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ON THE INTEGRABILITY OF THE SIRD EPIDEMIC MODEL

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Abstract. In this paper, Lie symmetry and Painlevé Techniques are applied to the SIRD (Susceptible, Infected, Recovered and Dead) model. A demonstration of the integrability of the model is provided to present an explicit solution. The study revealed a complex chaotic behaviour at a specific value of constant constraints. However, the system fails to pass the Painlevé test while constraints reach values equivalent to the corresponding complex chaotic behaviour. The two-dimensional Lie symmetry algebra and the commutator table of the infinitesimal generators are obtained. Lie symmetry analysis serves to linearize the nonlinear system and find the corresponding exact solution.

Keywords: Lie symmetry and Painlevé techniques; SIRS model; integrability.

2010 AMS Subject Classification: 34C14, 34M55.

1. Introduction

In this paper, we modified the classical SIR model of Kermack and McKendrick [5] by assuming that an individual can be born infected. We assumed that after the recovery process, the disease person becomes resistant and the number of individuals died from the disease are counted. The model divides the total population into three different classes: The susceptible class, S, are those who can get infected with the disease; the infective class, I, are those who

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can spread the disease after getting infected; the removed class, R, are those individuals who recovered from the disease, resistant or sequestered while waiting for recovery; and the death class, D, are those who die from the disease. Most of the infantile viruses, essentially measles, have a removed and death class [14]. The model flow diagram which represented the disease is given by

$$S \to I \to R \to D.$$

The model is governed by the following nonlinear system of first order ordinary differential equations [8]

$$\dot{S} = -\beta SI + \gamma I - \mu_1 S + \nu_1 S \tag{1}$$

$$\dot{I} = \beta SI - (\alpha + \gamma)I - \mu_2 I + \nu_2 I \tag{2}$$

$$\dot{R} = \alpha I - \mu_3 R + \nu_3 R \tag{3}$$

$$\dot{D} = \mu_1 S + \mu_2 I + \mu_3 R \tag{4}$$

the dot represents differentiation with respect to time, t, the vulnerable (or susceptible) population is denoted by $S(t)$, $I(t)$ represents the infected population by the disease, the population group cured by the disease are represented by $R(t)$, $D(t)$ denotes dead population due to infectious disease, μ_1 represents the natural death rate of vulnerable group of population, a dead rate due to infectious disease is represented by μ_2, the rate at which a rescue (or recovered) individual may die is denoted by μ_3, the natural birth rate is represented by ν_1, an individual may born infected at rate ν_2, the proportionate birth rate of the recovered individual is denoted by ν_3, the rate in which a recovered individual becomes immune is denoted by α, the infection rate is represented by γ, while β denotes the rate of infected individual becoming susceptible, after efficient treatment.

The discussion in the present paper is organised as follows. In Section 2, we reduced the four-dimensional system (1)-(4) into a single ordinary differential equation of second-order. The painlevé-analysis was performed for the solutions of nonlinear second order differential equation in Section 3. The Painlevé property is carried out in In Section 4, and we found that the values of parameters of the system (1)-(4) has no movable critical singular points. In Section

5, we performed a Lie symmetry method of the reduced equations. The explicit and numerical solutions were established in Section 6.

2. The Reduced Form of the Nonlinear System (1)-(4)

In this Section we reduce the four-dimensional system (1)-(4) to a one-dimension second order ordinary differential equation. Since equations (1) and (2) does not dependent on R and D, therefore, we can find the number of individual who are recovered, R once we know the infected individual I, hereafter we can excluded R in any consequent analysis of the system. From (2) we have

$$S = \frac{1}{\beta}\frac{\dot{I}}{I} + \frac{\alpha+\gamma+\mu_2-\nu_2}{\beta}. \tag{5}$$

The derivative of (5) gives

$$\dot{S} = \frac{1}{\beta}\left(\frac{I\ddot{I}-\dot{I}^2}{I^2}\right). \tag{6}$$

The substitution of (5) and (6) into (1) gives

$$\begin{aligned} I\ddot{I}-\dot{I}^2 &= -\beta\dot{I}I^2 - \beta\left(\alpha+\gamma+\mu_2-\nu_2\right)I^3 + \gamma\beta I^3 - \mu_1\dot{I}I \\ &- \mu_1\left(\alpha+\gamma+\mu_2-\nu_2\right)I^2 + \nu_1\dot{I}I + \nu_1\left(\alpha+\gamma+\mu_2-\nu_2\right)I^2. \end{aligned} \tag{7}$$

We have after some arrangement

$$I\ddot{I}-\dot{I}^2+\beta\dot{I}I^2 = -\beta\left(\alpha+2\gamma+\mu_2-\nu_2\right)I^3 + (\nu_1-\mu_1)\dot{I}I + (\nu_1-\mu_1)(\alpha+\gamma+\nu_2-\mu_2)I^2. \tag{8}$$

We may attain the following simplification by means of the given change of variable:

$$I = \frac{u}{\beta}. \tag{9}$$

The substitution of (9) to (8) gives

$$u\ddot{u}-\dot{u}^2+\dot{u}u^2 = a\dot{u}u - (b+\gamma)\,u^3 + abu^2 \tag{10}$$

with

$$\nu_1-\mu_1 = a \tag{11}$$

$$\alpha+\gamma+\mu_2-\nu_2 = b. \tag{12}$$

3. Painlevé Analysis

The method of Painlevé Analysis was found by a Russian mathematician, Kowalevski [6]. She was very determined to know about the integrability conditions of the Euler equations which was in great importance throughout the historical time of La Belle Époque [16]. The technique of Painlevé Analysis have been contributed significantly in solving nonlinear differential equations. Concerning the methodology, We referred the interested reader to the book written by Tabor in [18] and later on explain by Ramani et al. in [17]. The quintessence of investigating an ordinary differential equation as well as a system of nonlinear ordinary differential equations from the view point of singularity analysis is the Willpower of the existence of isolated movable singularities whereby one may obtained a Laurent series expansion containing arbitrary coefficients which will be the same with the order of the corresponding differential equations [7]. However, the initial conditions of the system are the main aspect as far as the location of the singularity is concern. Nevertheless, a more complex equation (or system of equations) involving multifaceted arrangement retained more than one polelike singularity. Conversely, when it comes to the system of differential equations with many singularities, one need to assure the existence of a Laurent expansion containing an essential amount of arbitrary constant. Nevertheless, a counter example to this can be find in [7] Whereby the behaviour of the first form of singularity preserves a Laurent series expansion containing a precise amount of arbitrary parameters while the second does not preserve, nonetheless contain an uneven solution [13] of the category discussed by Ince [4]. Though, the general solution of the nonlinear system is remarkably explicit.

Ablowitz [1], developed a standard algorithm in order to analyse a differential equation from the view point of Painlevé. Even if there are some illustrations of specific significance of the Painlevé method in analysing a system of nonlinear first order differential equations which are common in mathematical modelling of epidemics, such that the tactical method approach promoted in [2] is considered. In this section, we will primarily, summarise the typical algorithm due to Ablowitz [1]. Furthermore, different approach will be provided. We will start by considering the following autonomous system of first-order ordinary differential equations

$$\Phi_i(x,\dot{x},\sigma) = 0,\ i = 1,n \tag{13}$$

with n dependent variables represented by x, the independent variable is denoted by t while σ is the conservative of constraints which constantly appears to increase a system commonly used in mathematical modelling of natural phenomena. The assumption made here is that the first derivative of the n functions Φ_i are linear and the later are polynomials functions in the dependent variables x. We have to bear in mind that those assumptions are not entirely necessary, nevertheless they do simplify the complexity of the model and explain the reality.

3.1. The Painlevé Test. In [15], Ove proved that if a differential equation admits solutions which are single value in a neighborhood of non characteristic movable singularity manifolds, then this equation is integrable and therefore possesses the Painlevé property. The author also claimed that the method described by Weiss and Carnevale [19] proposes a necessary condition of integrability, also known as the Painlevé test. However, while computing the Painlevé test, one seeks solution of a given rational differential equation in the form of a Laurent series (also known as the Painlevé expansion) [15].

The execution of Painlevé test suggests that solution of the following differential equation

$$F(x,u,u_{x1},u_{x2},\ldots) = 0 \tag{14}$$

with independent variable $x = (x_0,x_1,x_2,\ldots,x_{n-1})$ has the form below

$$u(x) = \phi^{-m}(x)\sum_{j=0}^{\infty} u_j(x)\phi^j(x) \tag{15}$$

where the functions ϕ, $u_j(u_0 \neq 0)$ are analytic of x around the region of $\phi(x) = 0$.

After substituting equation (15) into (14), if one obtains the correct number of arbitrary functions which are required by the Cauchy-Kovalevskaya expression, then the corresponding differential equation (14) passes the Painlevé test. The Cauchy-Kovalevskaya expression is represented by the expansion coefficients in (15), whereby ϕ is counted as one of the arbitrary functions. The exponents in the Painlevé expansion, where the arbitrary functions are to appear, are known as the resonances. If a given differential equation possesses a Painlevé test, then the construction of Bäcklund transformations to linear equations or other known integrable equations becomes possible. In this regards, the sufficient condition of integrability and the Painlevé property of the given equation is evident.

In the following subsection, Ablowitz's algorithm will be used to find the leading-order behaviour of the nonlinear system (1)-(4) and the reduced equation (10). We will start by substituting $x_i = \sigma_i \tau^{p_i},\ i = 1, n$, with $\tau = t - t_0$ and t_0 the assumed location of the movable singularity, into the system (13) and compare the resulting power.

3.2. Painlevé Analysis of the nonlinear system (1)-(4). We commence the Painlevé analysis of nonlinear system (1)-(4) in the customary manner by substituting

$$S = k_1 \tau^{q_1},\ I = k_2 \tau^{q_2}\ R = k_3 \tau^{q_3} \text{ and } D = k_4 \tau^{q_4}$$

into (1)-(4) and obtain the following

$$\begin{aligned} k_1 q_1 \tau^{q_1 - 1} &= -\beta k_1 k_2 \tau^{q_1 + q_2} + \gamma k_2 \tau^{q_2} + (\nu_1 - \mu_1) k_1 \tau^{q_1}, \\ k_2 q_2 \tau^{q_2 - 1} &= \beta k_1 k_2 \tau^{q_1 + q_2} - (\nu_2 - \mu_2 - \alpha - \gamma) k_2 \tau^{q_2}, \\ k_3 q_3 \tau^{q_3 - 1} &= \alpha k_2 \tau^{q_2} - (\nu_3 - \mu_3) k_3 \tau^{q_3}, \\ k_4 q_4 \tau^{q_4 - 1} &= \mu_1 k_1 \tau^{q_1} + \mu_2 k_2 \tau^{q_2} + \mu_3 k_3 \tau^{q_3}. \end{aligned} \tag{16}$$

At

$$q_1 = q_3 = q_4 = -1 \text{ and } q_2 = -2,$$

the leading order behaviour is as follows

$$S = -\frac{2}{\beta}\tau^{-1},\ I = \frac{2}{\beta\gamma}\tau^{-2},\ R = -\frac{2\alpha}{\beta\gamma}\tau^{-1} \text{ and } D = -\frac{2\mu_2}{\beta\gamma}\tau^{-1}. \tag{17}$$

The determination of the resonances is find by setting [9]

$$
\begin{aligned}
S &= -\frac{2}{\beta}\tau^{-1} + n_1\tau^{p-1} \\
I &= \frac{2}{\beta\gamma}\tau^{-2} + n_2\tau^{p-2} \\
R &= -\frac{2\alpha}{\beta\gamma}\tau^{-1} + n_3\tau^{p-1} \\
D &= -\frac{2\mu_2}{\beta\gamma}\tau^{-1} + n_4\tau^{p-1}
\end{aligned} \tag{18}
$$

In such a way that arbitrary constants of integration are obtained. However, n_1, n_2, n_3, n_4 are found by using the dominant terms obtained in (17):

$$
\begin{pmatrix} p-1 & -\gamma & 0 & 0 \\ \frac{2}{\gamma} & p-2 & 0 & 0 \\ 0 & \alpha & -(p-1) & 0 \\ 0 & \mu_2 & 0 & p-1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{19}
$$

3.3. Painlevé Analysis of the reduced equation (10).

Theorem 1. *labeltheo The system (1)-(4) passes the Painlevé test under parameter values $a = \gamma + b$ and $\nu_1 - \mu_1 = \nu_2 - \mu_2 + \alpha + 2\gamma$.*

Proof of Theorem 3.1. In order to obtain the formal Laurent series expansion, we substitute equation

$$
u = \sum_{i=0}^{\infty} \sigma_i \tau^{i-1} \tag{20}
$$

into equation (10) which gives the following equation:

$$
\begin{aligned}
&\sigma_i\sigma_j(i-1)(i-2)\tau^{i+j-4} - (i-1)(i-1)\sigma_i\sigma_j\tau^{i+j-4} + (i-1)\sigma_i\sigma_j\sigma_k\tau^{i+j+k-4} \\
&\quad = a(i-1)\sigma_i\sigma_j\tau^{i+j-3} - (b+\gamma)\sigma_i\sigma_j\sigma_k\tau^{i+j+k-3} + ab\sigma_i\sigma_j\tau^{i+j-2}.
\end{aligned}
$$

for $i = j = k = 0, 1, 2, ...$ At τ^{-4} we require

$$2\sigma_0^2 - \sigma_0^2 - \sigma_0^3 = 0.$$

Therefore

$$\sigma_0 = 1.$$

We move to the next power, τ^{-3}, and find

$$2\sigma_0\sigma_1 - \sigma_0\sigma_1 - \sigma_0^2\sigma_1 = -a\sigma_0^2 - (b+\gamma)\sigma_0^3.$$

Since $\sigma_0 = 1$, this gives an arbitrary σ_1 only if

$$a = \gamma + b. \tag{21}$$

From (11) and (21) we have:

$$\nu_1 - \mu_1 = \nu_2 - \mu_2 + \alpha + 2\gamma. \tag{22}$$

□

Theorem 2. *If $a = \gamma + b$ and $\nu_1 - \mu_1 = \nu_2 - \mu_2 + \alpha + 2\gamma$. Then the system of first order differential equations (1)-(4) is a Painlevé-type and does not possess chaotic behaviour.*

Theorem 3. *Equation (10) is equivalent to the equation*

$$yy'' - y'^2 + y'y^2 + y'y + \frac{b+\gamma}{a}y^3 + \frac{b}{a}y^2 = 0. \tag{23}$$

with y and t the new dependent and independent variables respectively such that

$$u = ay \text{ and } t = \frac{x}{a} \tag{24}$$

Theorem 4.
- *Equation (23) under parameters values $a \neq b$ possesses chaotic behavior and does not pass the Painlevé test.*
- *Equation (23) passes Painlevé test under parameters values $a = b$ and $\gamma = 0$ does not possess chaotic behavior.*

4. Lie Symmetry Analysis

A second order ordinary differential equation

$$u_t - F(t,u,u_{(1)}) = 0 \tag{25}$$

admits a one-parameter Lie group of transformations

$$\begin{aligned} \bar{t} &\approx t + a\xi^0(t,u) \\ \bar{u} &\approx u + a\eta(t,u) \end{aligned} \tag{26}$$

with infinitesimal generator

$$X = \xi^0(t,u)\frac{\partial}{\partial t} + \eta(t,u)\frac{\partial}{\partial u} \tag{27}$$

if

$$\bar{u}_{\bar{t}} - F(\bar{t},\bar{u},\bar{u}_{(1)}) = 0 \tag{28}$$

The group transformations $\bar{t}$ and $\bar{u}$ are obtained by solving the following Lie equations [3, 10]

$$\begin{aligned} \frac{d\bar{t}}{da} &= \xi^0(\bar{t},\bar{u}) \\ \frac{d\bar{u}}{da} &= \eta(\bar{t},\bar{u}) \end{aligned} \tag{29}$$

with initial conditions

$$\bar{t}\mid_{a=0} = t, \bar{u}\mid_{a=0} = u. \tag{30}$$

The infinitesimal form of $\bar{u}_{\bar{t}}, \bar{u}_{(1)}$ are found by the given formulas [3, 11]:

$$\begin{aligned} \bar{u}_{\bar{t}} &\approx u_t + a\zeta_0(t,u,u_t,u_{(1)}) \\ \bar{u}_{\bar{x}^i} &\approx u_{x^i} + a\zeta_i(t,u,u_t,u_{(1)}) \end{aligned} \tag{31}$$

The functions ζ_0 and ζ_i are found by using the prolongation formulas below [12]

$$\begin{aligned} \zeta_0 &= D_t(\eta) - u_t D_t(\xi^0) \\ \zeta_i &= D_i(\eta) - u_t D_i(\xi^0) \end{aligned} \tag{32}$$

In [9], Matadi claimed that the equation (25) possesses the symmetry (group generator)

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{33}$$

iff

$$X^{[N]}N|_{N=0} = 0 \tag{34}$$

with $X^{[N]}$ the n-th extension of G.

Since equation (23) under parameters values $a = b$ and $\gamma = 0$ does not possess chaotic behavior and pass the Painlevé test, the infinitesimal symmetry of equation (23) has coefficient functions of the form

$$\xi(x,y) = c_1 + c_2 e^x \tag{35}$$

$$\eta(x,y) = -c_2 y e^x \tag{36}$$

where c_1 and c_2 are arbitrary constants. Thus the Lie algebra of equation (23) is spanned by the following two infinitesimal generator:

$$X_1 = \partial_x \tag{37}$$

$$X_2 = e^x(\partial_x - y\partial_y). \tag{38}$$

Computing the Lie bracket we obtain the given commutator table:

	X_1	X_2
X_1	0	X_2
X_2	0	0

TABLE 1. The commutator table of the infinitesimal generator

From the commutator table, we conclude that the reduction of (23) can be made by X_2 only. The Lagrange's system associated to X_2 is given by

$$\frac{dx}{1} = \frac{dy}{-y} = \frac{dy'}{-2y' - y} \tag{39}$$

Solving equation (39) we obtain the new dependent variable, X, and independent variable, Y, namely:

$$X = x + \log y,\ Y = \frac{y'}{y^2} + \frac{1}{y} \tag{40}$$

Therefore equation (23) becomes

$$\frac{dY}{dX} + Y + 1 = 0 \tag{41}$$

the integration of equation (41) gives

$$(Y+1)\exp[X] = A. \tag{42}$$

The substitution of (40) into (42) gives

$$\frac{y'}{y} + y + 1 = A\exp[-x]. \tag{43}$$

The integration of (43) gives

$$y = \frac{C\exp[-x]}{D\exp[A\exp[-x]] + B}. \tag{44}$$

Substituting (11) and (44) into (24) we have

$$u(t) = \frac{(\mu_1 - \nu_1)C\exp[-(\mu_1 - \nu_1)t]}{D\exp[A\exp[-(\mu_1 - \nu_1)t]] + B}. \tag{45}$$

The number of infected population is obtain by subtituting (45) into (9)

$$I(t) = \frac{(\mu_1 - \nu_1)C\exp[-(\mu_1 - \nu_1)t]}{\beta[D\exp[A\exp[-(\mu_1 - \nu_1)t]] + B]}. \tag{46}$$

The substitution of (46) into (5) gives

$$S(t) = \frac{1}{\beta}(\mu_1 - \nu_1) + \frac{1}{\beta}(\mu_1 - \nu_1)\exp[-(\mu_1 - \nu_1)t] + \frac{\alpha + \mu_2 - \nu_2}{\beta}. \tag{47}$$

5. The General Solutions

In this Section we obtain the analytical solution of the system (1)-(4) by combining the first two equations.

$$\begin{aligned} \dot{S}+\dot{I} &= (\nu_1-\mu_1)S+(\nu_2-\mu_2-\alpha)I \\ &= (\nu_1-\mu_1)(S+I). \end{aligned} \tag{48}$$

Let

$$N = S+I. \tag{49}$$

Equation (48) becomes

$$\dot{N} = (\nu_1-\mu_1)N. \tag{50}$$

The solution of (50) is

$$N(t) = N(0)\exp[(\nu_1-\mu_1)t]. \tag{51}$$

From (49) and (51) we have

$$S = N(0)\exp[(\nu_1-\mu_1)t]-I. \tag{52}$$

Equation (4.5.1b) becomes

$$\begin{aligned} \dot{I} &= -\beta I^2+\beta I N(0)\exp[(\nu_1-\mu_1)t]-(\alpha+\gamma+\mu_2-\nu_2)I \\ \frac{\dot{I}}{I^2} &= -\beta+\frac{\beta}{I}N(0)\exp[(\nu_1-\mu_1)t]-\frac{(\alpha+\gamma+\mu_2-\nu_2)}{I}. \end{aligned} \tag{53}$$

With the use of the transformation

$$u = \frac{1}{I} \tag{54}$$

equation (53) becomes:

$$\dot{u} = \beta-\beta N(0)\exp[(\nu_1-\mu_1)t]u+(\alpha+\gamma+\mu_2-\nu_2)u. \tag{55}$$

Since

$$\nu_1-\mu_1 = a$$

and

$$\alpha+\gamma+\mu_2-\nu_2=b,$$

equation (55) gives

$$\dot{u}+(\beta N(0)\exp[at]-b)\,u=\beta \tag{56}$$

which has the integrating factor

$$\exp\left[\int(\beta N(0)\exp[at]-b)\mathrm{d}t\right].$$

The solution of (56) is

$$\begin{aligned} u &= A\exp\left[-\int(\beta N(0)\exp[at]-b)\mathrm{d}t\right] \\ &+ \exp\left[-\int(\beta N(0)\exp[at]-b)\mathrm{d}t\right]\int\beta\exp\left[\int(\beta N(0)\exp[at]-b)\mathrm{d}t\right]\mathrm{d}t. \end{aligned}$$

From (54) we have

$$I(t)=\frac{\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]}{A+\int\beta\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]} \tag{57}$$

and from (52), we have

$$S(t)=N(0)\exp[(\nu_1-\mu_1)t]-\frac{\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]}{A+\int\beta\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]}. \tag{58}$$

From (4.5.1c) we obtain

$$\dot{R}-(\nu_3-\mu_3)R=\alpha I. \tag{59}$$

Equation (59) has the integrating factor $\exp[(\nu_3-\mu_3)t]$. Therefore

$$R(t)=\exp[(\nu_3-\mu_3)t]\left[B+\int\frac{\alpha\exp[-(\nu_3-\mu_3)t]\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]}{A+\int\beta\exp[\int(\beta N(0)\exp[at]-b\mathrm{d}t]}\mathrm{d}t\right] \tag{60}$$

and from (4) we obtain the death component of the population to be

$$\begin{aligned} D(t) &= \mu_1 N(0)\exp[(\nu_1-\mu_1)t] \\ &+ (\mu_2-\mu_1)\frac{\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]}{A+\int\beta\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]} \\ &+ \mu_3\exp[(\nu_3-\mu_3)t]\left[B+\int\frac{\alpha\exp[-(\nu_3-\mu_3)t]\exp[\int(\beta N(0)\exp[at]-b)\mathrm{d}t]}{A+\int\beta\exp[\int(\beta N(0)\exp[at]-b\mathrm{d}t]}dt\right]. \end{aligned}$$

6. Discussion

In this section we give a numerical result based on the Susceptibles and Infected component of the population. The parameters are chosen as $\nu_1 = 0.0003$, $\nu_2 = 0.0001$, $\nu_3 = 0.0003$, $\mu_1 = 0.0002$, $\mu_2 = 0.0003$, $\mu_3 = 0.0002$, $\alpha = 0.01$, $\beta = 0.04$, $\gamma = 0.04$. Figure 1 suggests that the solution is globally asymptotically stable.

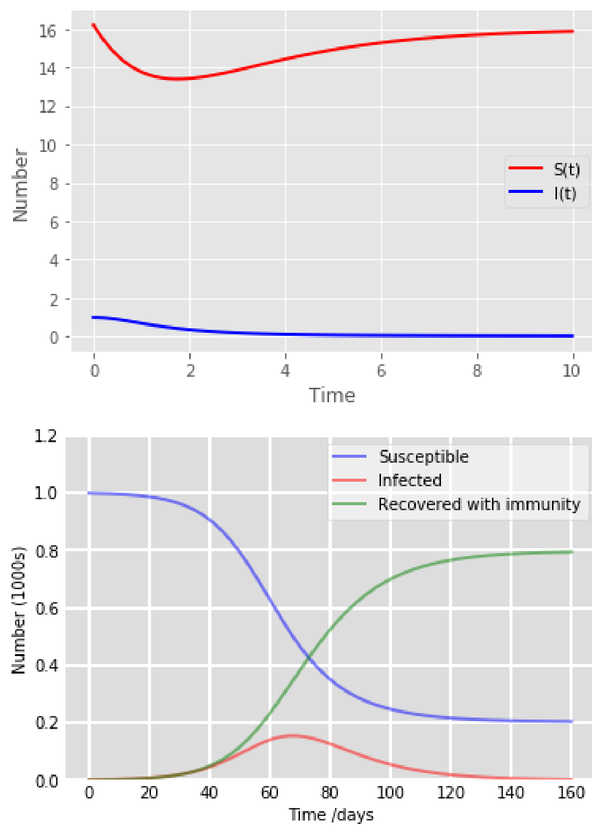

FIGURE 1. Plotting the Susceptibles, Infected and Recovered components with the use of Python

7. Conclusion

In order to understand physical model, the analysis of a nonlinear differential play an essential role. Ove [15] stated that the by finding a closed form solution of a nonlinear differential, one can arrive at a complete understanding of the phenomena which are modeled. In this paper, four dimensional system of the SIRD epidemial model is reduced into a one dimensional second order differential equation. The Painlevé-analysis was performed for solutions of nonlinear second order differential equation. When pameters attain the values corresponding to complex chaotic behavior, equation (23) possesses chaotic behavior if $a \neq b$, consequently it does not pass the Painlevé test. The result revealed that under parameters values $a = b$ and $\gamma = 0$, equation (23) possesses chaotic behavior and does pass the Painlevé test. The techniques of Symmetry Analysis is performed to reduce equation and obtain the combinations of parameters which lead to the possibility of the linearisation of the system and provide the corresponding solutions.

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ABBREVIATIONS

The following abbreviations are used in this manuscript:

SIR Model	Susceptible, Infected, Recovered Model
SIRD Model	Susceptible, Infected, Recovered and Dead Model

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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