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## **DYNAMICAL ANALYSIS OF A PREDATOR-PREY EPIDEMIOLOGICAL MODEL WITH DENSITY DEPENDENT DISEASE RECOVERY**

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**Abstract.** An epidemiological predator-prey model with a predating scavenger species is proposed and analysed. The intermediate predator community is assumed to have a disease and is classified into infected and susceptible. The recovery of infected predators into susceptible predator is considered to be density-dependent. The role of the crowding factor of the predator population is discussed in the case of all the equilibrium points. The stability analysis for the positive equilibrium is done with the help of Routh–Hurwitz criteria. It is observed that increasing the crowding factor of the predator population promotes the stability of the positive equilibrium. A Period doubling cascade is observed for the increasing mortality rate of scavenger species. The variation of stocks of all the species is observed when mortality rates are increased. A positive effect on the biomass of the scavenger species occurs when scavenger species are removed, culled, or harvested. Finally, the proposed model is modified into a harvesting model by ignoring the mortality rate of susceptible predator and scavenger. The associated control problem has been analyzed for optimal harvesting with help of Pontryagin’s maximum principle.

**Keywords:** epidemiological model; density dependent recovery; stability and period doubling cascade; hydra effect; harvesting and optimal control.

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## 1. INTRODUCTION

Chaotic dynamics in tri-trophic food web continuous-time ecological model was first observed by Hasting and Powel [1]. Although the functional responses of both the predator species were nonlinear and saturating [2, 3], they considered a simple linear food chain and neglected the omnivorous nature of the predators. This aspect was considered by [4] where the authors numerically demonstrated the appearance of Hopf bifurcation and period-doubling cascades while considering a tri-trophic food web with an omnivore. Following [4], Previtte and Hoffman [5] introduced a third scavenger species to the classical predator-prey system in a biologically feasible way showing off-on-off chaos having infinitely many bounded paired cascades and at most finitely many unbounded cascades. After that, in recent years, many researchers investigated the tri-trophic scenario with additional effects like harvesting [6, 7, 8], the presence of small immigrants in all the species [9], etc. Another important effect that is considered in many ecological modeling is the ‘Hydra effect’. Recent theoretical [10, 11, 12] and empirical [13] research have shown that increasing mortality can induce a positive impact on the stock of the same species. This paradoxical result is coined as ‘hydra effect’ by Abrams and Matsuda [14, 15]. Cortez and Abrams (2016) [16] investigated different predator-prey and food chain systems where the ‘Hydra effect’ was present. Recently Pal et al. (2019) [18] ensures the existence of ‘Hydra effect’ in Rosenzweig-Mac Arthur type food chain. They considered a trophic level of more than four and displayed a complete scenario for stock variations in a table for harvesting individual trophic levels.

Many articles can be found describing the symptoms, diagnosis, treatment, and prevention of bacterial and parasitic diseases in aquaculture [23, 24, 25]. Most of the diseases are infectious in nature, and this aspect has drawn considerable attention from the researchers in recent years. Therefore the effect of disease in the ecological system is a significant issue from a mathematical and ecological perspective. After the pioneering work of Kermack and Mc Kendrick, epidemiological models of SIRS type have received much attention from scientists [26]. Many researchers have merged the scenario of epidemiology in the prey-predator system in a different biologically feasible way [Anderson and May(1986) [27]; Haderler and Freedman(1989) [28];

Venturino(1995) [29]; Chattopadhyay and Arino(1999) [30]]. Density-dependent demographics in epidemiological models were first studied by Hethcote and Gao [31, 32].

However, it can not be ruled out that some infectious parasitic and bacterial diseases also occur due to stressed environments like temperature fluctuation [21]. In some situations, the presence of microbes in the environment is constant and the infection spreads unrestricted only when weakness occurs in fish due to a stressed environment (in the form of high population density) [34]. Moreover, there are various non-infectious diseases caused due to stressed environments, for example, Gas bubble disease and disease caused by pollutants like carcinogenic polycyclic aromatic hydrocarbon (PAH) 7, 12-dimethylbenzanthracene (DMBA), Bayluscide, etc. [21]. Agricultural run-offs, including pesticides, fertilizer residues, etc., decomposition of biological waste also increases the ammonia level in the water resulting in ammonia poisoning/toxicity [21, 22]. Such water pollutants have a very harmful impact on Tilapias. It is observed that stress in the form of increasing density worsens ammonia toxicity in rainbow trout [21]. Though there are several theoretical ways to reduce ammonia concentration but for large ponds used in commercial aquaculture, most of the approaches seem to be impractical [33]. Due to the high density of the fish population, the recovery rate becomes slow as high density increases the stress, resulting in ammonia poisoning. Due to inexperienced farmers, sometimes it is impossible to detect the problem, which ultimately results in a slow recovery due to high population density. Such problems were discussed in [24]. Although sometimes removal of susceptible to an infection-free region can effectively be achieved [34], in general, either for large ponds used in commercial aquaculture or due to lack of experiences of farmers, such techniques seem to be impractical [24, 33].

In this paper, the authors consider a Shrimp-Tilapia polyculture in a stressed environment in the form of high density with a pre-existed non-infectious disease viz. ammonia toxicity for which the newborn predator, say, Tilapia offspring are harmed directly. Shrimp is often a choice as a primary or secondary species in polyculture due to its high adaptability in food habits. It can be omnivorous scavengers, detritus feeders, and predators [19, 20]. Similarly, Tilapia is also one of the favorable options in polyculture due to its high growth factor. Shrimps are introduced as scavengers which are assumed to be unaffected by the disease. Recovery of

the infected predator population is considered to be density-dependent because of the stressed environment.

The paper is organized as follows: In section 2, we propose a continuous-time predator-prey model with a non-infectious disease under density-dependent recovery. All the assumptions considered in developing the model are described therein. In section 3, the biologically feasible equilibrium points and their local stability are analyzed. The role of the density-dependent term is analyzed therein. In Section 4, numerical simulations are done with the help of a hypothetical parameter set which validates our analytical findings. In section 5, a numerical scheme is described which represents the behaviour of scavengers on system dynamics. In section 6, mortality of predators towards system dynamics is analysed numerically. In section 7, the linear mortality is considered as linear harvesting in susceptible predator and scavenger species and the associated control problem is discussed with the help of Pontryagin's Maximum Principle. Section 8 consists of a brief discussion of the outcomes of our proposed model.

## 2. MODEL FORMULATION

In this section, an eco-epidemiological system consisting of prey–predator–scavenger is proposed mathematically with a disease in the second species in the food web. In order to formulate the dynamics of such a real life eco-epidemiological system the following hypothetical assumptions are adopted:

- (1) We consider that predator species have two categories, susceptible and infected.
- (2) Let  $x(t)$ ,  $y(t)$ ,  $z(t)$  and  $w(t)$  represent the densities of the prey, infected predator, susceptible predator and scavenger at time  $t$  respectively. It is assumed that the prey species grows logistically with natural growth rate  $r_1 > 0$  while per capita probability to disease susceptibility is  $r_2 > 0$ . Per capita death rate of the prey is assumed to be  $c_1 > 0$ .
- (3) We assume that only the susceptible predator class take part in predation in our model. Preys are consumed by the susceptible predator according to Holling-type II functional (and numerical) response [36] and by the scavengers in Lotka Volterra type of functional responses. We assume  $\alpha > 0$  as the attack rate of prey species by susceptible predator,  $\beta > 0$  as the conversion coefficient and  $h$  as the half saturation constant. Conversion

coefficient of the scavenger due to predation is considered to be  $d > 0$ . Sustainable energy of the predator together with their growth in terms of offspring has been considered implicitly in the energy conversion factor.

- (4) We have not considered here any horizontal transmission from the same species. The surrounding environment having pathogens, have been considered as responsible for disease transmission.
- (5) In our model, the recovery from infected to susceptible predator is density-dependent term  $by(1 - c_2y)$ , where  $b$  is the maximum per capita recovery rate of the infected predator and  $c_2$  is the (crowding factor). The density dependent recovery is not used till now so far as author's knowledge. However, density dependent conversion in stage structured species is done by Abrams et al. [17]. This is quite practical where recovery is dependent on treatment from outside and there is a chance of density may exceed the carrying capacity of the treatment facility and some has to depend his own capability of recovery resulting in lower recovery as desired. We are assuming that normal recovery is prolonged and  $m_1$  is the mortality due to disease. The death rate is considered as constant.
- (6) The scavenger species scavenges the predator species and itself a predator of the prey species. The population of scavenger benefits from naturally died predator with benefit rate  $e > 0$  while the predation rate towards the prey species is  $p > 0$ .
- (7) Intrinsic death rate of the infected predator population is considered to be  $m_1 > 0$ , while in the absence of the prey species, the susceptible predators decays exponentially with intrinsic death rate  $m_2 > 0$ . The scavenger in the absence of prey species and all other resources of food also decays exponentially with natural death rate  $\mu > 0$ .
- (8) The strength of the intra-specific competition among scavengers is considered to be  $n > 0$ , where the term  $nw^2$  is referred to the crowding effect among scavenger class.

According to the above hypothesis the dynamics of the above food web system can be describe mathematically as follows:

$$(1) \quad \begin{aligned} \frac{dx}{dt} &= r_1x(1 - c_1x) - \frac{\alpha xz}{h+x} - pxw \\ \frac{dy}{dt} &= r_2z - by(1 - c_2y) - m_1y \\ \frac{dz}{dt} &= by(1 - c_2y) + \frac{\beta \alpha xz}{h+x} - m_2z \\ \frac{dw}{dt} &= dxw + ew(y+z) - \mu w - nw^2 \end{aligned}$$

with initial conditions:  $x(t) > 0, y(t) > 0, z(t) > 0, w(t) > 0$ .

**Theorem 2.1.** *All the solutions of the system (1) with the given initial condition, which initiate in  $\mathbf{R}_+^4$  are uniformly bounded.*

*Proof.* Define a function,  $G(t) = \frac{x(t)}{r_1} + y(t) + z(t) + w(t)$  and then by taking the derivative along the solution of system (1), we get

$$\frac{dG}{dt} \leq x(1 - c_1x) - by(1 - c_2y) - m_1y - m_2z - \mu w - nw^2$$

where  $\eta = \min \{m_1, m_2, \mu\}$  then we get

$$\frac{dG}{dt} + \eta G \leq x\left(1 + \frac{\eta}{r_1} - c_1x\right) - by(1 - c_2y) - (m_1 - \eta)y - (m_2 - \eta)z - (\mu - \eta)w - nw^2$$

$$\frac{dG}{dt} + \eta G \leq x\left(1 + \frac{\eta}{r_1} - c_1x\right)$$

$$\frac{dG}{dt} + \eta G \leq \frac{(1 + \frac{\eta}{r_1})^2}{4c_1}$$

$$\frac{dG}{dt} + \eta G \leq K$$

where  $K = \frac{(1 + \frac{\eta}{r_1})^2}{4c_1}$

Now, by using Gronwall lemma,

$$0 < G(t) \leq G(0)e^{-\eta t} - \frac{K}{\eta}(e^{-\eta t} - 1)$$

Thus  $G(t) \leq \frac{K}{\eta}$  as  $t \rightarrow \infty$  that is independent of initial conditions and hence the system (1) is bounded.  $\square$

### 3. DYNAMICAL SCENARIO OF THE MODEL

The dynamical scenario of the model is discussed by studying the evolution of the population trajectory of the system with the help of equilibrium points and their stability. The effect of stress due to crowding which hinders the recovery of the infected predator population is also analysed. We assume  $\frac{dx}{dt} = \Psi_1$ ,  $\frac{dy}{dt} = \Psi_2$ ,  $\frac{dz}{dt} = \Psi_3$ ,  $\frac{dw}{dt} = \Psi_4$  for discussion in equilibrium points and their existences.

**3.1. Equilibrium points.** The ecologically feasible possible equilibria of system (1) are

- (i) The trivial equilibrium  $E_0(0, 0, 0, 0)$  where all populations are extinct, which always exists.
- (ii) The axial equilibrium  $E_1\left(\frac{1}{c_1}, 0, 0, 0\right)$  which always exists without any parameter condition.
- (iii) Predator free equilibrium  $E_2\left(-\frac{\mu(-p)-nr_1}{c_1nr_1+dp}, 0, 0, -\frac{c_1\mu r_1-dr_1}{c_1nr_1+dp}\right)$  which also exists under the sufficient condition  $c_1\mu < d$ .
- (iv) **Positive equilibrium:** We discuss the feasibility of positive equilibrium  $E^*(x^*, y^*, z^*, w^*)$ , where  $w^* = \sigma_1$ ,  $z^* = \sigma_2$ ,  $y^* = \sigma_3$  are obtained by solving nullclines of scavenger, susceptible predator, infected predator respectively and  $x^*$  is obtained from the cubic equation

$$\phi_1 x^3 + \phi_2 x^2 + \phi_3 x + \phi_4 = 0$$

where  $\phi_1 = \sigma_4, \phi_2 = \sigma_5, \phi_3 = \sigma_6, \phi_4 = \sigma_7$  obtained from prey nullcline. Although, it is desired to know population status in parametric form, we bring in the whole tuple

$(x, y, z, w, r_1, c_1, \alpha, h, p, r_2, b, c_2, m_1, \beta, m_2, d, e, \mu, n)$  to explain the existence of positive equilibrium. Let

$$R = \{\hat{x} = (x^*, y^*, z^*, w^*, r_1, c_1, \alpha, h, p, r_2, b, c_2, m_1, \beta, m_2, d, e, \mu, n) : \hat{x} \neq 0, \Psi_i(\hat{x}) = 0, i = 1, 2, 3, 4\}$$

and  $R_+ = \{\hat{x} \in R : \hat{x} > 0\}$ . Let  $I$  be the ordered set of all components of  $\hat{x}$ , which are independent in the model and  $D$  be the corresponding dependent one. Clearly,  $D = (x_1, x_2, x_3, x_4)$  where  $x_1 = x^*, x_2 = y^*, x_3 = z^*, x_4 = w^*$  and the tuple  $I$  consists of remaining components (parameters) of  $\hat{x}$ . For  $x_i = x_i(I)$ ,  $i = 1, 2, 3, 4$  and if  $D > 0$  for a suitable choice of  $I > 0$ , then  $\hat{x} \in R_+$ . To clarify that  $R_+ \neq \emptyset$ , we consider  $B = \{\hat{x} \in R | z^* = x^* + h\}$  and  $B_+ = \{\hat{x} \in B | \hat{x} > 0\} \subseteq R_+$ .

Solving  $\Psi_i(\hat{x}) = 0, i = 1, 2, 4$  one can get  $x^* = \sigma_8, y^* = \sigma_9, w^* = \sigma_{10}$ . Again  $\Psi_3(\hat{x}) = 0$  is solved for  $\beta$ .  $\beta$  so obtained along with  $x^*, y^*, w^*$  results  $\hat{x} \in B$ . We consider this  $\hat{x}, D = (x_1, x_2, x_3, x_4)$  where  $x_1 = x^*, x_2 = y^*, x_3 = \beta, x_4 = w^*$  and the tuple  $I$  contains remaining parameters of the system. We want to find suitable  $I > 0$  such that  $D > 0$ , because if  $D > 0, z^* > 0$  then ultimately  $\hat{x} \in B_+ \subseteq R_+$ .

If  $x_1 > 0$  then  $y^* > \frac{1}{c_2}$  then  $x_3 > 0$ . Interchanging  $y^*$  and  $\mu$  among the sets  $D$  and  $I$ , and solving the equation  $\Psi_2(\hat{x}) = 0$  for  $\mu$ , one can get  $x_2 = \mu = \sigma_{11}$ . Now,  $x_2 > 0$  iff  $n < \frac{\Delta_1}{\Delta_2}$  under  $\Delta_2 < 0$ . Further  $\Delta_1 < 0$  imply that  $0 < n < \frac{\Delta_1}{\Delta_2}$ . Therefore,  $\Delta_1 < 0, \Delta_2 < 0$  together with  $y^* > \frac{1}{c_2}$  are implied by  $\sigma_{12}$  which is a sufficient condition for existence of the positive equilibrium.

Here,

$$\begin{aligned} \sigma_1 &= \frac{dx + ey + ez - \mu}{n} \\ \sigma_2 &= \frac{by(c_2y - 1)(h + x)}{-hm_2 - m_2x + \alpha\beta x} \\ \sigma_3 &= \frac{b(-m_2(h + x) + r_2(h + x) + \alpha\beta x) + m_1(\alpha\beta x - m_2(h + x))}{bc_2(-m_2(h + x) + r_2(h + x) + \alpha\beta x)} \\ \sigma_4 &= -bc_2(\alpha\beta - m_2 + r_2)^2(c_1nr_1 + dp) \\ \sigma_5 &= bc_2(\alpha\beta - m_2 + r_2)(m_2(nr_1(2c_1h - 1) + p(2dh - \mu)) + nr_1(\alpha\beta + r_2(1 - 2c_1h))) \\ &\quad + p(\alpha\beta\mu + r_2(\mu - 2dh)) - ep(\alpha\beta + m_1 - m_2 + r_2)(b(\alpha\beta - m_2 + r_2) + m_1(\alpha\beta - m_2)) \\ \sigma_6 &= -bh(m_2 - r_2)(c_2(m_2(nr_1(c_1h - 2) + p(dh - 2\mu)) + nr_1(2\alpha\beta + r_2(2 - c_1h))) \\ &\quad + p(2\alpha\beta\mu + r_2(2\mu - dh)) - 2ep(\alpha\beta - m_2 + r_2) - m_1(-m_2(2behp + \alpha bn + 2\alpha\beta eh p + 2ehpr_2) \\ &\quad + \alpha b\beta(ehp + \alpha n) + r_2(2behp + \alpha bn + \alpha\beta eh p) + 2ehm_2^2p + m_1^2(m_2(2ehp + \alpha n) - \alpha\beta(ehp + \alpha n))) \\ \sigma_7 &= h(-bh(m_2 - r_2)^2(ep - c_2(nr_1 + \mu p)) + m_1(m_2 - r_2)(behp + \alpha bn - ehm_2p) + m_2m_1^2(ehp + \alpha n)) \\ \sigma_8 &= \frac{-ehp - epy - \alpha n + nr_1 + \mu p}{c_1nr_1 + dp + ep} \\ \sigma_9 &= \frac{\sqrt{\frac{(b(c_1nr_1 + p(d+e)) + m_1(c_1nr_1 + p(d+e)))^2 - 4bc_2r_2(c_1nr_1 + p(d+e))(r_1(c_1hn + n) + dhp - \alpha n + \mu p)}{(c_1nr_1 + p(d+e))^2}}}{2bc_2} + \\ &\quad \frac{b + \frac{epr_2}{c_1nr_1 + p(d+e)} + m_1}{2bc_2} \\ \sigma_{10} &= -\frac{\alpha + c_1r_1x - r_1}{p} \\ \sigma_{11} &= \frac{-bc_2y^2(c_1nr_1 + p(d+e)) + by(c_1nr_1 + p(d+e)) - m_2(nr_1(c_1h + 1) + p(dh - ey + \mu)) + \alpha(-n)}{\alpha(ep(h+y) + \alpha n - nr_1 - \mu p)} \end{aligned}$$



$$\begin{aligned} \sigma_{12} &= r_1 > 0 \wedge 0 < \alpha < r_1 \wedge c_2 > 0 \wedge y > \frac{1}{c_2} \wedge m_1 > 0 \wedge 0 < b < \frac{m_1}{c_2 y - 1} \wedge r_2 > 0 \wedge 0 < h < \\ &\frac{-bc_2 y^2 + by + m_1 y}{r_2} \wedge e > 0 \wedge d > 0 \wedge 0 < c_1 < \frac{\alpha r_2 - r_1 r_2}{bc_2 r_1 y^2 - br_1 y + hr_1 r_2 - m_1 r_1 y} \wedge p > 0 \wedge 0 < n \\ &< \frac{-bc_2 d p y^2 - bc_2 e p y^2 + b d p y + b e p y - d h p r_2 + d m_1 p y + e m_1 p y + e p r_2 y}{bc_1 c_2 r_1 y^2 - bc_1 r_1 y + c_1 h r_1 r_2 - c_1 m_1 r_1 y - \alpha r_2 + r_1 r_2} \\ \Delta_1 &= -bc_2 d p y^2 - bc_2 e p y^2 + b d p y + b e p y - d h p r_2 + d m_1 p y + e m_1 p y + e p r_2 y \\ \Delta_2 &= bc_1 c_2 r_1 y^2 - bc_1 r_1 y + c_1 h r_1 r_2 - c_1 m_1 r_1 y - \alpha r_2 + r_1 r_2 \end{aligned}$$

When  $c_2 = 0$ , the system in the immature form i.e., the infected population is not too high to bring the crowding stress effect  $c_2$ . Taking  $c_2 = 0$ , we get the positive equilibrium  $E^* (x^*, y^*, z^*, w^*)$  where  $z^* = \frac{y^*(b+m_1)}{r_2}$  and  $w^* = -\frac{\alpha b y^* + r_1 r_2 (c_1 x^* - 1)(h+x^*) + \alpha m_1 y^*}{p r_2 (h+x^*)}$

We determine  $x^*$  from the nullcline of the adult predator species. Thus,  $x^*$  is a positive solution of the equation:

$$(2) \quad y^* \left( \frac{(b+m_1)(\alpha\beta x^* - m_2(h+x^*))}{r_2(h+x^*)} + b \right) = 0$$

which is found to be  $x^* = -\frac{h(bm_2 - br_2 + m_1 m_2)}{\alpha(-b)\beta + bm_2 - br_2 - \alpha\beta m_1 + m_1 m_2}$

provided that  $\alpha b \beta + \alpha \beta m_1 > m_2 (b + m_1)$

After computing  $x^*$ , we determine  $y^*$  from the nulcline of the scavenger species. Thus  $y^*$  is a positive solution of the equation

$$(3) \quad \frac{(\alpha(-b)y + r_1 r_2 (h+x^*) - \alpha m_1 y)(by(ep(h+x^*) + \alpha n) + r_2(h+x^*)(p(dx^* + ey - \mu) - nr_1) + m_1 y(ep(h+x^*) + \alpha n))}{p^2 r_2^2 (h+x^*)^2} = 0$$

which is found to be

$$y^* = \frac{\beta h r_2 (b(-m_2(p(dh+\mu) + nr_1) + p(\alpha\beta\mu + r_2(dh+\mu)) + nr_1(\alpha\beta + r_2)) + m_1(\alpha\beta(nr_1 + \mu p) - m_2(p(dh+\mu) + nr_1)))}{(b(\alpha\beta - m_2 + r_2) + m_1(\alpha\beta - m_2))(b\beta e h p - b m_2 n + \alpha b \beta n + b n r_2 + m_1(\beta e h p - m_2 n + \alpha \beta n) + \beta e h p r_2)}$$

Substituting  $x^*$  in  $w^*$ ,

$$w^* = \frac{b m_2 y^* - b r_2 y^* + \alpha(-b)\beta y^* + \beta h r_1 r_2 + m_1 y^* (m_2 - \alpha\beta)}{\beta h p r_2}$$

$w^*$  is positive for the necessary condition  $h\beta r_1 - b y^* > 0$ .

$x^*, y^*, z^*, w^*$  are positive under the necessary condition,

$$(4) \quad \frac{m_1(\alpha\beta(nr_1 + \mu p) - m_2(p(dh + \mu) + nr_1))}{m_2(p(dh + \mu) + nr_1) - p(\alpha\beta\mu + r_2(dh + \mu)) - nr_1(\alpha\beta + r_2)} < b < -\frac{m_1 m_2}{m_2 - r_2}$$

(v) **The Scavenger free equilibrium:**

Case (a):  $c_2 \neq 0$

We discuss the feasibility of scavenger free equilibrium  $E_3(\bar{x}, \bar{y}, \bar{z}, 0)$ . If

$$R = \{\hat{x} = (\bar{x}, \bar{y}, \bar{z}, 0, r_1, c_1, \alpha, h, p, r_2, b, c_2, m_1, \beta, m_2, d, e, \mu, n) : \hat{x} \neq 0, \Psi_i(\hat{x}) = 0, i = 1, 2, 3, 4\}$$

Then  $R_+ = \{\hat{x} \in R : \hat{x} > 0\}$  is the set of scavenger free equilibrium points.

If  $\hat{x} \in R$  then  $\bar{z} = t_1$ ,  $\bar{y} = t_2$  and  $\bar{x}$  can be solved from the equation

$$(5) \quad \phi_1 x^3 + \phi_2 x^2 + \phi_3 x + \phi_4 = 0$$

where  $\phi_1 = t_3$ ,  $\phi_2 = t_4$ ,  $\phi_3 = t_5$ ,  $\phi_4 = t_6$ .

Now,  $\hat{x} \in R_+$  iff  $D = (x_1, x_2, x_3) > 0$  where  $x_1 = \bar{x}$ ,  $x_2 = \bar{y}$ ,  $x_3 = \bar{z}$  for a suitable choice of the tuple  $I > 0$  consisting of remaining components of  $\hat{x}$ . By solving equation(5) for  $\beta$ , and interchanging  $x, \beta$  between  $D$  and  $I$ , implies  $x_1 = \beta$ . Clearly  $D > 0$  iff

$$(i) \quad 0 < c_1 < \frac{1}{x}, \frac{\alpha\beta x}{h+x} + r_2 > m_2, 0 < \alpha < \frac{m_2(h+x)}{\beta x}, 0 < r_1 < -\frac{\alpha m_1}{c_2(c_1 x - 1)(-m_2(h+x) + r_2(h+x) + \alpha\beta x)}$$

or

$$(ii) \quad 0 < c_1 < \frac{1}{x}, \alpha > \frac{m_2(h+x)}{\beta x}, r_1 > -\frac{\alpha m_1}{c_2(c_1 x - 1)(-m_2(h+x) + r_2(h+x) + \alpha\beta x)}$$

In a similar procedure like positive equilibrium, we consider

$$B = \{\hat{x} \in R | \bar{z} = \bar{x} + h\} \text{ and } B_+ = \{\bar{x} \in B | \hat{x} > 0\} \subseteq R_+$$

If  $\hat{x} \in B$ , then  $\bar{x} = t_7$ ,  $\bar{y} = t_8$  obtained from  $\Psi_i(\hat{x}) = 0$ ,  $i = 1, 2$  and  $\beta = t_9$ , obtained from  $\Psi_3(\hat{x}) = 0$ .

We find a condition for  $\hat{x} \in B_+$ . Here  $D = (x_1, x_2, x_3)$ ,  $x_1 = \bar{x}$ ,  $x_2 = \bar{y}$ ,  $x_3 = \beta$ .

Further interchanging  $\bar{y}$  and  $m_1$  among  $D$  and  $I$  by solving  $\Psi_2(\hat{x}) = 0$  for  $m_2$ ,

we get  $x_2 = m_1 = t_{10}$ . It can be easily checked that  $D > 0$  iff  $t_{11}$  holds.

$$t_1 = \frac{by(c_2 y - 1)(h+x)}{-hm_2 - m_2 x + \alpha\beta x}$$

$$t_2 = \frac{-bhm_2 + bhr_2 - bm_2 x + br_2 x + \alpha b\beta x - hm_1 m_2 + \alpha\beta m_1 x - m_1 m_2 x}{bc_2(-hm_2 + hr_2 - m_2 x + r_2 x + \alpha\beta x)}$$

$$t_3 = -bc_1 c_2 r_1 (\alpha\beta - m_2 + r_2)^2$$

$$t_4 = bc_2 r_1 (\alpha\beta - m_2 + r_2) (\alpha\beta + m_2 (2c_1 h - 1) + r_2 (1 - 2c_1 h))$$

$$t_5 = -bc_2 hr_1 (m_2 - r_2) (2\alpha\beta + m_2 (c_1 h - 2) + r_2 (2 - c_1 h)) - \alpha b m_1 (\alpha\beta - m_2 + r_2) + \alpha m_1^2 (m_2 - \alpha\beta)$$

$$t_6 = h (bc_2 hr_1 (m_2 - r_2)^2 + \alpha b m_1 (m_2 - r_2) + \alpha m_2 m_1^2)$$

$$t_7 = \frac{r_1 - \alpha}{c_1 r_1}$$

$$t_8 = \frac{\sqrt{(-b-m_1)^2 - 4bc_2 \left( -\frac{\alpha r_2}{c_1 r_1} + \frac{r_2}{c_1} + hr_2 \right)} + b + m_1}{2bc_2}$$

$$t_9 = \frac{bc_1 r_1 y (1 - c_2 y) + m_2 (\alpha - r_1 (c_1 h + 1))}{\alpha (\alpha - r_1)}$$

$$t_{10} = \frac{bc_1 c_2 r_1 y^2 - bc_1 r_1 y + c_1 hr_1 r_2 - \alpha r_2 + r_1 r_2}{c_1 r_1 y}$$

$$t_{11} = y \geq \frac{1}{c_2} \text{ or } 0 < y < \frac{1}{c_2}, 0 < b < \frac{-c_1 hm_2 r_1 + \alpha m_2 - m_2 r_1}{c_1 c_2 r_1 y^2 - c_1 r_1 y}, r_2 > \frac{bc_1 r_1 y - bc_1 c_2 r_1 y^2}{-\alpha + c_1 hr_1 + r_1}$$

Case (b):  $c_2 = 0$

In the immature form of the system, the infected population is not too high to bring the crowding stress effect  $c_2$ . Taking  $c_2 = 0$ , we get the scavenger free equilibrium  $E_3 (x_3, y_3, z_3, 0)$

where

$$x_3 = -\frac{h(m_2(b+m_1)-br_2)}{m_1(m_2-\alpha\beta)-b(\alpha\beta-m_2+r_2)}, y_3 = \frac{\beta hr_1 r_2 (b(\alpha\beta-m_2(c_1 h+1)+c_1 hr_2+r_2)+m_1(\alpha\beta-m_2(c_1 h+1)))}{(b(\alpha\beta-m_2+r_2)+m_1(\alpha\beta-m_2))^2}$$

$$z_3 = \frac{\beta hr_1 (b+m_1)(b(\alpha\beta-m_2(c_1 h+1)+c_1 hr_2+r_2)+m_1(\alpha\beta-m_2(c_1 h+1)))}{(b(\alpha\beta-m_2+r_2)+m_1(\alpha\beta-m_2))^2}$$

which exists for  $r_2 > m_2$  and  $\frac{c_1 hm_2 m_1 - \alpha \beta m_1 + m_2 m_1}{\alpha \beta - c_1 hm_2 + c_1 hr_2 - m_2 + r_2} < b < -\frac{m_1 m_2}{m_2 - r_2}$ .

### 3.2. Stability analysis of the equilibria: Jacobian Matrix for the model is

$$\begin{pmatrix} -pw + \frac{xz\alpha}{(h+x)^2} - xc_1 r_1 + (1 - xc_1) r_1 - \frac{z\alpha}{h+x} & 0 & -\frac{x\alpha}{h+x} & -px \\ 0 & byc_2 - b(1 - yc_2) - m_1 & r_2 & 0 \\ \frac{z\alpha\beta}{h+x} - \frac{xz\alpha\beta}{(h+x)^2} & b(1 - yc_2) - byc_2 & \frac{x\alpha\beta}{h+x} - m_2 & 0 \\ dw & ew & ew & -2mw + dx + e(y+z) - \mu \end{pmatrix}$$

#### (i) Stability of Vanishing equilibrium:

Eigen values of the Jacobian matrix at  $E_0$  are:

$$-\mu, r_1, \frac{1}{2}(-b - m_1 - m_2 - \phi_1), \frac{1}{2}(-b - m_1 - m_2 + \phi_1)$$

where,

$$\phi_1 = \sqrt{(b + m_1 + m_2)^2 - 4(bm_2 - br_2 + m_1 m_2)} \text{ and } \phi_1 > 0$$

Clearly one of the eigen value  $r_1 > 0$ , therefore the vanishing equilibrium  $E_0$  is a saddle point.

#### (ii) Stability of the axial equilibrium:

Eigen values of the stability matrix at  $E_1$  are  $\frac{d-c_1\mu}{c_1}$ ,  $-r_1$  and

$$\frac{1}{2c_1(c_1 h + 1)} \left( -bc_1^2 h - bc_1 + \alpha\beta c_1 - c_1^2 hm_1 - c_1^2 hm_2 - c_1 m_1 - c_1 m_2 \pm \sqrt{\bar{\omega}} \right)$$

where

$$\bar{\omega} = c_1^2 \left( (\alpha\beta + c_1 h (b + m_1 - m_2) + b + m_1 - m_2)^2 + 4br_2 (c_1 h + 1)^2 \right)$$

Since  $\bar{\omega} > 0$  therefore the axial equilibrium is always locally asymptotically unstable.

**(iii) Stability of predator free equilibrium:**

Eigen values of the stability matrix at  $E_2$  are

$$\begin{aligned} & \frac{-(h+x_2)(r_1(2c_1x_2-1)-dx_2+\mu+w_2(2n+p)) \pm \sqrt{\omega_1}(h+x_2)}{2(h+x_2)} \\ & \frac{-bh-bx_2-hm_1-hm_2-m_1x_2-m_2x_2+\alpha\beta x_2 \pm \sqrt{\omega_2}}{2(h+x_2)} \\ \omega_1 &= -2r_1(2c_1x_2-1)(-dx_2+\mu+w_2(2n-p)) + r_1^2(1-2c_1x_2)^2 - \\ & 2w_2(dx_2(2n+p)+\mu(p-2n)) + (\mu-dx_2)^2 + w_2^2(p-2n)^2 \\ \omega_2 &= (h(b+m_1+m_2)+x_2(-\alpha\beta+b+m_1+m_2))^2 - 4(h+x_2)(m_2(b+m_1)(h+x_2) \\ & -br_2(h+x_2) - \alpha\beta x_2(b+m_1)) \end{aligned}$$

where  $x_2 = -\frac{\mu(-p)-nr_1}{c_1nr_1+dp}$ ,  $w_2 = -\frac{c_1\mu r_1-dr_1}{c_1nr_1+dp}$ . The predator free equilibrium  $E_2$  is locally asymptotically stable under the conditions:  $2r_1c_1x_2 > dx_2 + r_1$ ;  $b > \alpha\beta$  and  $\omega_{1,2} < 0$  along with the positiveness of  $x_2$  and  $w_2$ .

**(iv) Stability of scavenger free equilibrium:**

Jacobian matrix at scavenger free equilibrium  $E_3$  is

$$\begin{pmatrix} \frac{xz\alpha}{(h+x)^2} - \frac{z\alpha}{h+x} - xc_1r_1 + (1-xc_1)r_1 & 0 & -\frac{x\alpha}{h+x} & -px \\ 0 & byc_2 - b(1-yc_2) - m_1 & r_2 & 0 \\ \frac{z\alpha\beta}{h+x} - \frac{xz\alpha\beta}{(h+x)^2} & b(1-yc_2) - byc_2 & \frac{x\alpha\beta}{h+x} - m_2 & 0 \\ 0 & 0 & 0 & dx + e(y+z) - \mu \end{pmatrix}$$

The characteristic polynomial of the Jacobian matrix at  $E_3$  is  $a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$  where one of the eigen values is  $\lambda_1 = dx + e(y+z) - \mu$ . According to Routh Hurwitz criteria  $E_3$  will be locally asymptotically stable iff  $a_1, a_2 > 0$  and  $\Delta = a_3(a_1a_2 - a_0a_3) > 0$ . To show  $\Delta > 0$  it is sufficient to show  $a_3 > 0$ .

Putting  $z = \frac{by(c_2y-1)(h+x)}{-hm_2-m_2x+\alpha\beta x}$ ;  $y = \frac{-bhm_2+bhr_2-bm_2x+br_2x+\alpha\beta x-hm_1m_2+\alpha\beta m_1x-m_1m_2x}{bc_2(-hm_2+hr_2-m_2x+r_2x+\alpha\beta x)}$  in the nullcline of susceptible predator we obtain  $b$ . With the positiveness condition of  $z, y$  and  $b$ ,  $\lambda_1$  is negative for the sufficient conditions:

$$m_2 < \frac{\alpha\beta x}{h+x} + r_2; d < \frac{\mu}{x}; x > 0$$

$$\frac{\alpha m_1(dx - \mu)}{r_1(c_1x - 1)(m_1(h+x) - m_2(h+x) + hr_2 + r_2x + \alpha\beta x)} > e$$

$$c_1 < \frac{1}{x}; \alpha < \frac{m_2(h+x)}{\beta x}$$

$$\frac{\alpha m_1}{r_1(c_1x - 1)(-m_2(h+x) + r_2(h+x) + \alpha\beta x)} + c_2 < 0$$

Now,  $a_1 > 0$  under the sufficient condition,

$$0 < t_6 < 1; t_1 > 0; 0 < t_3 < 1; t_2 > 0; \frac{1}{2} \leq t_4 < 1; x > 0; 0 < t_5 < 1$$

Under the positivity condition of  $a_1 > 0, a_2 > 0$  under the sufficient conditions:

$$r_1 > 0; \frac{m_2(t_5 - 1)(h+x)}{2ht_4 + 4t_4x - 2x} + r_1 < 0; t_1 > 0$$

$$t_6 > 0; t_6 \leq \frac{r_1(x - t_4(h+2x))}{m_2(t_5 - 1)(h+x) + r_1(ht_4 + (2t_4 - 1)x)}$$

$$t_1 > 0; \frac{m_2(t_5 - 1)(h+x)}{2ht_4 + 4t_4x - 2x} + r_1 \geq 0; t_6 > 0; t_6 < 1$$

Now  $\Delta = a_3(a_1a_2 - a_0a_3) > 0$  for the sufficient condition

$$\frac{-ht_4t_6 - 2t_4t_6x + t_6x}{ht_4 - ht_6 - h - 2t_4t_6x + t_6x} \leq t_5 < 1; 0 < t_3 < 1; 0 < t_6 < 1; \frac{1}{2} \leq t_4 < 1; t_1 > 0$$

for  $r_1 = t_7 - \frac{m_2(t_5 - 1)(h+x)}{2ht_4 + 4t_4x - 2x}$  where  $t_7 > 0$ .

(v) **Stability of positive equilibrium:** The Jacobian matrix  $J(A_{ij})$  at  $E^*$  is

$$\begin{pmatrix} -pw^* + (1 - 2x^*c_1)r_1 - \frac{hz^*\alpha}{(h+x^*)^2} & 0 & -\frac{x^*\alpha}{h+x^*} & -px^* \\ 0 & -b - m_1 & r_2 & 0 \\ \frac{hz^*\alpha\beta}{(h+x^*)^2} & b & \frac{x^*\alpha\beta}{h+x^*} - m_2 & 0 \\ dw^* & ew^* & ew^* & -2nw^* + dx^* + e(y^* + z^*) - \mu \end{pmatrix}$$

The characteristic equation corresponding to the Jacobian matrix is

$$\lambda^4 + D_1\lambda^3 + D_2\lambda^2 + D_3\lambda + D_4 = 0$$

Here,

$$\begin{aligned}
D_1 &= \sum_{i=1}^4 A_{ii} \\
D_2 &= \sum_{i=1}^3 \sum_{j=i+1}^4 A_{ii}A_{jj} - \sum_{i=1}^3 \sum_{j=i+1}^4 A_{ij}A_{ji} \\
D_3 &= \sum_{i=1}^4 \sum_{j \neq i}^3 \sum_{k=j+1, k \neq i}^4 A_{ii}A_{jk}A_{kj} - \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{k=j+1}^4 A_{ii}A_{jj}A_{kk} - \sum_{i=1}^2 \sum_{j=i+1}^4 \sum_{k \neq j, k > i}^4 A_{ij}A_{jk}A_{ki}
\end{aligned}$$

According to Routh–Hurwitz criteria, the positive equilibrium is locally asymptotically stable iff

$$D_1, D_3, D_4 > 0 \text{ and } \Delta_2 = D_1 D_2 D_3 - D_3^2 - D_1^2 D_4 > 0$$

#### 4. NUMERICAL SIMULATION

In this section, numerical simulations are conducted to validate the analytical findings of system (1) numerically. For this purpose, we have adopted the following hypothetical parameter values:

$$\begin{aligned}
r_1 &= 1; r_2 = 0.5; p = 0.4; d = 1.5; m_1 = 0.3; h = 1; m_2 = 0.2; \alpha = 0.3; \beta = 0.2; d = 0.3; e = 0.4; \\
c_1 &= 0.04; \mu = 0.2.
\end{aligned}$$

In case of the positive equilibrium  $E^*$  one can numerically verify that  $\lim_{c_2 \rightarrow 0} (x^*, y^*, z^*, w^*) = (\hat{x}, \hat{y}, \hat{z}, \hat{w})$  i.e., decreasing the parameter value  $c_2$  which is actually the crowding factor of predators, the positive equilibrium of system (1) tends to positive equilibrium of the system without crowding factor ( $c_2 = 0$ ).

It can be observed from Figure(1) that as stress value  $c_2$  decreases, the equilibrium value of  $x^*$  and  $w^*$  decreases and the equilibrium value of  $y^*$  and  $z^*$  increases. It is clear that the equilibrium value of  $x^*$  and  $w^*$  becomes minimum at  $c_2 = 0$ , and the equilibrium value of  $y^*$  and  $z^*$  maximum at  $c_2 = 0$ .

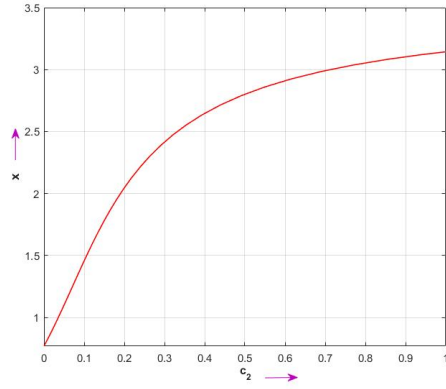
In similar way for scavenger free equilibrium one can numerically verify that  $\lim_{c_2 \rightarrow 0} (\bar{x}, \bar{y}, \bar{z}) = (\hat{x}, \hat{y}, \hat{z})$  It can be observed from Figure(2) that as stress value  $c_2$  decreases, the equilibrium value of  $\bar{x}$  and  $\bar{z}$  decreases and attains a minimum value at  $c_2 = 0$ .

Now, using the parameter values mentioned in the beginning of Section 4, from the threshold condition (4) of existence of the positive equilibrium under  $c_2 = 0$  we obtain

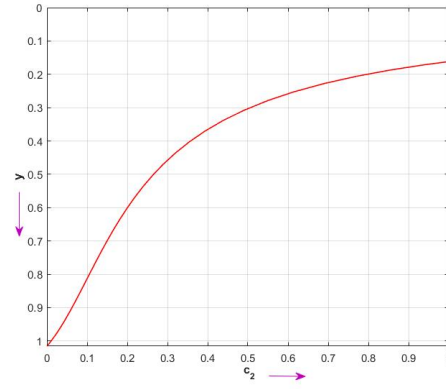
$$0.131034 < b < 0.2$$

Clearly for  $0.131034 < b < 0.2$ ,  $D_1 > 0, D_2 > 0, D_3 > 0$ . All the population persist when  $b \in (0.131034, 0.2)$ .

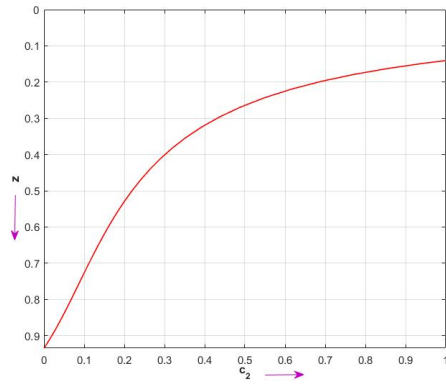
Within this range we compute the values of  $D_4$  and  $D_1 D_2 D_3 - D_3^2 - D_1^2 D_4$ . It is observed that within the range  $D_4$  is always positive but  $(D_1 D_2 D_3 - D_3^2 - D_1^2 D_4)$  changes its stability (Figure3).



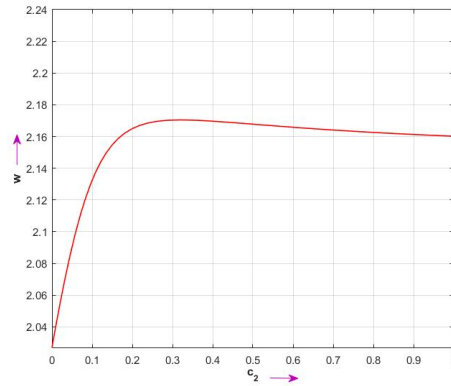
(A)



(B)



(C)



(D)

FIGURE 1. Depicts  $\lim_{c_2 \rightarrow 0}(x^*, y^*, z^*, w^*) = (\hat{x}, \hat{y}, \hat{z}, \hat{w})$ ; (A)  $x^*$  Vs  $c_2$ , (B)  $y^*$  Vs  $c_2$ , (C)  $z^*$  Vs  $c_2$  (D)  $w^*$  Vs  $c_2$

$(D_1 D_2 D_3 - D_3^2 - D_1^2 D_4)$  remains positive within  $b \in (0.131034, 0.188049)$  and become negative otherwise. Thus all the population coexist at stable steady state for all  $b \in (0.131034, 0.188049)$ . In particular, when the  $b = 0.186$  the coexisting equilibrium  $(0.1682, 1.1005, 1.0697, 1.7964)$  is stable and the population experience cyclic dynamics at  $b = 0.193491$ .

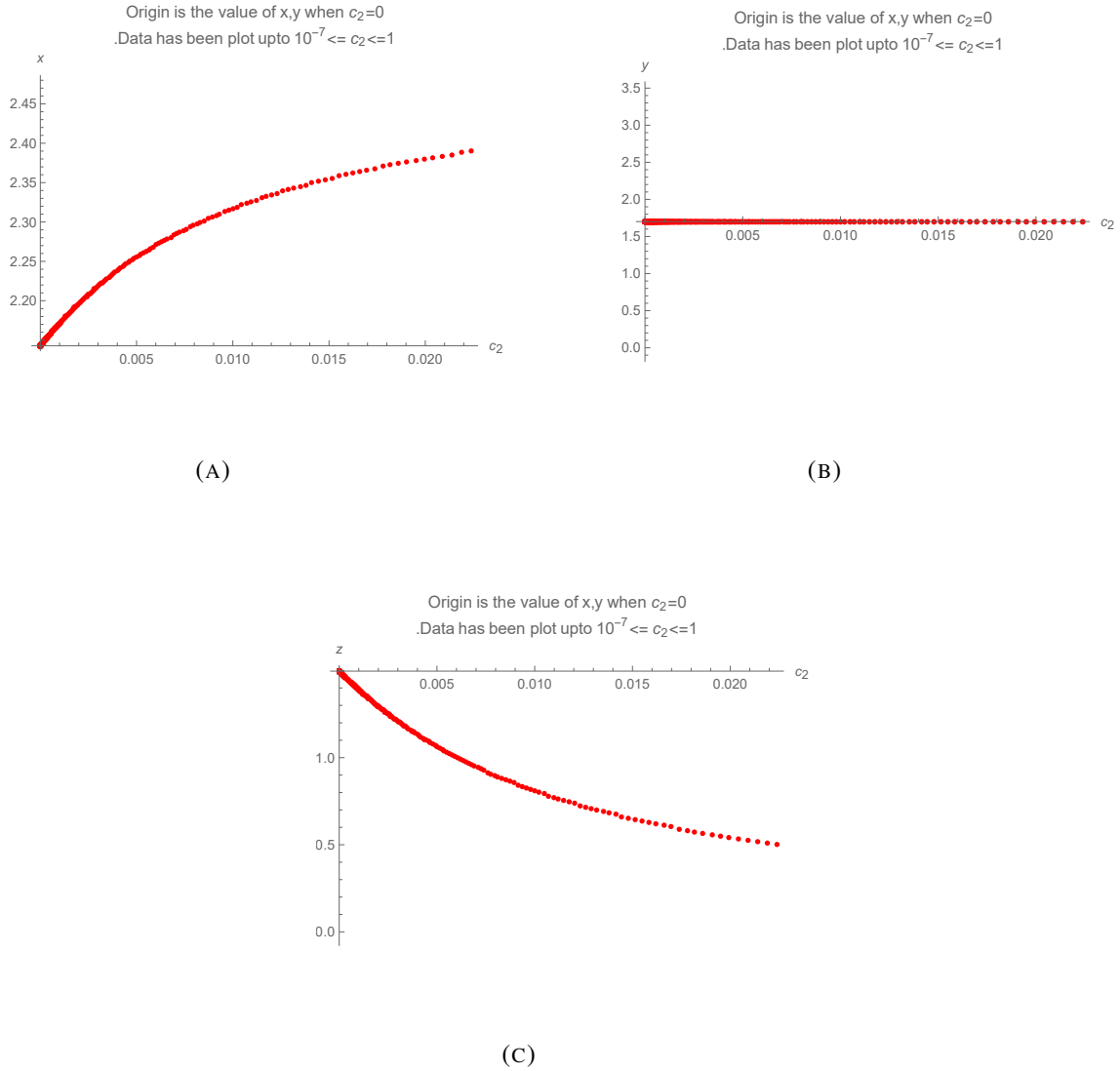


FIGURE 2. (A)  $\bar{x}$  Vs  $c_2$ , (B)  $\bar{y}$  Vs  $c_2$ , (C)  $\bar{z}$  Vs  $c_2$

We fix  $b = 0.2$  along with the other parameter values mentioned in the beginning of Section 4. Clearly, the corresponding equilibrium is unstable (Figure:3). Now, we check the stability nature of the equilibrium of the complex model when  $c_2$  is increased. Clearly,  $D_1, D_2, D_3 > 0$ . All the population coexists for  $c_2 \in [0.007, 1]$  (Figure:4). It is also found that  $D_1, D_2, D_3, D_4$  remains positive within this interval. Moreover, in this interval  $\Delta_2 = (D_1 D_2 D_3 - D_3^2 - D_1^2 D_4) > 0$ . Therefore, we can conclude that increasing crowding factor of the predators has a stabilizing effect on the system dynamics.



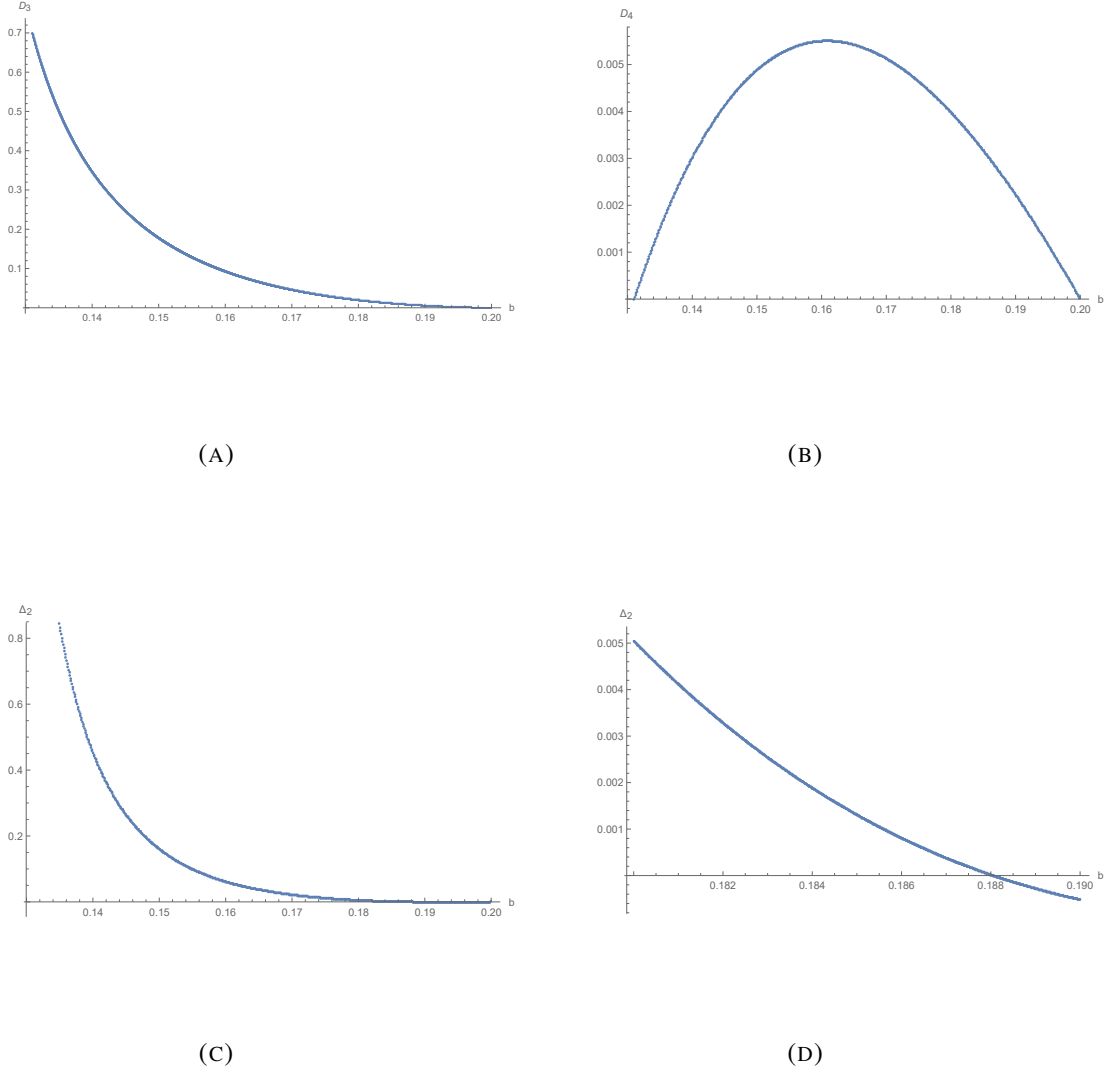


FIGURE 3. (A)  $b$  Vs  $D_3$ ,(B)  $b$  Vs  $D_4$ ,(C)  $b$  Vs  $\Delta_2$  (D)  $b$  Vs  $\Delta_2$ (enlarge)

## 5. BEHAVIOUR OF SCAVENGER

**5.1. Effect of recovery rate and mortality rate of scavengers : a period doubling cascade:** We fix the parameters  $c_2 = 0.06$ ;  $b = 0.21$  keeping the other parameters fixed as mentioned in section 4. With these parameter values all the species coexists with equilibrium  $(0.1569, 1.1005, 1.0714, 1.7897)$ . Increasing the death rate of scavengers i.e.,  $\mu$  (keeping other parameters fixed), all of the species undergoes oscillatory behaviour. All the species enters a cyclic dynamics in the region when  $\mu \in (0.722050, 3.175599)$ . In Figure 5 the points  $H1 = 0.722050, H2 = 3.175599$  are Hopf bifurcation points which are calculated

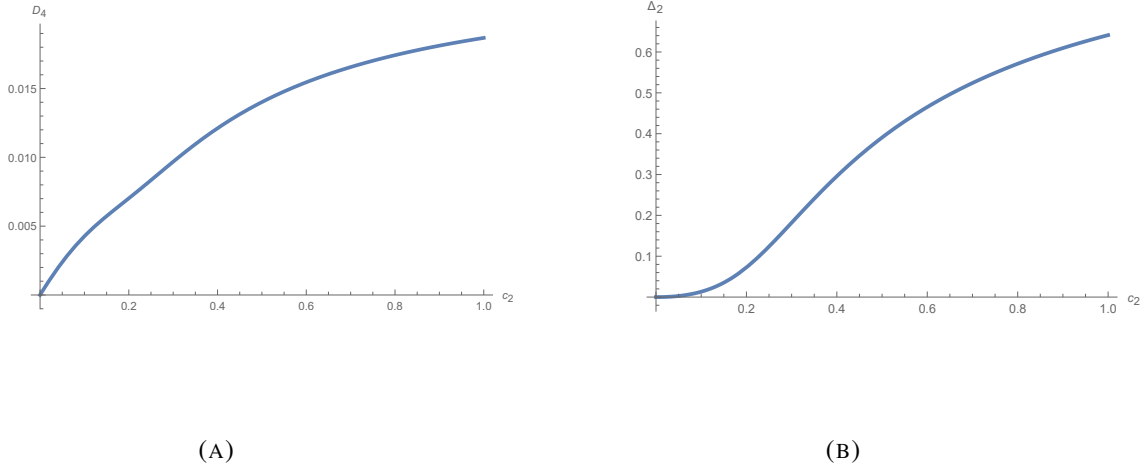


FIGURE 4. (A)  $c_2$  Vs  $D_4$ , (B)  $c_2$  Vs  $\Delta_2$

using MatCont 6.0[39]. Here, our main attention is to examine the effect of recovery rate of infected predator on all of the species. Now setting  $b \approx 0.21$  and varying  $\mu$  (keeping other parameters fixed) all the species undergoes a period doubling bifurcation which results a chaotic dynamics. Figure 6 depicts the coexisting equilibrium undergoes a Hopf bifurcation at  $\mu = 0.5190769$  and  $\mu = 4.078909$  (H1,H2 respectively). Further it undergoes a period doubling bifurcation at  $\mu = 2.552255$  and  $\mu = 3.787953$  (PD1,PD2 respectively) which results a chaotic dynamics. For  $\mu = 0.4$ , the all the population coexists (Figure:7). For  $\mu \in (0.5190769, 2.552255)$  the system undergoes period-2 bifurcation. Figure 8 depicts period-2 bifurcation for  $\mu = 1.5$ . Continuous increase in the magnitude of  $\mu$  results in successive period doubling and ultimately we find chaotic attractor. We find periodic attractor with period-4, period-8 and period-16 for  $\mu \in (2.552255, 2.72545)$ ,  $\mu \in (2.72545, 2.77202)$ ,  $\mu \in (2.77202, 2.78322)$  respectively [Figure (9)-Figure (11)]. Further increase in the magnitude of  $\mu$  results chaotic dynamics which can be seen in Figure (12) and Figure(13) respectively. The whole period doubling cascade for varying  $\mu$  can be observed in Figure(14).

**5.2. Effect of predation rate of scavengers towards prey:** We fix the parameter  $d = 0.17$  and keeping other parameters as mentioned in section 4. It is observed that all the species enters a cyclic dynamics when  $\mu \in (-0.027483, 3.558740)$  (Points H1,H2 in Figure15). Since all the parameter values must be positive so we can consider the region as  $\mu \in (0, 3.558740)$ . Here our main attention is to examine the effect of predation rate of scavengers towards prey and it's natural mortality on the whole system.

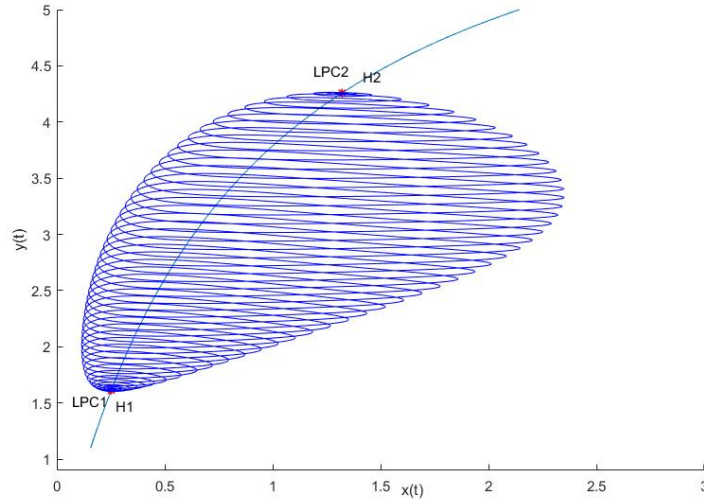


FIGURE 5. Hopf bifurcation with respect to the bifurcation parameter  $\mu$

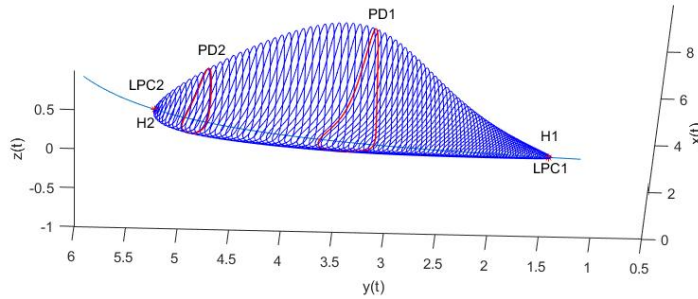
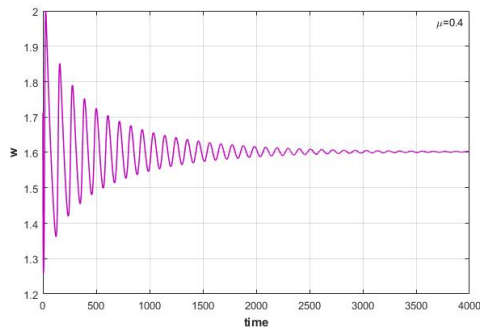


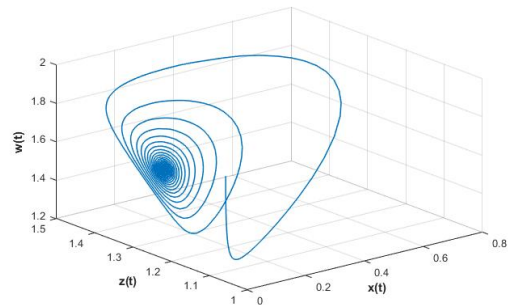
FIGURE 6. Hopf and period doubling bifurcation with respect to the bifurcation parameter  $\mu$  for  $b \approx 0.21$

Now setting  $d = 0.17$  and increasing  $\mu$  (keeping other parameter values fixed) it is observed that at  $\mu = 3.017307$  the system undergoes period doubling bifurcation. Numerical calculations reveals that on further increasing  $\mu$  the system undergoes another period doubling at  $\mu \approx 3.088$ . On further increasing  $\mu$  the system becomes stable again at  $\mu = 3.558740$  (Figure 16).

**5.3. Removing scavenger population.** In this subsection, we examine whether the mortality of scavenger species can result a positive hydra effect. First we check for the system with  $c_2 = 0$ . We fix

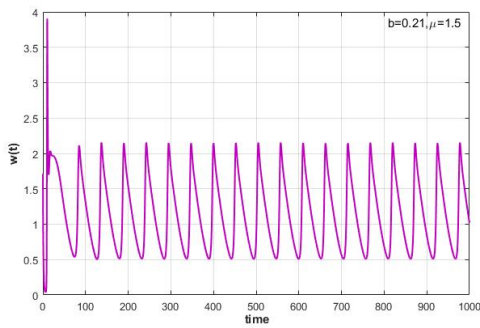


(A)

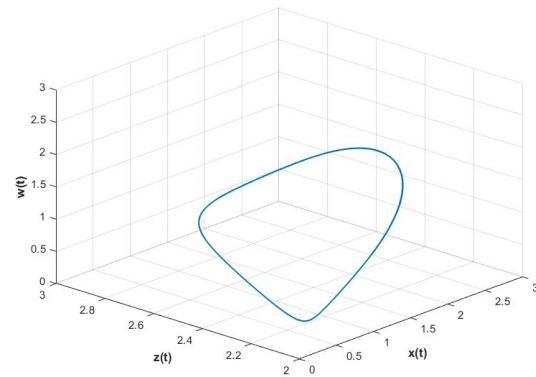


(B)

FIGURE 7. For  $\mu = 0.4$ : (A) Time series for stable solution  $w$ , (B) Stable solution in the positive  $xz$ -octant

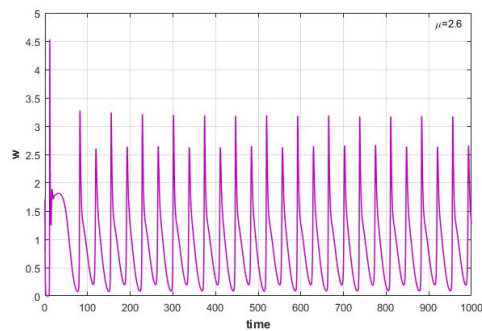


(A)

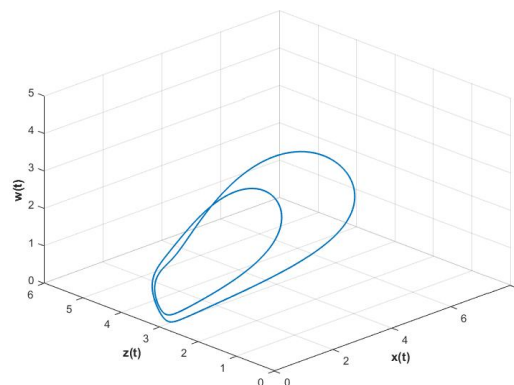


(B)

FIGURE 8. For  $\mu = 1.5$ : (A) Time series for periodic solution  $w$ , (B) Periodic solution in the positive  $xz$ -octant.

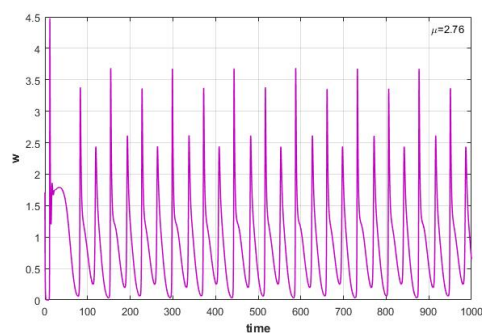


(A)

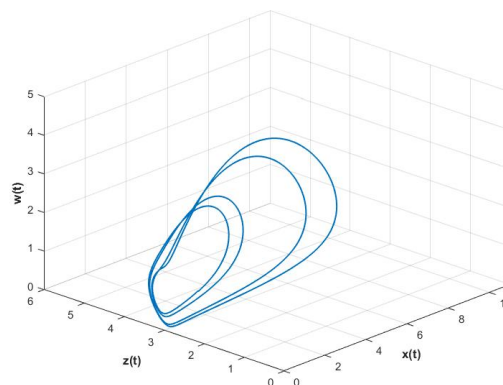


(B)

FIGURE 9. For  $\mu = 2.6$ : (A) Time series for periodic solution  $w$ , (B) Periodic solution in the positive  $xzw$ -octant.



(A)



(B)

FIGURE 10. For  $\mu = 2.76$ : (A) Time series for periodic solution  $w$ , (B) Periodic solution in the positive  $xzw$ -octant.

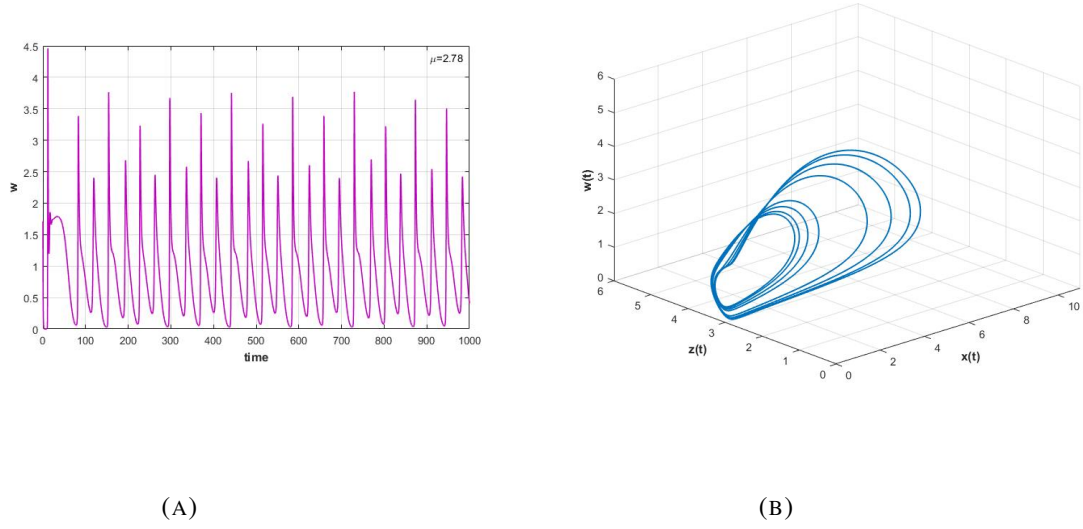


FIGURE 11. For  $\mu = 2.78$ : (A) Time series for periodic solution  $w$ , (B) Periodic solution in the positive  $xzw$ -octant.

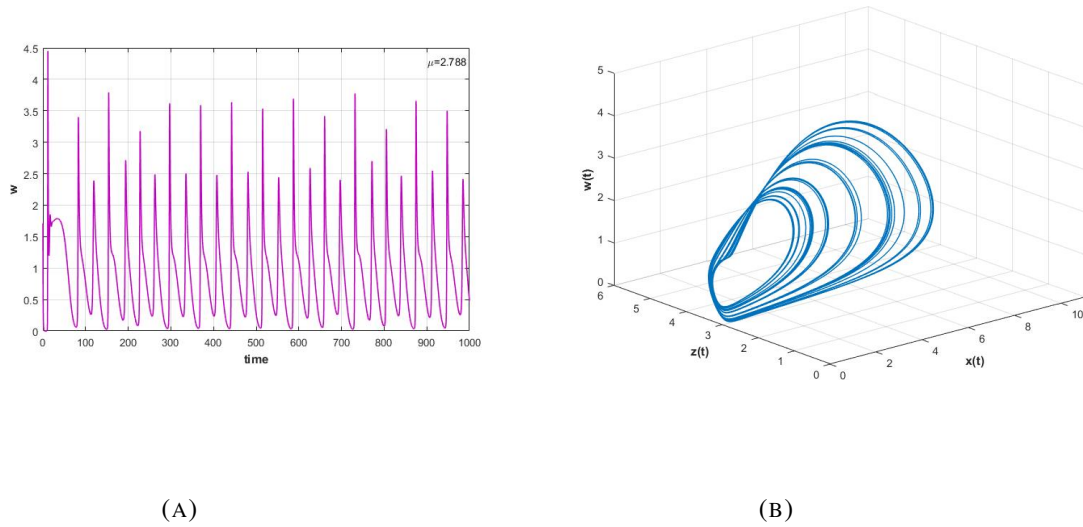


FIGURE 12. For  $\mu = 2.788$ : (A) Time series for periodic solution  $w$ , (B) Periodic solution in the positive  $xzw$ -octant.

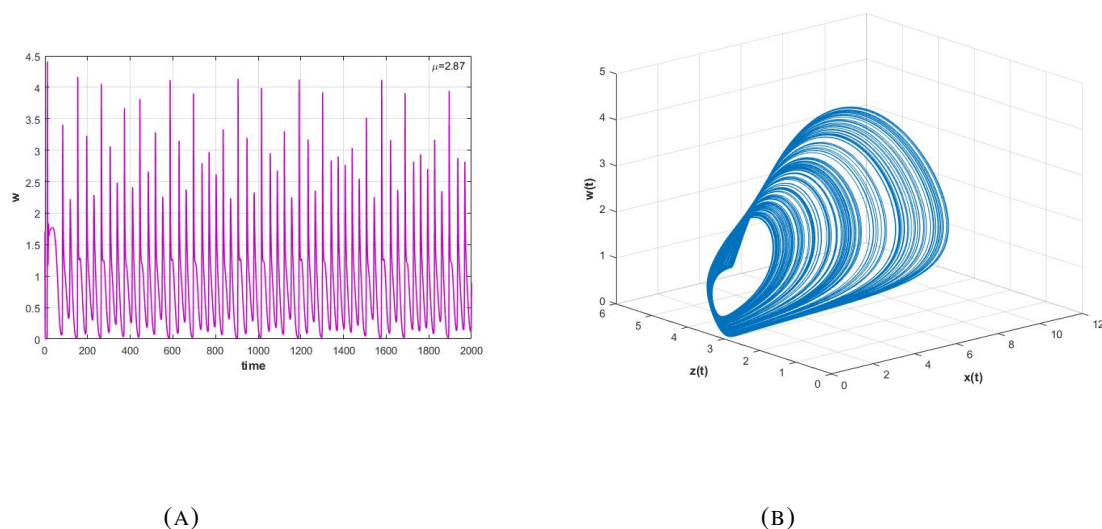


FIGURE 13. For  $\mu = 2.87$ : (A) Time series for periodic solution  $w$ , (B) Periodic solution in the positive  $xzw$ -octant.

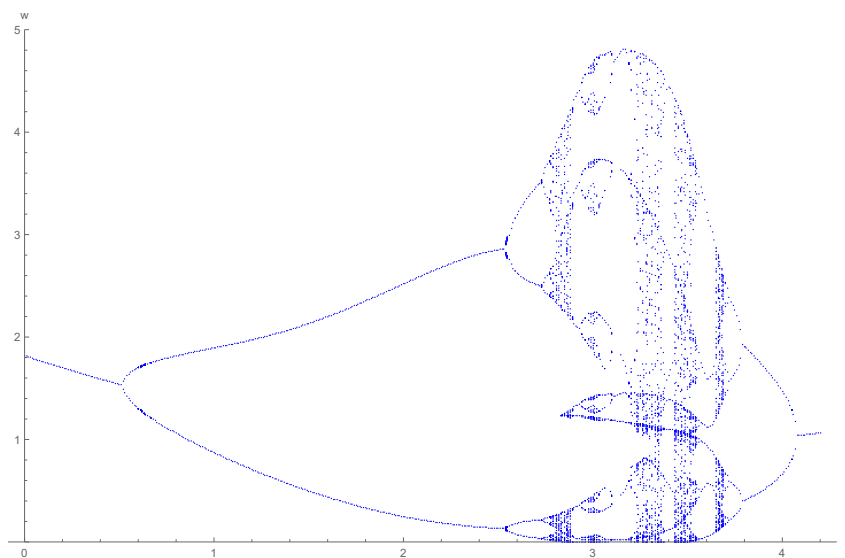


FIGURE 14. Bifurcation diagram with respect to the parameter  $\mu$

$b = 0.186$  along with the other parameters as mentioned in the beginning of section 4 for which all the species coexists for  $\mu \in (0, 0.257135]$ . With these parameter values when mortality rate of scavenger species is increased it is observed that there is no hydra effect appears in scavenger population. Next for the system with  $c_2 \neq 0$  again we set  $b = 0.186$  and  $c_2 = 0.06$  along with other parameter as above. With

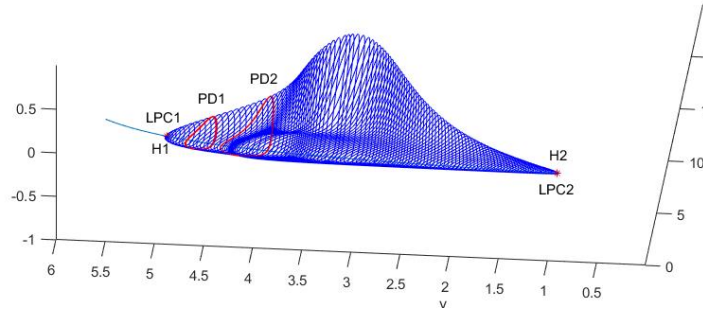


FIGURE 15. Period doubling scenario with respect to the parameter  $\mu$

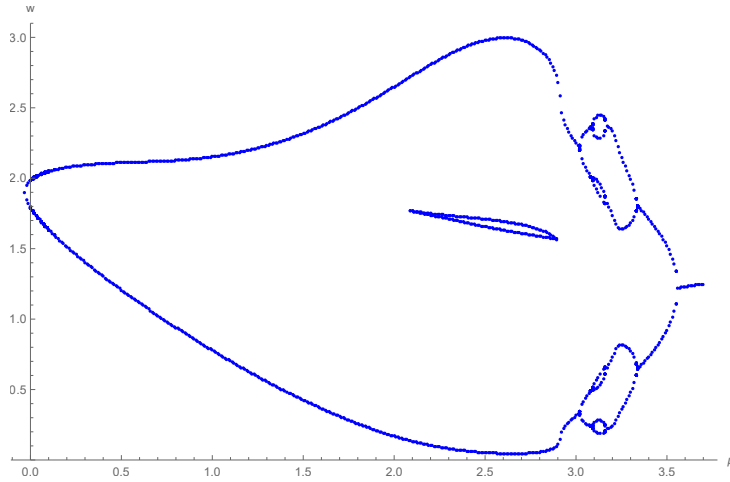


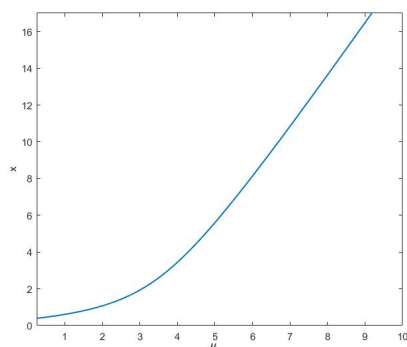
FIGURE 16. Bifurcation diagram with respect to the parameter  $\mu$

these parameter values all the species coexists when  $\mu \in (0, 11.422205)$ . Calculations reveals that there is a possibility of hydra effect when  $\mu \in (2.56, 4.292)$ . On increasing the death rate ( $\mu$ ) from  $\mu = 2.56$  to  $\mu = 4.292$  the biomass of the scavenger species increases which confirms that there is a positive hydra effect (Figure 17).

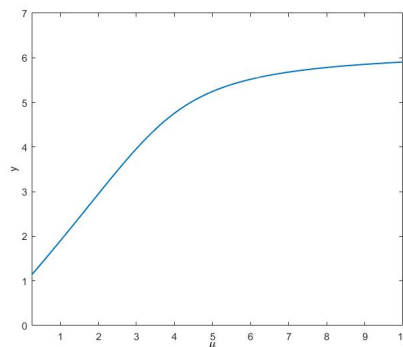
## 6. PREDATOR BEHAVIOUR

**6.1. Impact of mortality of infected predators.** In this subsection, we examine whether the mortality of infected predator species can result a positive hydra effect. For  $b = 0.186$  and  $c_2 = 0$  along with the other parameter values fixed, it is observed that all the populations coexists for  $m_1 \in (0.279, 0.426)$ . It

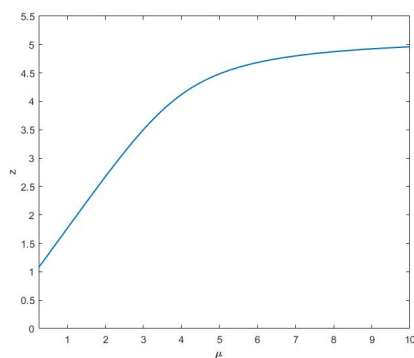




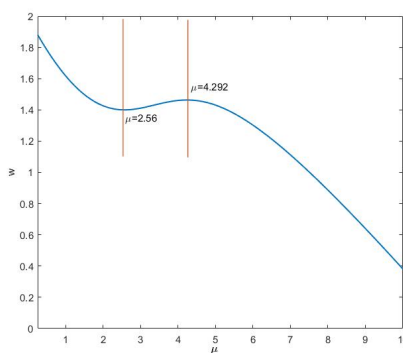
(A)



(B)



(C)



(D)

FIGURE 17. Depicts stable stocks for all the population when  $\mu$  is increased. Biomass of all the population are increased for  $\mu \in (2.56, 4.292)$ , (D) confirms the existence of hydra effect.

can be seen that with increasing mortality of the susceptible predator, biomass both the prey and scavenger population are increased, while biomass of both the predator decreases. There is no increase in the biomass of the infected predator population is observed. Thus no hydra effect appears in infected predator (Figure 18).

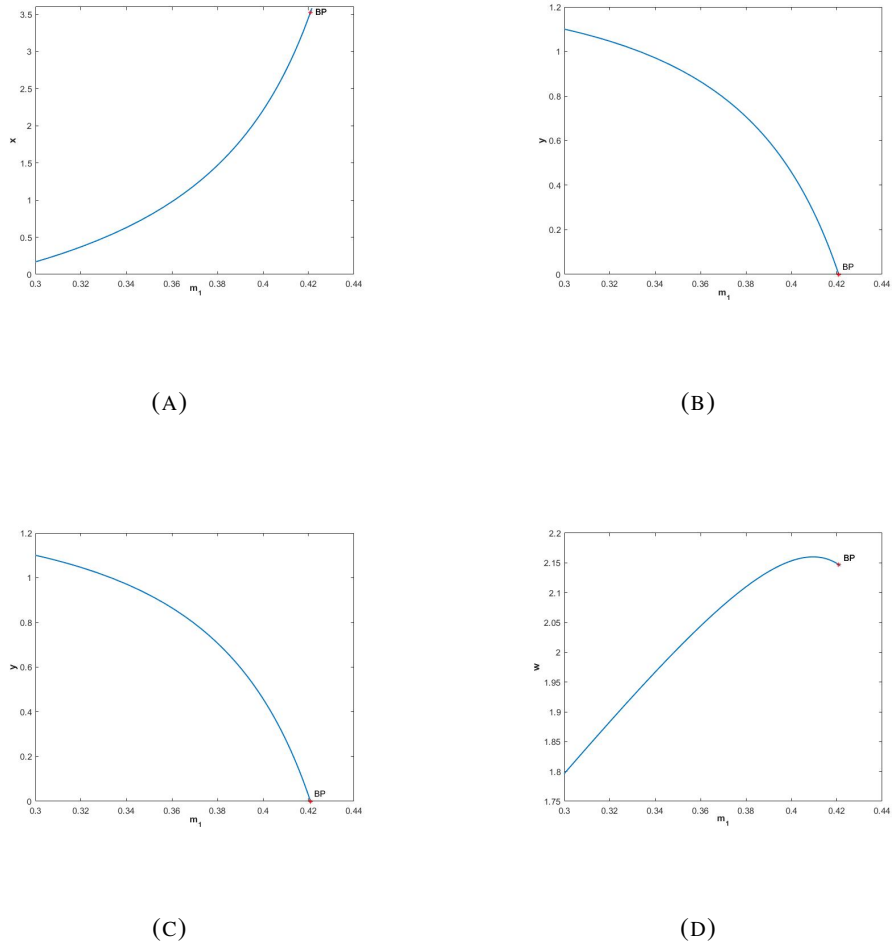
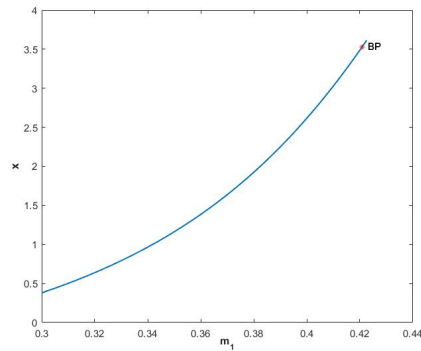


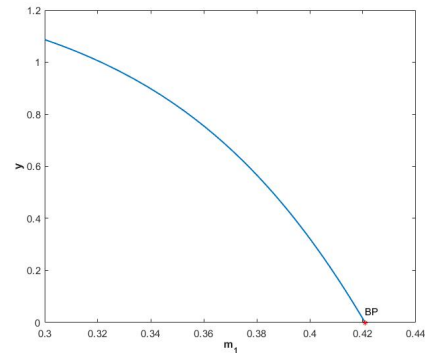
FIGURE 18. Depicts stable stocks for all the population when  $m_1$  is increased. Biomass of all the population are coexists for  $m_1 \in (0.279, 0.426)$ . Hydra effect cannot be appeared at stable biomass on infected predator.

Next, for the system under  $c_2 \neq 0$  we set  $b = 0.186$  and  $c_2 = 0.06$  along with the other parameter values fixed. It is observed that all the populations coexists for  $m_1 \in (0.258, 0.421)$ . It can be seen that with the increasing mortality rate of susceptible predator, biomass of both the prey and scavenger population are increased, while biomass of both the predator decreases. Thus, no hydra effect appears in infected predator(Figure19).

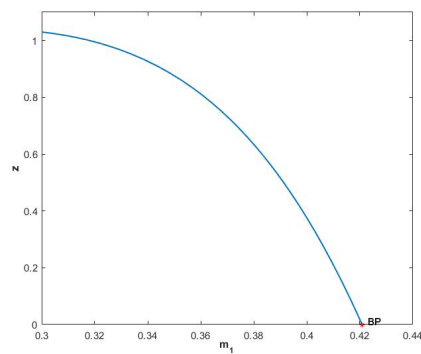
**6.2. Impact of mortality of susceptible predators.** In this subsection, we examine whether the mortality of susceptible predator species can result a positive hydra effect. For  $b = 0.186$  and  $c_2 = 0$



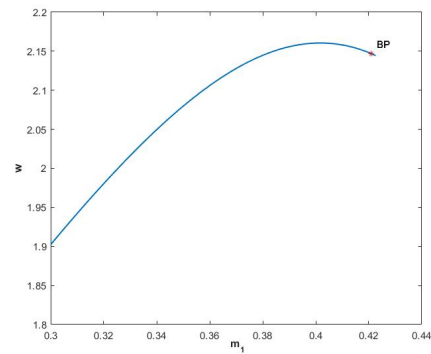
(A)



(B)



(C)



(D)

FIGURE 19. Depicts stable stocks for all the population when  $m_1$  is increased. Biomass of all the population are coexists for  $m_1 \in (0.258, 0.421)$ . Hydra effect cannot be appeared at stable biomass on infected predator.

along with the other parameter values fixed, it is observed that all the populations coexists for  $m_2 \in (0.191, 0.239)$ . It can be seen that with increasing mortality rate of susceptible predator, biomass of both the prey and scavenger population are increased, while biomass of both the predator decreases. Thus, no hydra effect appears in susceptible predator (Figure20).

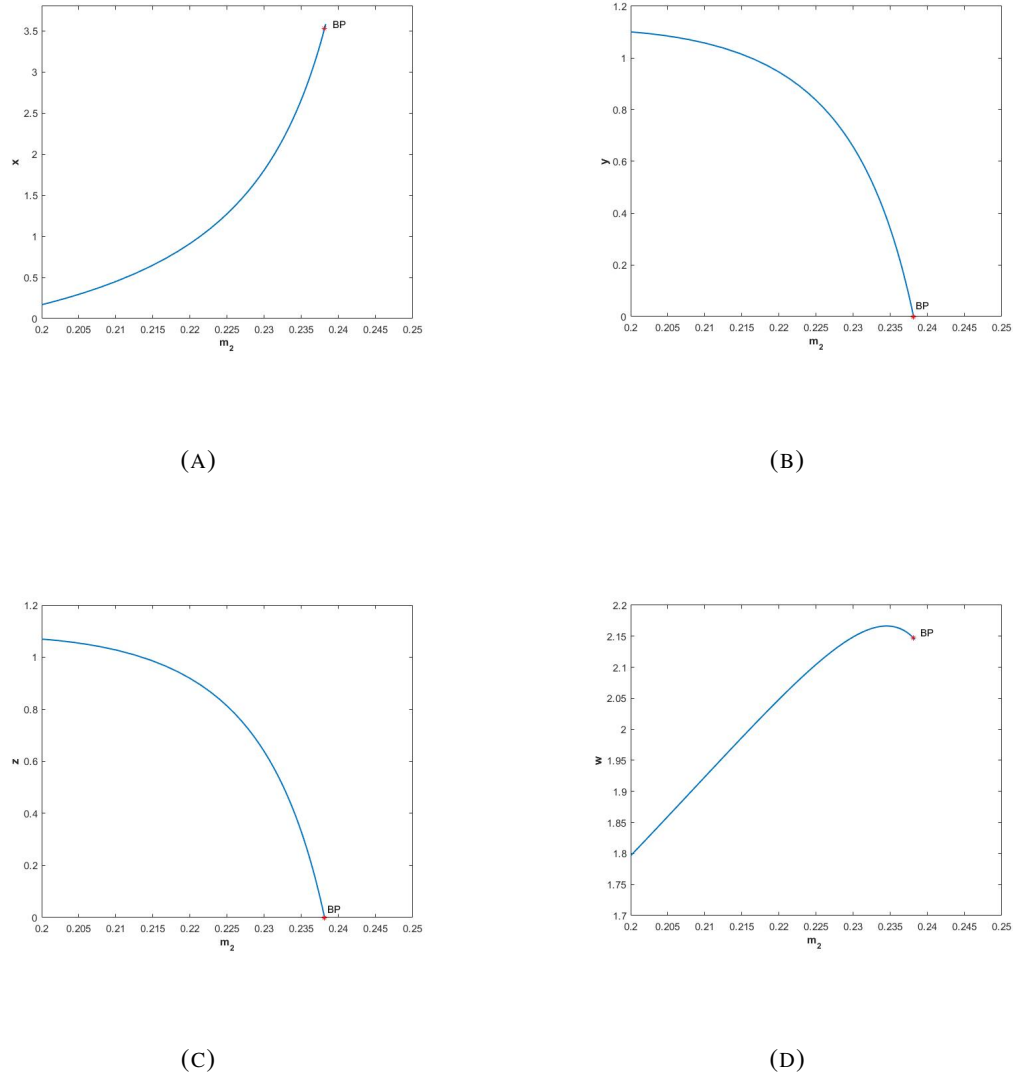
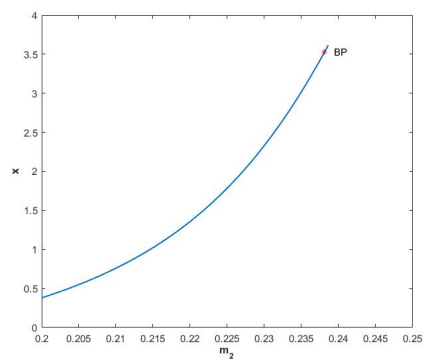


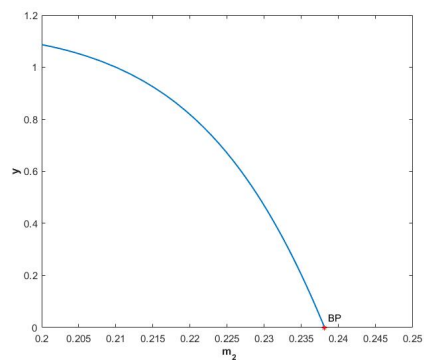
FIGURE 20. Depicts the variations of all the population. The stock corresponding to the stable equilibrium exists when  $m_2 \in (0.191, 0.239)$ . Hydra effect cannot be appeared at stable biomass on susceptible predator.

Next, for the system under  $c_2 \neq 0$  we set  $b = 0.186$  and  $c_2 = 0.06$  along with the other parameter values fixed. It is observed that all the populations coexists for  $m_2 \in [0.2, 0.238111]$ . It can be seen that with the increasing mortality rate of susceptible predator, biomass of both the prey and scavenger population are increased, while biomass of both the predator decreases. Thus, no hydra effect appears in

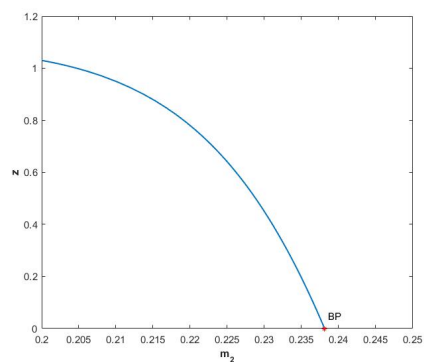
infected predator (Figure 21).



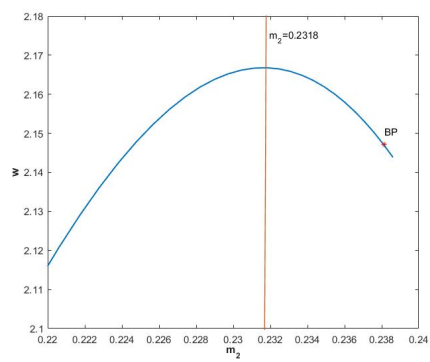
(A)



(B)



(C)



(D)

FIGURE 21. Depicts the variations of all the population. The stock corresponding to the stable equilibrium exists when  $m_2 \in [0.2, 0.238111]$ . Hydra effect cannot be appeared at stable biomass on susceptible predator.

## 7. HARVESTING OF SUSCEPTIBLE PREDATORS AND SCAVENGERS

In such environment no disease free equilibrium is obtained i.e., the system will not become free of disease until the whole susceptible predator population is transferred into another disease free environment. The natural death rates can be considered as harvesting rate. Here we are providing a scenario by adopting the mortality rate due to harvesting. Ignoring the natural mortality rates of the susceptible predators and scavengers, a harvesting policy is proposed by considering  $m_2 = q_1 \xi_1$  and  $\mu = q_1 \xi_2$ , where  $q_1, q_2$  are catchability coefficients for susceptible predator and scavenger species and  $\xi_1, \xi_2$  are respectively the efforts while harvesting them. Then the system (1) becomes,

$$(6) \quad \begin{aligned} \frac{dx}{dt} &= r_1 x(1 - c_1 x) - \frac{\alpha x z}{h + x} - p x w \\ \frac{dy}{dt} &= r_2 z - b y(1 - c_2 y) - m_1 y \\ \frac{dz}{dt} &= b y(1 - c_2 y) + \frac{\beta \alpha x z}{h + x} - q_1 \xi_1 z \\ \frac{dw}{dt} &= d x w + e w(y + z) - q_2 \xi_2 w - n w^2 \end{aligned}$$

**7.1. Optimal Harvesting Policy.** In this section, we discuss to obtain an optimum revenue by harvesting susceptible predator and scavenger species. For this we follow the process adopted by Pal et al. and Gupta et al. [[6, 38]]. Let  $h_1, h_2$  are the price of per unit biomass of the predator and scavenger species respectively and  $k_1, k_2$  are the harvesting cost per unit of the corresponding. Then the net economic revenue is given by,

$$P(t, z, w, \xi_1, \xi_2) = (h_1 q_1 z - k_1) \xi_1(t) + (h_2 q_2 w - k_2) \xi_2(t)$$

which is optimized by the objective function

$$(7) \quad J(\xi_1, \xi_2) = \int_0^{\infty} e^{-\delta t} \{P(t, z, w, \xi_1, \xi_2)\} dt$$

such that the harvesting species donot become extinct. Here  $\delta$  is the continuous annual discount rate which is fixed by harvesting companies. Pontryagin's maximum principle is used to determine the optimal solution of this problem. The associated Hamiltonian function is given by,

$$(8) \quad \begin{aligned} H(t, x, y, z, w, \xi_1, \xi_2) &= (h_1 q_1 z - k_1) \xi_1 e^{-\delta t} + (h_2 q_2 w - k_2) \xi_2 e^{-\delta t} + \\ &+ \lambda_1 [r_1 x(1 - c_1 x) - \frac{\alpha x z}{h + x} - p x w] + \lambda_2 [r_2 z - b y(1 - c_2 y) - m_1 y] + \\ &+ \lambda_3 [b y(1 - c_2 y) + \frac{\beta \alpha x z}{h + x} - q_1 \xi_1 z] + \lambda_4 [d x w + e w(y + z) - q_2 \xi_2 w - n w^2] \end{aligned}$$

where  $\lambda_i = \lambda_i(t)$ ,  $i = 1, 2, 3, 4$  are adjoint corresponding to the variables  $x, y, z, w$  respectively. We have,

$$(9) \quad \begin{aligned} \frac{\partial H}{\partial \xi_1} &= e^{\delta(-t)} (h_1 q_1 z - k_1) - \lambda_3 q_1 z = \psi_1(t) \\ \frac{\partial H}{\partial \xi_2} &= e^{\delta(-t)} (h_2 q_2 w - k_2) - \lambda_4 q_2 w = \psi_2(t) \end{aligned}$$

The corresponding optimal harvest policy is

$$\Gamma_i(t) = \begin{cases} \Gamma_i^{\max}(t) & \text{when } \psi_i(t) > 0 \\ 0 & \text{when } \psi_i(t) < 0, \\ \Gamma_i^*(t) & \text{when } \psi_i(t) = 0 \end{cases}$$

For  $i = 1, 2$  and the singular control is  $\Gamma_i^*(t)$  such that  $0 < \Gamma_i^*(t) < \Gamma_i^{\max}(t)$

For singular control we have  $\psi_i(t) = 0, i = 1, 2$  hence from (8), we get,

$$(10) \quad \begin{aligned} \lambda_3 &= e^{\delta t} \left( h_1 - \frac{k_1}{q_1 z} \right) \\ \lambda_4 &= e^{\delta t} \left( h_2 - \frac{k_2}{q_2 w} \right) \end{aligned}$$

In order to find a singular control, Pontryagin's Maximum Principle [37] is utilized and the adjoint variables must satisfy the adjoint equations given by

$$(11) \quad \begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial x} \\ \frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial y} \\ \frac{d\lambda_3}{dt} &= -\frac{\partial H}{\partial z} \\ \frac{d\lambda_4}{dt} &= -\frac{\partial H}{\partial w} \end{aligned}$$

From (8) and (11) we have

$$(12) \quad \begin{aligned} \frac{d\lambda_1}{dt} &= \lambda_1 \left( - \left( -c_1 r_1 x + r_1 (1 - c_1 x) + \frac{\alpha x z}{(h+x)^2} - \frac{\alpha z}{h+x} - p w \right) \right) - d\lambda_4 w - \lambda_3 \left( \frac{\alpha \beta z}{h+x} - \frac{\alpha \beta x z}{(h+x)^2} \right) \\ \frac{d\lambda_2}{dt} &= \lambda_2 (- (bc_2 y - b(1 - c_2 y) - m_1)) - \lambda_3 (b(1 - c_2 y) - bc_2 y) - e\lambda_4 w \end{aligned}$$

Using equilibrium conditions, (12) becomes

$$(13) \quad \begin{aligned} \frac{d\lambda_1}{dt} &= \lambda_3 \left( - \left( \frac{\alpha \beta z}{h+x} - \frac{\alpha \beta x z}{(h+x)^2} \right) \right) - d\lambda_4 w \\ \frac{d\lambda_2}{dt} &= \lambda_3 (- (b(1 - c_2 y) - bc_2 y)) - e\lambda_4 w \end{aligned}$$

Now using (10), the eq. (13) becomes,

$$(14) \quad \begin{aligned} \frac{d\lambda_1}{dt} &= \frac{de^{\delta(-t)}(k_2 - h_2q_2w)}{q_2} - \frac{e^{\delta(-t)}(k_1 - h_1q_1z) \left( \frac{\alpha\beta xz}{(h+x)^2} - \frac{\alpha\beta z}{h+x} \right)}{q_1z} \\ \frac{d\lambda_2}{dt} &= \frac{ee^{\delta(-t)}(k_2 - h_2q_2w)}{q_2} - \frac{e^{\delta(-t)}(bc_2y - b(1 - c_2y))(k_1 - h_1q_1z)}{q_1z} \end{aligned}$$

Integrating (14) we get,

$$(15) \quad \begin{aligned} \lambda_1 &= - \frac{e^{\delta(-t)}(dk_2q_1(h+x)^2 - q_2(q_1(dh_2w(h+x)^2 + \alpha\beta hh_1z) - \alpha\beta hk_1))}{\delta q_1 q_2 (h+x)^2} \\ \lambda_2 &= - \frac{e^{\delta(-t)}(ek_2q_1z - q_2(q_1z(h_1(b - 2bc_2y) + eh_2w) + bk_1(2c_2y - 1)))}{\delta q_1 q_2 z} \end{aligned}$$

Now using (11) and (15) we get,

$$(16) \quad \begin{aligned} \frac{d\lambda_3}{dt} &= \frac{r_2e^{\delta(-t)}(ek_2q_1z - q_2(q_1z(h_1(b - 2bc_2y) + eh_2w) + bk_1(2c_2y - 1)))}{\delta q_1 q_2 z} \\ &+ \frac{ee^{\delta(-t)}(k_2 - h_2q_2w)}{q_2} - h_1\xi_1q_1e^{\delta(-t)} + \frac{\alpha\lambda_1x}{h+x} \\ \frac{d\lambda_4}{dt} &= - \frac{pxe^{\delta(-t)}(dk_2q_1(h+x)^2 - q_2(q_1(dh_2w(h+x)^2 + \alpha\beta hh_1z) - \alpha\beta hk_1))}{\delta q_1 q_2 (h+x)^2} \\ &- h_2\xi_2q_2e^{\delta(-t)} \end{aligned}$$

Integrating (16) we get,

$$(17) \quad \begin{aligned} \lambda_3 &= \frac{r_2e^{\delta(-t)}(-2bc_2h_1q_2q_1yz + 2bc_2k_1q_2y + bh_1q_2q_1z - bk_1q_2 + eh_2q_2q_1wz - ek_2q_1z)}{\delta^2 q_1 q_2 z} \\ &+ \frac{ee^{\delta(-t)}(h_2q_2w - k_2)}{\delta q_2} + \frac{h_1\xi_1q_1e^{\delta(-t)}}{\delta} + \frac{\alpha\lambda_1tx}{h+x} \\ \lambda_4 &= - \frac{e^{\delta(-t)}(q_2(q_1(h_2(h+x)^2(dpwx - \delta\xi_2q_2) + \alpha\beta hh_1pxz) - \alpha\beta hk_1px) - dk_2pq_1x(h+x)^2)}{\delta^2 q_1 q_2 (h+x)^2} \end{aligned}$$

Eqs. (10) and (17) give the optimal harvesting efforts as,

$$(18) \quad \begin{aligned} \xi_1 &= - \frac{2bc_2k_1r_2y}{\delta h_1q_1^2z} + \frac{2bc_2r_2y}{\delta q_1} + \frac{bk_1r_2}{\delta h_1q_1^2z} - \frac{br_2}{\delta q_1} + \frac{ek_2r_2}{\delta h_1q_1q_2} + \frac{ek_2}{h_1q_1q_2} - \frac{eh_2r_2w}{\delta h_1q_1} \\ &- \frac{eh_2w}{h_1q_1} - \frac{\delta k_1}{h_1q_1^2z} - \frac{\alpha\delta\lambda_1txe^{\delta t}}{h_1q_1(h+x)} + \frac{\delta}{q_1} \\ \xi_2 &= - \frac{dk_2px}{\delta h_2q_2^2} + \frac{dpwx}{\delta q_2} - \frac{\alpha\beta hk_1px}{\delta h_2q_1q_2(h+x)^2} - \frac{\delta k_2}{h_2q_2^2w} + \frac{\alpha\beta hh_1pxz}{\delta h_2q_2(h+x)^2} + \frac{\delta}{q_2} \end{aligned}$$

Hence solving steady state equations together with  $\xi_1$  and  $\xi_2$  we get the optimal solution  $(x_\delta, y_\delta, z_\delta, w_\delta)$  and optimal harvesting efforts  $\xi_1$  and  $\xi_2$ .



## 8. CONCLUSION AND DISCUSSION

We have proposed a tri-trophic predator-prey community model with a scavenger species. We have considered that the predator species has a non-infectious disease that can be occurred due to a stressed environment. As an outcome, one can see that there is no occurrence of disease-free equilibrium. The main purposes of the paper are to examine (i) the effect of density-dependent recovery on the stability of the system, (ii) the effect of the mortality rate of scavengers with a different recovery rate of the infected predator as well as predation rate of scavengers towards prey on the coexistence of the system, (iii) Positive impacts of increasing population mortality on the stock size. Local stability of the equilibrium points are discussed analytically. The stability nature of the steady states are examined using Routh–Hurwitz criteria. Among the equilibrium points the scavenger free equilibrium  $E_3$  and positive equilibrium  $E^*$  involves the term  $c_2$  which is the crowding factor of predators. Numerical simulations are conducted with a hypothetical set of parameter values to validate the analytical findings. From Figure (1), it is observed that decreasing the crowding factor  $c_2$  to 0 the positive equilibrium tends to  $(0.7690, 1.0156, 0.9343, 2.0266)$  which is the positive equilibrium under  $c_2 = 0$ . In aquaculture increasing the crowding factor increases the social stress in the environments, which is responsible for slower recovery rate in fish. Practically without reducing the stress environment farmers unable to treat the fish diseases that occur because of the stress environment. For example, in the case of ammonia poisoning in ponds or fisheries with a high density of fish. Farmers cannot treat the disease without reducing the density of the fish population. From Figure(1), one can see a continuous dynamics while increasing the stress from  $c_2 = 0$  to  $c_2 = 1$ . A similar scenario can be observed in the case of scavenger free system Figure (2). From Figure(3), one can observe that all the population coexists for a small range of recovery rate  $b$  under  $c_2 = 0$ . From Figure(4), it is observed that an unstable equilibrium under zero density ( $c_2 = 0$ ), which was unstable for a smaller value of disease recovery rate, could be stable for increasing the value of the density or crowding factor of the predators. It is seen that with a small disease recovery rate, all the species coexists for a limited rate of crowding factor. We may also conclude that increasing the crowding factor of the predators has a stabilizing effect on the system dynamics. Besides predation of the prey, scavengers feed on carcasses of both the predator species. So there is a possible effect of the disease recovery rate of infected predators on the scavenger species. From Figure(5-14)it is observed that scavengers along with all the other species appear a period-doubling cascade with an increasing mortality rate of the scavenger species. All the population undergoes a period-doubling cascade and becomes stable again. Similar cascades are also observed Figure(15-16) for the mortality rate of scavenger and

it's predation rate for the prey. We have examined the influence of linear mortality for both predator and scavenger species. This mortality could be treated as the harvesting in fishery science. Numerical simulation in our analysis reveals that the scavenger stock (Figure 17) is increased at a stable state on increasing its own mortality rate under the density-dependent factor. That is a positive hydra effect appears on scavenger species. Moreover, we have discussed the scenario by considering the linear mortality rate of susceptible predator and scavenger species as linear harvesting. The associated The control problem of the model is discussed with the help of Pontryagin's maximum principle together with an optimal control policy.

### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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