Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2021, 2021:33
https://doi.org/10.28919/cmbn/4409
ISSN: 2052-2541

LOCAL STABILITY OF A FRACTIONAL ORDER SIS EPIDEMIC MODEL WITH SPECIFIC NONLINEAR INCIDENCE RATE AND TIME DELAY

MOUHCINE NAIM[1], GHASSANE BENRHMACH[1,*], FOUAD LAHMIDI[1], ABDELWAHED NAMIR[2]

[1]Laboratory of Analysis, Modeling and Simulation, Faculty of Sciences Ben M'sik, Hassan II University, P.O. Box 7955, Sidi Othman, Casablanca, Morocco

[2]Laboratory of Information Technology and Modeling, Faculty of Sciences Ben M'sik, Hassan II University, P.O. Box 7955, Sidi Othman, Casablanca, Morocco

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we study the stability of a fractional order SIS epidemic model with specific functional response and time delay, where the fractional derivative is defined in the Caputo sense. Using the theory of stability of differential equations of delayed fractional order systems, we prove that the disease-free equilibrium is locally asymptotically stable when the basic reproduction number $R_0 < 1$. Also, we show that if $R_0 > 1$, the endemic equilibrium is locally asymptotically stable. Numerical simulations are presented to illustrate the theoretical results of this work.

Keywords: fractional derivative; SIS epidemic model; time delay; stability.

2010 AMS Subject Classification: 34A08, 37C75, 92D30.

1. INTRODUCTION

Epidemiology is the study of the spread of diseases in human populations and the factors that are responsible for or contribute to their occurrence. Consequently, it has been investigated

*Corresponding author
E-mail address: ghassane.benrhmach@gmail.com
Received December 3, 2019

by several researchers through study the dynamical behavior of infectious diseases by mathematical models (see, e.g., [1, 2, 3, 4]). Particularly, the SIS (susceptible-infected-susceptible) epidemic model is often used to model the dynamics of the diseases such as the bacterial diseases and some sexually transmitted diseases where infection with the disease does not confer permanent immunity against re-infection so that those who survived the infection revert to the class of wholly-susceptible individuals [5].

Fractional calculus is the field of mathematical analysis aiming at the investigation of integrals and derivatives of arbitrary (non integer) orders. The main advantage of fractional order derivative in comparison in integer order is that fractional order derivative can be describe the memory and hereditary effects in various substances. Therefore, many applied researchers have treat many real processes using the fractional derivative such as botanical electrical impedances [6], viscoelasticity of cancellous bone [7], human root dentin [8], financial processes [9], $PI^{\lambda}D^{\mu}$ controller [10], and so on.

Due to the memory effects which is has an important role on the spread of an infectious disease, many investigators have started to study the fractional order epidemic models, see, e.g., [11, 12, 13, 14]. In 2014, El-Saka in [14] introduced a fractional order SIS model with variable population size where the author study the stability of equilibrium points. Our aim in this present work is to extend the model presented in [14] to a model with specific functional response and time delay. In this way, we propose the following fractional order SIS epidemic model

$$
\begin{cases}
D^{\alpha}S(t) = & \Lambda - \mu S(t) - \frac{\beta S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)+\alpha_3 S(t)I(t-\tau)} + rI(t), \\
D^{\alpha}I(t) = & \frac{\beta S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)+\alpha_3 S(t)I(t-\tau)} - (\mu+a+r)I(t),
\end{cases}
\tag{1}
$$

where $\alpha \in (0,1]$ is the order of the fractional derivative, $S(t)$ is the proportion of susceptible individuals at time t, $I(t)$ is the proportion of infected individuals at time t, Λ is the recruitment rate of the susceptible, μ is the natural death rate of the population, a is the death rate due to disease, r is the recovery rate of infective individuals, $\beta > 0$ is the contact transmission coefficient, which measures the infection force of the disease and $\alpha_1, \alpha_2, \alpha_3 \geq 0$ are the saturation factors measuring the psychological or inhibitory effect. The constant $\tau \geq 0$ is the time delay in which the infectious individuals develop in the vector and it is only after that time that the infected vector can infect a susceptible individual (see, e.g., [15], [16]).

The fractional order derivative used in model (1) is in the sense of Caputo definition, which is a modification of the Riemann-Liouville integral definition, and has the advantage that the initial values for fractional differential equations with Caputo derivatives take the same form as that for integer order differential equations. Also, another advantage of this definition is that the Caputo derivative of a constant is zero.

The Riemann-Liouville fractional integral and Caputo fractional derivative are defined respectively as follows [17, 18].

Definition 1. *The Riemann-Liouville integral of order $\alpha > 0$ for an integrable function $f : \mathbb{R}_+ \mapsto \mathbb{R}$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the Gamma function defined by the integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Definition 2. *The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in \mathscr{C}^n(\mathbb{R}_+, \mathbb{R})$ is defined as*

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where n is a positive integer such that $\alpha \in (n-1, n]$. In particular, when $\alpha \in (0,1]$, one has

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds.$$

The rest of this paper is organized as follows. In the next section, we investigate the existence and the local stability of equilibria. In Section 3, we present the numerical simulation to illustrate our results, and finally we gave our conclusion in Section 4.

2. Stability Analysis

In this section, we discuss the existence and the local stability of the equilibria of system (1). In this sense, we define the basic reproduction number of model (1) as follows

$$R_0 = \frac{\beta\Lambda}{(\mu + \alpha_1\Lambda)(\mu + a + r)}.$$

From biological point of view, R_0 represents the average number of secondary infections that occur when one infectious individual is introduced into a completely susceptible population [19].

The equilibrium of model (1) is obtained by setting $D^{\alpha}S = D^{\alpha}I = 0$. Then, system (1) always has a disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0)$. Further, if $R_0 > 1$, then system (1) has a unique endemic equilibrium $E^* = (S^*, I^*)$, where

$$S^* = \frac{\Lambda - (\mu + a)I^*}{\mu},$$

$$I^* = \frac{2\varpi(\mu + \alpha_1\Lambda)(R_0 - 1)}{(\mu + a)(\beta - \alpha_1\varpi) + \varpi(\alpha_2\mu + \alpha_3\Lambda) + \sqrt{\Delta}},$$

with $\varpi = a + \mu + r$ and

$$\begin{aligned} \Delta &= [(\mu + a)(\beta - \alpha_1\varpi) + \varpi(\alpha_2\mu + \alpha_3\Lambda)]^2 - 4\alpha_3(\mu + a)\varpi\,[\beta\Lambda - (\mu + \alpha_1\Lambda)\varpi] \\ &= [(\mu + a)(\beta - \alpha_1\varpi) + \varpi(\alpha_2\mu - \alpha_3\Lambda)]^2 + 4\alpha_3\mu\varpi^2(\mu + a + \alpha_2\Lambda). \end{aligned}$$

Consider the following linear delayed fractional differential system

$$D^{\alpha}x(t) = Ax(t) + Bx(t - \tau), \;\; t \geq 0, \tag{2}$$

where $\alpha \in (0, 1]$, $x(t) \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n\times n}$ and $\tau \geq 0$. The characteristic equation of system (2) is

$$\Delta(s) = \det(s^{\alpha}I_n - A - Be^{-s\tau}) = 0.$$

If $\tau = 0$, system (2) can be expressed as

$$D^{\alpha}x(t) = Mx(t),$$

where the coefficient matrix $M = A + B$.

In the case of $A = 0$, Deng et al. in [20] obtained the following two stability results.

Lemma 1. *If all the roots of the characteristic equation $\Delta(s) = 0$ have negative real parts, then the zero solution of system (2) is Lyapunov globally asymptotically stable.*

Lemma 2. *If all the eigenvalues λ of B satisfy $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, and the characteristic equation $\Delta(s) = 0$ has no purely imaginary roots for any $\tau > 0$, then the zero solution of system (2) is Lyapunov globally asymptotically stable.*

If $A \neq 0$, according to [21], we have the following conclusion.

Lemma 3. *If all the eigenvalues λ of M satisfy $|\arg(\lambda)| > \frac{\pi}{2}$ and the characteristic equation $\Delta(s) = 0$ has no purely imaginary roots for any $\tau > 0$, then the zero solution of system (2) is Lyapunov globally asymptotically stable.*

Remark 1. *If $A \neq 0$, the stability of system (2) is not guaranteed under conditions that the eigenvalues of M are satisfied $|\arg(\lambda)| > \frac{\alpha\pi}{2}$. In fact, when the eigenvalues of M are satisfied $\frac{\alpha\pi}{2} < |\arg(\lambda)| \leq \frac{\pi}{2}$, and the characteristic equation $\Delta(s) = 0$ has no purely imaginary roots for any $\tau > 0$, the zero solution has unstable situation (see Section 5 in [21]).*

Remark 2. *To study the local asymptotic stability of equilibria of nonlinear fractional order systems, we investigate the stability of the linearized systems of such nonlinear systems around these equilibria based on the previous lemmas.*

2.1. Stability of the disease-free equilibrium. This subsection is devoted to studying the stability of the diseases-free equilibrium E_0 of system (1). For this, let $x(t) = S(t) - \frac{\Lambda}{\mu}$ and $y(t) = I(t)$. Then the linearized system of (1) around E_0 takes the following form

$$\begin{cases} D^\alpha x(t) = & -\mu x(t) - \frac{\beta\Lambda}{\mu+\alpha_1\Lambda} y(t-\tau) + r y(t), \\ D^\alpha y(t) = & \frac{\beta\Lambda}{\mu+\alpha_1\Lambda} y(t-\tau) - (\mu + a + r) y(t). \end{cases} \tag{3}$$

The associated characteristic equation of system (3) can be described as

$$\Delta(s) = \det \begin{pmatrix} s^\alpha + \mu & \frac{\beta\Lambda}{\mu+\alpha_1\Lambda} e^{-s\tau} - r \\ 0 & s^\alpha - \frac{\beta\Lambda}{\mu+\alpha_1\Lambda} e^{-s\tau} + \mu + a + r \end{pmatrix} = 0,$$

which leads to

$$\Delta(s) = (s^\alpha + \mu)\left[s^\alpha + (\mu + a + r)(1 - R_0 e^{-s\tau})\right] = 0. \tag{4}$$

Theorem 1. *If $R_0 < 1$, then the disease-free equilibrium E_0 is asymptotically stable for all $\tau \geq 0$ and $\alpha \in (0,1]$. E_0 is unstable if $R_0 > 1$.*

Proof. When $\tau = 0$, the coefficient matrix M of system (3) satisfies

$$M = \begin{pmatrix} -\mu & -\frac{\beta\Lambda}{\mu+\alpha_1\Lambda} + r \\ 0 & \frac{\beta\Lambda}{\mu+\alpha_1\Lambda} - (\mu+\alpha+r) \end{pmatrix}.$$

The eigenvalues of the coefficient matrix M are $\lambda_1 = -\mu < 0$ and $\lambda_2 = (\mu+\alpha+r)(R_0-1) < 0$ if $R_0 < 1$. Whence $|\arg\lambda_i| = \pi > \frac{\pi}{2}$ $(i = 1,2)$, so that all the eigenvalues λ of M satisfy $|\arg(\lambda)| > \frac{\pi}{2}$ if $R_0 < 1$. If $R_0 > 1$, then $\lambda_2 > 0$ and consequently E_0 is unstable [22].

Now, we research the circumstance of delay $\tau > 0$. We only need to analyze the second factor of (4) as it contains τ, so substituting $s = iw = w(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ in the second factor of (4), with $w > 0$. Then

$$w^{\alpha}(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}) + (\mu+a+r)\left[1 - R_0(\cos w\tau - i\sin w\tau)\right] = 0.$$

Separating real and imaginary parts gives

$$\begin{cases} (\mu+a+r)R_0\cos w\tau = & w^{\alpha}\cos\frac{\alpha\pi}{2} + (\mu+a+r), \\ (\mu+a+r)R_0\sin w\tau = & -w^{\alpha}\sin\frac{\alpha\pi}{2}. \end{cases}$$

Thus

$$w^{2\alpha} + 2w^{\alpha}(\mu+a+r)\cos\frac{\alpha\pi}{2} + (\mu+a+r)^2(1-R_0^2) = 0. \tag{5}$$

Obviously, since $(\mu+a+r)\cos\frac{\alpha\pi}{2} \geq 0$ for $\alpha \in (0,1]$ and our assumption that $R_0 < 1$, then the Eq. (5) has no positive roots. Which ensures that Eq. (4) has no purely imaginary roots. According to Lemma 3, the equilibrium E_0 is asymptotically stable for any delay $\tau \geq 0$ and $\alpha \in (0,1]$ if $R_0 < 1$. The proof is completed. □

2.2. Stability of the endemic equilibrium. In this subsection, we analyse the stability of the endemic equilibrium of the system (1). To begin with, we linearise the system about the endemic equilibrium E^*. Let the transformation $x(t) = S(t) - S^*$ and $y(t) = I(t) - I^*$. Then by linearizing system (1) around $E^* = (S^*, I^*)$, we get the following system

(6)
$$\begin{cases} D^{\alpha}x(t) = & -m_1x(t) - m_2y(t-\tau) + ry(t), \\ \\ D^{\alpha}y(t) = & m_3x(t) + m_2y(t-\tau) - m_4y(t), \end{cases}$$

where

$$\begin{aligned} m_1 &= \mu + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} > 0, \\ m_2 &= \frac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} > 0, \\ m_3 &= \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} > 0, \\ m_4 &= \mu + a + r > 0. \end{aligned}$$

Characteristic equation which is associated with system (6) is given by

$$\Delta(s) = \det \begin{pmatrix} s^{\alpha} + m_1 & m_2 e^{-s\tau} - r \\ -m_3 & s^{\alpha} - m_2 e^{-s\tau} + m_4 \end{pmatrix} = 0.$$

Hence, the above equation can be rewritten equivalently as

(7)
$$\Delta(s) = s^{2\alpha} + a_1 s^{\alpha} + a_2 - (a_3 s^{\alpha} + a_4)e^{-s\tau} = 0,$$

where

$$\begin{aligned} a_1 &= m_1 + m_4 > 0, \\ a_2 &= m_1 m_4 - r m_3 = (\mu + a)m_3 + \mu m_4 > 0, \\ a_3 &= m_2 > 0, \\ a_4 &= \mu m_2 > 0. \end{aligned}$$

Theorem 2. *If $R_0 > 1$, then the endemic equilibrium E^* is asymptotically stable for all $\tau \geq 0$ and $\alpha \in (0,1]$.*

Proof. When $\tau = 0$, the characteristic equation of the coefficient matrix M of system (6) is

$$\lambda^2 + (a_1 - a_3)\lambda + (a_2 - a_4) = 0. \tag{8}$$

Since

$$\begin{aligned} m_4 - m_2 &= (\mu + a + r) - \frac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} \\ &= \frac{\beta S^*}{1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*} - \frac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} \\ &= \frac{\beta S^* I^*(\alpha_2 + \alpha_3 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} \geq 0, \end{aligned} \tag{9}$$

then $a_1 - a_3 = m_1 + (m_4 - m_2) > 0$ and $a_2 - a_4 = (\mu + a)m_3 + \mu(m_4 - m_2) > 0$. Hence the two roots λ_i $(i = 1,2)$ of the Eq. (8) have negative real parts, so that all the eigenvalues of M of system (6) satisfy $|\arg(\lambda)| > \frac{\pi}{2}$ if $R_0 > 1$.

For $\tau > 0$, let $s = iw$ is a root of Eq. (7), with $w > 0$. Substituting s into (7) gives

$$\begin{aligned} & w^{2\alpha}(\cos\alpha\pi + i\sin\alpha\pi) + w^{\alpha}a_1(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}) + a_2 - (w^{\alpha}a_3\cos\frac{\alpha\pi}{2} + a_4 \\ & + iw^{\alpha}a_3\sin\frac{\alpha\pi}{2})(\cos w\tau - i\sin w\tau) = 0. \end{aligned}$$

We separate the real and imaginary parts to have

$$\begin{cases} w^{2\alpha}\cos\alpha\pi + w^{\alpha}a_1\cos\frac{\alpha\pi}{2} + a_2 = & w^{\alpha}a_3\cos\left(\frac{\alpha\pi}{2} - w\tau\right) + a_4\cos w\tau, \\ \\ w^{2\alpha}\sin\alpha\pi + w^{\alpha}a_1\sin\frac{\alpha\pi}{2} = & w^{\alpha}a_3\sin\left(\frac{\alpha\pi}{2} - w\tau\right) - a_4\sin w\tau. \end{cases} \tag{10}$$

Squaring and adding the two equations in (10), we obtain

$$w^{4\alpha} + \eta_1 w^{3\alpha} + \eta_2 w^{2\alpha} + \eta_3 w^{\alpha} + \eta_4 = 0, \tag{11}$$

where

$$\begin{aligned} \eta_1 &= 2a_1\cos\frac{\alpha\pi}{2}, \\ \eta_2 &= a_1^2 - a_3^2 + 2a_2\cos\alpha\pi, \\ \eta_3 &= 2(a_1a_2 - a_3a_4)\cos\frac{\alpha\pi}{2}, \\ \eta_4 &= a_2^2 - a_4^2. \end{aligned}$$

Since $\alpha \in (0,1]$ and $a_1 > 0$, then $\eta_1 \geq 0$. In addition, we have

$$a_2^2 - a_4^2 = (a_2 - a_4)(a_2 + a_4) > 0,$$

and

$$a_1 a_2 - a_3 a_4 = a_1 (a_2 - a_4) + a_4 (a_1 - a_3) > 0,$$

since $a_2 - a_4 > 0$, $a_1 - a_3 > 0$ and $a_1, a_2, a_4 > 0$. Then $\eta_3 \geq 0$ and $\eta_4 > 0$. And since $a_2 > 0$, we have

$$\begin{aligned} \eta_2 &= a_1^2 - a_3^2 + 2a_2 \cos \alpha\pi \\ &\geq a_1^2 - a_3^2 - 2a_2 \\ &= (m_1 + m_4)^2 - m_2^2 - 2(m_1 m_4 - r m_3) \\ &= m_1^2 + 2r m_3 + m_4^2 - m_2^2. \end{aligned}$$

From (9), we have $m_4^2 - m_2^2 \geq 0$ since $m_4 + m_2 > 0$. Hence $\eta_2 > 0$. Therefore the Eq. (11) has no positive real roots, which implies that Eq. (7) has no purely imaginary roots. Thus, according to Lemma 3, the equilibrium point E^* is asymptotically stable for delay $\tau \geq 0$ and $\alpha \in (0,1]$. This concludes the proof. □

3. Numerical Simulations

In this section, we give some numerical simulations in order to illustrate our theoretical results.

Consider the following parameters $\Lambda = 0,95$, $\beta = 0.1$, $\mu = 0.2$, $a = 0.03$, $r = 0.3$, $\alpha_1 = 0.1$, $\alpha_2 = 0.03$, $\alpha_3 = 0.05$. By calculation, we obtain $R_0 = 0.63 < 1$, then, by Theorem 1, E_0 is asymptotically stable for different values of $\tau \geq 0$ and $\alpha \in (0,1]$ (see Figs. 1,2 and 3).

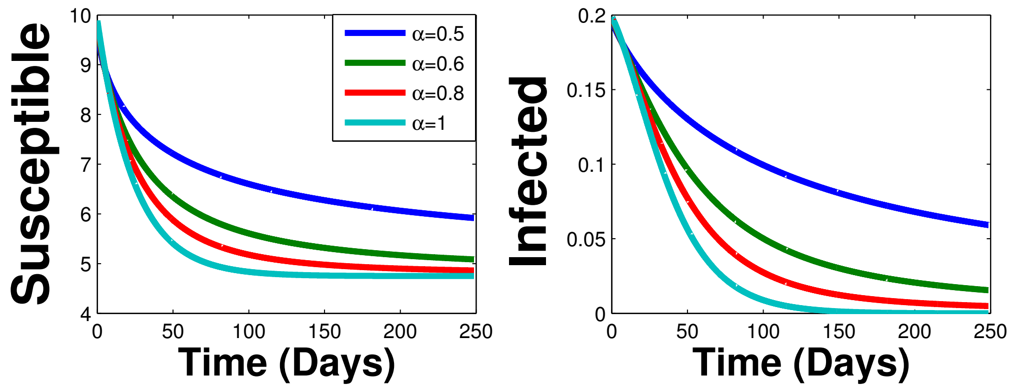

FIGURE 1. Stability of the disease-free equilibrium E_0 and $\tau = 2$

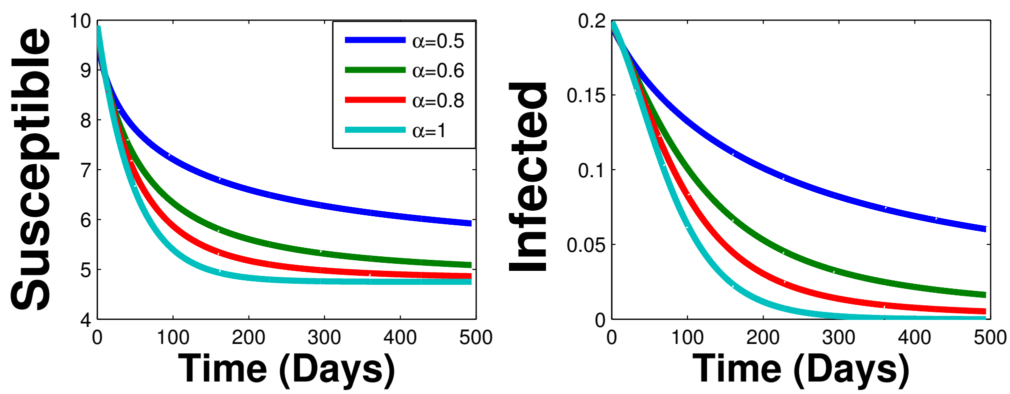

FIGURE 2. Stability of the disease-free equilibrium E_0 and $\tau = 5$

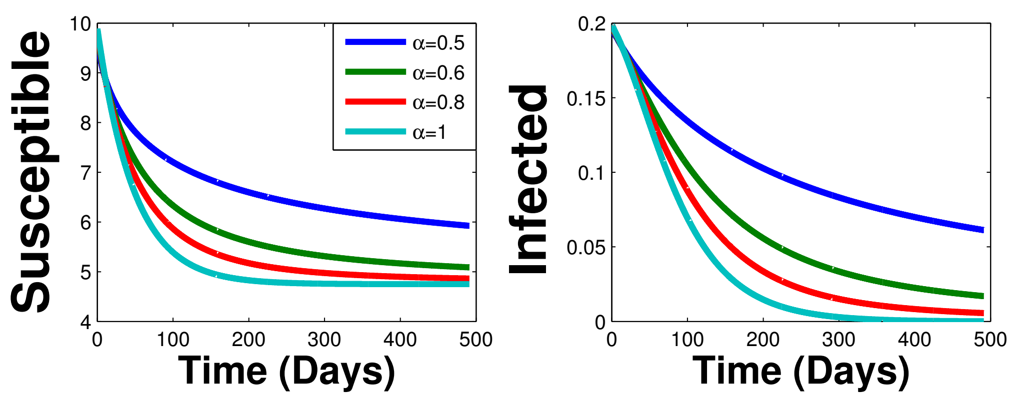

FIGURE 3. Stability of the disease-free equilibrium E_0 and $\tau = 8$

Now, we keep all the parameter values except that β is increased to 0.3 from 0.1. In this case, we have $R_0 = 1.78 > 1$. Hence, we can conclude, by Theorem 2, that E^* is asymptotically stable for different values of $\tau \geq 0$ and $\alpha \in (0,1]$ (see Figs. 4,5 and 6).

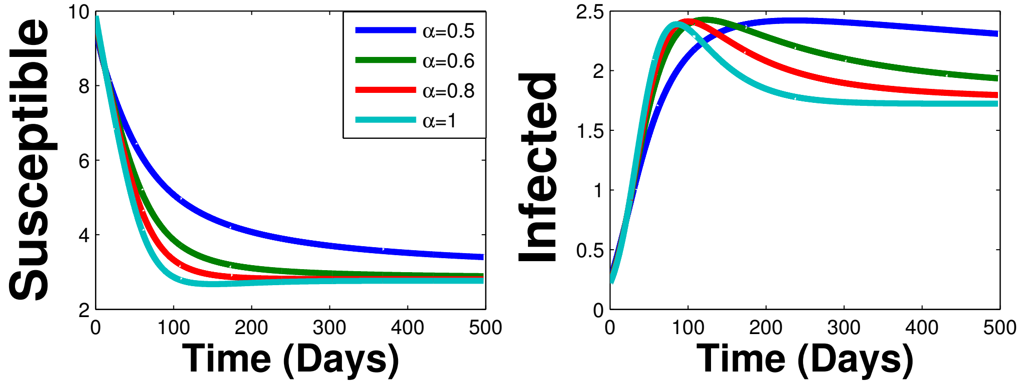

FIGURE 4. Stability of the endemic equilibrium E^* and $\tau = 2$

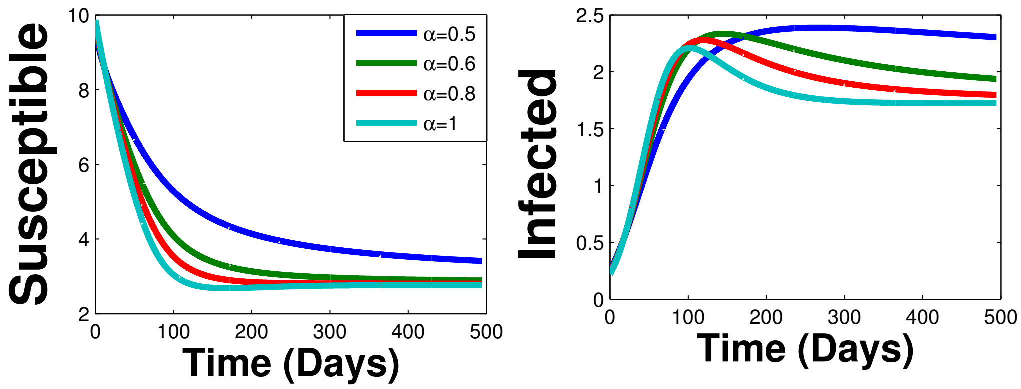

FIGURE 5. Stability of the endemic equilibrium E^* and $\tau = 5$

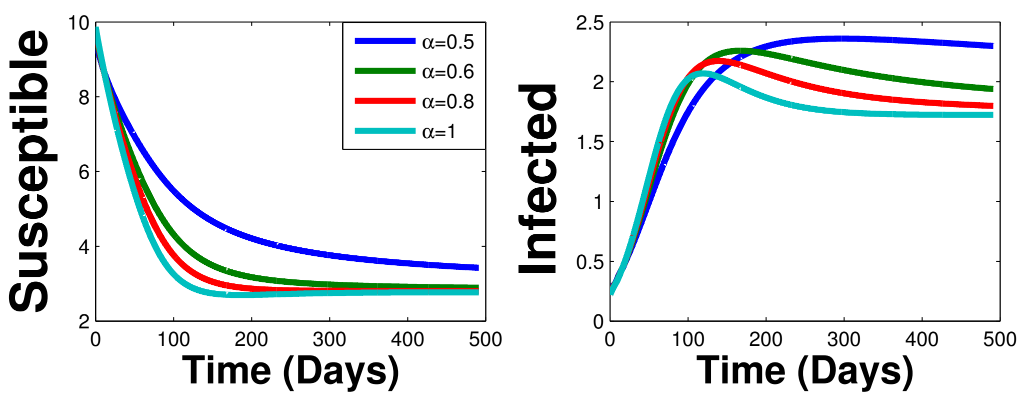

FIGURE 6. Stability of the endemic equilibrium E^* and $\tau = 8$

4. Conclusion

In this paper, we have presented a fractional order SIS epidemic model with the Caputo fractional derivative and a specific functional response with delay given by $\frac{\beta S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)+\alpha_3 S(t)I(t-\tau)}$. We show that if the basic reproduction number R_0, is less than one, the disease-free equilibrium is locally asymptotically stable for all $\tau \geq 0$ and $0 < \alpha \leq 1$, which means that the disease will go to extinction. Moreover, we prove that if $R_0 > 1$, the endemic equilibrium is locally asymptotically stable, so the disease will be persistent at the unique endemic equilibrium. In the end some numerical simulations are given to illustrate the results. From, our theoretical and numerical analysis, we can observe that the different values of α and τ have no effect on the stability of both equilibria but affect the time to reach the steady states.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] A. Korobeinikov, Lyapunov functions and global stability for SIR and SIRS epidemiologcal models with non-linear transmission, Bull. Math. Biol. 68 (2006), 615-626.

[2] J. Li, Z. Ma, Stability analysis for SIS epidemic models with vaccination and constant population size, Discrete Contin. Dyn. Syst., Ser. B. 4 (2004), 635-642.

[3] J. Li, Z. Ma, Qualitative analyses of SIS epidemic model with vaccination and varying total population size, Math. Computer Model. 35(11-12) (2002), 1235-1243.

[4] H.F. Huo, G.M. Qiu, Stability of a mathematical model of malaria transmission with relapse, Abstr. Appl. Anal. 2014 (2014), 289349.

[5] A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A stochastic differential equation SIS epidemic model, SIAM J. Appl. Math. 71(3) (2011), 876-902.

[6] I.S. Jesus, J.A.T. Machado, J.B. Cunha, Fractional electrical impedances in botanical elements, J. Vib. Control. 14 (2008), 1389-1402.

[7] J.G. Liu, M.Y. Xu, Study on the Viscoelasticity of Cancellous Bone Based on Higher-Order Fractional Models, in: 2008 2nd International Conference on Bioinformatics and Biomedical Engineering, IEEE, Shanghai, China, 2008: pp. 1733–1736.

[8] L.M. Petrovic, D.T. Spasic, T.M. Atanackovic, On a mathematical model of a human root dentin, Dent. Mater. 21 (2005), 125–128.

[9] J. Korbel, Y. Luchko, Modelling of financial processes with a space-time fractional diffusion equation of varying order, Fract. Calc. Appl. Anal. 19(6) (2016), 1414-1433.

[10] I. Podlubny, Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers, IEEE Trans. Autom. Control, 44(1) (1999), 208-214.

[11] A.M. Yousef, S.M. Salman, Backward Bifurcation in a Fractional-Order SIRS Epidemic Model with a Nonlinear Incidence Rate, Int. J. Nonlinear Sci. Numer. Simul. 17 (2016), 401–412.

[12] I. Ameen, P. Novati, The solution of fractional order epidemic model by implicit Adams methods, Appl. Math. Model. 43 (2017), 78-84.

[13] D. Rostamy, E. Mottaghi, Forward and backward Bifurcation in a fractional-order SIR epidemic model with vaccination, Iran. J. Sci. Technol. 42(2) (2018), 663-671.

[14] H.A.A. El-Saka, The fractional-order SIS epidemic model with variable population size, J. Egypt. Math. Soc. 22 (2014), 50-54.

[15] M.A. Wanbiao, M.E.I, Song, Y. Takeuchi, Global stability of an SIR epidemic model with time delay, Appl. Math. Lett. 17 (2004), 1141-1145.

[16] R. Xu, Z. Ma, Stability of a delayed SIRS epidemic model with a nonlinear incidence rate, Chaos Solitons Fractals. 41 (2009), 2319–2325.

[17] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA, 1999.

[18] I. Petras, Fractional-order nonlinear systems: Modeling, Analysis and Simulation, Springer, New York, 2011.

[19] O. Diekmann, J.A.P. Heesterbeek, Mathematical epidemiology of infectious diseases, John Wiley & Son, Ltd, Hoboken, 2000.

[20] W. Deng, C. Li, J. Lü, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dyn. 48 (2007), 409–416.

[21] H. Wang, Y. Yu, G. Wen, S. Zhang, Stability Analysis of Fractional-Order Neural Networks with Time Delay, Neural Process, Lett. 42 (2015), 479–500.

[22] N.D. Cong, T.S. Doan, S. Siegmund, H.T. Tuan, An instability theorem for nonlinear fractional differential systems, Discrete Contin. Dyn. Syst., Ser. B. 22 (2017), 3079–3090.