RELIABLE ITERATIVE METHODS FOR MATHEMATICAL BIOLOGY MODEL

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Abstract. In this paper, three iterative methods will be used to solve the nonlinear differential equation systems that represents a model for HIV infection of $CD4^+T$ cells. The iterative methods are the Daftardar-Jafari method namely (DJM), Temimi-Ansari method namely (TAM) and Banach contraction method namely (BCM) to get the approximate solutions. Numerical solutions were also obtained by using the Runge-Kutta (RK4) method and good agreement have achieved. In addition, the convergence of proposed methods is present based on Banach's fixed point theorem. Also, comparisons are made between the proposed methods and the Adomian decomposition method (ADM), and the Variational iteration method (VIM), a good agreement have achieved without required to evaluate the Adomian polynomials to handle the nonlinear terms as in the ADM, does not require to calculate Lagrange multiplier as in the VIM and time saver. Moreover, the results of the maximum error remainder values are obtained and show that the proposed methods are effective and reliable. Our calculations were performed using the MATHEMATICA®10.

Keywords: HIV infection of $CD4^+T$ cells; semi-analytical method; Daftardar-Jafari method; Temimi-Ansari method; Banach contraction method.

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1. INTRODUCTION

More than 40 million people are infected by human immunodeficiency virus (HIV) and about 16 million deaths caused by this disease. Scientists have made a lot of efforts to get rid of this disease but there was no cure it and have less resistance to the immune system. Therefore, the body cannot fight other infectious diseases [1]. Regulated Tcell in the peripheral blood are in the average person in between 800 and 1,200mm^{-3} [2] and the number of cells in people with this virus decreases. In 1993, Perelson presented a fitted model based on an old model which appeared in [3]. All types of the HIV infection models of CD4$^+$T cells have been studied over the past 20 years these models are described by non-linear difference equations.

In this paper, three iterative methods will be used to solve HIV of CD4$^+$T cells to obtain new approximate solutions. The first one is proposed by the Daftardar-Gejji and Jafari in 2006 namely (DJM) [4]. Many authors solved different kinds of differential equations by using the DJM [5, 6], solving the Laplace equation [7], solving the Volterra integro-differential equations with some applications for the Lane-Emden equations of the first kind [8]. The second iterative method is suggested in 2011 by the Temimi-Ansari method namely (TAM) [9]. Al-Jawary *et al.* [10] have successfully applied the TAM for Duffing equation, and the nonlinear Burgers and advection–diffusion equations [11]. The third iterative method is the Banach contraction method namely (BCM), it has been presented by Daftardar-Gejji and Bhalekar in 2009 [12]. Al-Jawary *et al*. [13] have solve the nonlinear thin film flows of non-Newtonian fluids by using the BCM.

This paper is organized as follows: Section 2, the mathematical biology model will be presented. Section 3, the fundamentals concepts of the three iterative methods will be given. Section 4, solve a model of HIV for CD4$^+$T cells by using the suggested methods are given. The numerical simulations and error analyses of the approximate solutions are shown in section 5. In section 6, the convergence of the used methods is presented. The conclusion will be given in section 7.

2. THE MATHEMATICAL BIOLOGY MODEL OF HUMAN IMMUNODEFICIENCY VIRUS (HIV) OF CD4$^+$T CELLS

2.1. Description of the Model

The model of HIV infection of CD4$^+$T is given by the following system of the nonlinear differential equation as [14]

$$\frac{dT}{dt} = \lambda - \alpha T + rT\left(1 - \frac{T+I}{T_{max}}\right) - kVT,$$

$$\frac{dI}{dt} = kVT - \beta I,$$

(1) $$\frac{dV}{dt} = \mu\beta I - \gamma V,$$

with the initial condition as follows:

(2) $$T(0) = r_1, \quad I(0) = r_2 \quad \text{and} \quad V(0) = r_3.$$

where $T(t), I(t)$ and $V(t)$ demonstrate the concentration of susceptible CD4$^+$T cells, CD4$^+$T cells infected by the HIV and free HIV particles in the blood at time t, respectively. Also, α, β, and γ represent natural turnoverrates of uninfected T cells, infected T cells, and virus particles, respectively. $k(> 0)$ is the infection rate , λ denotes a rate at which the body produces CD4$^+$T cell from precursors in the bone narrow and thymus, r represents a rate at which T cells multiply through mitosis when the T cells are stimulated by antigen or mitogen. μ means the virus particles that each infected CD4$^+$T cell produces during its life. Including all its daughter cells, T_{max} indicates themaximum CD4$^+$T cells concentration in the body. Also $\left(1 - \frac{T+I}{T_{max}}\right)$ describes the logistic growth of healthy CD4+T cells [15].

3. THE FUNDAMENTALS OF THE THREE ITERATIVE METHODS

In this section, the basic concepts of the DJM, BCM and TAM will be presented.

3.1. The basic idea of the DJM

Let us solve the following functional equation [16, 17]

(3) $$T = f + L(T) + N(T),$$

where f is a known analytic function, L and N is a linear and nonlinear operators, and T is unknow

function which can be decomposed in the form:

(4) $$T = \sum_{i=0}^{\infty} T_i$$

Therefore, define [18]

(5) $$u_0 = N(T_0),$$

(6) $$u_n = N(\sum_{i=0}^{n} T_i) - N(\sum_{i=0}^{n-1} T_i), \qquad n \geq 1.$$

So, $N(T)$ you will analyze as follows:

$$N\left(\sum_{i=0}^{\infty} T_i\right) = \underbrace{N(T_0)}_{u_0} + \underbrace{[N(T_0 + T_1) - N(T_0)]}_{u_1} + \underbrace{[N(T_0 + T_1 + T_2) - N(T_0 + T_1)]}_{u_2}$$

(7) $$+ \underbrace{[N(T_0 + T_1 + T_2 + T_3) - N(T_0 + T_1 + T_2)]}_{u_3} + \cdots,$$

Also, the relation is defined with recurrences so that:

(8) $$T_0 = f,$$

$$T_1 = L(T_0) + u_0,$$

$$T_2 = L(T_1) + u_1,$$

$$\vdots$$

(9) $$T_{n+1} = L(T_n) + u_n.$$

Then L represents a linear operator $\sum_{i=0}^{n} L(T_i) = L(\sum_{i=0}^{n} T_i)$ we write it

(10) $$\sum_{i=1}^{n+1} T_i = (\sum_{i=0}^{n} L(T_i)) + N(\sum_{i=0}^{n} T_i) = L(\sum_{i=0}^{n} T_i) + N(\sum_{i=0}^{n} T_i),$$

and,

(11) $$\sum_{i=0}^{\infty} T_i = f + L(\sum_{i=0}^{\infty} T_i) + N(\sum_{i=0}^{\infty} T_i).$$

The approximate solution in Eqs.(10) and (11) it is given by $T = T_0 + T_1 + \dots + T_{n-1}$, more details

can be found in [19].

3.2. The basic idea of the TAM

Let us consider the following nonlinear differential equation [20]:

(12) $$L(T) + N(T) + f = 0,$$

with initial conditions

(13) $$T(0) = r,$$

where L is a linear operator, N is a nonlinear operator, f is the given function. T is the unknown

this method can be implemented as follows, let T_0 is the initial approximation solve by the initial

problem

(14) $$L(T_0) + f = 0 \quad \text{and} \quad T_0 = r,$$

The next approximate solution, by solving the following problem

(15) $$L(T_1) + f + N(T_0) = 0 \quad \text{and} \quad T_1 = r,$$

Thus, we have a simple iterative step procedure that can be used to solve a set of linear problems

(16) $$L(T_{n+1}) + f + N(T_n) = 0 \quad \text{and} \quad T_n = r.$$

T_n is an approximate solution to the Eq. (12), then solution for the problem can be given by [21, 22]

(17) $$T = \lim_{n \to \infty} T_n.$$

3.3. The basic idea of the BCM

Let us first consider the nonlinear functional equation [23, 24].

(18) $$T = f + N(T) \,,$$

where T is an unknown function, f represents a given function, N represents a nonlinear

operator for the functional Eq.(18). Now we can define some successive approximations as:

(19) $$T_0 = f,$$

$$T_1 = T_0 + N(T_0),$$

$$T_2 = T_0 + N(T_1),$$

$$T_3 = T_0 + N(T_2),$$

$$\vdots$$

(20)
$$T_n = T_0 + N(T_{n-1}), \qquad n = 1,2,\dots$$

So, the solution for the relation (18) obtained by

(21)
$$T = \lim_{n\to\infty} T_n.$$

4. SOLVING A MODEL OF HIV OF CD4$^+$T CELLS

In this section, the three iterative methods presented in section three will be used to solve the systems of nonlinear differential Eq.(1) with the initial condition given in Eq. (2).

4.1. Solving a model of Human immunodeficiency virus (HIV) of CD4$^+$T cells by the DJM

To solve the systems of nonlinear differential Eq.(1) with the initial conditions given in Eq.(2) by the DJM, let us re-write the Eq.(1) as:

$$T'(t) = N_1\left(\lambda - \alpha T(t) + rT(t)\left(1 - \frac{T(t)+I(t)}{T_{max}}\right) - kV(t)I(t)\right),$$

$$I'(t) = N_2(kV(t)T(t) - \beta I(t)),$$

(22)
$$V'(t) = N_3\big(\mu \beta I(t) - \gamma V(t)\big),$$

Integrating Eq. (22) from 0 to t, and use initial conditions, we get

$$T(t) = r_1 + \lambda t + \int_0^t (-\alpha T(s) + rT(s)(1 - \frac{T(s)+I(s)}{T_{max}}) - kV(s)I(s))ds,$$

$$I(t) = r_2 + \int_0^t (kV(s)T(s) - \beta I(s))ds,$$

(23)
$$V(t) = r_3 + \int_0^t (\mu \beta I(s) - \gamma V(s))ds,$$

Then, we have

$$\overline{T_0}(t) = r_1 + \lambda t,$$

$$\overline{I_0}(t) = r_2,$$

(24)
$$\overline{V_0}(t) = r_3,$$

In general,

$$\overline{T_{n+1}}(t) = \int_0^t N_1(\sum_{i=0}^n \overline{T_i}(s))\, ds - \int_0^t N_1(\sum_{i=0}^{n-1} \overline{T_i}(s))ds,$$

$$\overline{I_{n+1}}(t) = \int_0^t N_2\left(\sum_{i=0}^n \overline{I_i}(s)\right) ds - \int_0^t N_2\left(\sum_{i=0}^{n-1} \overline{I_i}(s)\right) ds,$$

(25)
$$\overline{V_{n+1}}(t) = \int_0^t N_3\left(\sum_{i=0}^n \overline{V_i}(s)\right) ds - \int_0^t N_3\left(\sum_{i=0}^{n-1} \overline{V_i}(s)\right) ds, \quad n \geq 1$$

By applying the DJM, we get

$$\overline{T_0}(t) = r_1 + \lambda t,$$

$$\overline{I_0}(t) = r_2,$$

(26)
$$\overline{V_0}(t) = r_3,$$

Also,

$$\overline{T_1}(t) = \frac{1}{2}rt^2\lambda - \frac{1}{2}t^2\alpha\lambda - \frac{rt^3\lambda^2}{3T_{max}} + rtr_1 - t\alpha r_1 - \frac{rt^2\lambda r_1}{T_{max}} - \frac{rtr_1^2}{T_{max}}$$

$$- \frac{rt^2\lambda r_2}{2T_{max}} - \frac{rtr_1r_2}{T_{max}} - \frac{1}{2}kt^2\lambda r_3 - ktr_1r_3,$$

$$\overline{I_1}(t) = -t\beta r_2 + \frac{1}{2}kt^2\lambda r_3 + ktr_1r_3,$$

(27)
$$\overline{V_1}(t) = t(\mu\beta r_2 - \gamma r_3),$$

and,

$$\overline{T_2}(t) = \frac{1}{6}r^2t^3\lambda - \frac{1}{3}rt^3\alpha\lambda + \frac{1}{6}t^3\alpha^2\lambda - \frac{r^2t^4\lambda^2}{3T_{max}} - \frac{r^3t^5\lambda^2}{20T_{max}} + \frac{rt^4\alpha\lambda^2}{3T_{max}} +$$

$$\frac{r^2t^5\alpha\lambda^2}{10T_{max}} - \frac{rt^5\alpha^2\lambda^2}{20T_{max}} + \frac{2r^2t^5\lambda^3}{15T_{max}^2} + \frac{r^3t^6\lambda^3}{18T_{max}^2} - \frac{r^2t^6\alpha\lambda^3}{18T_{max}^2} - \frac{r^3t^7\lambda^4}{63T_{max}^3} +$$

$$\frac{1}{2}r^2t^2r_1 - rt^2\alpha r_1 + \frac{1}{2}t^2\alpha^2 r_1 - \frac{4r^2t^3\lambda r_1}{3T_{max}} - \frac{r^3t^4\lambda r_1}{4T_{max}} + \frac{4rt^3\alpha\lambda r_1}{3T_{max}} + \cdots,$$

$$\overline{I_2}(t) = \frac{1}{2}t^2\beta^2 r_2 + \frac{1}{3}k\mu t^3\beta\lambda r_2 + \frac{1}{8}k\mu rt^4\beta\lambda r_2 - \frac{1}{8}k\mu t^4\alpha\beta\lambda r_2 - \frac{k\mu rt^5\beta\lambda^2 r_2}{15T_{max}}$$

$$+ \frac{1}{2}k\mu t^2\beta r_1 r_2 + \frac{1}{3}k\mu rt^3\beta r_1 r_2 - \frac{1}{3}k\mu t^3\alpha\beta r_1 r_2 \frac{k\mu rt^4\beta\lambda r_1 r_2}{4T_{max}} - \cdots$$

(28)
$$\overline{V_2}(t) = -\frac{1}{2}\mu t^2\beta^2 r_2 - \frac{1}{2}\mu t^2\beta\gamma r_2 + \frac{1}{2}t^2\gamma^2 r_3 + \frac{1}{6}k\mu t^3\beta\lambda r_3 + \frac{1}{2}k\mu t^2\beta r_1 r_3,$$

Then, according to Eq. (11), we get:

$$T_1(t) = \overline{T_0}(t) + \overline{T_1}(t) = t\lambda + \frac{1}{2}rt^2\lambda - \frac{1}{2}t^2\alpha\lambda - \frac{rt^3\lambda^2}{3T_{max}} + r_1 + rtr_1 -$$

$$t\alpha r_1 - \frac{rt^2\lambda r_1}{T_{max}} - \frac{rtr_1^2}{T_{max}} - \frac{rt^2\lambda r_2}{2T_{max}} - \frac{rtr_1r_2}{T_{max}} - \frac{1}{2}kt^2\lambda r_3 - ktr_1r_3,$$

$$I_1(t) = \overline{I_0}(t) + \overline{I_1}(t) = r_2 - t\beta r_2 + \frac{1}{2}kt^2\lambda r_3 + ktr_1r_3,$$

(29)
$$V_1(t) = \overline{V_0}(t) + \overline{V_1}(t) = t\mu\beta r_2 + r_3 - t\gamma r_3,$$

$$T_2(t) = \overline{T_0}(t) + \overline{T_1}(t) + \overline{T_2}(t) = t\lambda + \frac{1}{2}rt^2\lambda + \frac{1}{6}r^2t^3\lambda - \frac{1}{2}t^2\alpha\lambda -$$

$$\frac{1}{3}rt^3\alpha\lambda + \frac{1}{6}t^3\alpha^2\lambda - \frac{rt^3\lambda^2}{3T_{max}} - \frac{r^2t^4\lambda^2}{3T_{max}} - \frac{r^3t^5\lambda^2}{20T_{max}} + \frac{rt^4\alpha\lambda^2}{3T_{max}} +$$

$$\frac{r^2t^5\alpha\lambda^2}{10T_{max}} - \frac{rt^5\alpha^2\lambda^2}{20T_{max}} + \frac{2r^2t^5\lambda^3}{15T_{max}^2} + \frac{r^3t^6\lambda^3}{18T_{max}^2} - \frac{r^2t^6\alpha\lambda^3}{18T_{max}^2} - \frac{r^3t^7\lambda^4}{63T_{max}^3} + r_1 +$$

$$rtr_1 + \frac{1}{2}r^2t^2r_1 - t\alpha r_1 - \cdots,$$

$$I_2(t) = \overline{I_0}(t) + \overline{I_1}(t) + \overline{I_2}(t) = r_2 - t\beta r_2 + \frac{1}{2}t^2\beta^2 r_2 + \frac{1}{3}k\mu t^3\beta\lambda r_2 +$$

$$\frac{1}{8}k\mu rt^4\beta\lambda r_2 - \frac{1}{8}k\mu t^4\alpha\beta\lambda r_2 - \frac{k\mu rt^5\beta\lambda^2 r_2}{15T_{max}} + \frac{1}{2}k\mu t^2\beta r_1 r_2 +$$

$$\frac{1}{3}k\mu rt^3\beta r_1 r_2 - \frac{1}{3}k\mu t^3\alpha\beta r_1 r_2 - \frac{k\mu rt^4\beta\lambda r_1 r_2}{4T_{max}} - \cdots,$$

(30)
$$V_2(t) = \overline{V_0}(t) + \overline{V_1}(t) + \overline{V_2}(t) = \mu\, t\beta r_2 - \frac{1}{2}\mu t^2\beta^2 r_2 - \frac{1}{2}\mu t^2\beta\gamma r_2 + r_3 -$$

$$t\gamma r_3 + \frac{1}{2}t^2\gamma^2 r_3 + \frac{1}{6}k\mu t^3\beta\lambda r_3 + \frac{1}{2}k\mu t^2\beta r_1 r_3,$$

So, we continue to obtain the approximations at $n = 6$ for $T_n(t)$, $I_n(t)$ and $V_n(t)$ were also calculated but for brevity not listed here.

4.2. Solving a model of Human immunodeficiency virus (HIV) of CD4$^+$T cells by the TAM

We use the TAM to solve the systems of nonlinear differential equations given in Eqs. (1) and (2). We have the following form

$$L_1(T(t)) = T'(t),$$
$$L_2(I(t)) = I'(t),$$
$$L_3(V(t)) = V'(t),$$

and,

$$N_1(T(t)) = -\left(-\alpha T + rT\left(1 - \frac{T+I}{T_{max}}\right) - kVT\right),$$

$$N_2\big(I(t)\big) = -(kVT - \beta I),$$

$$N_3\big(V(t)\big) = -(\mu \beta I - \gamma V),$$

Also,

$$f_1(t) = \lambda,$$

$$f_2(t) = 0,$$

(31) $$f_3(t) = 0.$$

with the initial conditions (2), we get the following initial problem

$$L_1\big(T_0(t)\big) = 0, \quad T_0(0) = r_1,$$

$$L_2\big(I_0(t)\big) = 0, \quad I_0(0) = r_2,$$

(32) $$L_3\big(V_0(t)\big) = 0, \quad V_0(0) = r_3,$$

In general, we get

$$L_1\big(T_{n+1}(t)\big) + N_1\big(T_n(t)\big) + f_1(t) = 0, \quad T_{n+1}(0) = r_1,$$

$$L_2\big(I_{n+1}(t)\big) + N_2\big(I_n(t)\big) + f_2(t) = 0, \quad I_{n+1}(0) = r_2,$$

(33) $$L_3\big(V_{n+1}(t)\big) + N_3\big(V_n(t)\big) + f_3(t) = 0, \quad V_{n+1}(0) = r_3,$$

The following initial problem must be solved to get the initial approximation:

$$T_0'(t) = 0,$$

$$I_0'(t) = 0,$$

(34) $$V_0'(t) = 0,$$

By integrating both sides of Eq. (34) and using the initial conditions

$$T_0(0) = r_1, \quad I_0(0) = r_2, \quad V_0(0) = r_3.$$

We get,

$$T_0(t) = r_1 + \lambda t,$$

$$I_0(t) = r_2,$$

$$V_0(t) = r_3.$$

In the second step, we will solve the following problem:

$$T_1'(t) = N_1\big(T_0(t)\big), \quad T_1(0) = r_1,$$

$$I_1'(t) = N_2\big(I_0(t)\big), \quad I_1(0) = r_2,$$

(35) $$V_1'(t) = N_3\big(V_0(t)\big), \quad V_1(0) = r_3,$$

This produces,

$T_1(t), I_1(t)$ and $V_1(t)$, similarly, to Eq. (29).

Similarly, we get $T_2(t), I_2(t)$ and $V_2(t)$, which means solving the following problem:

(36) $$T_2'(t) = N_1\big(T_1(t)\big), \quad T_2(0) = r_1,$$

(37) $$I_2'(t) = N_2\big(I_1(t)\big), \quad I_2(0) = r_2,$$

(38) $$V_2'(t) = N_3\big(V_1(t)\big), \quad V_2(0) = r_3,$$

We get:

$T_2(t), I_2(t)$ and $V_2(t)$, similarly, to Eq. (30)

So, we continue to obtain the approximations at $n = 6$ for $T_n(t)$, $I_n(t)$ and $V_n(t)$ were also calculated but for brevity not listed here.

4.3. Solving a model of Human immunodeficiency virus (HIV) of $CD4^+T$ cells by the BCM

Consider the Eqs. (1) and (2), by following the similar procedure as given for the DJM in the subsection (4.1), we have the Eqs. (23) and (24). So according to the BCM, we get:

$$T_0(t) = r_1 + \lambda t,$$

$$I_0(t) = r_2,$$

$$V_0(t) = r_3.$$

In general, we have

$$T_n(t) = T_0(t) + \int_0^t (-\alpha T_{n-1}(s) + r T_{n-1}(s)(1 - \tfrac{T_{n-1}(s)+I_{n-1}(s)}{T_{max}}) -$$

$$kV_{n-1}(s)T_{n-1}(s))ds,$$

$$I_n(t) = I_0(t) + \int_0^t (kV_{n-1}T_{n-1}(s) - \beta I_{n-1}(s))ds,$$

(39) $$V_n(t) = V_0(t) + \int_0^t \big(\mu\, \beta I_{n-1}(s) - \gamma V_{n-1}(s)\big)ds, \quad n \in \mathbb{N}$$

Also, we have

$T_1(t)$, $I_1(t)$ and $V_1(t)$, similarly to Eq. (29) .

Moreover, $T_2(t)$, $I_2(t)$ and $V_2(t)$, similarly to Eq. (30).

Therefore, we continue to obtain the approximations at $n = 6$ for $T_n(t)$, $I_n(t)$ and $V_n(t)$ were also calculated but for brevity not listed here.

It is clear that the obtained nth approximate solution by the DJM for Eq. (30) is the same to the nth iteration obtained by the TAM or by the BCM. This means that the solutions resulted by the three proposed methods are exactly the same.

5. THE NUMERICAL RESULTS

The effect of the proposed methods: DJM, TAM and BCM for HIV of CD4$^+$T cells are examined to evaluate the accuracy of these approximate methods of solution. We can find appropriate approximate solutions when determining $\lambda, r, \alpha, \beta, \gamma, k, \mu, T_{max}$ values, the following two cases will be studied:

Case1:

We will examine the system given in Eqs. (1) and (2)when $r_1 = 0.1$, $r_2 = 0$ and $r_3 = 0.1$ by setting $\lambda = 0.5$, and the others parameters are taking from[25], $r = 3$, $\alpha = 0.02$, $\beta = 0.3$, $\gamma = 2.4$, $k = 0.0027$, $T_{max} = 1500$, $\mu = 10$, and approximations are calculated for $T(t)$, $I(t)$ and $V(t)$, respectively. The results for all the proposed methods are the same. Hence we have selected the BCM for solving this case.

By applying the BCM, we get,

$$T_0(t) = 0.1 + 0.5t,$$

$$I_0(t) = 0,$$

$$V_0(t) = 0.1,$$

Also, we have

$$T_1(t) = 0.1 + 0.797953\, t + 0.744833\, t^2 - 0.000166667\, t^3,$$

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$$I_1(t) = 0. + 0.000027t + 0.0000675t^2,$$

$$V_1(t) = 0.1 - 0.24t,$$

Moreover,

$$T_2(t) = 0.1 + 0.797953\,t + 1.18872\,t^2 + 0.739448\,t^3 -$$

$$0.000597854\,t^4 - 0.000221845\,t^5 + 8.2762910^{-8}t^6 -$$

$$7.9365110^{-12}t^7,$$

$$I_2(t) = 0. + 0.000027\,t + 0.0000712737\,t^2 -$$

$$0.000112073\,t^3 - 0.000120674\,t^4 + 2.1610^{-8}t^5,$$

$$V_2(t) = 0.1 - 0.24t + 0.288041t^2 + 0.0000675t^3,$$

So, we continue to obtain the approximation at $n = 6$ for $T_n(t), I_n(t)$ and $V_n(t)$ were also calculated but for brevity are not listed here

Case2:

In this case we will examine the HIV model in Eq. (1) and (2) by setting $\lambda = 0.5, r = 1$, and the others parameters are taking from [25], $\beta = 0.3$, $\alpha = 0.02, \gamma = 2.4$, $k = 0.0027$, $T_{max} = 1500$, $\mu = 10$, and in particular sixth approximations are calculated for $T(t), I(t)$ and $V(t)$, respectively. The results for all the proposed methods are the same. Hence we have selected the DJM for solving this case.

By applying the DJM, we get

$$\overline{T_0}(t) = 0.1 + 0.5t,$$

$$\overline{I_0}(t) = 0,$$

$$\overline{V_0}(t) = 0.1,$$

and,

$$\overline{T_1}(t) = 0. + 0.0979663\,t + 0.244899\,t^2 - 0.0000555556\,t^3,$$

$$\overline{I_1}(t) = 0. + 0.000027\,t + 0.0000675\,t^2,$$

$$\overline{V_1}(t) = -0.24\,t,$$

Also,

$$\overline{T_2}(t) = 0. + 0.04801612t^2 + 0.0800727\,t^3 - 0.0000227535\,t^4$$
$$- 7.9972910^{-6}t^5 + 3.0238610^{-9}t^6 - 2.9394510^{-13}t^7,$$

$$\overline{I_2}(t) = \; 0. - 0.0000232245\,t^2 - 0.00011387\,t^3 - 0.0000396774\,t^4$$
$$+ 7.210^{-9}t^5,$$

$$\overline{V_2}(t) = 0. + 0.288041\,t^2 + 0.0000675\,t^3,$$

Then,

$$T_1(t) = \overline{T_0}(t) + \overline{T_1}(t) = \; 0.1 + 0.597966\,t + 0.244899\,t^2$$
$$- 0.0000555556\,t^3,$$

$$I_1(t) = \overline{I_0}(t) + \overline{I_1}(t) = \; 0. + 0.000027\,t + 0.0000675t^2,$$

$$V_1(t) = \overline{V_0}(t) + \overline{V_1}(t) = 0.1 - 0.24t,$$

and,

$$T_2(t) = \overline{T_0}(t) + \overline{T_1}(t) + \overline{T_2}(t) = 0.1 + 0.597966\,t + 0.292915\,t^2$$
$$+ 0.0800172\,t^3 - 0.0000227535\,t^4 - 7.9972910^{-6}t^5$$
$$3.0238610^{-9}t^6 - 2.9394510^{-13}t^7,$$

$$I_2(t) = \overline{I_0}(t) + \overline{I_1}(t) + \overline{I_2}(t) = 0.000027\,t + 0.0000442755\,t^2$$
$$- 0.00011387\,t^3 - 0.0000396774\,t^4 + 7.210^{-9}t^5,$$

$$V_2(t) = \overline{V_0}(t) + \overline{V_1}(t) + \overline{V_2}(t) = 0.1 - 0.24\,t + 0.288041\,t^2 + 0.0000675\,t^3,$$

$$\vdots$$

So, we continue to obtain the approximation at $n = 6$ for $T_n(t), I_n(t)$ and $V\;(t)$ were also calculated but for brevity are not listed here.

We observed that the numerical results obtained using the three suggested methods are similar to each other. The error remainder function is calculated by [26]

$$ER_{1n}(t) = T_n' - \left(\lambda - \alpha T_n + rT_n\left(1 - \frac{T_n + I_n}{T_{max}}\right) - kV_nT_n\right),$$

$$ER_{2n}(t) = I_n' - (kV_nT_n - \beta I_n),$$

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(40) $$ER_{3n}(t) = V_n' - (\mu \beta I_n - \gamma V_n),$$

and the MER_{in} is:

(41) $$MER_{in} = \max_{0 \leq t \leq 0.1} |ER_{in}(t)| \, , i = 1,2,3$$

The maximal error remainder MER_{in} values for the numerical solutions obtained by the proposed methods, the ADM [27, 28] and the VIM [29] for Eqs. (1) and (2) for case.1, is presented in Figs.1-3.
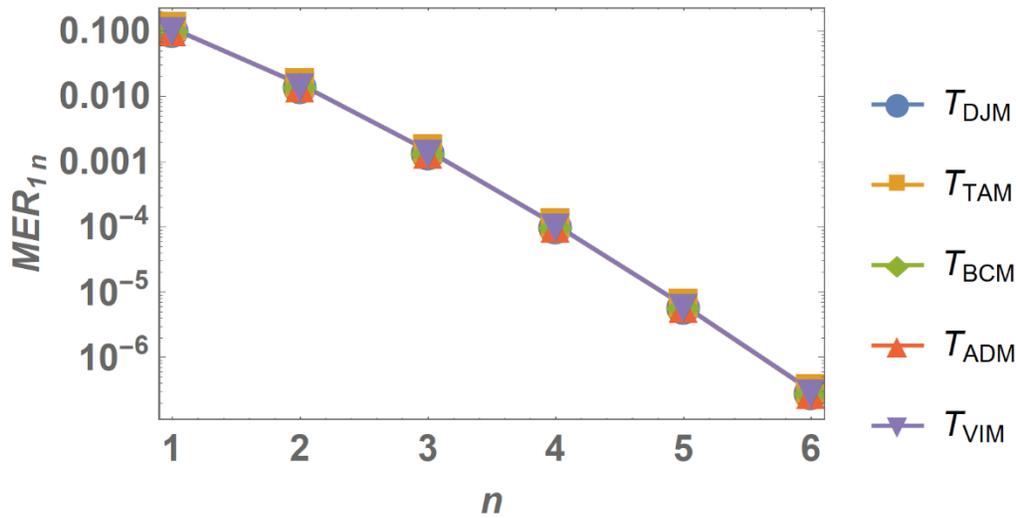

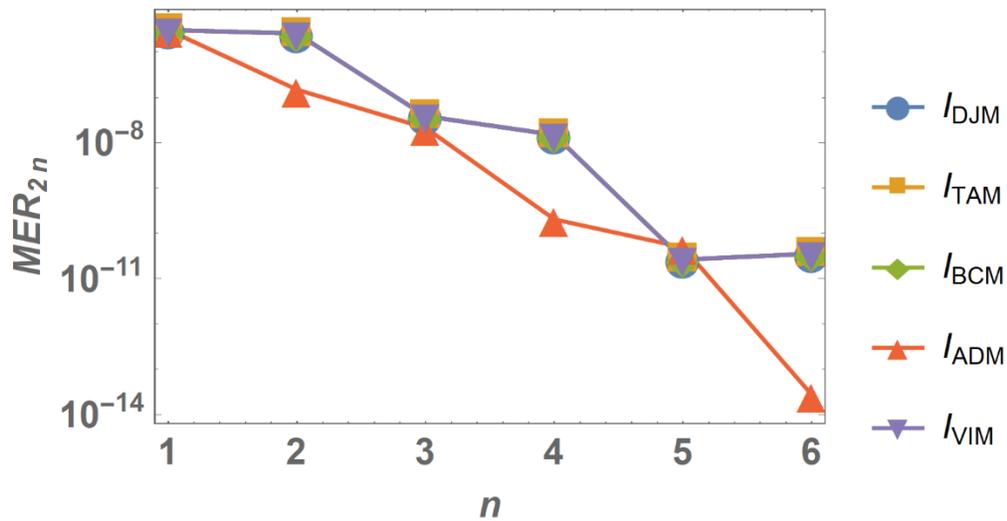**Fig. 1:** The MER_{1n} Comparison of proposed methods solution for case 1 with ADM and VIM method for $T(t)$.

Fig. 2: The MER_{2n} Comparison of proposed methods solution for case 1 with ADM and VIM method for $I(t)$.
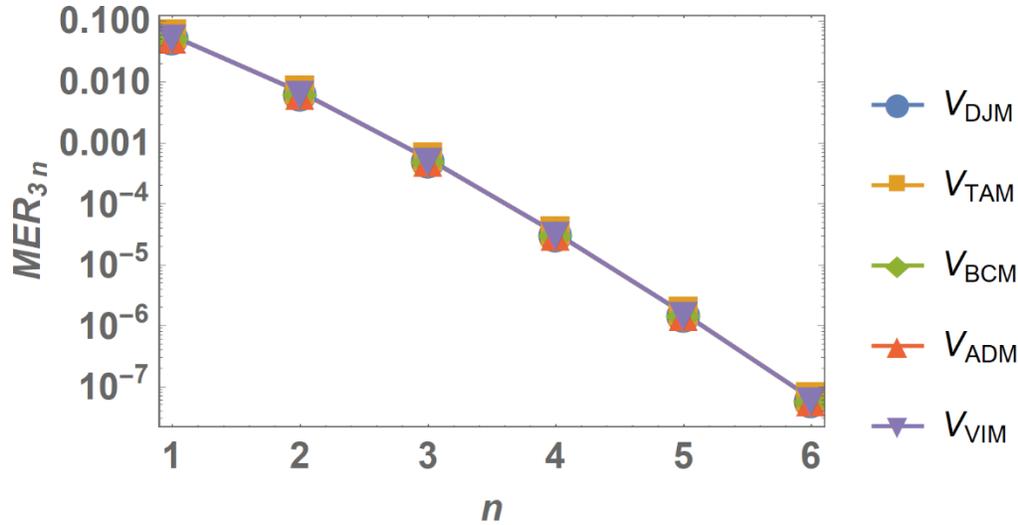
Fig. 3: The MER_{3n} Comparison of proposed methods solution for case 1 with ADM and VIM method for $V(t)$.

It can be seen clearly that the error decreases when increases the number of iteration for $T(t)$, $I(t)$ and $V(t)$. A good agreement have achieved between the proposed methods ADM and VIM without required to evaluate the Adomian polynomials to handle the nonlinear terms as in the ADM, does not require to calculate Lagrange multiplier as in the VIM and time saver.

Moreover, in Figs.4-6, the approximate solutions for case 2 are presented. Once again, good agreements have achieved between the proposed methods, ADM and VIM.
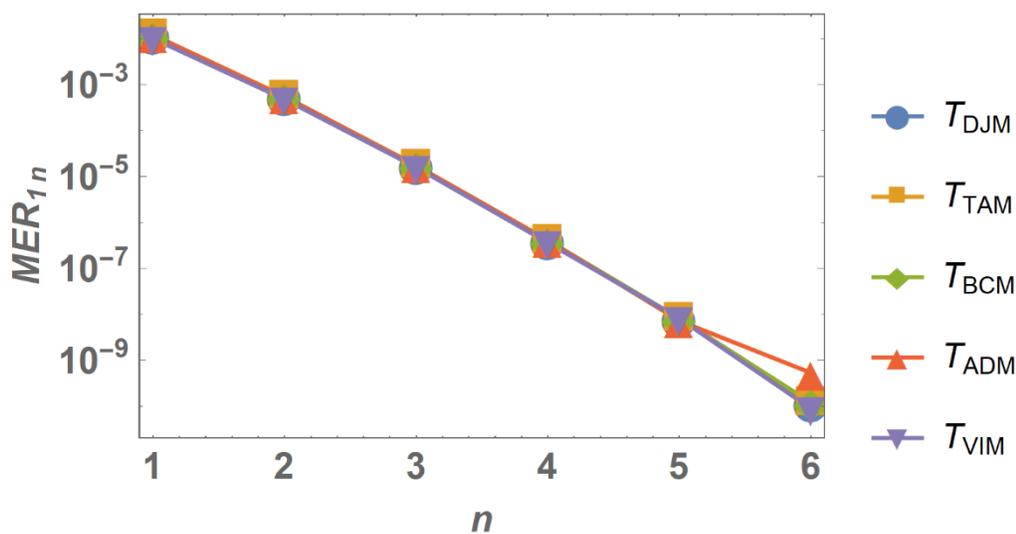

Fig. 4: The MER_{1n} Comparison of proposed methods solution for case 2 with ADM and VIM method for $T(t)$.
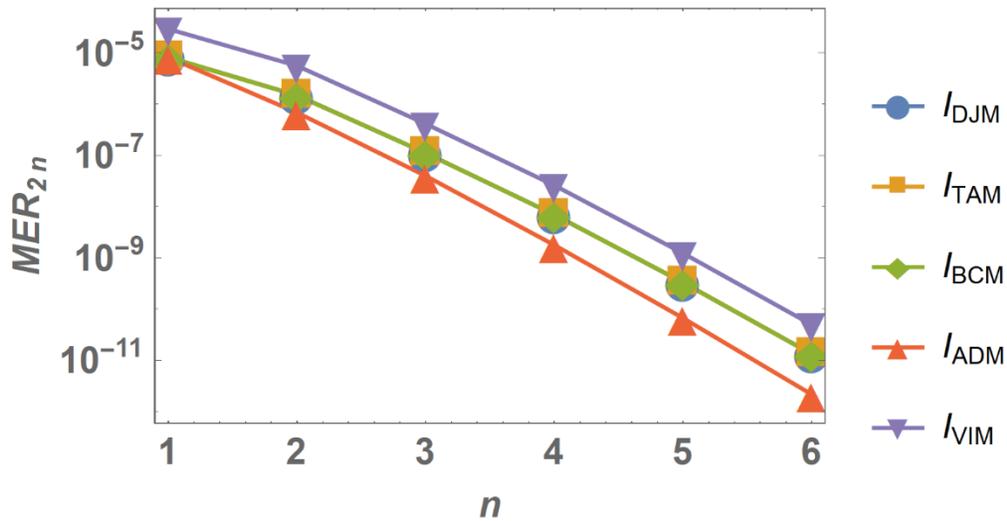
Fig. 5: The MER_{2n} Comparison of proposed methods solution for case 2 with ADM and VIM method for $I(t)$.
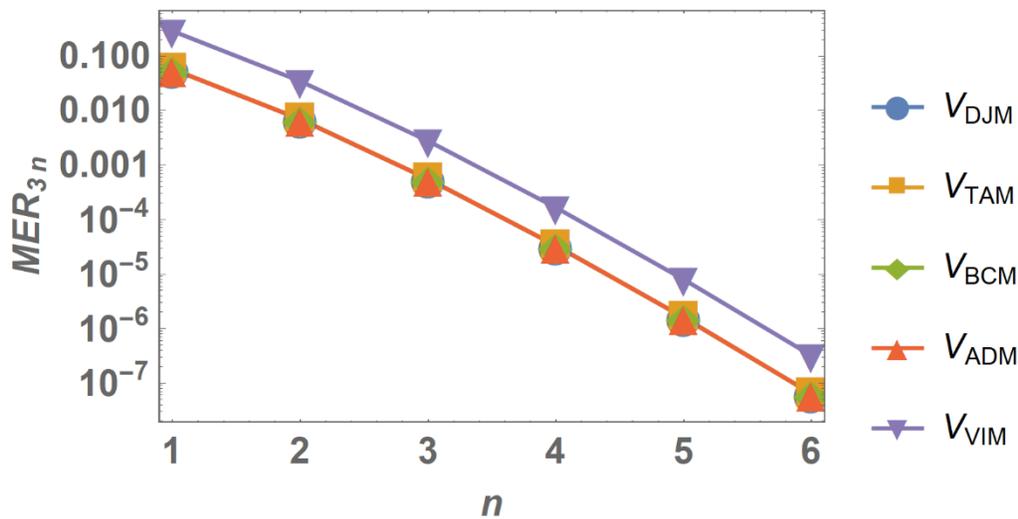

Fig. 6: The MER_{3n} Comparison of proposed methods solution for case 2 with ADM and VIM method for $V(t)$.

It can be seen that the error decreases when the number of iteration increases for $T(t), I(t)$ and $V(t)$.

Moreover, the effect of parameters γ, α and k are presented in Figs. 7- 9, for case 1. It can be noticed, when the values of γ and k are increase, the error will be increases. Furthermore, when the value of α increased the error will be decrease.
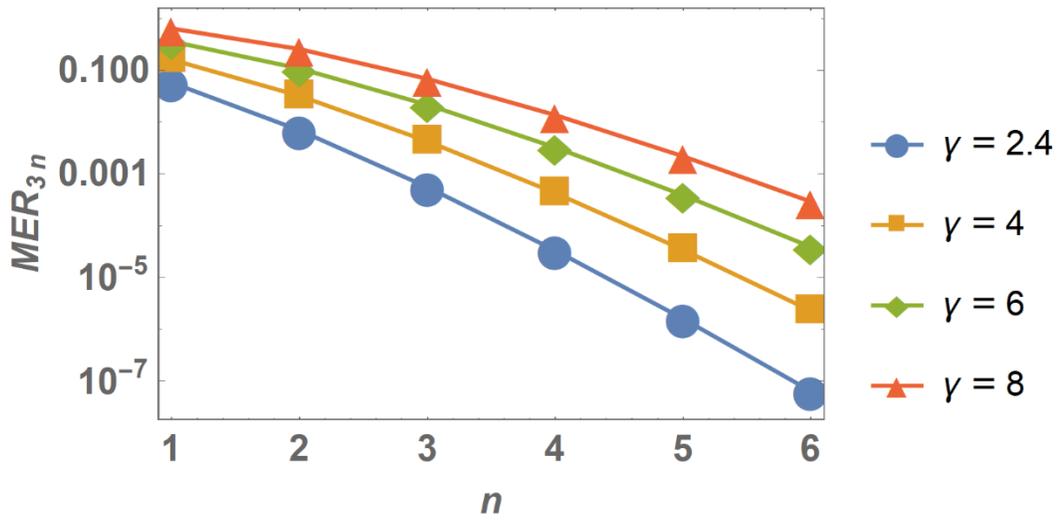
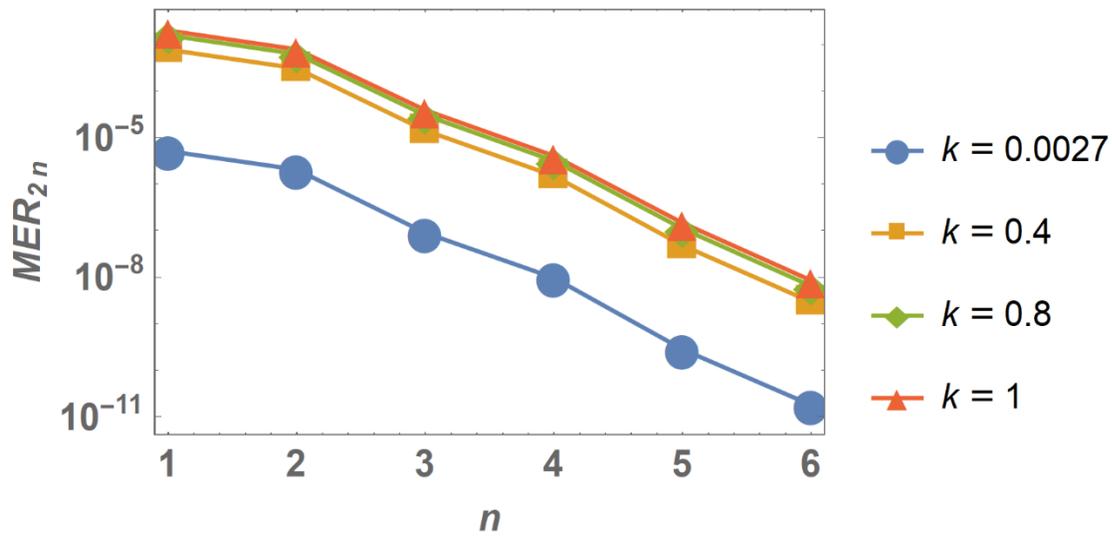**Fig. 7:** The MER_{3n} curves for different values of γ for case 1.

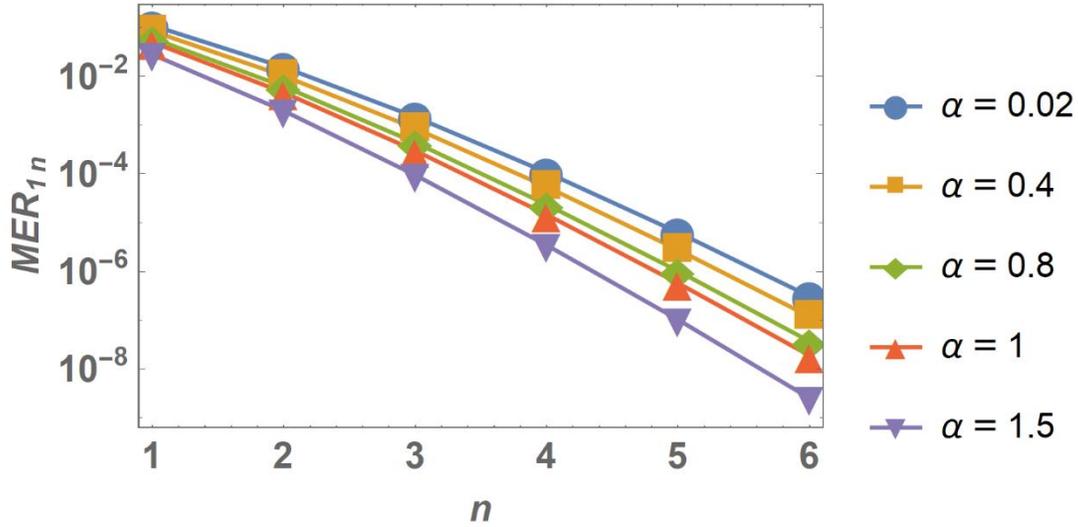**Fig. 8:** The MER_{2n} curves for different values of k for case 1.

Fig. 9: The MER_{1n} curves for different values of α for case 1.

Moreover, we will examine the maximal error remainder MER_{in} by setting $\lambda = 0.1$, $r = 3$, $\alpha = 0.02$, $\beta = 0.3$, $\gamma = 2.4$, $k = 0.0027$, $T_{max} = 1500$, $\mu = 10$ [25], the maximal error remainder MER_{in} values by our proposed methods are also shown in Tables 1, 2. It can be seen clearly that by enlarging the interval of t, the accuracy deteriorates and the MER_{in} is increases. Since these approaches lead to an expansion of the solution, when we enlarge the interval of t, in fact we go farther from the initial point. Thus, the accuracy of the proposed methods diminishes as in Taylor expansion. Thus in fact the limitation of the proposed methods.

Table 1: The maximal error remainder : MER_{in}, where $0 \leq t \leq 0.5$ by proposed methods, where $n=1,..,6$.

n	$MER_{1n}T(t)$	$MER_{2n}I(t)$	$MER_{3n}V(t)$
1	0.555412	0.00471494	0.29175
2	0.378658	0.00646718	0.177082
3	0.183793	0.00271458	0.0732605
4	0.0655693	0.00117611	0.0225898
5	0.0194399	0.000235087	0.00565843
6	0.0046969	0.0000658559	0.00116137

Table 2: The maximal error remainder : MER_{in}, where $0 \leq t \leq 0.7$ by proposed methods, where $n=1,..,6$.

n	$MER_{1n}T(t)$	$MER_{2n}I(t)$	$MER_{3n}V(t)$
1	0.842655	0.0100828	0.40887
2	0.782371	0.0175283	0.349544
3	0.530668	0.0128055	0.204047
4	0.259976	0.00731101	0.00938235
5	0.108652	0.00253588	0.00938235
6	0.0362165	0.000856801	0.0093823

Further, investigation can be done by applying the classical Runge–Kutta method (RK4). Using the MATHEMATICA's code (see appendix A) as a benchmark to assess the performance of the proposed methods.

In Figs 10-12, for case 1, good agreement is observed between proposed methods and RK4, for $T(t)$, $I(t)$ and $V(t)$.
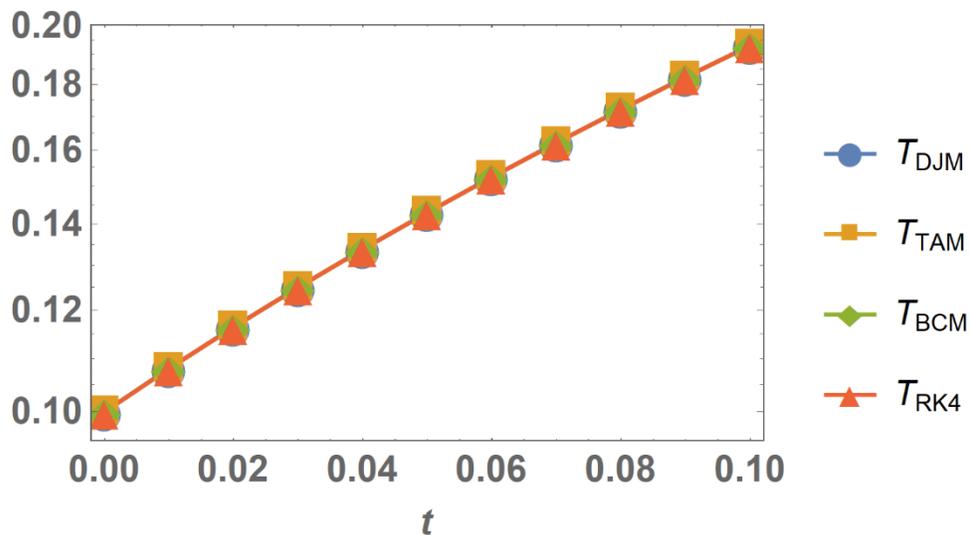

Fig. 10: Comparison of proposed methods solution for case 1 with RK4 method for $T(t)$.
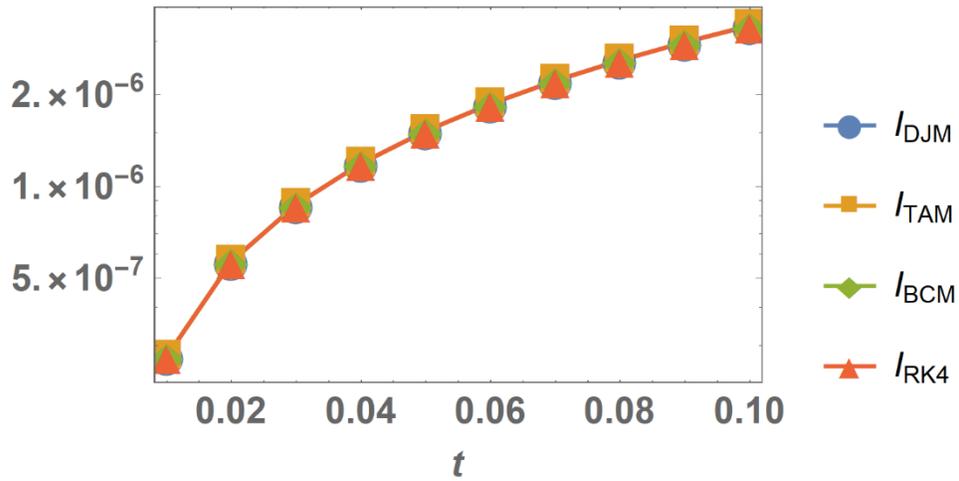
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Fig. 11: Comparison of proposed methods solution for case 1 with RK4 method for $I(t)$.
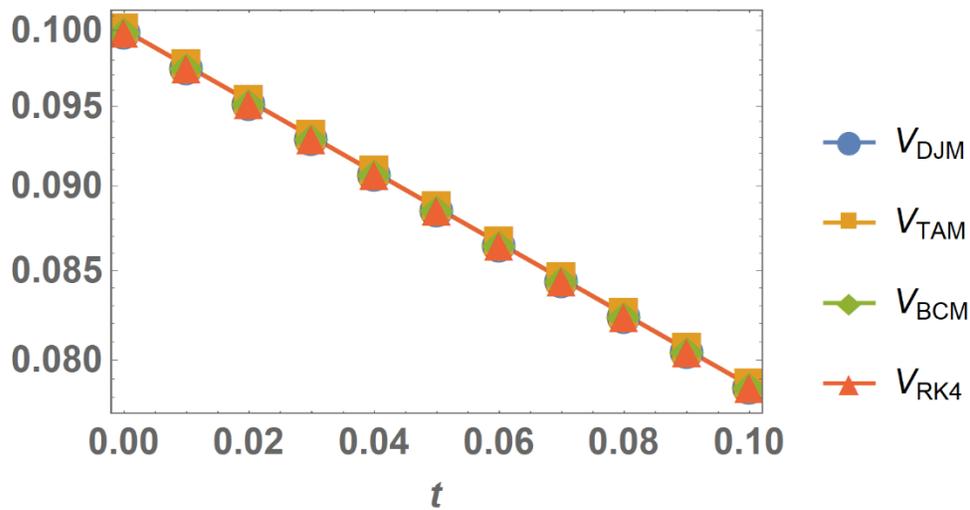

Fig. 12: Comparison of proposed methods solution for case1 with RK4 method for $V(t)$.

Finally, Figs. 13-15, for case 2, a good agreement the approximate solution achieved by proposed methods proposed and RK4 for $T(t),\ I(t)$ and $V(t)$.
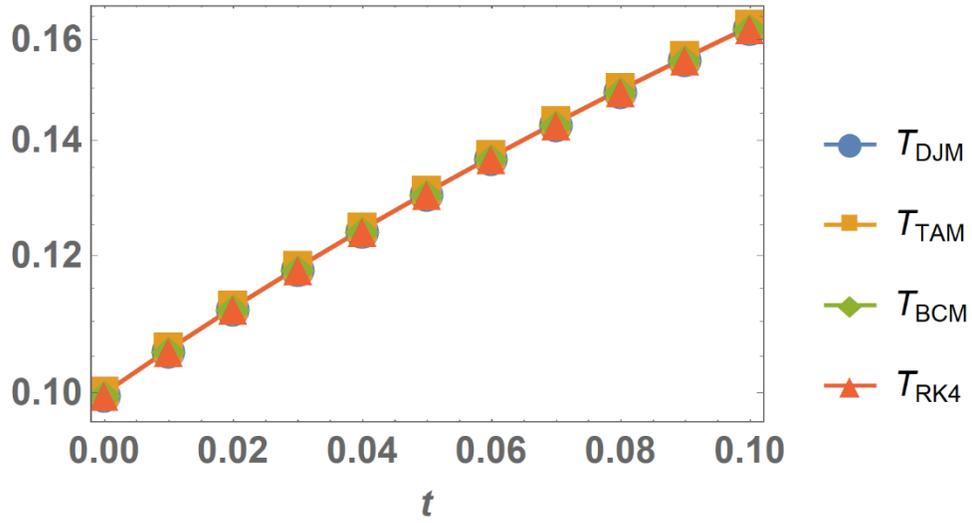
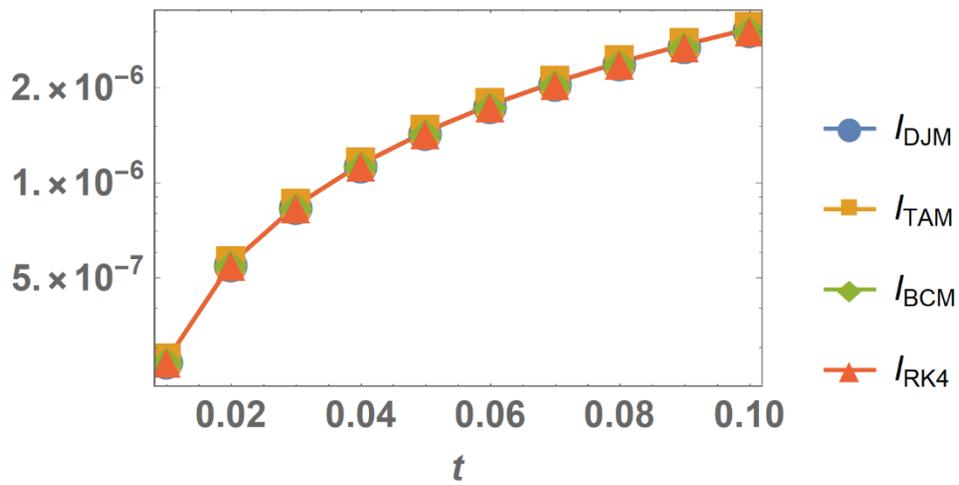**Fig. 13.** Comparison of proposed methods solution for case 2 with RK4 method for $T(t)$.

Fig. 14. Comparison of proposed methods solution for case 2 with RK4 method for $I(t)$.
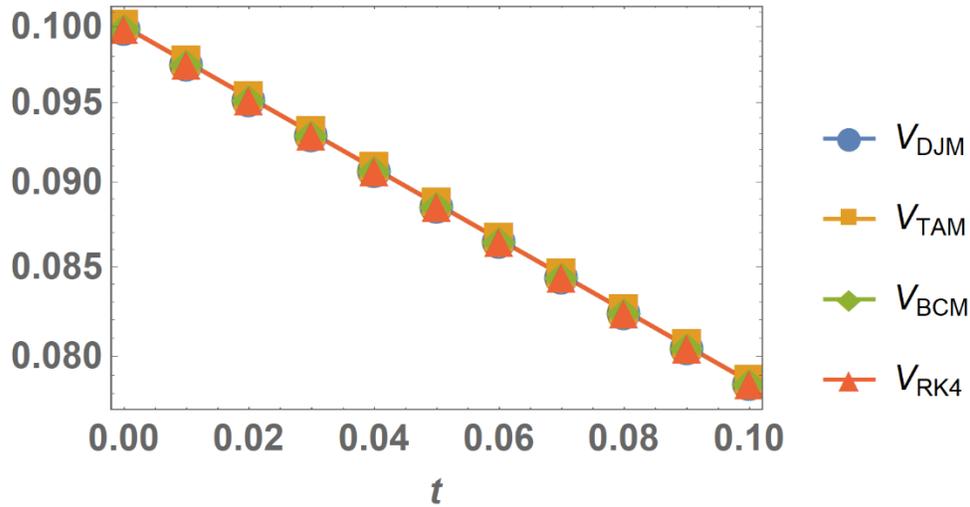
Fig. 15. Comparison of proposed methods solution for case 2 with RK4 method for $V(t)$.

It should be noted that the proposed methods have many advantages such as being derivative-free and overcoming the calculating Adomian polynomials to handle the non-linear terms in the ADM. It does not need evaluating the multiplier of Lagrange as in the VIM in which the terms of the sequence are shifted to be complex after several iterations, thus, analytical evaluation of terms becomes very difficult or impossible in the VIM.

6. THE CONVERGENCE OF THE PROPOSED TECHNIQUES

In this section, we will provide basic concepts and theorems for the convergence of proposed iterative methods [30].

To prove the convergence for the iterations given by the DJM, TAM and BCM, for the DJM, it can be directly proved the convergence. However, for the TAM and BCM, we should follow some steps as below

(42)
$$\begin{aligned}
w_0 &= T_0(t), \\
w_1 &= F[w_0], \\
w_2 &= F[w_0 + w_1], \\
&\vdots \\
w_{n+1} &= F[w_0 + w_1 + \cdots + w_n].
\end{aligned}$$

where F is the operator that can be defined as

(43)
$$F[w_m] = S_m - \sum_{j=0}^{m-1} w_j(t), m = 1,2, ...$$

The term S_m represents the solution for one of the following problems

For the TAM:

(44)
$$L(w_m(t)) + f(t) + N\left(\sum_{j=0}^{m-1} w_j(t)\right) = 0, \quad m = 1,2, ...$$

For the BCM:

(45)
$$w_m = w_0 + N\left[\sum_{j=0}^{m-1} w_j(t)\right], \quad m = 1,2, ...$$

The same given conditions with the used iterative method will be used. Thus, by this manner, we have $T(t) = \lim_{n\to\infty} T_n(t) = \sum_{n=0}^{\infty} w_n$. So, by using (42) and (43), we can get the following solution in a series form

(46)
$$T(t) = \sum_{j=0}^{\infty} w_j(t).$$

According to the recursive algorithm of the DJM, TAM and BCM, the enough conditions for conducting convergence of these methods in the following theorems.

Theorem 6.1. Let F defined in (43), be an operator from a Hilbert space H to H. The series solution $T_n(t) = \sum_{j=0}^{n} w_j(t)$ converges if $\exists\, 0 < \xi < 1$ such that $\|F[w_0 + w_1 + \cdots + w_{j+1}]\| \leq \xi\|F[w_0 + w_1 + \cdots + w_j]\|$ (such that $\|w_{j+1}\| \leq \xi\|w_j\|$) $\forall j = 0,1,2, ...$

This theorem is just a special case of Banach's fixed point theorem which is a sufficient condition to study the convergence of our proposed iterative techniques.

Proof: See [30].

Theorem.6.2. If the series solution $T(t) = \sum_{j=0}^{\infty} w_j(t)$ is convergent, then this series represents the exact solution of the current nonlinear problem.

Proof: See [30].

Theorem.6.3. Assume that the series solution $\sum_{j=0}^{\infty} w_j(t)$ which is defined in (46) be convergent to the solution $T(t)$. If the truncated series $\sum_{j=0}^{n} w_j(t)$ is used as an approximation to the solution of the current problem, then the maximum error $E_n(t)$ is estimated by

(47) $$E_n(t) \leq \frac{1}{1-\xi} \xi^{n+1} \|w_0\|.$$

Proof: See [30].

Theorems 6.1 and **6.2** state that the solutions obtained by one of the presented methods, the DJM given in (9), the TAM given in (16), or the solution of the BCM given in (20), for the nonlinear Eqs. (1) and (2), converges to the exact solution under the condition $\exists\, 0 < \xi < 1$ such that $\|F[w_0 + w_1 + \cdots + w_{j+1}]\| \leq \xi \|F[w_0 + w_1 + \cdots + w_j]\|$ (that is $\|w_{j+1}\| \leq \xi \|w_j\|$) $\forall j = 0,1,2, \ldots$. In another meaning, for each rank i, if we define the parameters

(48) $$\delta_j^i = \begin{cases} \frac{\|w_{j+1}\|}{\|w_j\|}, & \|w_j\| \neq 0 \\ 0, & \|w_j\| = 0 \end{cases}$$

Then the series solution $\sum_{j=0}^{\infty} w_j(x)$ of Eqs. (1) and (2), converges to the exact solution $T(t)$, when $0 \leq \delta_j^i < 1, \forall j = 0,1,2, \ldots, i = 1,2,3$. Also, as in **Theorem 6.3**, the maximum truncation error is estimated to be $\|T(t) - \sum_{j=0}^{n} w_j\| \leq \frac{1}{1-\delta} \delta^{n+1} \|w_0\|$, where $\delta = \max\{\delta_j, j = 0,1, \ldots, n\}$, we evaluate the δ_j^i, where δ_j^1 for $T(t)$, δ_j^2 for $I(t)$ and δ_j^3 for $V(t)$.

6.1. Convergence and error analysis of a model HIV of CD4$^+$T cells by DJM, TAM, and BCM

To prove the convergence of the DJM, TAM and BCM, the Eqs. (1) and (2), we do the following:

Applying the TAM, the S_m represents the following problem

$$T_m'(t) = N_1\left(\sum_{j=0}^{m-1} T_j(t)\right), \qquad T_m(0) = 0.1, \qquad m = 1,2, \ldots$$

$$I_m'(t) = N_2\left(\sum_{j=0}^{m-1} I_j(t)\right), \qquad I_m(0) = 0, \qquad m = 1,2, \ldots$$

(49) $$V_m'(t) = N_3\left(\sum_{j=0}^{m-1} V_j(t)\right), \qquad V_m(0) = 0.1, \qquad m = 1,2, \ldots$$

Moreover, while applying BCM, the S_m represents for the following problem

$$T_m = T_0 + N_1\left[\sum_{j=0}^{m-1} T_j(t)\right], \quad m = 1,2, \ldots$$

$$I_m = I_0 + N_2\left[\sum_{j=0}^{m-1} I_j(t)\right], \quad m = 1,2, \dots$$

(50)
$$V_m = V_0 + N_3\left[\sum_{j=0}^{m-1} V_j(t)\right], \quad m = 1,2, \dots$$

As presented in the proof of the convergence of the proposed methods, the terms given by the series $T(t) = \sum_{j=0}^{\infty} w_j(t)$ in (46) we get the convergent conditions by (42).

case 1:

For $T(t)$, we get

$$\delta_0^1 = \frac{\|T_1\|}{\|T_0\|} = 0.24829 \ < 1,$$

$$\delta_1^1 = \frac{\|T_2\|}{\|T_1\|} = \ 0.139041 < 1,$$

$$\delta_2^1 = \frac{\|T_3\|}{\|T_2\|} = \ 0.0957501 < 1,$$

$$\delta_3^1 = \frac{\|T_4\|}{\|T_3\|} = 0.0728296 < 1,$$

$$\delta_4^1 = \frac{\|T_5\|}{\|T_4\|} = 0.0586768 < 1,$$

$$\delta_5^1 = \frac{\|T_6\|}{\|T_5\|} = 0.0491099 < 1.$$

and, for $I(t)$, we have

$$\delta_0^2 = \frac{\|I_1\|}{\|I_0\|} = 0 < 1,$$

$$\delta_1^2 = \frac{\|I_2\|}{\|I_1\|} = 0.0256011 < 1,$$

$$\delta_2^2 = \frac{\|I_3\|}{\|I_2\|} = \ 0.93412 < 1,$$

$$\delta_3^2 = \frac{\|I_4\|}{\|I_3\|} = 0.00692434 < 1,$$

$$\delta_4^2 = \frac{\|I_5\|}{\|I_4\|} = 0.519441 < 1,$$

$$\delta_5^2 = \frac{\|I_6\|}{\|I_5\|} = 0.00323865 < 1.$$

Moreover, for $V(t)$, we get

$$\delta_0^3 = \frac{\|V_1\|}{\|V_0\|} = 0.24 < 1,$$

$$\delta_1^3 = \frac{\|V_2\|}{\|V_1\|} = 0.12002 < 1,$$

$$\delta_2^3 = \frac{\|V_3\|}{\|V_2\|} = 0.0800014 < 1,$$

$$\delta_3^3 = \frac{\|V_4\|}{\|V_3\|} = 0.0600245 < 1,$$

$$\delta_4^3 = \frac{\|V_5\|}{\|V_4\|} = 0.0480004 < 1,$$

$$\delta_5^3 = \frac{\|V_6\|}{\|V_5\|} = 0.0400201 < 1,$$

where, the δ_j^i values for $j \geq 1$ and $0 \leq t \leq 0.1$, are less than 1, so the proposed iterative methods are convergent.

case2:

For $T(t)$, we get
$$\delta_0^1 = \frac{\|T_1\|}{\|T_0\|} = 0.0816371 < 1,$$

$$\delta_1^1 = \frac{\|T_2\|}{\|T_1\|} = 0.0457498 < 1,$$

$$\delta_2^1 = \frac{\|T_3\|}{\|T_2\|} = 0.0314182 < 1,$$

$$\delta_3^1 = \frac{\|T_4\|}{\|T_3\|} = 0.0240797 < 1,$$

$$\delta_4^1 = \frac{\|T_5\|}{\|T_4\|} = 0.0190236 < 1,$$

$$\delta_5^1 = \frac{\|T_6\|}{\|T_5\|} = 0.0167228 < 1.$$

and, for $I(t)$, we have

$$\delta_0^2 = \frac{\|I_1\|}{\|I_0\|} = 0 < 1,$$

$$\delta_1^2 = \frac{\|I_2\|}{\|I_1\|} = 0.103728 < 1,$$

$$\delta_2^2 = \frac{\|I_3\|}{\|I_2\|} = 0.129104 < 1,$$

$$\delta_3^2 = \frac{\|I_4\|}{\|I_3\|} = 0.0565231 < 1,$$

$$\delta_4^2 = \frac{\|I_5\|}{\|I_4\|} = 0.0508092 < 1,$$

$$\delta_5^2 = \frac{\|I_6\|}{\|I_5\|} = 0.0398715 < 1,$$

Finally, for $V(t)$, we get

$$\delta_0^3 = \frac{\|V_1\|}{\|V_0\|} = 0.24 < 1,$$

$$\delta_1^3 = \frac{\|V_2\|}{\|V_1\|} = 0.12002 < 1,$$

$$\delta_2^3 = \frac{\|V_3\|}{\|V_2\|} = 0.0800106 < 1,$$

$$\delta_3^3 = \frac{\|V_4\|}{\|V_3\|} = 0.0600132 < 1,$$

$$\delta_4^3 = \frac{\|V_5\|}{\|V_4\|} = 0.0480096 < 1,$$

$$\delta_5^3 = \frac{\|V_6\|}{\|V_5\|} = 0.0400084 < 1,$$

Here, δ_j^i values are also less than 1, for $j \geq 1$ and $0 \leq t \leq 0.1$, for three methods (since the approximate solutions are the same). Hence, the proposed methods are converges.

7. CONCLUSION

In this paper, three iterative methods namely the DJM, TAM and BCM were used to solve a model of HIV for $CD4^+T$ cells. The approximate solutions were obtained in a converged series. It is showed that when the number of iterations increased, the remaining maximum error values are decreased. The results obtained using the Runge-Kutta (RK4) method were compared with the numerical results obtained by the proposed methods, VIM and ADM, good agreements have been achieved. Thus, the proposed methods produced accurate, reliable, effective results and can be applied to the other types of linear and nonlinear problems.

APPENDIX A: MATHEMATICA [CODE FOR A PPLYING THE RK4 FOR THE Eqs. (1) AND (2) FOR CASE (1)].

```
ClearAll["Global`*"]
λ=0.5;α=0.02;β=0.3;r=3;γ=2.4;k=0.0027;T_max =1500;μ=10;
ClassicalRungeKuttaCoefficients[4,prec_]:=With[{amat={{1/2},{0,1/2},{0,0,1}},bvec={1/6,1/3
,1/3,1/6},cvec={1/2,1/2,1}},N[{amat,bvec,cvec},prec]]
{Tf,Iif,Vf}={T,Ii,V}/.First@NDSolve[{T'[t]==λ-α*T[t]+r*T[t](1-(T[t]+Ii[t])/Tmax)-
k*V[t]*T[t],Ii'[t]==k*V[t]*T[t]-β*Ii[t],V'[t]== n*β*Ii[t]-
γ*V[t],T[0]==V[0]==0.1,Ii[0]==0},{T,Ii,V},{t,0,0.1},Method-
>{"ExplicitRungeKutta","DifferenceOrder"->4,"Coefficients"-
>ClassicalRungeKuttaCoefficients},StartingStepSize->0.01];
xl=MapThread[Append,{Tf["Grid"],Tf["ValuesOnGrid"]}]
yl=MapThread[Append,{Iif["Grid"],Iif["ValuesOnGrid"]}]
zl=MapThread[Append,{Vf["Grid"],Vf["ValuesOnGrid"]}]
```

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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