sigma_1 - 2bd^2 v - cn_1 n_2 - den_1 - dn_2 \sigma_2, \\ b_3 & = & -2adv + an_1 n_2 - b^2 d^2 v + b^2 dn_1 n_2 + 4bcdv - 2bcn_1 n_2 - 2bden_1 + cen_1 + cn_2 \sigma_2 \\ & + & de\sigma_2 - 2n_1 \sigma_1 \sigma_2, \\ b_2 & = & -2abdvz_5 + abn_1 n_2 z_5 + 2acvz_5 - aen_1 z_5 - an_2 \sigma_2 z_5 + 2b^2 cdv - b^2 dn_2 \sigma_2 + b^2 n_1^2 \sigma_1 \\ & + & 2bcen_1 + 2bcn_2 \sigma_2 + 2bde\sigma_2 - 4bn_1 \sigma_1 \sigma_2 - ce\sigma_2 + \sigma_1 \sigma_2^2, \\ b_1 & = & -a^2 vz_5^2 + 2abcvz_5 - aben_1 z_5 - abn_2 \sigma_2 z_5 + ae\sigma_2 z_5 - b^2 c^2 v + b^2 cen_1 + b^2 cn_2 \sigma_2 \\ & + & b^2 de\sigma_2 - 2b^2 n_1 \sigma_1 \sigma_2 - 2bce\sigma_2 + 2b\sigma_1 \sigma_2^2, \\ b_0 & = & abe\sigma_2 z_5 - b^2 ce\sigma_2 + b^2 \sigma_1 \sigma_2^2 - b^2 den_1 - 2bc^2 v. \end{array}$$

where

$c = r_1 - \sigma_1$, $d = u + \frac{r_1}{K}$, $e = r_2 - \sigma_2$.

Using criteria of Descartes [3] it is necessary to impose that: The above equation (14) had a unique positive solution if the following inequalities hold. We find that $b_5 < 0$ and $b_i > 0$, for $i = 0, \dots, 4$.

According to $f_1(x_5, y_5, z_5, 0, (E_2)_5) = 0$, we get

$$y_5 = \frac{1}{\sigma_2 - n_1 x_5} \left((u + \frac{r_1}{K})x_5^2 + \frac{ax_5 z_5}{b + x_5} - (r_1 - \sigma_1)x_5 \right) > 0, \tag{16}$$

if

$$\begin{aligned} &\sigma_2 - n_1 x_5 > 0 \quad \text{and} \quad z_5 > \tfrac{b+x_5}{a}(r_1 - \sigma_1 - (u+\tfrac{r_1}{K})x_5), \\ &\text{or } \sigma_2 - n_1 x_5 < 0 \quad \text{and} \quad z_5 < \tfrac{b+x_5}{a}(r_1 - \sigma_1 - (u+\tfrac{r_1}{K})x_5). \end{aligned} \tag{17}$$

Using $f_3(x_5, y_5, z_5, 0, (E_2)_5) = 0$, we get $(E_2)_5 = \frac{1}{q_2}\left(\frac{\beta a x_5}{b+x_5} - d - w\right) > 0$, if $x_5 > \frac{b(d+w)}{\beta a - d - w}$.

6- In the interior of the equilibrium $P_6(x_6, y_6, z_6, (E_1)_6, (E_2)_6)$, i.e. $f_i(x_6, y_6, z_6, (E_1)_6, (E_2)_6) = 0$, for $i = 1, \dots, 5$, we get a positive solution:

$$\begin{cases} x_6 &= \dfrac{c_1}{q_1(m_1 - \tau_1)}, \\ y_6 &= \dfrac{(r_2 - \sigma_2 - n_2 x_6) + \sqrt{(r_2 - \sigma_2 - n_2 x_6)^2 + 4v\sigma_1 x_6}}{2v}, \\ z_6 &= \dfrac{c_2}{q_2(m_2 - \tau_2)}, \\ (E_1)_6 &= \frac{1}{q_2 x_6}\left((r_1 - \sigma_1)x_6 + \sigma_2 y_6 - (u + \frac{r_1}{K})x_6^2 - \frac{a x_6 z_6}{b+x_6} - n_1 x_6 y_6\right) > 0, \\ (E_2)_6 &= \frac{1}{q_2}\left(\frac{\beta a x_6}{b+x_6} - d - w\right) > 0. \end{cases} \tag{18}$$

which is positive if

$$y_6(\sigma_2 - n_1 x_6) > \left((u + \frac{r_1}{K})x_6 + \frac{a z_6}{b + x_6} - (r_1 - \sigma_1)\right) x_6 \quad \text{and} \quad x_6 > \frac{b(d+w)}{\beta a - d - w}. \tag{19}$$

Now, we discuss the local stability of this six equilibria.

Theorem 3.1. *The equilibrium $P_1(0,0,0,0,0)$ of the system (1) is unstable.*

Proof. The characteristic equation of P_1 is:

$$(X + d + w)(X + \lambda_1 c_1)\left(X + \lambda_2 c_2)(X^2 - (r_1 - \sigma_1 + r_2 - \sigma_2)X + (r_2 - \sigma_2)(r_1 - \sigma_1) - \sigma_2\sigma_1\right) = 0.$$

It is easy to verify $X_1 = -(d+w) < 0, X_2 = -\lambda_1 c_1, X_3 = -\lambda_2 c_2$. Let X_4 and X_5 be the two other eigenvalues. Obviously $X_4 + X_5 = r_1 - \sigma_1 + r_2 - \sigma_2 > 0$.

Therefore X_4 and X_5 have one positive value. Hence, P_1 is unstable. □

Theorem 3.2. *The equilibrium point $P_2(x_2, y_2, 0, 0, 0)$ of the system (1) is locally asymptotically stable if $x_2 < \min\left(\frac{c_1}{q_1(m_1 - \tau_1)}, \frac{b(d+w)}{\beta a - d - w}\right)$.*

Proof. The characteristic equation at P_2 is:

$(X+\lambda_2 c_2)\,(X-\lambda_1(q_1x_2(m_1-\tau_1)-c_1))\,(X-\frac{\beta a x_2}{b+x_2}-d-w)\times$
$\Big((X+\sigma_2\frac{y_2}{x_2}+(\frac{r_1}{K}+u)x_2)(X+\sigma_1\frac{x_2}{y_2}+vy_2)-(\sigma_2-n_1x_2)(\sigma_1-n_2y_2)\Big)=0,$

The eigenvalues of P_2:

$$\begin{aligned} X_1 &= -\lambda_2c_2<0,\\ X_2 &= \lambda_1(q_1x_2(m_1-\tau_1)-c_1)<0 \quad \text{if} \quad x_2<\frac{c_1}{q_1(m_1-\tau_1)},\\ X_3 &= \frac{\beta a x_2}{b+x_2}-(d+w)<0 \quad \text{if} \quad x_2<\frac{b(d+w)}{\beta a-d-w},\end{aligned}$$

on the other hand $X_4+X_5=-\Big(\sigma_2\frac{y_2}{x_2}+(u+\frac{r_1}{K})x_2+\sigma_1\frac{x_2}{y_2}+vy_2\Big)<0,$

$X_4X_5=\Big(\sigma_2\frac{y_2}{x_2}+(u+\frac{r_1}{K})x_2\Big)\Big(\sigma_1\frac{x_2}{y_2}+vy_2\Big)-(\sigma_2-n_1x_2)(\sigma_1-n_2y_2)>0$. Therefore $X_4,X_5<0$.

Hence, P_2 is locally asymptotically stable if $x_2<\min\Big(\frac{c_1}{q_1(m_1-\tau_1)},\frac{b(d+w)}{\beta a-d-w}\Big)$. □

Theorem 3.3. *The equilibrium point $P_3(x_3,y_3,z_3,0,0)$ is locally asymptotically stable if $x_3<\frac{c_1}{q_1(m_1-\tau_1)}$, $z_3<\frac{c_2}{q_2(m_2-\tau_2)}$, $d_0,d_1,d_2>0$ and $d_2d_1-d_0>0$.*

Proof. The characteristic equation at P_3:

$(X-\lambda_2(q_2z_3(m_2-\tau_2)-c_2))(X-\lambda_1(q_1x_3(m_1-\tau_1)-c_1))(X^3+d_2X^2+d_1X+d_0)=0,$

where: $X_1=\lambda_2(q_1x_3(m_1-\tau_1)-c_1)<0$ if $x_3<\frac{c_1}{q_1(m_1-\tau_1)}$,

$X_2=\lambda_2(q_2z_3(m_2-\tau_2)-c_2)<0$ if $z_3<\frac{c_2}{q_2(m_2-\tau_2)}$,

$$\begin{aligned} d_2 &= -(r_1-\sigma_1-2x_3\frac{r_1+Ku}{K}-n_1y_3-\frac{abz_3}{(b+x_3)^2}+r_2-\sigma_2-2vy_3-n_2x_3),\\ d_1 &= (r_1-\sigma_1-2x_3\frac{r_1+Ku}{K}-n_1y_3-\frac{abz_3}{(b+x_3)^2})(r_2-\sigma_2-2vy_3-n_2x_3)-(\sigma_1-n_2y_3)(\sigma_2-n_1x_3)\\ &+ \frac{\beta a^2bx_3z_3}{(b+x_3)^3},\\ d_0 &= -(r_2-\sigma_2-2vy_3-n_2x_3)\frac{\beta a^2bx_3z_3}{(b+x_3)^3}.\end{aligned}$$

Using criteria of Herwitz [11], P_3 is locally asymptotically stable if $x_3<\frac{c_1}{q_1(m_1-\tau_1)}$, $z_3<\frac{c_2}{q_2(m_2-\tau_2)}$, d_0 ,d_1, $d_2>0$ and $d_2d_1-d_0>0$. □

Theorem 3.4. *The equilibrium point $P_4(x_4,y_4,0,(E_1)_4,0)$ is asymptotically stable if $\phi_0,\phi_1,\phi_2,\phi_3>0$ and $\phi_2\phi_3-\phi_1>\frac{\phi_0\phi_3^2}{\phi_1}$.*

Proof. The characteristic equation at P_4 is:

$(X+\lambda_2c_2)(X^4+\phi_3X^3+\phi_2X^2+\phi_1X+\phi_0)=0,$

where:

$$(20)\quad \begin{aligned} \phi_3 &= -(A_1+A_2+A_3),\\ \phi_2 &= A_1A_2+A_3(A_1+A_2)+q_1^2x_4(m_1-\tau_1)(E_1)_4-(\sigma_1-n_2y_4)(\sigma_2-n_1x_4),\\ \phi_1 &= -A_3\left(A_1A_2-(\sigma_2-n_1x_4)(\sigma_1-n_2y_4)\right)-(A_2+A_3)q_1^2x_4(m_1-\tau_1)(E_1)_4,\\ \phi_0 &= q_1^2x_4(m_1-\tau_1)(E_1)_4A_1A_2. \end{aligned}$$

Where:

$$\begin{aligned} A_1 &= r_1-\sigma_1-2(\tfrac{r_1}{K}+u)x_4-q_1(E_1)_4-n_1y_4,\\ A_2 &= r_2-\sigma_2-2vy_4-n_2x_4,\\ A_3 &= \tfrac{\beta a x_4}{b+x_4}-d-w. \end{aligned}$$

From the characteristic equation of P_4 we get $X_1=-\lambda_2c_2<0$.

Using Herwitz criteria [11], P_4 is asymptotically stable if $\phi_0,\phi_1,\phi_2,\phi_3>0$ and $\phi_2\phi_3-\phi_1>\frac{\phi_0\phi_3^2}{\phi_1}$. □

Theorem 3.5. *The equilibrium point $P_5(x_5,y_5,z_5,0,(E_2)_5)$ is asymptotically stable if $x_5<\frac{c_1}{q_1(m_1-\tau_1)}$, $\varphi_0,\varphi_1,\varphi_2,\varphi_3>0$ and $\varphi_2\varphi_3-\varphi_1>\frac{\varphi_0\varphi_3^2}{\varphi_1}$.*

Proof. The characteristic equation at P_5 is:

$(X-i_5)(X^4+\varphi_3X^3+\varphi_2X^2+\varphi_1X+\varphi_0)=0$, where:

$$(21)\quad \begin{aligned} \varphi_3 &= -(a_5+f_5),\\ \varphi_2 &= a_5f_5-b_5e_5-c_5g_5-h_6j_6,\\ \varphi_1 &= a_5h_5j_5+c_5f_5g_5+f_5h_5j_5,\\ \varphi_0 &= b_5e_5h_5j_5-a_5f_5h_5j_5. \end{aligned}$$

Where:

$$\begin{aligned} a_5 &= r_1-\sigma_1-2(\tfrac{r_1}{K}+u)x_5-n_1y_5-\tfrac{abz_5}{(b+x_5)^2},\\ b_5 &= \sigma_2-n_1x_5,\\ c_5 &= \tfrac{-ax_5}{b+x_5},\\ d_5 &= -q_1x_5,\\ e_5 &= \sigma_1-n_2y_5,\\ f_5 &= r_2-\sigma_2-2vy_5-n_2x_5, \end{aligned}$$

$$
\begin{aligned}
g_5 &= \tfrac{\beta a b z_5}{(b+x_5)^2},\\
h_5 &= -q_2 z_5\\
i_5 &= \lambda_1(q_1 x_5(m_1-\tau_1)-c_1),\\
j_5 &= \lambda_2 q_2(m_2-\tau_2)(E_2)_5.
\end{aligned}
$$

From the characteristic equation of P_5 we get $X_1 = i_5 < 0$ if $x_5 < \frac{c_1}{q_1(m_1-\tau_1)}$. Using Herwitz criteria [11], P_5 is asymptotically stable if $x_5 < \frac{c_1}{q_1(m_1-\tau_1)}$, $\varphi_0, \varphi_1, \varphi_2, \varphi_3 > 0$ and
$\varphi_2\varphi_3 - \varphi_1 > \frac{\varphi_0\varphi_3^2}{\varphi_1}$. □

Theorem 3.6. *The equilibrium point $P_6(x_6, y_6, z_6, (E_1)_6, (E_2)_6)$ is asymptotically stable if $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4 > 0$, $\mu_2 - \mu_3\mu_4 < 0$, $\mu_0 - \mu_1\mu_4 < 0$ and $\mu_4(\mu_0 - \mu_1\mu_4) > \mu_2(\mu_2 - \mu_3\mu_4)$.*

Proof. The characteristic equation at P_6 is:
$X^5 + \mu_4 X^4 + \mu_3 X^3 + \mu_2 X^2 + \mu_1 X + \mu_0 = 0$,
where:

$$
\begin{aligned}
\mu_4 &= -(a_6+f_6),\\
\mu_3 &= a_6 f_6 - b_6 e_6 - c_6 g_6 - i_6 d_6 - h_6 j_6,\\
\mu_2 &= a_6 h_6 j_6 + c_6 f_6 g_6 + i_6 d_6 f_6 + f_6 h_6 j_6,\\
\mu_1 &= -a_6 f_6 h_6 j_6 + b_6 e_6 h_6 j_6 + i_6 d_6 h_6 j_6,\\
\mu_0 &= -i_6 d_6 f_6 h_6 j_6.
\end{aligned}
\tag{22}
$$

Where:

$$
\begin{aligned}
a_6 &= r_1 - \sigma_1 - 2(\tfrac{r_1}{K}+u)x_6 - q_1(E_1)_6 - n_1 y_6 - \tfrac{abz_6}{(b+x_6)^2},\\
b_6 &= \sigma_2 - n_1 x_6,\\
c_6 &= \tfrac{-ax_6}{b+x_6},\\
d_6 &= -q_1 x_6,\\
e_6 &= \sigma_1 - n_2 y_6,\\
f_6 &= r_2 - \sigma_2 - 2v y_6 - n_2 x_6,\\
g_6 &= \tfrac{\beta a b z_6}{(b+x_6)^2},\\
h_6 &= -q_2 z_6,\\
i_6 &= \lambda_1 q_1(m_1-\tau_1)(E_1)_6,\\
j_6 &= \lambda_2 q_2(m_2-\tau_2)(E_2)_6.
\end{aligned}
\tag{23}
$$

From the characteristic equation of P_6, using Herwitz criteria [11], P_6 is asymptotically stable if $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4 > 0$, $\mu_2 - \mu_3\mu_4 < 0$, $\mu_0 - \mu_1\mu_4 < 0$ and $\mu_4(\mu_0 - \mu_1\mu_4) > \mu_2(\mu_2 - \mu_3\mu_4)$. □

Using Lyaponuv functions, we study the global stability of each equilibrium points.

Theorem 3.7. *The equilibrium $P_2(x_2, y_2, 0, 0, 0)$ is globally asymptotically stable if* $n_1 + \frac{\sigma_2 n_2 y_2}{\sigma_1 x_2} < 2\min\left(\frac{\sigma_2 y_2 v}{\sigma_1 x_2}, \frac{r_1}{K} + u\right)$.

Proof. The Lyapunov function of equilibria $P_2(x_2, y_2, 0, 0, 0)$ is given by:
$V_2(x,y) = \left(x - x_2 - x_2 \ln\left(\frac{x}{x_2}\right)\right) + \frac{\sigma_2 y_2}{\sigma_1 x_2}\left(y - y_2 - y_2 \ln\left(\frac{y}{y_2}\right)\right)$.
Differentiating V_2 respect to time t, we obtain:

$$\begin{aligned} \frac{dV_2}{dt} &= (x - x_2)\left(-\left(\frac{r_1 + Ku}{K}\right)(x - x_2) + \sigma_2\left(\frac{y}{x} - \frac{y_2}{x_2}\right) - n_1(y - y_2)\right) \\ &+ \frac{\sigma_2 y_2}{\sigma_1 x_2}(y - y_2)\left(-v(y - y_2) + \sigma_1\left(\frac{x}{y} - \frac{x_2}{y_2}\right) - n_2(x - x_2)\right), \end{aligned}$$

we find,

$$\begin{aligned} \frac{dV_2}{dt} &= -\left(\frac{r_1 + Ku}{K}\right)(x - x_2)^2 - \frac{v\sigma_2 y_2}{\sigma_1 x_2}(y - y_2)^2 - \frac{\sigma_2}{x x_2 y}(y x_2 - y_2 x)^2 - \left(n_1 + \frac{\sigma_2 y_2 n_2}{\sigma_1 x_2}\right)(x - x_2)(y - y_2), \\ &< \left(\frac{1}{2}\left(n_1 + \frac{\sigma_2 y_2 n_2}{\sigma_1 x_2}\right) - \frac{r_1 + Ku}{K}\right)(x - x_2)^2 - \left(\frac{1}{2}\left(n_1 + \frac{\sigma_2 y_2 n_2}{\sigma_1 x_2}\right) - \frac{v\sigma_2 y_2}{\sigma_1 x_2}\right)(y - y_2)^2 - \frac{\sigma_2}{x x_2 y}(y x_2 - y_2 x)^2. \end{aligned}$$

Therefore, $\frac{dV_2}{dt} < 0$ if $n_1 + \frac{\sigma_2 n_2 y_2}{\sigma_1 x_2} < 2\min\left(\frac{\sigma_2 y_2 v}{\sigma_1 x_2}, \frac{r_1}{K} + u\right)$. □

Theorem 3.8. *The equilibrium $P_3(x_3, y_3, z_3, 0, 0)$ is globally asymptotically stable if* $n_1 + \frac{\sigma_2 n_2 y_3}{\sigma_1 x_3} < 2\min\left(\frac{\sigma_2 y_3 v}{\sigma_1 x_3}, \frac{r_1}{K} + u - \frac{a z_3}{b(b + x_3)}\right)$.

Proof. By constructing a Lyapunov function to prove this theorem. The Lyapunov function of $P_3(x_3, y_3, z_3, 0, 0)$ is given by:
$V_3(x,y,z) = \left(x - x_3 - x_3 \ln\left(\frac{x}{x_3}\right)\right) + \frac{\sigma_2 y_3}{\sigma_1 x_3}\left(y - y_3 - y_3 \ln\left(\frac{y}{y_3}\right)\right) + \frac{b + x_3}{b\beta}\left(z - z_3 - z_3 \ln\left(\frac{z}{z_3}\right)\right)$.
Differentiating V respect to time t, we obtain:

$$\begin{aligned} \frac{dV_3}{dt} &= (x - x_3)\left(-\left(\frac{r_1 + Ku}{K}\right)(x - x_3) + \sigma_2\left(\frac{y}{x} - \frac{y_3}{x_3}\right) - n_1(y - y_3) - a\left(\frac{z}{b + x} - \frac{z_3}{b + x_3}\right)\right) \\ &+ \frac{\sigma_2 y_3}{\sigma_1 x_3}(y - y_3)\left(-v(y - y_3) + \sigma_1\left(\frac{x}{y} - \frac{x_3}{y_3}\right) - n_2(x - x_3)\right) + \frac{b + x_3}{b\beta}\beta a(z - z_3)\left(\frac{x}{b + x} - \frac{x_3}{b + x_3}\right), \end{aligned}$$

After the simplification we get:

$$
\begin{aligned}
\frac{dV_3}{dt} &= \left(\frac{az_3}{(b+x)(b+x_3)} - \frac{r_1+Ku}{K}\right)(x-x_3)^2 - \frac{v\sigma_2 y_3}{\sigma_1 x_3}(y-y_3)^2 - \frac{\sigma_2}{xx_3 y}(yx_3-y_3x)^2 \\
&- (n_1 + \frac{\sigma_2 y_3 n_2}{\sigma_1 x_3})(x-x_3)(y-y_3), \\
&< \left(\frac{1}{2}(n_1+\frac{\sigma_2 y_3 n_2}{\sigma_1 x_3}) - \frac{r_1+Ku}{K} + \frac{az_3}{b(b+x_3)}\right)(x-x_3)^2 - \left(\frac{1}{2}(n_1+\frac{n_2\sigma_2 y_3}{\sigma_1 x_3}) - \frac{v\sigma_2 y_3}{\sigma_1 x_3}\right)(y-y_3)^2 \\
&- \frac{\sigma_2}{xx_3 y}(yx_3-y_3x)^2.
\end{aligned}
$$

Therefore, $\frac{dV_3}{dt} < 0$ if $n_1 + \frac{\sigma_2 n_2 y_3}{\sigma_1 x_3} < 2\min\left(\frac{\sigma_2 y_3 v}{\sigma_1 x_3}, \frac{r_1}{K} + u - \frac{az_3}{b(b+x_3)}\right)$. □

Theorem 3.9. *The equilibrium $P_4(x_4, y_4, 0, (E_1)_4, 0)$ is globally asymptotically stable if $n_1 + \frac{\sigma_2 n_2 y_4}{\sigma_1 x_4} < 2\min\left(\frac{\sigma_2 y_4 v}{\sigma_1 x_4}, \frac{r_1}{K} + u\right)$.*

Proof. The Lyapunov function of equilibria $P_4(x_4, y_4, 0, (E_1)_4, 0)$ is given by:

$$
\begin{aligned}
V_4(x,y,E_1) &= \left(x - x_4 - x_4\ln\left(\frac{x}{x_4}\right)\right) + \frac{\sigma_2 y_4}{\sigma_1 x_4}\left(y - y_4 - y_4\ln\left(\frac{y}{y_4}\right)\right) \\
&+ \frac{1}{\lambda_1(m_1-\tau_1)}\left(E_1 - (E_1)_4 - (E_1)_4\ln\left(\frac{E_1}{(E_1)_4}\right)\right).
\end{aligned}
$$

Differentiating V_4 respect to time t, we obtain:

$$
\begin{aligned}
\frac{dV_4}{dt} &= (x-x_4)\left(-\left(\frac{r_1+Ku}{K}\right)(x-x_4) + \sigma_2\left(\frac{y}{x} - \frac{y_4}{x_4}\right) - n_1(y-y_4) - q_1(E_1-(E_1)_4)\right) \\
&+ \frac{\sigma_2 y_4}{\sigma_1 x_4}(y-y_4)\left(-v(y-y_4) + \sigma_1(\frac{x}{y} - \frac{x_4}{y_4}) - n_2(x-x_4)\right) \\
&+ \frac{1}{\lambda_1(m_1-\tau_1)}(E_1-(E_1)_4)\lambda_1 q_1(m_1-\tau_1)(x-x_4),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV_4}{dt} &= -\left(\frac{r_1+Ku}{K}\right)(x-x_4)^2 - \frac{v\sigma_2 y_4}{\sigma_1 x_4}(y-y_4)^2 - \frac{\sigma_2}{xx_4 y}(yx_4-y_4x)^2 - (n_1+\frac{\sigma_2 y_4 n_2}{\sigma_1 x_4})(x-x_4)(y-y_4), \\
&< \left(\frac{1}{2}(n_1+\frac{\sigma_2 y_4 n_2}{\sigma_1 x_4}) - \frac{r_1+Ku}{K}\right)(x-x_4)^2 - \left(\frac{1}{2}(n_1+\frac{\sigma_2 y_4 n_2}{\sigma_1 x_4}) - \frac{v\sigma_2 y_4}{\sigma_1 x_4}\right)(y-y_4)^2 - \frac{\sigma_2}{xx_4 y}(yx_4-y_4x)^2.
\end{aligned}
$$

Therefore, $\frac{dV_4}{dt} < 0$ if $n_1 + \frac{\sigma_2 n_2 y_4}{\sigma_1 x_4} < 2\min\left(\frac{\sigma_2 y_4 v}{\sigma_1 x_4}, \frac{r_1}{K} + u\right)$. □

Theorem 3.10. *The equilibrium $P_5(x_5, y_5, z_5, 0, (E_2)_5)$ is globally asymptotically stable if $n_1 + \frac{\sigma_2 n_2 y_5}{\sigma_1 x_5} < 2\min\left(\frac{\sigma_2 y_5 v}{\sigma_1 x_5}, \frac{r_1}{K} + u - \frac{az_5}{b(b+x_5)}\right)$.*

Proof. The Lyapunov function of equilibria $P_5(x_5, y_5, z_5, 0, (E_2)_5)$ is given by:

$$
\begin{aligned}
V_5(x,y,z,E_2) &= \left(x - x_5 - x_5\ln\left(\frac{x}{x_5}\right)\right) + \frac{\sigma_2 y_5}{\sigma_1 x_5}\left(y - y_5 - y_5\ln\left(\frac{y}{y_5}\right)\right) \\
&+ \frac{b+x_5}{b\beta}\left(z - z_5 - z_5\ln\left(\frac{z}{z_5}\right)\right) + \frac{b+x_5}{b\beta\lambda_2(m_2-\tau_2)}\left(E_2 - (E_2)_5 - (E_2)_5\ln\left(\frac{E_2}{(E_2)_5}\right)\right).
\end{aligned}
$$

Differentiating V_5 respect to time t, we obtain:

$$\begin{aligned}
\frac{dV_5}{dt} &= (x-x_5)\left(-\left(\frac{r_1+Ku}{K}\right)(x-x_5)+\sigma_2\left(\frac{y}{x}-\frac{y_5}{x_5}\right)-n_1(x-x_5)-a\left(\frac{z}{b+x}-\frac{z_5}{b+x_5}\right)\right)\\
&+ \frac{\sigma_2 y_5}{\sigma_1 x_5}(y-y_5)\left(-v(y-y_5)+\sigma_1\left(\frac{x}{y}-\frac{x_5}{y_5}\right)-n_2(x-x_5)\right)\\
&+ \frac{b+x_5}{b\beta}(z-z_5)\left(\frac{\beta a x}{b+x}-\frac{\beta a x_5}{b+x_5}-q_2(E_2-(E_2)_5)\right)\\
&+ \frac{b+x_5}{b\beta\lambda_2(m_2-\tau_2)}(E_2-(E_2)_5)\lambda_2 q_2(m_2-\tau_2)(z-z_5),
\end{aligned}$$

we find,

$$\begin{aligned}
\frac{dV_5}{dt} &= \left(\frac{az_5}{(b+x)(b+x_5)}-\frac{r_1+Ku}{K}\right)(x-x_5)^2-\frac{v\sigma_2 y_5}{\sigma_1 x_5}(y-y_5)^2-\frac{\sigma_2}{xx_5y}(yx_5-y_5x)^2\\
&- \left(n_1+\frac{\sigma_2 y_5 n_2}{\sigma_1 x_5}\right)(x-x_5)(y-y_5),\\
&< \left(\frac{1}{2}\left(n_1+\frac{\sigma_2 y_5 n_2}{\sigma_1 x_5}\right)-\frac{r_1+Ku}{K}+\frac{az_5}{b(b+x_5)}\right)(x-x_5)^2+\left(\frac{1}{2}\left(n_1+\frac{n_2\sigma_2 y_5}{\sigma_1 x_5}-\frac{v\sigma_2 y_5}{\sigma_1 x_5}\right)(y-y_5)^2\right.\\
&- \frac{\sigma_2}{xx_5y}(yx_5-y_5x)^2.
\end{aligned}$$

Therefore, $\frac{dV_5}{dt}<0$ if $n_1+\frac{\sigma_2 n_2 y_5}{\sigma_1 x_5}<2\min\left(\frac{\sigma_2 y_5 v}{\sigma_1 x_5},\frac{r_1}{K}+u-\frac{az_5}{b(b+x_5)}\right)$. □

Theorem 3.11. *The equilibrium $P_6(x_6,y_6,z_6,(E_1)_6,(E_2)_6)$ is globally asymptotically stable if $n_1+\frac{\sigma_2 n_2 y_6}{\sigma_1 x_6}<2\min\left(\frac{\sigma_2 y_6 v}{\sigma_1 x_6},\frac{r_1}{K}+u\right)$.*

Proof. By constructing a Lyapunov function to prove this theorem. The Lyapunov function is given by:

$$V_6(x,y,z,E_1,E_2)=\left(x-x_6-x_6\ln\left(\frac{x}{x_6}\right)\right)+\frac{\sigma_2 y_6}{\sigma_1 x_6}\left(y-y_6-y_6\ln\left(\frac{y}{y_6}\right)\right)+\frac{b+x_6}{b\beta}\left(z-z_6-z_6\ln\left(\frac{z}{z_6}\right)\right)$$
$$+\frac{1}{\lambda_1(m_1-\tau_1)}\left(E_1-(E_1)_6-(E_1)_6\ln\left(\frac{E_1}{(E_1)_6}\right)\right)+\frac{b+x_6}{b\beta(\lambda_2(m_2-\tau_2))}\left(E_2-(E_2)_6-(E_2)_6\ln\left(\frac{E_2}{(E_2)_6}\right)\right).$$

Differentiating V respect to time t, we obtain:

$$\begin{aligned}
\frac{dV_6}{dt} &= (x-x_6)\left(-\left(\frac{r_1+Ku}{K}\right)(x-x_6)+\sigma_2\left(\frac{y}{x}-\frac{y_6}{x_6}\right)-a\left(\frac{z}{b+x}-\frac{z_6}{b+x_6}\right)-q_1(E_1-(E_1)_6)-n_1(y-y_6)\right)\\
&+ \frac{\sigma_2 y_6}{\sigma_1 x_6}(y-y_6)\left(-v(y-y_6)+\sigma_1\left(\frac{x}{y}-\frac{x_6}{y_6}\right)-n_2(x-x_6)\right)\\
&+ \frac{b+x_6}{b\beta}(z-z_6)\left(\frac{\beta a x}{b+x}-\frac{\beta a x_6}{b+x_6}-q_2(E_2-(E_2)_6)\right)+\frac{1}{\lambda_1(m_1-\tau_1)}(E_1-(E_1)_6)\lambda_1(q_1x(m_1-\tau_1)-c_1)\\
&+ \frac{b+x_6}{b\beta(\lambda_2(m_2-\tau_2))}(E_2-(E_2)_6)\lambda_2(q_2z(m_2-\tau_2)-c_2),
\end{aligned}$$

we find,

$$\begin{aligned}
\frac{dV_6}{dt} &= \left(\frac{1}{2}\left(n_1+\frac{\sigma_2 y_6 n_2}{\sigma_1 x_6}\right)+\frac{az_6}{(b+x)(b+x_6)}-\frac{r_1+Ku}{K}\right)(x-x_6)^2-\frac{v\sigma_2 y_6}{\sigma_1 x_6}(y-y_6)^2-\frac{\sigma_2}{xx_6y}(yx_6-y_6x)^2\\
&- \left(n_1+\frac{\sigma_2 y_6 n_2}{\sigma_1 x_6}\right)(x-x_6)(y-y_6),\\
&< \left(\frac{1}{2}\left(n_1+\frac{\sigma_2 y_6 n_2}{\sigma_1 x_6}\right)-\frac{r_1+Ku}{K}+\frac{az_6}{b(b+x_6)}\right)(x-x_6)^2+\left(\frac{1}{2}\left(n_1+\frac{n_2\sigma_2 y_6}{\sigma_1 x_6}\right)-\frac{v\sigma_2 y_6}{\sigma_1 x_6}\right)(y-y_6)^2\\
&- \frac{\sigma_2}{xx_6y}(yx_6-y_6x)^2.
\end{aligned}$$

Therefore, $\frac{dV_6}{dt}<0$ if $n_1+\frac{\sigma_2 n_2 y_6}{\sigma_1 x_6}<2\min\left(\frac{\sigma_2 y_6 v}{\sigma_1 x_6},\frac{r_1}{K}+u-\frac{az_6}{b(b+x_6)}\right)$. □

4. Optimal Harvesting Policy

In this section, we utilize the Pontryagin's Principle [13] in the presence of predator is discussed. Let c_1 be the fishing cost per unit for prey species, c_2 be the fishing cost per unit for predator species, p_1 is the constant price per unit biomass of the prey, p_2 is the constant price per unit biomass of the predator. Then net economic revenue at any time t is given by:

$$\Pi(x,y,z,E_1,E_2) = (q_1p_1x - c_1)E_1 + (q_2p_2z - c_2)E_2.$$

The present value I of a continuous time-stream of revenues is given by:

$$I = \int_{t_0}^{t_f} \Pi(x,y,z,E_1,E_2)e^{-\delta t}\,\mathrm{d}t = \int_{t_0}^{t_f} \left((q_1p_1x - c_1)E_1 + (q_2p_2z - c_2)E_2\right)e^{-\delta t}\,\mathrm{d}t,$$

where δ is the instantaneous discount rate, and one unit of time is required to change a harvesting strategy to maximize the total discounted net revenue earned from the system (1) and to the control constraints:

$$0 < \tau_i(t) < (\tau_i)_{max} \quad (i = 1,2).$$

The associated Hamiltonian of the problem is given by:

$$\begin{aligned} H &= e^{-\delta t}\left((q_1p_1x - c_1)E_1 + (q_2p_2z - c_2)E_2\right) \\ &+ \gamma_1(t)\left((r_1 - \sigma_1)x - (\tfrac{r_1}{K} + u)x^2 + \sigma_2 y - \tfrac{axz}{b+x} - n_1xy - q_1E_1x\right) \\ &+ \gamma_2(t)\left((r_2 - \sigma_2)y + \sigma_1 x - vy^2 - n_2xy\right) + \gamma_3(t)\left(\tfrac{\beta axz}{b+x} - (d + w + q_2E_2)z\right) \\ &+ \gamma_4\left(\lambda_1(q_1x(m_1 - \tau_1) - c_1)E_1\right) + \gamma_5\left(\lambda_2(q_2z(m_2 - \tau_2) - c_2)E_2\right), \end{aligned}$$

where γ_i $(i = 1,\ldots,5)$ are the adjoint variables.

According to Pontryagin's maximum principle [13], we get:

$$\text{(24)} \qquad \begin{cases} \frac{\partial H}{\partial \tau_1} = 0, \frac{\partial H}{\partial \tau_2} = 0, \frac{d\gamma_1}{dt} = -\frac{\partial H}{\partial x}, \frac{d\gamma_2}{dt} = -\frac{\partial H}{\partial y}, \\ \frac{d\gamma_3}{dt} = -\frac{\partial H}{\partial z}, \frac{d\gamma_4}{dt} = -\frac{\partial H}{\partial E_1}, \frac{d\gamma_5}{dt} = -\frac{\partial H}{\partial E_2}. \end{cases}$$

$$\text{(25)} \qquad \frac{\partial H}{\partial \tau_1} = 0 \Longrightarrow \gamma_4(t) = 0 \Longrightarrow \gamma_1(t) = e^{-\delta t}\left(p_1 - \frac{c_1}{q_1x}\right),$$

$$\text{(26)} \qquad \frac{\partial H}{\partial \tau_2} = 0 \Longrightarrow \gamma_5(t) = 0 \Longrightarrow \gamma_3(t) = e^{-\delta t}\left(p_2 - \frac{c_2}{q_2z}\right),$$

Using (24), the adjoint equation are:

$$
\begin{aligned}
\frac{d\gamma_1}{dt} &= -\frac{\partial H}{\partial x} = -e^{-\delta t}p_1q_1E_1 - \gamma_1\left(r_1 - \sigma_1 - q_1E_1 - 2\left(\frac{r_1+Ku}{K}\right)x - \frac{abz}{(b+x)^2} - n_1y\right)\\
&- \gamma_2(\sigma_1 - n_2y) - \gamma_3\frac{\beta abz}{(b+x)^2} - \gamma_4(\lambda_1q_1(m_1-\tau_1)E_1),\\
\frac{d\gamma_2}{dt} &= -\frac{\partial H}{\partial y} = -\gamma_1(\sigma_2 - n_1x) - \gamma_2(r_2 - \sigma_2 - 2vy - n_2x),\\
\frac{d\gamma_3}{dt} &= -\frac{\partial H}{\partial z} = -e^{-\delta t}p_2q_2E_2 + \gamma_1\frac{ax}{b+x} - \gamma_3\left(\frac{\beta ax}{b+x} - (d+w+q_2E_2)\right),\\
\frac{d\gamma_4}{dt} &= -\frac{\partial H}{\partial E_1} = -e^{-\delta t}(p_1q_1x - c_1) + \gamma_1q_1x - \gamma_4(\lambda_1(q_1x(m_1-\tau_1)-c_1)),\\
\frac{d\gamma_5}{dt} &= -\frac{\partial H}{\partial E_2} = -e^{-\delta t}(p_2q_2z - c_2) + \gamma_3q_2z - \gamma_5\lambda_2(q_2z(m_2-\tau_2)-c_2).
\end{aligned}
\tag{27}
$$

From (27) we get $\frac{d\gamma_2}{dt} = \gamma_2F_1 + e^{-\delta t}F_2$

where:

$F_1 = (n_1x - \sigma_2)\left(p_1 - \frac{c_1}{q_1x}\right),$

$F_2 = 2vy + n_2x + \sigma_2 - r_2.$

Then, $\gamma_2(t) = -\frac{F_1}{F_2+\delta}e^{-\delta t},$

Similarly, $\frac{d\gamma_1}{dt} = \gamma_1F_4 + e^{-\delta t}F_3,$

where:

$F_3 = \frac{F_1}{F_2+\delta}(\sigma_1 - n_2y) - q_1p_1E_1 - (p_2 - \frac{c_2}{q_2z})\frac{\beta azb}{(b+x)^2},$

$F_4 = 2(u + \frac{r_1}{K})x + \frac{azb}{(b+x)^2} + n_1y - (r_1 - \sigma_1 - q_1E_1).$

Then, $\gamma_1(t) = -\frac{F_3}{F_4+\delta}e^{-\delta t}.$

Similarly, $\frac{d\gamma_3}{dt} = \gamma_3F_6 + e^{-\delta t}F_5,$

where:

$F_5 = -\frac{F_3ax}{(F_4+\delta)(b+x)} - p_2q_2E_2,$

$F_6 = -\frac{\beta ax}{b+x} + d + w + q_2E_2.$

Then, $\gamma_3(t) = -\frac{F_5}{F_6+\delta}e^{-\delta t}$

$$
\begin{aligned}
p_1 - \frac{c_1}{q_1x} + \frac{F_3}{F_4+\delta} &= 0,\\
p_2 - \frac{c_2}{q_2z} + \frac{F_5}{F_6+\delta} &= 0.
\end{aligned}
\tag{28}
$$

From (28), the optimal equilibrium value of prey population and predator population (x_δ, z_δ) can be obtained for any particular value of δ. We get:

$$
\begin{aligned}
y_\delta &= \frac{(r_2-\sigma_2-n_2x_\delta)+\sqrt{(r_2-\sigma_2-n_2x_\delta)^2+4v\sigma_1x_\delta}}{2v},\\
(E_1)_\delta &= \frac{1}{q_1x_\delta}\left((r_1-\sigma_1)x_\delta-(\frac{r_1}{K}+u)x_\delta^2+\sigma_2y_\delta-\frac{ax_\delta z_\delta}{b+x_\delta}-n_1x_\delta y_\delta\right),\\
(E_2)_\delta &= \frac{1}{q_2z_\delta}\left(\frac{\beta ax_\delta z_\delta}{b+x_\delta}-(d+w)z_\delta\right),\\
(\tau_1)_\delta &= m_1-\frac{c_1}{q_1x_\delta},\\
(\tau_2)_\delta &= m_2-\frac{c_2}{q_2z_\delta}.
\end{aligned}
$$

From the above analysis, we observe the following assertions:

(1) From (25) and (26), we concludes that $\gamma_i e^{\delta t}(i=1,3)$ is independent of time in an optimum equilibrium. Hence they remain bounded as $t \longrightarrow \infty$.

(2) Equation (28) leads to the result: $\begin{cases} \frac{\partial \Pi}{\partial E_1} = p_1q_1x-c_1=-\frac{F_3q_1x}{F_4+\delta},\\ \frac{\partial \Pi}{\partial E_2} = p_2q_2z-c_2=-\frac{F_5q_2z}{F_6+\delta}.\end{cases}$

Then, $\Pi(x_\delta,y_\delta,z_\delta,(E_1)_\delta,(E_2)_\delta)=0$ as $\delta \longrightarrow \infty$.

This leads to an infinite discount rate leading to the loss of economic revenues. If the discount rate is zero, the current value of the time stream reaches its maximum value.

5. Numerical Simulations

To illustrate our results from our model (1), we take hypothetical data as follows:

In this exemple, the parameter values are chosen as:

$r_1=6, r_2=8, K=4, \sigma_1=2.3, \sigma_2=2, q_1=0.1, q_2=0.2, n_1=0.5, n_2=0.3, a=1,$

$b=0.45, u=0.4, v=0.4, \beta=0.8, d=0.85, w=0.4, \lambda_1=2, \lambda_2=2.1, m_1=3.5,$

$m_2=2.3, \tau_1=1, \tau_2=1.21, c_1=4, c_2=4,$ with initial conditions $(x_0,y_0,z_0,(E_1)_0,(E_2)_0)=(10,10,10,10,10)$.

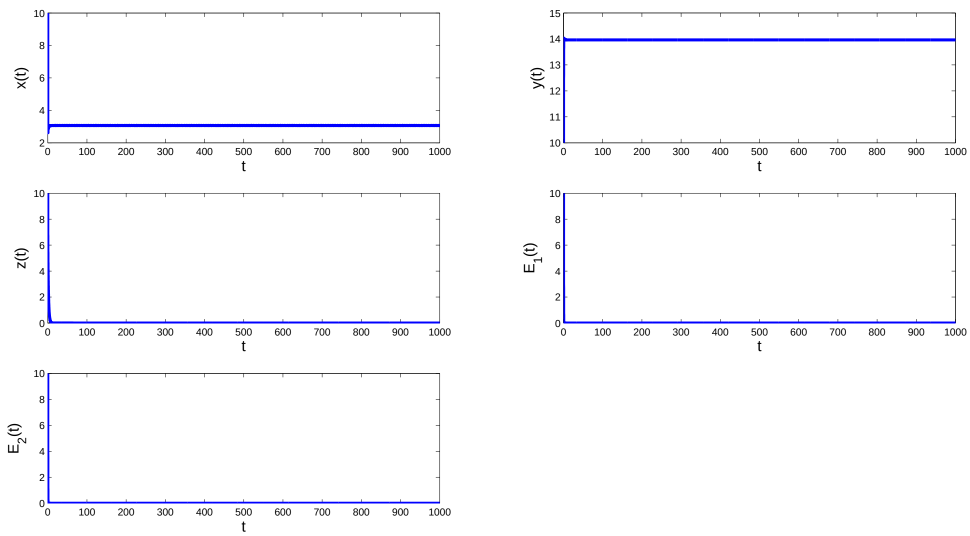

FIGURE 1. Solution curves corresponding to the set values parameters of the system (1) of equilibrium P_2.

As it's shown in this example the parameter values are chosen as:
$r_1 = 6, r_2 = 8, K = 5, \sigma_1 = 2, \sigma_2 = 2, q_1 = 0.1, q_2 = 0.2, n_1 = 0.5, n_2 = 0.3, a = 1,$
$b = 0.45, u = 0.0001, v = 0.4, \beta = 0.8, d = 0.3, w = 0.4, \lambda_1 = 2, \lambda_2 = 2.1, m_1 = 3.5,$
$m_2 = 2.3, \tau_1 = 1, \tau_2 = 1.21, c_1 = 4, c_2 = 4,$ with initial conditions $(x_0, y_0, z_0, (E_1)_0, (E_2)_0) = (10, 10, 10, 10, 10)$.

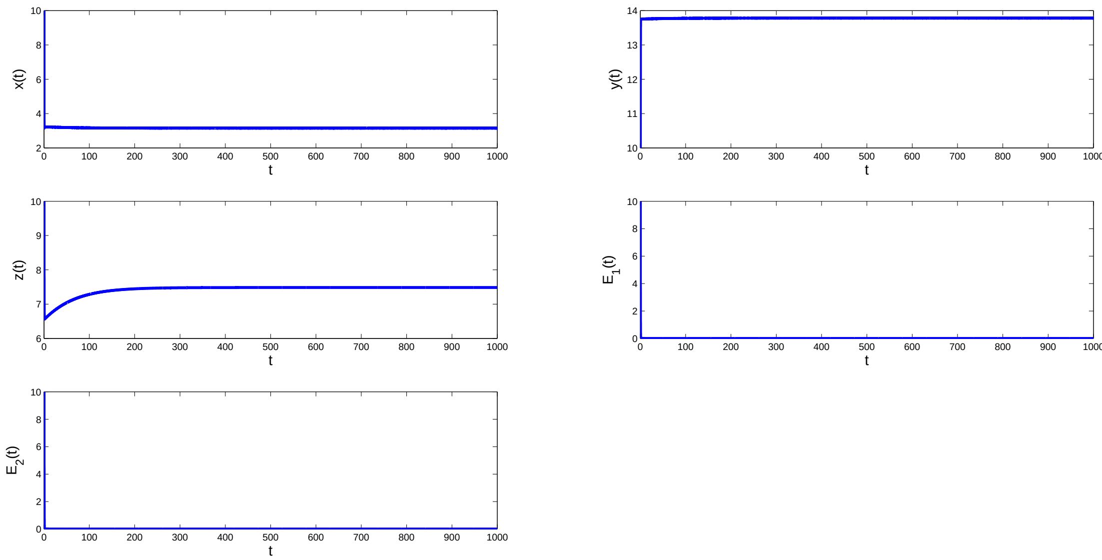

FIGURE 2. Solution curves corresponding to the set values parameters of the system (1) of equilibrium P_3.

As it's shown in this example the parameter values are chosen as:
$r_1 = 6, r_2 = 8, K = 5, \sigma_1 = 2.3, \sigma_2 = 2, q_1 = 0.1, q_2 = 0.2, n_1 = 0.5, n_2 = 0.3, a = 1, b = 0.45,$
$u = 0.4, v = 0.4, \beta = 0.8, d = 0.85, w = 0.4, \lambda_1 = 2, \lambda_2 = 2.1, m_1 = 3.5, m_2 = 2.3, \tau_1 = 1,$
$\tau_2 = 1.21, c_1 = 4, c_2 = 4$, with initial conditions $(x_0, y_0, z_0, (E_1)_0, (E_2)_0) = (10, 10, 10, 10, 10)$.

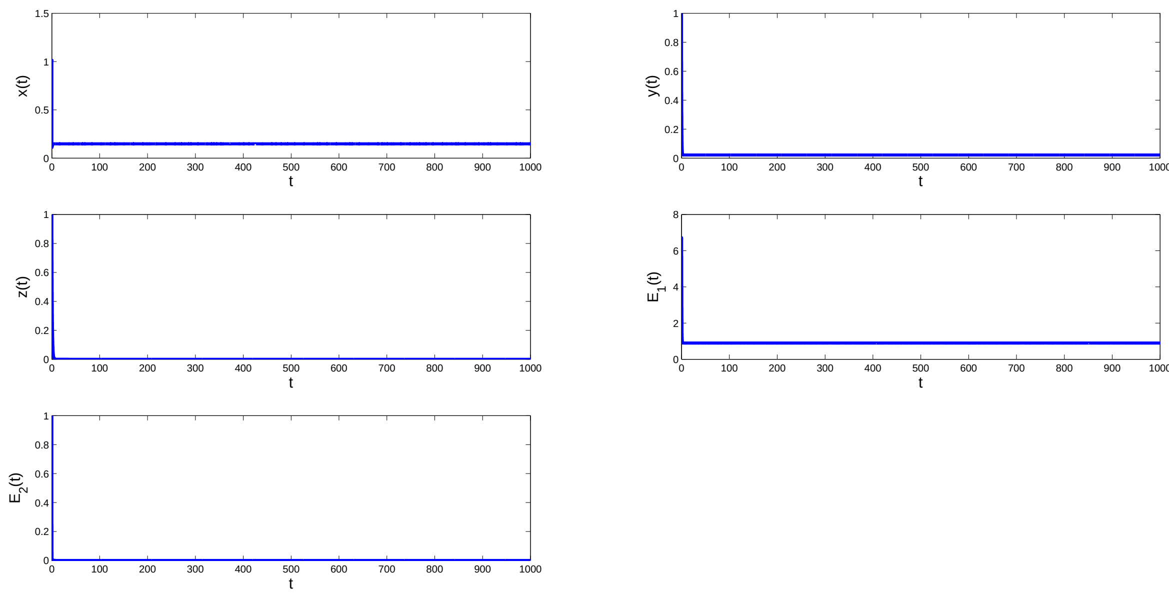

FIGURE 3. Solution curves corresponding to the set values parameters of the system (1) of equilibrium P_4.

In this example, the parameter values are:
$r_1 = 10, r_2 = 8, K = 5, \sigma_1 = 2, \sigma_2 = 8, q_1 = 0.1, q_2 = 0.2, n_1 = 0.5, n_2 = 0.3, a = 1,$
$b = 1.1, u = 0.01, v = 0.4, \beta = 1.1, d = 0.3, w = 0.4, \lambda_1 = 2, \lambda_2 = 4.1, m_1 = 1,$
$m_2 = 3.3, \tau_1 = 1, \tau_2 = 1.21, c_1 = 4, c_2 = 4,$ with initial conditions $(x_0, y_0, z_0, (E_1)_0, (E_2)_0) = (10, 10, 10, 10, 10)$.

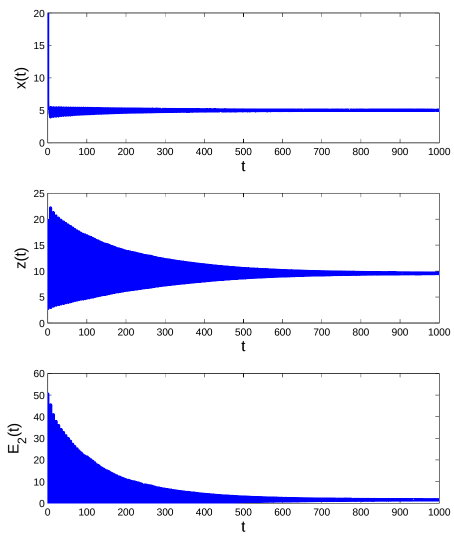

FIGURE 4. Solution curves corresponding to the set values parameters of the system (1) of equilibrium P_5.

Choosing the parameter values:
$r_1 = 5, r_2 = 1, K = 4, \sigma_1 = 1, \sigma_2 = 1, q_1 = 0.1, q_2 = 0.2, n_1 = 0.5, n_2 = 0.3, a = 0.94,$
$b = 0.7, u = 0.0001, v = 0.333, \beta = 0.998, d = 0.03, w = 0.00003, \lambda_1 = 2, \lambda_2 = 2.1,$
$m_1 = 3.5, m_2 = 2.3, \tau_1 = 1, \tau_2 = 1.21, c_1 = 0.5, c_2 = 0.3.$

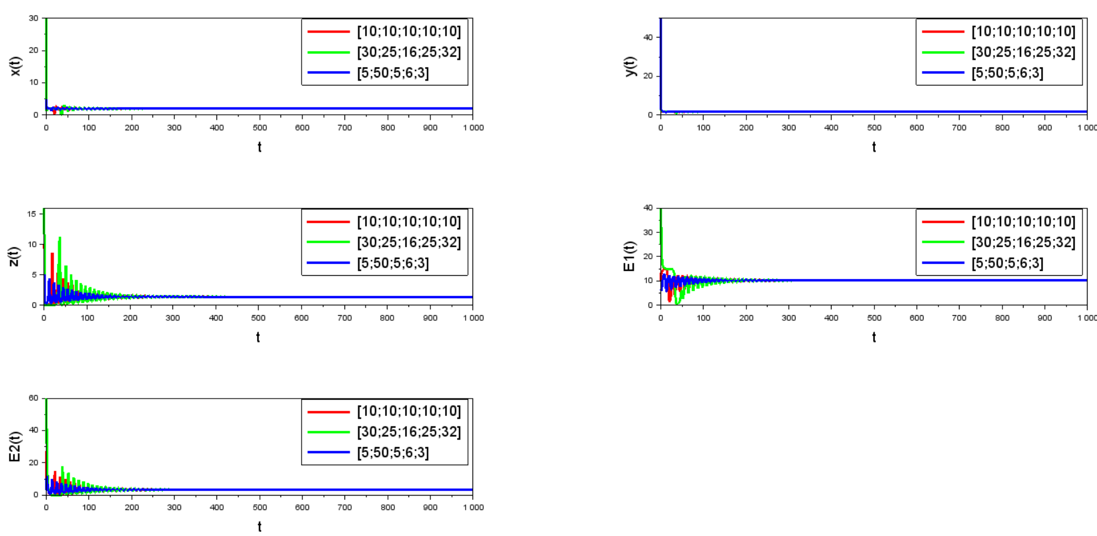

FIGURE 5. The equilibrium point P_6 of is globally asymptotically stable. x, y, z, E_1, E_2 states for different initial points of the system (1).

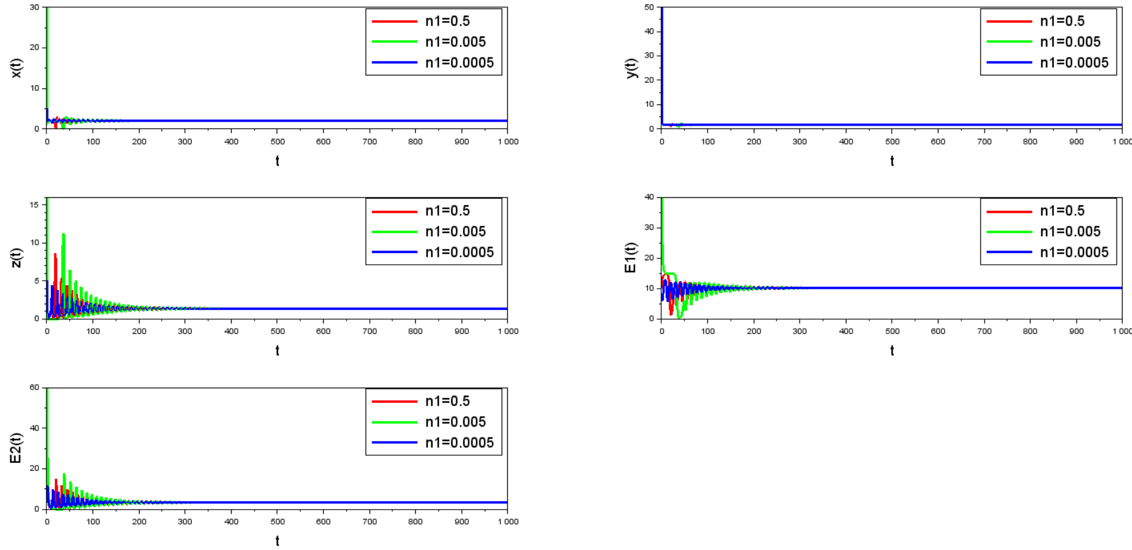

FIGURE 6. Plot of x, y, z, E_1, E_2 with respect to time t for different values of n_1 of the system (1).

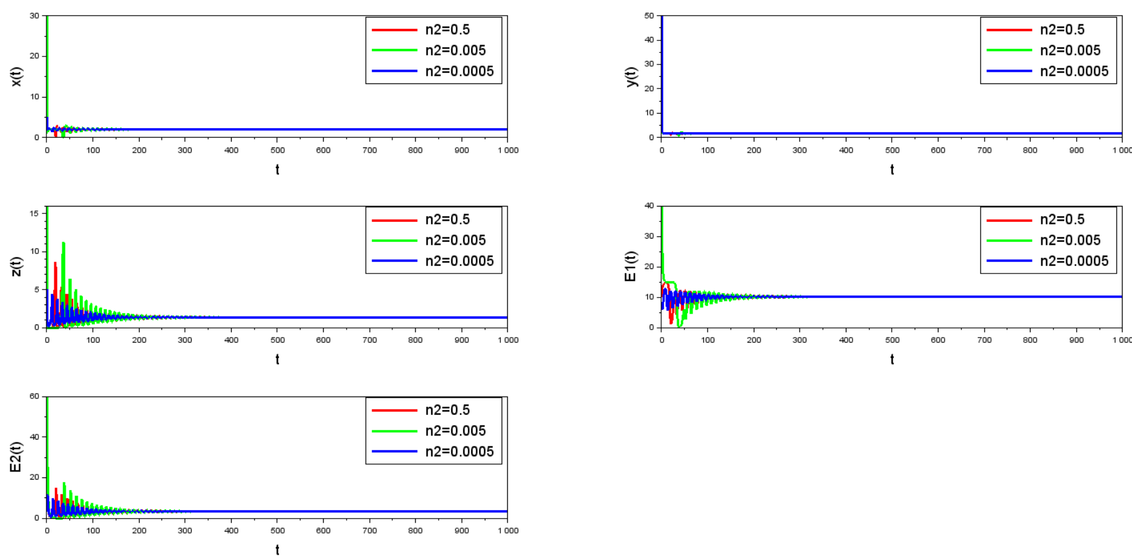

FIGURE 7. Plot of x, y, z, E_1, E_2 with respect to time t for different values of n_2 of the system (1).

Figures (1–4) show that system solutions converge to the equilibria point P_i for $i = 2, \dots, 5$, respectively. In Figure 1, x and y increases a short time, z, E_1 and E_2 decreases a short time and attain the equilibrium P_2. In Figure 2, the system has periodic solution and converges to the stable equilibrium point P_3. In Figure 3, x, z and E_2 decreases a short time, y and E_1 increases a short time and attain the equilibrium P_4. In Figure 4 the system has periodic solution and converges to the stable equilibrium point P_5.

In Figure 5, we show the behavior of x, y, z, E_1, and E_2 with several different initial values. From this figure, we see that all trajectories converge to P_6. Thus, P_6 is globally asymptotically stable.

It can be seen that n_1 and n_2 have a strong influence on the dynamics of the system (1). From Figure 6, we notice that x and y decrease rapidly and increase after n_1 increases, but z decreases as n_1 increases. From Figure 7, we notice that x decreases in a short time and increases after n_2 increases, but y and z decrease when n_2 increases.

6. CONCLUSION

In this paper, we have proposed and analyzed a mathematical model describing the exploitation of fishing resources with the reserve area in the presence of predators from which two different fishing efforts were considered by integrating toxic substances taking into account the competition factor. It is assumed that the aquatic ecosystem comprises two areas: a free fishing zone and another prohibited zone where fishing is strictly prohibited. We observe that our system admits six equilibrium of which P_1 which is unstable, concerning the other points of equilibrium the local stability of our system (1) was demonstrated under certain conditions, then the global stability of the equilibrium internal has been demonstrated using Lyapunov function. Using the maximum Pontryagin principle, optimal harvest is discussed. Numerical simulations are carried out to confirm the equilibrium stability as well as their stability properties. On the other hand, we have considered the reserve area as a means of control to regulate prey capture, and we have incorporated two variable fishing efforts into our model. In the optimally managed fishery, the marine reserve may or may not increase the fishing rent. In many cases, marine reserves are optimal depending on the availability of marine reserves, which may or may not increase fishing rents, depending on their optimality, and the existence of equilibria may not be sufficient to ensure that stocks of biological resources are not used excessively without loss of productivity.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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