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# GLOBAL STABILITY OF A DISCRETE SIR EPIDEMIC MODEL WITH SATURATED INCIDENCE RATE AND DEATH INDUCED BY THE DISEASE

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**Abstract.** We discuss a numerical discretization for an SIR epidemic model, where the incidence rate is assumed to be saturated. The numerical discretization is performed by employing a nonstandard finite difference (NSFD) method to discretize the continuous SIR epidemic model where the denominator function is chosen such that the scheme maintains the population conservation law. This discretization leads to a numerical scheme which can be considered as a discrete system. The dynamics of the obtained discrete system is then analyzed. It is found that the disease-free equilibrium of the discrete system is globally asymptotically stable if the basic reproduction number is less than or equals to one. On the other hand, if the basic reproduction number is greater than one, then the endemic equilibrium is globally asymptotically stable. Such global stability properties have been confirmed by our numerical simulations. Furthermore, our numerical simulations show that the proposed conservative NSFD is more accurate than the NSFD without considering the population conservation law.

**Keywords**: NSFD Method, SIR epidemic model, basic reproduction number

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## 1. INTRODUCTION

Since the classical epidemic model was proposed by Kermack and McKendrick [1], epidemic models are continuously developed and studied to understand the patterns of spread of and to control various infectious diseases. One of important factors in the spread of infectious disease is the incidence rate. In the classical Kermack and McKendrick epidemic model [1], the bilinear incidence rate is used, which means that the disease transmission continues to increase along with the increase in infective individuals. The bilinear incidence rate is considered reasonable when the number of infective individuals is small, but it will not make sense when the number of infective individuals becomes large. To overcome this, Capasso and Serio [2] applied a saturated incidence rate. The saturated incidence rate can describe the inhibition of disease transmission due to changes in people's behavior when the number of infective individual increases. By considering a saturated incidence rate and the disease may cause a death, the SIR epidemic model is written as

$$\frac{dS}{dt} = A - \frac{bSI}{1+aI} - mS \tag{1}$$

$$\frac{dI}{dt} = \frac{bSI}{1+aI} - (m+g+d)I \tag{2}$$

$$\frac{dR}{dt} = gI - mR, \tag{3}$$

where *S*, *I* and *R* denote the susceptible, infective and recovered sub-classes, respectively. *A*,*b*,*m* are respectively the recruitment rate of population, the disease transmission rate and the natural death rate. *a* is a constant that corresponds to inhibitory effect due to the increasing number of infective individuals, $g$ is the recovery rate and *d* is the death rate caused by the disease. Notice that system (1-3) is a special case of model proposed by Pathak et al. [3] and Kaddar [4-5]. It is shown in [3-5] that system (1-3) has always a disease free equilibrium (DFE) $E_0 = (A/m, 0, 0)$ which is asymptotically stable if $\mathcal{R}_0 \leq 1$. Here $\mathcal{R}_0$ is the basic reproduction number and is given by

$$\mathcal{R}_0 = \frac{Ab}{m(m+g+d)}. \tag{4}$$

Furthermore, if $\mathcal{R}_0 > 1$ then DFE becomes unstable and there appears an endemic equilibrium (EE) $E^* = (S^*, I^*, R^*)$ which is asymptotically stable, where $S^* = \frac{A(a\, m\mathcal{R}_0 + b)}{m\, \mathcal{R}_0 (a\, m + b)}$, $I^* = \frac{m(\mathcal{R}_0 - 1)}{a\, m + b}$ and $R^* = \frac{g\, I^*}{m}$.

Note that the SIR epidemic model (1-3) takes the form of a system of nonlinear differential equations where the exact analytical solutions are difficult to obtain. The solutions of the system is then usually obtained numerically. In the literature, various numerical methods have been introduced to solve the system of nonlinear differential equations. Some standard numerical methods, for example the Euler or Runge-Kutta methods, may fail to maintain the properties of the solution such as positivity, stability of the equilibrium point, etc., see e.g. [6-8]. One of well-known methods that can maintain the dynamics of the model is the nonstandard finite difference (NSFD) method [6-16]. The authors in [6] have derived a NSFD scheme for a SIR epidemic model with modified saturated incidence rate and performed local analysis for the scheme. In this paper, we revisit the NSFD scheme derived in [6]. We specifically highlight the conservative property of the NSFD scheme and apply this property to prove the global stability of the equilibrium points.

## 2. CONSERVATIVE NSFD SCHEME

Following [6], if we implement the NSFD scheme to the system (1-3), then we get the following discrete SIR epidemic model

$$\frac{S_{n+1} - S_n}{\varphi(h)} = A - \frac{bS_{n+1}I_n}{1 + \alpha I_n} - mS_{n+1} \tag{5}$$

$$\frac{I_{n+1} - I_n}{\varphi(h)} = \frac{bS_{n+1}I_n}{1 + \alpha I_n} - (m + g + d)I_{n+1} \tag{6}$$

$$\frac{R_{n+1} - R_n}{\varphi(h)} = gI_{n+1} - mR_{n+1}, \tag{7}$$

with the denominator function

$$\varphi(h) = \frac{\exp(mh) - 1}{m}. \tag{8}$$

The scheme (5-7) is called the NSFD since it applies a nonlocal discretization for the right hand sides. Moreover, the first order derivative is approximated by the generalization of forward difference scheme, where $h$ is the time step of integration. The denominator function (8) is determined using the exact population conservation law; see [8-11] for the detail terminology of population conservation law. Henceforth, we call scheme (5-8) as the conservative finite difference (CNSFD) scheme. In next section, the population conservation law will be used to prove the global stability of the equilibrium points, which has not been done in [6]. We notice that Hattaf et al. [12] has proposed a backward finite difference scheme for system (1-3) but only for the case of $d = 0$. In addition, numerical scheme in [12] is similar to the CNSFD scheme (5-8) except that the first derivative is approximated by the forward Euler, i.e. $\varphi(h) = h$. This scheme is still considered as a NSFD scheme since it applies nonlocal approximation, but it does not consider the conservation law. Hence, in this paper we call this scheme as the NSFD scheme.

## 3. BASIC PROPERTIES OF DISCRETE SYSTEM

First, we notice that the CNSFD scheme (5-7) has two equilibrium points which are exactly the same as those of system (1-3), i.e. DFE $E_0 = \left(\frac{A}{\mu}, 0, 0\right)$ and EE $E^* = (S^*, I^*, R^*)$. The same as in the continuous case, i.e. system (1-3), DFE exists unconditionally and is locally asymptotically stable if $\mathcal{R}_0 \leq 1$. The EE exists and is locally asymptotically stable only if $\mathcal{R}_0 > 1$, see [6]. By performing some algebraic manipulation, the CNSFD scheme (5-7) can be written as

$$S_{n+1} = \frac{S_n + A\varphi(h)}{1 + \varphi(h)(m + b\phi(I_n))} \tag{9}$$

$$I_{n+1} = \frac{I_n + b\varphi(h)\phi(I_n)S_{n+1}}{1 + \varphi(h)(m + g + d)} \tag{10}$$

$$R_{n+1} = \frac{R_n + g\ \varphi(h)\ I_{n+1}}{1 + m\ \varphi(h)}, \tag{11}$$

where $\phi(x) = x/(1+ax)$. Because all parameters in the CNSFD scheme (5-7) or equivalently system (9-11) are positive, it can be easily checked that $S_n, I_n$ and $R_n$ will always be non-negative for any $n$ and for any positive initial value $S_0 > 0,\ I_0 > 0$ and $R_0 > 0$. Furthermore, from the CNSFD (5-7) we get

$$\frac{T_{n+1} - T_n}{\varphi(h)} = A - mT_{n+1} - dI_{n+1} \leq A - mT_{n+1}\,, \tag{12}$$

where $T_n = S_n + I_n + R_n$. The solution of equation (12) satisfies

$$T_n \leq \frac{1}{m}[A - (A - m\,T_0)\exp(-mnh)], \tag{13}$$

where $T_0 = S_0 + I_0 + R_0$. Clearly that $\lim_{n\to\infty} T_n \leq A/m$. Hence the CNSFD scheme (5-7) or (9-11) satisfies the continuous population conservation law. In addition, we can easily show that for the case of $d = 0$, the total population $(T_n)$ obtained by the CNSFD scheme is exactly the same as that obtained from the continuous system (1-3) for any $n$. Such population conservation law is not satisfied by the NSFD scheme in [12].

## 4. GLOBAL STABILITY ANALYSIS

It was shown in [6] that the DFE of the CNSFD scheme (5-7) is locally asymptotically stable if $\mathcal{R}_0 \leq 1$. Furthermore if $\mathcal{R}_0 > 1$ then the DFE becomes unstable and the EE is locally asymptotically stable. In this section we show the global stability of both DFE and EE.

### 4.1 Global Stability of DFE

The global stability of DFE is stated in the following theorem.

**Theorem 1**. *If $\mathcal{R}_0 \leq 1$ then the disease free equilibrium $E_0 = (A/m, 0, 0)$ is globally asymptotically stable.*

**Proof**. If $I_0 = 0$, then equation (10) gives $I_n = 0$ for $n \geq 0$. Hence, equation (9) and equation (11) respectively lead to $\lim_{n\to\infty} S_n = A/m$ and $\lim_{n\to\infty} R_n = 0$, and we have the result. Now we suppose that $I_0 > 0$. Since $x > \phi(x)$ for any $x > 0$, equation (10) leads to

$$\frac{I_{n+1}}{I_n} < \frac{1+\varphi(h)bS_{n+1}}{1+\varphi(h)(m+g+d)}. \tag{14}$$

From the population conservation law, there exists $n_1 \geq 0$ such that $T_n \leq A/m$ for any $n \geq n_1$, and so $S_{n+1} \leq A/m$. Therefore inequality (14) gives

$$\frac{I_{n+1}}{I_n} < \frac{1+\frac{\varphi(h)bA}{m}}{1+\varphi(h)(m+g+d)} = \frac{1+\varphi(h)\mathcal{R}_0(m+g+d)}{1+\varphi(h)(m+g+d)}.$$

It is clear that if $\mathcal{R}_0 \leq 1$ then $I_{n+1}$ is a monotone deacreasing sequence, i.e. $I_{n+1}/I_n < 1$, showing that $\lim_{n\to\infty} I_n = \hat{I} \geq 0$ exists. If $\hat{I} = 0$, then from equation (9) and (11) we get that $\lim_{n\to\infty} S_n = A/m$ and $\lim_{n\to\infty} R_n = 0$. We now assume that $\hat{I} > 0$. The population conservation law (12) shows that

$$T_{n+1} = \frac{T_n - (A - dI_{n+1})}{1+m\varphi(h)}.$$

Therefore there exists a $\lim_{n\to\infty} T_n = (A - d\hat{I})/m = \hat{T} \geq 0$. From equation (11) we also have $\lim_{n\to\infty} R_n = g\hat{I}/m = \hat{R} > 0$. Finally, from the definition of total population we get $\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n - I_n - R_n = \hat{T} - \hat{I} - \hat{R} = \hat{S} \geq 0$. As a result, the solution of system ( 9-11) is convergent to $(\hat{S},\ \hat{I},\ \hat{R})$. This contradicts the fact that $E_0 = (A/m, 0, 0)$ is the only equilibrium. Therefore we get $\hat{I} = 0$, and using the previous argument we can conclude that the DFE $E_0 = (A/m, 0, 0)$ is globally asymptotically stable.□

We remark that the global stability of DFE is a direct consequence of population conservation law where the conservation law for the CNSFD scheme applies because of the clever choice of the denominator function (8).

**4.2 Global Stability of EE**

As mentioned previously, if $\mathcal{R}_0 > 1$ the system (9-11) also has EE, in addition to DFE. In the following we prove the global stability of EE. The global stability of EE is proven by using a Lyapunov function of discrete system, which is adopted from Enatsu et al. [17].

**Theorem 2**. *If $\mathcal{R}_0 > 1$ then the endemic equilibrium $E^* = (S^*, I^*, R^*)$ is globally asymptotically stable.*

**Proof**. For any $s_1 > 0, s_2 > 0$ and $s_2 \geq s_1$, we have that

$$\int_{s_1}^{s_2} \frac{1}{s} ds \geq \int_{s_1}^{s_2} \frac{1}{s_2} ds.$$

Then we obtain that

$$\ln \frac{s_2}{s_1} \geq \frac{s_2 - s_1}{s_2}. \tag{15}$$

Following Enatsu et al. [17], we consider the following sequence $\{V(n)\}_{n=0}^{+\infty}$ defined by

$$V(n) = \frac{V_1(n)}{b\varphi(h)S^* \phi(I^*)} + \frac{I^* V_2(n)}{b\varphi(h)S^* \phi(I^*)} + \frac{V_3(n)}{\varphi(h)}.$$

Here $V_1(n) = S^* f\left(\frac{S_n}{S^*}\right), V_2(n) = f\left(\frac{I_n}{I^*}\right), V_3(n) = f\left(\frac{\phi(I_n)}{\phi(I^*)}\right)$,

where $f(x) = x - 1 - \ln(x) \geq f(1) = 0$. From the definition of $V_1(n)$ and using equation (15) we get

$$\begin{aligned}
V_1(n+1) - V_1(n) &= S^* \left[ f\left(\frac{S_{n+1}}{S^*}\right) - f\left(\frac{S_n}{S^*}\right) \right] \\
&= S_{n+1} - S_n - S^* \ln\left(\frac{S_{n+1}}{S_n}\right) \\
&\leq S_{n+1} - S_n - S^* \frac{S_{n+1} - S_n}{S_{n+1}} \\
&= \left(1 - \frac{S^*}{S_{n+1}}\right)(S_{n+1} - S_n) \\
&= \varphi(h)\left(1 - \frac{S^*}{S_{n+1}}\right)(A - b\phi(I_n)S_{n+1} - mS_{n+1}) \\
&= -\frac{m\varphi(h)}{S_{n+1}}(S_{n+1} - S^*)^2 + b\varphi(h)S^*\phi(I^*)\left(1 - \frac{S^*}{S_{n+1}}\right)\left(1 - \frac{\phi(I_n)S_{n+1}}{\phi(I^*)S^*}\right).
\end{aligned}$$

Notice that in the last equation, we have substituted $A = b\phi(I^*)S^* + mS^*$. Similarly, we calculate $V_2(n+1) - V_2(n)$ and $V_3(n+1) - V_3(n)$ as follows.

$$V_2(n+1) - V_2(n) = \frac{I_{n+1} - I_n}{I^*} - \ln\left(\frac{I_{n+1}}{I_n}\right)$$

$$\leq \frac{I_{n+1}-I_n}{I^*} - \frac{I_{n+1}-I_n}{I_{n+1}}$$

$$= \frac{1}{I^*}\left(1-\frac{I^*}{I_{n+1}}\right)(I_{n+1}-I_n)$$

$$= \frac{\varphi(h)}{I^*}\left(1-\frac{I^*}{I_{n+1}}\right)(b\phi(I_n)S_{n+1}-(m+g+d)I_{n+1})$$

$$= \frac{\varphi(h)}{I^*}\left(1-\frac{I^*}{I_{n+1}}\right)(b\phi(I_n)S_{n+1}-bS^*\phi(I^*)I_{n+1})$$

$$= \frac{b\varphi(h)S^*\phi(I^*)}{I^*}\left(1-\frac{I^*}{I_{n+1}}\right)\left(\frac{S_{n+1}}{S^*}\frac{\phi(I_n)}{\phi(I^*)}-\frac{I_{n+1}}{I^*}\right),$$

$V_3(n+1)-V_3(n) = \frac{\phi(I_{n+1})-\phi(I_n)}{\phi(I^*)} - \ln\left(\frac{\phi(I_{n+1})}{\phi(I^*)}\right) + \ln\left(\frac{\phi(I_n)}{\phi(I^*)}\right)$.

Let $X_n = S_n/S^*$, $Y_n = I_n/I^*$ and $Z_n = \phi(I_n)/\phi(I^*)$. Then we obtain

$$V(n+1)-V(n) \leq -\frac{mS_{n+1}}{bS^*\,\phi(I^*)}(1-X_{n+1})^2 + \left(1-\frac{1}{X_{n+1}}\right)(1-X_{n+1}Z_n)$$

$$+\left(1-\frac{1}{Y_{n+1}}\right)(X_{n+1}Z_n - Y_{n+1}) + Z_{n+1} - Z_n - \ln(Z_{n+1}) + \ln(Z_n)$$

$$= -\frac{mS_{n+1}}{bS^*\,\phi(I^*)}(1-X_{n+1})^2 + \left(f(Z_{n+1}) - f(Y_{n+1})\right)$$

$$-\left(f\left(\frac{1}{X_{n+1}}\right) + f\left(\frac{X_{n+1}Z_n}{Y_{n+1}}\right)\right).$$

By some algebraic manipulation, we can show that

$$f(Z_{n+1}) - f(Y_{n+1}) = \frac{\phi(I_{n+1})}{\phi(I^*)} - \frac{I_{n+1}}{I^*} + \ln\left(\frac{I_{n+1}}{I^*}\frac{\phi(I^*)}{\phi(I_{n+1})}\right)$$

$$\leq \frac{\phi(I_{n+1})}{\phi(I^*)} - \frac{I_{n+1}}{I^*} + \frac{I_{n+1}}{I^*}\frac{\phi(I^*)}{\phi(I_{n+1})} - 1$$

$$= -\frac{a(I_{n+1}-I^*)^2}{I^*(1+aI_{n+1})(1+aI^*)}.$$

Thus, $V(n+1)-V(n) \leq 0$ for any $n \geq 0$, i.e. $V(n)$ is monotone decreasing sequence. Because $V(n) \geq 0$, $\lim_{n\to\infty} V(n) \geq 0$ exists and hence $\lim_{n\to\infty} V(n+1)-V(n) = 0$. Then we obtain $\lim_{n\to\infty} S_{n+1} = S^*$, $\lim_{n\to\infty} I_{n+1} = I^*$ and $\lim_{n\to\infty} R_{n+1} = R^*$. We conclude that EE is globally asymptotically stable. □.

## 5. NUMERICAL SIMULATIONS

In this section we present some numerical results using the CNSFD scheme (9-11) with hypothetical values of parameter as given in Table 1. Using these parameter values, the CNSFD scheme has two equilibrium points: $E_0 = (2.5, 0, 0)$ and $E^* = (S^* = 1.4706, I^* = 0.4118, R^* = 0.5147)$. Since $\mathcal{R}_0 = 1.875 > 1$, from previous section we now that $E_0$ is unstable while $E^*$ is globally asymptotically stable. The global stability of $E^*$ is confirmed by our numerical simulations depicted in Figure 1, where all numerical solutions with some different initial values are convergent to $E^*$ and the total population for these cases are convergent to $T^* = S^* + I^* + R^* = 2.397 \leq A/m = 2.5$. Hence the population conservation law is also satisfied.

**Table 1.** Parameter values used for simulations

| $A$ | $m$ | $A$ | $b$ | $d$ | $g$ |
|---|---|---|---|---|---|
| 0.1 | 0.04 | 0.25 | 0.75 | 0.01 | 0.05 |

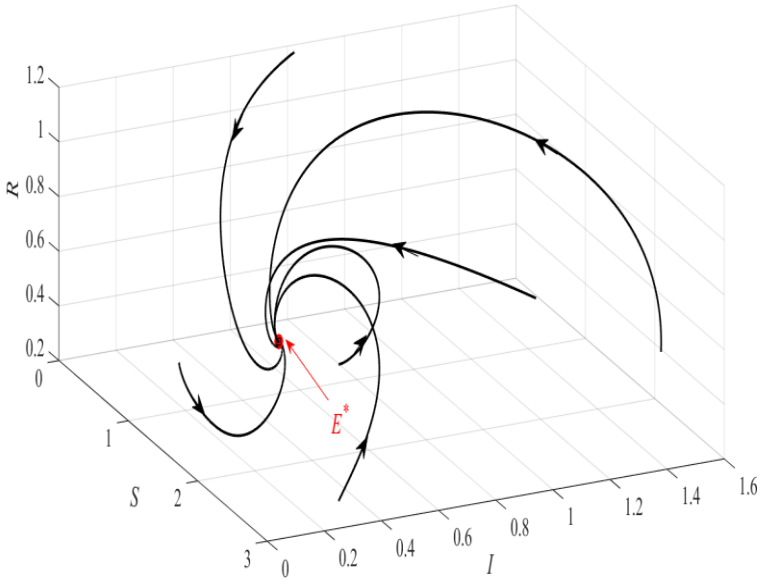


**Figure 1.** Phase portrait of system (1-3) obtained numerically by the CNSFD scheme using parameter values in Table 1.

Next we evaluate the numerical population conservation law in more detail. For this aim, we use parameter values as in Table 1, but with $d = 0$. Using these parameter values, we obtain that the exact total population of the SIR epidemic model (1-3) is given by

$$T(t) = \frac{1}{m}[A - (A - m\,T_0)\exp(-mt)]. \tag{16}$$

In Figure 2 we plot the absolute error of total population (as function *t*), obtained by the NSFD scheme [12] and the CNSFD scheme (9-11). It can be seen in Figure 2(a) that the error of total population produced by the NSFD scheme is relatively large, i.e. in the order of $10^{-4}$, where the smaller *h* leads to smaller absolute error. However, if we implement the CNSFD scheme then we get much smaller error absolute, i.e. in the order of $10^{-14}$; indicating that the population conservation law is maintained by the CNSFD scheme.

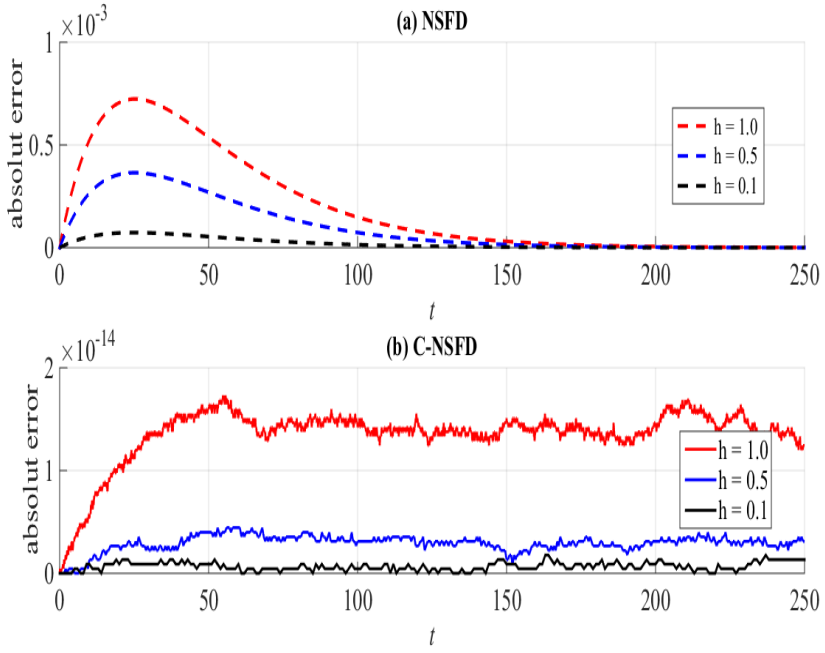


**Figure 2.** Absolute error of the total population as function of *t* obtained by (a) the NSFD scheme and (b) the CNSFD scheme.

## 6. CONCLUSIONS

We have discussed a discrete SIR epidemic model with saturated incidence rate and death caused by the disease. The discretization is performed by applying the NSFD method with a special denominator function. The choice of denominator function results in the fulfillment of population conservation law. Based on the population conservation law, the global stability of the disease free equilibrium is proven. Furthermore, the global stability of the endemic equilibrium has also been shown using a Lyapunov function. The global stability of equilibrium point as well as the population conservation law have been confirmed by some numerical simulations.

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## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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