



Available online at <http://scik.org>

Commun. Math. Biol. Neurosci. 2020, 2020:79

<https://doi.org/10.28919/cmbn/5017>

ISSN: 2052-2541

## FRACTIONAL-ORDER DYNAMICS OF RIFT VALLEY FEVER IN RUMINANT HOST WITH VACCINATION

ASIF JAN<sup>1</sup>, RASHID JAN<sup>2,\*</sup>, HASSAN KHAN<sup>3</sup>, M. S. ZOBAER<sup>4</sup>, RASOOL SHAH<sup>3</sup>

<sup>1</sup>Department of Pathogenic Microbiology & Immunology, School of Basic Medical Sciences, Xi'an Jiaotong University, Health Science Center 76 West Yanta Road, Xi'an 710061, PR China

<sup>2</sup>Department of Mathematics and Statistics, Bacha Khan University, Charsadda 24420, Pakistan

<sup>3</sup>Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan

<sup>4</sup>McGovern Medical School, The University of Texas Health Science Center at Houston, Texas, USA

Copyright © 2020 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Abstract.** Rift Valley fever is the most terrifying animal disease around the globe, which transfers through mosquitoes and caused by a virus. This infection is life-threatening and heavily affects the economic sectors. Therefore, it is valuable to conceptualize the dynamics of Rift Valley fever to understand its transmission pathway to provide better control policies. Here, we construct an epidemic model for Rift Valley fever with vaccination through fractional derivatives. Firstly, we present the proposed Rift Valley fever dynamics in the Caputo framework. The basic knowledge of fractional calculus is used to determine the rudimentary properties of the proposed fractional model, which include positivity, uniqueness, and boundedness of the solutions. We investigate our constructed model of Rift Valley fever for equilibria and determined the basic reproduction number of the system through next-generation technique, indicated by  $\mathcal{R}_0$ . The stability results are established for the infection-free steady-state of the system. Numerical simulations are conducted and sensitivity analysis of  $\mathcal{R}_0$  through partial rank correlation coefficient (PRCC) method is carried out to show the importance of different parameters in  $\mathcal{R}_0$ . Then the Rift Valley fever model is analyzed in the Atangana-Baleanu framework, furthermore, we present a numerical scheme for the proposed fractional model to illustrate the solution pathway of the model. We notice that

---

\*Corresponding author

E-mail address: [rashid\\_ash2000@yahoo.com](mailto:rashid_ash2000@yahoo.com)

Received September 12, 2020

the fractional-order dynamics can explain the complex system of Rift Valley fever infection more precisely and accurately rather than the integer-order dynamics. It is also observed that the Atangana-Baleanu operator provides more accurate results than the Caputo fractional derivative.

**Keywords:** rift valley fever; mathematical model; fractional calculus; sensitivity analysis; iterative scheme; numerical simulations.

**2010 AMS Subject Classification:** 92D25.

## 1. INTRODUCTION

It is prominent that infectious diseases bring unbelievable damage to humans and animals life, these infections are produced by parasites, bacteria, fungi, viruses and other microorganisms with the pathogen. To be more specific, these infections transfer to animals and humans with different direct and indirect route. Number of data is available for Hydrophobia, Dengue infection, Measles, Plague, and for some other diseases, which transfer through mediums. Rift Valley Fever (RVF) virus is a significant vector-borne viral zoonosis in Kenya and North Africa, which spread through mosquitoes and is from the Phlebovirus genus in the Bunyaviridae family. In the 1930's [1], it was found in Kenya for the first time. It transfers by drinking the milk of infected animals, touching the blood of an infected animal, breathing near infected animals, and by the bite of mosquitoes. RVF may affect animals such as camels goats, sheep and cows [2, 3].

It is eminent that modeling of biological processes and infection diseases explore the dynamics of these infections and produce accurate information which is helpful for its prevention [4, 5, 6]. Several dynamical models for Rift Valley fever has been introduced in the literature to observe the transmission behavior of Rift Valley fever and to predict better control policies and better suggestions for the medications of the infection. Some mathematical models are studied in [7, 8, 9, 10, 11] to visualize the transmission pathway of RVF. Gaff et al. [12] introduced a dynamical model for Rift Valley fever and investigated the stability of the disease-free equilibria of the system. A Rift Valley fever model was introduced by Saul et al. [13] with the human host. They determined the threshold parameter  $\mathcal{R}_0$  of the system and established results for the local stability of the equilibria of the proposed system. Xue et al. [14] constructed a dynamical model of RVF in the framework of ODE's to assess disease spread in space and time. Recently, a three patch model for Rift Valley fever with spatial assumptions was formulated by Gao et al.

[15] to conceptualize the transmission of infection. They determined the threshold dynamics for each patch and visualize the dynamics.

Control through vaccination and treatment play a dominant role to protect from infectious diseases [16, 17]. Numerous vaccine has been developed to protect from infectious disease in endemic regions. It is true that all these vaccines are not completely effective and have some side effects, some are expensive. In the case of Rift Valley fever, the vaccine sometimes bring abnormalities in the pregnant animals [18]. A vaccinated model was introduced by Farida et al. [19] to investigate the role of vaccination in ruminant animals and some other models are present with continuous and impulsive vaccinations [20, 21]. However, the true role of vaccination is not completely explored in RVF's transmission to understand the dynamics of RVFV.

It is notorious that fractional-order systems offer more reliable, deeper, precious and accurate information about the dynamics of a system [22, 23, 24]. Description of memory and hereditary property make it superior to integer-order models [25]. Moreover, fractional-order models can easily explore and demonstrate the dynamics between two points. Existing studies predict that various theories and ideas regarding fractional derivatives have been introduced and developed, for instance, the fundamental notion and idea of FO derivative is introduced in [25]. In recent years, novel FO derivatives has been developed in [26, 27, 28]. These new ideas have been effectively utilized in modeling real-world problem in different fields, such as physics, engineering, biology and several other areas [28, 29, 30]. The classical FO derivative cannot describe properly the crossover and non-local dynamical structure of several naturalistic problems due to its singular kernel. The fractional framework can address these issues and produces accurate results in the modelling of different problem. Therefore, motivated by the accurate results of the above, we opt to conceptualize and explore the dynamics of RFV with the effect of vaccination in the framework of fractional derivatives.

The further article is structured as: We put forward a brief summary of the rudimentary concepts of fractional calculus in section two. In the third section, we construct a fractional-order model for Rift Valley fever with vaccination in the Caputo sense, in addition, we proved basic results for our proposed fractional-order model. We establish stability results for the steady-state in the fourth section and carried out sensitivity analysis through the PRCC technique to

point out the importance of the input parameter on the output of  $\mathcal{R}_0$ . Then, we present the Rift Valley fever dynamics in Atangana-Baleanu sense in Section 5 and establish the existence and uniqueness result. Moreover, a numerical scheme is presented to demonstrate the solution pathway of the proposed fractional dynamics. In the last of the article, ending remarks and conclusion of the overall analysis is presented.

## 2. FRACTIONAL CONCEPTS

In this section, we introduce some basic concepts and results of fractional calculus, which will be used in further analysis of the fractional-order Rift Valley fever model.

**Definition 1.** Let  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ , the fractional integral of  $g$  is mentioned below

$$(1) \quad I_t^\vartheta(g(t)) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-y)^{\vartheta-1} g(y) dy.$$

where  $\vartheta$  indicates the order of fractional integral and  $\Gamma$  represents the Gamma function.

**Definition 2.** Let  $g$  be a function, then the fractional derivative of  $g$  of order  $\vartheta$  in the Caputo form is given by

$$(2) \quad {}_a^C D_t^\vartheta(g(t)) = I^{n-\vartheta} D^n g(t) = \frac{1}{\Gamma(n-\vartheta)} \int_a^t (t-y)^{n-\vartheta-1} g^n(y) dy,$$

where  $n-1 < \vartheta < n, n \in \mathbb{N}$ .

Let  $\eta_1, \eta_2$  be two positive numbers, then the Mittag-Leffler function is as

$$(3) \quad E_{\eta_1, \eta_2}(s) = \sum_{k=0}^{k=\infty} \frac{s^k}{\Gamma(\eta_1 k + \eta_2)}.$$

Let  ${}_0^C D_t^\vartheta$  be the CFD of order  $\vartheta$ , then its laplace transform is define by

$$(4) \quad L[{}_0^C D_t^\vartheta g(t)] = u^\vartheta G(u) - \sum_{k=0}^{n-1} g^k(0) u^{\vartheta-k-1},$$

further, for the function  $t^{\eta_2-1} E_{\eta_1, \eta_2}(\pm \omega t^\vartheta)$ , the laplace transform is given by

$$(5) \quad L[t^{\eta_2-1} E_{\eta_1, \eta_2}(\pm \omega t^\vartheta)] = \frac{u^{\eta_1-\eta_2}}{u^{\eta_1} \pm \omega}.$$

The following equation hold true by Mittage-Leffler function given in [31]:

$$E_{\eta_1, \eta_2}(s) = s.E_{\eta_1, \eta_1+\eta_2}(s) + \frac{1}{\Gamma(\eta_2)}.$$

**Lemma 1.** [32]. *Let us define a function  $g : R^+ \times R^4 \rightarrow R^4$ , which fulfills the conditions:*

- *The function  $g(t, Y(t))$  is Lebesgue measurable regarding  $t$  in  $R^+$ ;*
- *The function  $g(t, Y(t))$  is continuous regarding  $Y(t)$  on  $R^4$ ;*
- *The function  $\frac{\partial g(t, Y(t))}{\partial Y}$  is continuous regarding  $Y(t)$  on  $R^4$ ;*
- *and  $\|g(t, Y(t))\| \leq \hbar + \aleph \|Y\|, \forall t \in R^+, Y \in R^4$ , where  $\hbar, \aleph$  are positive constants.*

*Then the fractional order system*

$$\begin{aligned} {}_a^C D_t^\vartheta Y(t) &= g(t, Y(t)), \\ Y(0) &= Y_0, \end{aligned}$$

*where  $0 < \vartheta \leq 1$ , have a unique solution.*

**Lemma 2.** [33]. *Let a function  $g(t) \in C[r, s]$  such that  ${}_r^C D_t^\vartheta g(t) \in C[r, s]$  for  $\vartheta \in (0, 1]$ , then we have*

$$g(t) = g(r) + \frac{1}{\Gamma(\vartheta)} {}_r^C D_t^\vartheta g(\varepsilon)(t-r)^\vartheta, \quad r < \varepsilon < t, \forall t \in (r, s].$$

**Remark 3.** *Let a function  $g \in C[r, s]$ , such that  ${}_r^C D_t^\vartheta g(t) \in C[r, s]$  for  $\vartheta \in (0, 1]$ . Then by Lemma (2) if  ${}_r^C D_t^\vartheta g(t) \leq 0, \forall t \in (r, s)$ , then  $g(t)$  is non-increasing function for all  $t \in [r, s]$ , and if  ${}_r^C D_t^\vartheta g(t) \geq 0, \forall t \in (r, s)$ , then  $g(t)$  is non-decreasing function for all  $t \in [r, s]$ .*

**Definition 3.** *Assume a function  $g$  in a manner that  $g \in H^1(u, v)$ ,  $v > u$ , and  $\vartheta \in [0, 1]$ , then the fractional derivative through Atangana-Baleanu (AB) is defined as*

$${}_{u}^{ABC} D_t^\vartheta g(t) = \frac{B(\vartheta)}{1-\vartheta} \int_u^t g'(\kappa) E_\vartheta \left( -\vartheta \frac{(t-\kappa)^\vartheta}{1-\vartheta} \right) d\kappa.$$

**Definition 4.** *The integral of AB derivative for a function  $g$  is denoted by  ${}_{u}^{ABC} I_t^\vartheta g(t)$  and is defined as*

$${}_{u}^{ABC} I_t^\vartheta g(t) = \frac{1-\vartheta}{B(\vartheta)} g(t) + \frac{\vartheta}{B(\vartheta)\Gamma(\vartheta)} \int_u^t g(\kappa)(t-\kappa)^{\vartheta-1} d\kappa.$$

*Evidently, the initial function achieved as the order  $\vartheta$  tends to 0.*

**Theorem 4.** *Assume a function  $g$  in a way that  $g \in C[u, v]$ , then the below inequality satisfies [27]:*

$$\|{}_{u}^{ABC} D_t^\vartheta (g(t))\| < \frac{B(\vartheta)}{1-\vartheta} \|g(t)\|,$$

where

$$\|g(t)\| = \max_{u \leq t \leq v} |g(t)|.$$

Moreover, the Lipschitz condition in the Atangana-Baleanu Caputo sense is

$$\|{}_{u}^{ABC}D_t^{\vartheta} g_1(t) - {}_{u}^{ABC}D_t^{\vartheta} g_2(t)\| < \sigma_1 \|g_1(t) - g_2(t)\|.$$

**Theorem 5.** [27]. Assume the fractional differential equation of the form

$${}_{u}^{ABC}D_t^{\vartheta} g(t) = w(t),$$

has a unique solution given by

$$g(t) = \frac{1 - \vartheta}{B(\vartheta)} w(t) + \frac{\vartheta}{B(\vartheta)\Gamma(\vartheta)} \int_u^t w(\kappa)(t - \kappa)^{\vartheta-1} d\kappa.$$

### 3. MODEL FORMULATION

In structure of the model, the total vector size  $M$  (female mosquitoes) is categorized into ( $S_m$ ) susceptible and ( $I_m$ ) infected compartments, while the total population of ruminant  $N_r$  is categorized into ( $S_r$ ) susceptible, ( $V_r$ ) vaccinated, ( $I_r$ ) infected, and ( $R_r$ ) recovered compartments.

The recruitment rate of the ruminant population and female mosquitoes are indicated by  $\Pi_r$  and  $\Pi_m$ , respectively. We denote the natural death rate of ruminant by  $d_r$  and the natural death rate of mosquitoes by  $d_m$ , moreover, the disease-induced death rate is indicated by  $\delta$  and the rate of recovery is denoted by  $\gamma$ . We consider that a fraction  $\nu$  of the susceptible population moves to the vaccination class after vaccination. The transmission probability from animals to vectors and from vectors to animals are given by  $\beta_r$  and  $\beta_m$ , respectively, while the efficacy of vaccination is denoted by  $\alpha$  and the biting rate of mosquitoes is represented by  $b$ . The transmission dynamics of Rift Valley fever is given by

$$(6) \quad \begin{cases} \frac{dS_r}{dt} = \Pi_r - b\beta_r S_r I_m - \nu S_r - d_r S_r, \\ \frac{dV_r}{dt} = \nu S_r - (1 - \alpha)b\beta_r V_r I_m - \rho V_r - d_r V_r, \\ \frac{dI_r}{dt} = b\beta_r S_r I_m + (1 - \alpha)b\beta_r V_r I_m - (d_r + \gamma + \delta)I_r, \\ \frac{dR_r}{dt} = \rho V_r + \gamma I_r - d_r R_r, \\ \frac{dS_m}{dt} = \Pi_m - b\beta_m S_m I_r - d_m S_m, \\ \frac{dI_m}{dt} = b\beta_m S_m I_r - d_m I_m, \end{cases}$$

with positive initial state values given by

$$S_m(0), I_m(0), S_r(0), V_r(0), I_r(0), R_r(0),$$

where the fraction  $(b\beta_r I_v)$  indicates the infection rate per susceptible ruminant, while  $(b\beta_m I_m)$  denotes the infection rate of per susceptible mosquitoes. As fractional-order models describe the non-local behavior of biological systems and possess hereditary property, moreover, it provides information about its past and present state for the future, therefore, we represent the dynamical system (6) of RVF in the framework of fractional order Caputo's derivative to conceptualize the transmission of RVF in a better way. Then, the system of the fractional system is presented by

$$(7) \quad \left\{ \begin{array}{l} {}_0^C D_t^\vartheta S_r = \Pi_r - b\beta_r S_r I_m - \nu S_r - d_r S_r, \\ {}_0^C D_t^\vartheta V_r = \nu S_r - (1 - \alpha) b\beta_r V_r I_m - \rho V_r - d_r V_r, \\ {}_0^C D_t^\vartheta I_r = b\beta_r S_r I_m + (1 - \alpha) b\beta_r V_r I_m - (d_r + \gamma + \delta) I_r, \\ {}_0^C D_t^\vartheta R_r = \rho V_r + \gamma I_r - d_r R_r, \\ {}_0^C D_t^\vartheta S_m = \Pi_m - b\beta_m S_m I_r - d_m S_m, \\ {}_0^C D_t^\vartheta I_m = b\beta_m S_m I_r - d_m I_m, \end{array} \right.$$

where  ${}_0^C D_t^\vartheta$  indicates Caputo's fractional derivative of order  $\vartheta$ , the order  $\vartheta$  indicates the index of memory in the system. Moreover, the total size of both the species are given by  $M = S_m + I_m$  and  $N = S_r + V_r + I_r + R_r$ . Adding the last two equations, we get

$${}_0^C D_t^\vartheta (S_m + I_m) \leq \Pi_m - d_m (S_m + I_m),$$

solving this inequality, we get

$$\left( S_m(t) + I_m(t) \right) \leq (S_m(0) + I_m(0)) E_{\vartheta, 1}(-d_m t^\vartheta) + \Pi_m t^\vartheta E_{\vartheta, \vartheta+1}(-d_m t^\vartheta),$$

by asymptotic behaviour of Mittag-Leffler function [34], we obtain

$$\left( S_m(t) + I_m(t) \right) \leq \frac{\Pi_m}{d_m}.$$

As a result,  $N_m(t) \rightarrow \frac{\Pi_m}{d_m} = \widehat{N}_m(t)$  as time approaches  $\infty$ . Then the fractional order systems (7) of Rift Valley fever takes the form

$$(8) \quad \begin{cases} {}_0^C D_t^\vartheta S_r &= \Pi_r - b\beta_r S_r I_m - \nu S_r - d_r S_r, \\ {}_0^C D_t^\vartheta V_r &= \nu S_r - (1 - \alpha)b\beta_r V_r I_m - \rho V_r - d_r V_r, \\ {}_0^C D_t^\vartheta I_r &= b\beta_r S_r I_m + (1 - \alpha)b\beta_r V_r I_m - (d_r + \gamma + \delta)I_r, \\ {}_0^C D_t^\vartheta I_m &= b\beta_m(\widehat{N}_m - I_m)I_r - d_m I_m. \end{cases}$$

In Caputo's form, the derivative of constant is equal to zero which is another advantage to make the system more reliable and flexible for analysis. Next, we will analyze the biologically feasible region of the fractional-order dengue model (8).

**Theorem 6.** *The closed set  $\Omega = \{(S_r, V_r, I_r, I_m) \in R_+^4 : 0 \leq S_r + V_r + I_r \leq K_1, 0 \leq I_m \leq K_2\}$  is a positive invariant set for the proposed fractional-order system (8).*

**Proof 6** To prove that the system of equations (8) has a non-negative solution, the system of equations (8) implies

$$(9) \quad \begin{cases} {}_0^C D_t^\vartheta S_r |_{S_r=0} &= \Pi_m > 0, \\ {}_0^C D_t^\vartheta V_r |_{V_r=0} &= \nu S_r > 0, \\ {}_0^C D_t^\vartheta I_r |_{I_r=0} &= b\beta_r S_r I_m + (1 - \alpha)b\beta_r V_r I_m \geq 0, \\ {}_0^C D_t^\vartheta I_m |_{I_m=0} &= b\beta_m S_m I_r \geq 0. \end{cases}$$

Thus, the fractional system (8) has non-negative solutions. In the end, from the first three equations of the fractional system (8), we obtain

$$(10) \quad \begin{aligned} {}_0^C D_t^\vartheta (S_r + V_r + I_r) &\leq \Pi_r - d_r S_r - d_r V_r - I_r(d_r + \delta), \\ &\leq \Pi_r - \mathcal{W}(S_r + V_r + I_r) \end{aligned}$$

where  $\mathcal{W} = \min(d_r, \delta + d_r)$ . Solving the above inequality, we obtain

$$\left( S_r(t) + V_r(t) + I_r(t) \right) \leq \left( S_r(0) + V_r(0) + I_r(0) - \frac{\Pi_r}{\mathcal{W}} \right) E_\vartheta(-\mathcal{W}t^\vartheta) + \frac{\Pi_r}{\mathcal{W}},$$

so by the asymptotic behaviour of Mittag-Leffler function [34], we obtain

$$\left( S_r(t) + V_r(t) + I_r(t) \right) \leq \frac{\Pi_r}{\mathcal{W}} \cong K_1,$$



taking the same steps for the last equation of system (8), we get  $I_m \leq \frac{\Pi_r}{d_m} \cong K_2$ . Hence, the closed set  $\Omega$  is a positive invariant region for the fractional-order Rift Valley Fever model (8).

**Theorem 7.** *The proposed fractional-order Rift Valley Fever system of equations (8) has a unique solution.*

**Proof 7** For the required statement of the theorem, we first prove that the fractional system (8) has a unique solution for all initial conditions in  $R^4$ . We consider  $z = (z_1, z_2, z_3, z_4)$ , where  $z_1 = S_r, z_2 = V_r, z_3 = I_r$ , and  $z_4 = I_m$ . Clearly, the first three conditions of Lemma (1) are hold true by the vector function  $g$  of the system (8). Next, to prove the last condition of Lemma (1), we rewrite system (8) as

$${}_0^C D_t^\vartheta z(t) = L + E_1 z(t) + E_2 z_1(t) z(t) + E_3 z_2(t) z(t) + E_4 z_3(t) z(t) + E_5 z_4(t) z(t),$$

where

$$L = \begin{bmatrix} \Pi_r \\ 0 \\ 0 \\ 0 \end{bmatrix}, E_1 = \begin{bmatrix} -(v + d_r) & 0 & 0 & 0 \\ v & -(\rho + d_r) & 0 & 0 \\ 0 & 0 & -(d_r + \gamma + \delta) & 0 \\ 0 & 0 & b\beta_m \widehat{N}_m & -d_m \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & -b\beta_r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1 - \alpha)b\beta_r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b\beta_m \end{bmatrix}, E_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b\beta_r & (1 - \alpha)b\beta_r & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Symbolize the above

$$g(t, z(t)) = L + E_1 z(t) + E_2 z_1(t) z(t) + E_3 z_2(t) z(t) + E_4 z_3(t) z(t) + E_5 z_4(t) z(t),$$

and taking norm, we get

$$\begin{aligned}
& \|h(t, z(t))\| \\
&= \|L + E_1 z(t) + E_2 z_1(t) z(t) + E_3 z_2(t) z(t) + E_4 z_3(t) z(t) + E_5 z_4(t) z(t)\| \\
&\leq \|L\| + \|E_1\| \|z(t)\| + \|E_2\| \|z_1(t)\| \|z(t)\| + \|E_3\| \|z_2(t)\| \|z(t)\| + \|E_4\| \|z_3(t)\| \|z(t)\| \\
&\quad + \|E_5\| \|z_4(t)\| \|z(t)\| \\
&\leq \|L\| + \|E_1\| \|z(t)\| + \|E_2\| \|z(t)\| + \|E_3\| \|z(t)\| + \|E_4\| \|z(t)\| + \|E_5\| \|z(t)\| \\
&= \|L\| + \left( \|E_1\| + \|E_2\| + \|E_3\| + \|E_4\| + \|E_5\| + \|E_6\| \right) \|z(t)\| \\
&= L + \hbar \|z(t)\|.
\end{aligned}$$

Hence, the requirements of Lemma (1) are fulfilled, therefore the system of equations (8) of Rift Valley fever has a unique solution.

#### 4. ANALYSIS OF THE MODEL

Here, we will investigate the fractional-order system of Rift Valley fever for disease-free and endemic steady-state. To obtain the infection-free steady-state, we set the fractional derivative  ${}^C_0 D_t^\vartheta S_r, {}^C_0 D_t^\vartheta V_r, {}^C_0 D_t^\vartheta I_r$ , and  ${}^C_0 D_t^\vartheta I_m$  to zero of the fractional system (8) without infection, and get

$$\mathcal{E}_0(S_r^0, V_r^0, I_r^0, I_m^0) = \left( \frac{\Pi_r}{v + d_r}, \frac{v\Pi_r}{(v + d_r)(\rho + d_r)}, 0, 0 \right).$$

Further, we will use the technique presented in [35, 36] for the reproduction number, which is calculated as

$$\mathcal{F} = \begin{bmatrix} b\beta_r S_r I_m + (1 - \alpha)b\beta_r V_r I_m \\ b\beta_m(\widehat{N}_m - I_m)I_r \end{bmatrix} \text{ and } \mathcal{V} = \begin{bmatrix} (d_r + \gamma + \delta)I_r \\ d_m I_m \end{bmatrix},$$

because there are two infected compartments in the system, which further implies

$$F = \begin{bmatrix} 0 & b\beta_r S_r^0 + (1-\alpha)b\beta_r V_r^0 \\ b\beta_m \widehat{N}_m & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} (d_r + \gamma + \delta) & 0 \\ 0 & d_m \end{bmatrix},$$

which gives

$$FV^{-1} = \begin{bmatrix} 0 & \left( \frac{b\beta_r S_r^0 + (1-\alpha)b\beta_r V_r^0}{d_m} \right) \\ \frac{b\beta_m \widehat{N}_m}{(d_r + \gamma + \delta)} & 0 \end{bmatrix}.$$

We denote the basic reproduction number of the fractional RVF model by  $\mathcal{R}_0$ , and is obtained through next-generation technique as

$$\rho(FV^{-1}) = \sqrt{\frac{b\beta_m \widehat{N}_m}{(d_r + \gamma + \delta)} \left( \frac{b\beta_r S_r^0 + (1-\alpha)b\beta_r V_r^0}{d_m} \right)},$$

$$\mathcal{R}_0 = \sqrt{\frac{b\beta_m \widehat{N}_m}{d_m} \left( \frac{b\beta_r S_r^0 + (1-\alpha)b\beta_r V_r^0}{(d_r + \gamma + \delta)} \right)}.$$

Next, we assume a fractional-order linear homogenous system of the following form

$$(11) \quad \begin{aligned} {}^C_0 D_t^\vartheta y(t) &= By(t), \\ y(0) &= y_0, \end{aligned}$$

where  $B \in M_{m \times m}(R)$  and  $0 < \vartheta \leq 1$ . The following theorems are on the stability of linear homogenous system (11).

**Theorem 8.** [42]. *The origin of the fractional dynamical system (11) is asymptotically stable if and only if  $|\arg(\lambda_i)| > \frac{\vartheta\pi}{2}$  is fulfilled for all eigenvalues  $\lambda_i$  of matrix  $B$ .*

Let us consider a general fractional order system

$$(12) \quad \begin{aligned} {}^C_0 D_t^\vartheta y(t) &= g(y), \\ y(0) &= y_0, \end{aligned}$$

where  $0 < \vartheta \leq 1$  and  $g : D \rightarrow R^m$  with  $D \subset R^m$ .

**Theorem 9.** [43]. *The steady-state of fractional system (12) is locally asymptotically stable (LAS) if  $|\arg(\lambda_i)| > \frac{\vartheta\pi}{2}$  for all eigenvalues  $\lambda_i$ 's of  $g(y)$  at steady state, otherwise unstable.*

**Theorem 10.** *The infection-free steady-state of the fractional-order system (8) is LAS if  $R_0 < 1$ , and is unstable in the other circumstances.*

**Proof 10** To obtain the demanded result for the fractional system (8), we take the Jacobian matrix of the system at infection-free steady-state as

$$J(\mathcal{E}_0) = \begin{bmatrix} -(v + d_r) & 0 & 0 & -b\beta_r S_r^0 \\ v & -(\rho + d_r) & 0 & -(1 - \alpha)b\beta_r V_r^0 \\ 0 & 0 & -(d_r + \gamma + \delta) & b\beta_r S_r^0 + (1 - \alpha)b\beta_r V_r^0 \\ 0 & 0 & b\beta_m \widehat{N}_m & -d_m \end{bmatrix}.$$

We can easily determine that  $\lambda = -(\rho + d_r)$  is the first eigenvalue of the above  $J(\mathcal{E}_0)$  and the remanning eigenvalues are the eigenvalues of the following

$$J_1(\mathcal{E}_0) = \begin{bmatrix} -(v + d_r) & 0 & -b\beta_r S_r^0 \\ 0 & -(d_r + \gamma + \delta) & b\beta_r S_r^0 + (1 - \alpha)b\beta_r V_r^0 \\ 0 & b\beta_m \widehat{N}_m & -d_m \end{bmatrix},$$

the second eigenvalue is  $\lambda = -(v + d_r)$  and the remaining are the eigenvalues of the below Jacobian matrix

$$J_2(\mathcal{E}_0) = \begin{bmatrix} -(d_r + \gamma + \delta) & b\beta_r S_r^0 + (1 - \alpha)b\beta_r V_r^0 \\ b\beta_m \widehat{N}_m & -d_m \end{bmatrix}.$$

To prove that the remaining eigenvalues of the Jacobian have negative real parts, we will prove that  $Tr(J_2) < 0$  and  $DetJ_2 > 1$  for  $\mathcal{R}_0$  less than one. It is obvious that  $Tr(J_2) < 0$ , next we have to check the second case

$$DetJ_2 = (d_r + \gamma + \delta)d_m - b\beta_m \widehat{N}_m \left( b\beta_r S_r^0 + (1 - \alpha)b\beta_r V_r^0 \right)$$

this implies that

$$DetJ_2 = (d_r + \gamma + \delta)d_m \left[ 1 - \frac{b\beta_m \widehat{N}_m}{d_m} \left( \frac{b\beta_r S_r^0 + (1 - \alpha)b\beta_r V_r^0}{(d_r + \gamma + \delta)} \right) \right].$$

Thus the  $DetJ_2 > 0$  for  $\mathcal{R}_0$  less than one, and hence the infection-free equilibrium is locally asymptotically stable.

**Theorem 11.** *The infection-free equilibrium of the fractional-order system (8) is GAS without vaccination, if  $R_0 < 1$ .*

**Proof 11** Let  $v = 0$ , and  $(S_r, V_r, I_r, I_m)$  be the solution of system (8) with suitable initial conditions  $(S_r(0), V_r(0), I_r(0), I_m(0))$  in  $\Omega$ . Here, it is clear that our proposed system has only one equilibrium  $\mathcal{E}_0$  on the boundary of  $\Omega$ . Therefore, to achieve the target, it is enough to prove that the solution  $(S_r, V_r, I_r, I_m)$  tends to the infection-free equilibrium as time tends to infinity. Then the dynamical system of Rift Valley fever implies that

$$(13) \quad \begin{cases} {}_0^C D_t^\vartheta I_r & \leq b\beta_r S_r^0 I_m + (1 - \alpha)b\beta_r V_r^0 I_m - (d_r + \gamma + \delta)I_r, \\ {}_0^C D_t^\vartheta I_m & \leq b\beta_m(\widehat{N}_m - I_m)I_r - d_m I_m. \end{cases}$$

Taking the auxiliary system

$$(14) \quad \begin{cases} {}_0^C D_t^\vartheta x_1 & \leq b\beta_r S_r^0 x_2 + (1 - \alpha)b\beta_r V_r^0 x_2 - (d_r + \gamma + \delta)x_1, \\ {}_0^C D_t^\vartheta x_2 & \leq b\beta_m(\widehat{N}_m - x_2)x_1 - d_m x_2. \end{cases}$$

this further implies

$${}_0^C D_t^\vartheta X = (F - V)X$$

the coefficient matrix of the above fractional system is  $F - V$ , and if  $\mathcal{R}_0 = \rho(FV^{-1}) < 1$ , then the eigenvalues of  $F - V$  lies in the left half-plane. As a result of this each positive solution of the fractional system (13) fulfills  $\lim_{t \rightarrow \infty} x_1 = 0$ , and  $\lim_{t \rightarrow \infty} x_2 = 0$  by Theorem 8. By the comparison theory of fractional differential equations [44], we have  $\lim_{t \rightarrow \infty} I_r = 0$  and  $\lim_{t \rightarrow \infty} I_m = 0$ . Then from the system (8), we have

$$(15) \quad \begin{aligned} {}_0^C D_t^\vartheta I_r &= \Pi_r - vS_r - d_r S_r, \\ {}_0^C D_t^\vartheta I_m &= vS_r - \rho V_r - d_r V_r, \end{aligned}$$

let  $X = (S_r, I_r)$ , then the above can be further converted into

$${}_0^C D_t^\vartheta X = A - BX,$$

having the solution

$$(16) \quad X(t) = t^\vartheta E_{\vartheta, \vartheta+1}(-Bt^\vartheta)A + E_{\vartheta, 1}(-Bt^\vartheta)X^0,$$

taking the asymptotic behavior of Mittag-Leffler function [34] as

$$E_{\vartheta, 1}(-\omega t^\vartheta) \approx_{t \rightarrow \infty} \frac{t^\vartheta}{\omega \Gamma(1 - \vartheta)}; \quad 0 < \vartheta < 1 \text{ and } \omega > 0,$$

also, the real parts of the eigenvalues of  $-B$  are negative, so it can be observed that  $X(t) \rightarrow X^0$  as  $t$  tends to infinity. Hence, the infection-free equilibrium of the system (8) is GAS without vaccination.

Next, we will demonstrate the persistence of infection in the fractional-order system. It describes the level of endemicity of infection in the system. Biologically speaking, the infection persists in the system if the level of infected fraction stays at a higher level for  $t$  large enough.

**4.1. Sensitivity analysis and numerical results.** Sensitivity analysis is used for measuring unpredictability in intricate models developed from real world-problems. The main goal of this analysis is to detect and measure the influence of input parameters on the output of the system [45]. To be more specific, it is used to know how the input parameters and initial values contribute to the output of a system. Mostly, when there is a little uncertainty in initial conditions and input parameters partial derivative of output functions are computed with respect to the input factors around the base values. This method is named as the local sensitivity analysis and relies on the variations of parameters close to the base values. This technique is not most suitable for epidemiological models due to the uncertainty in the input of the system. Therefore, global sensitivity analysis is preferred to perform this analysis and to provide more accurate results.

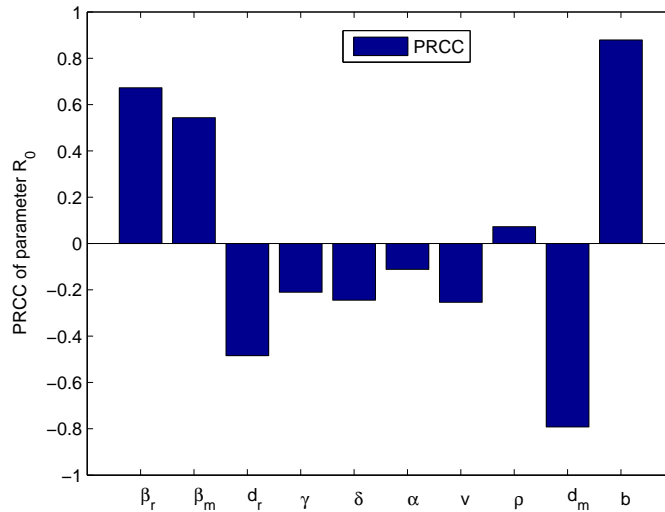


FIGURE 1. Sensitivity test for input factors of  $\mathcal{R}_0$  with PRCC outcomes.

Here, we used the PRCC method [46] for sensitivity analysis to point out the input parameters that highly influence the results of  $\mathcal{R}_0$ . It is an effective method and can successfully measure the monotonic, nonlinear relationship between input and output values of the system. PRCC analysis provides PRCC and p values for each factor involve therein, with which we can measure the contribution of each factor. More specifically, the input factors with sizeable PRCC and negligible p-values are considered to be highly effective factors in the system. In our analysis, we investigate all the parameters presented in Table 1 to know their contribution to the outcomes of  $\mathcal{R}_0$  and listed all the associated PRCC and p-values provided by the PRCC significance test. Figure 1 and Table 1 demonstrate that the parameters  $b$  and  $d_m$  are highly influential input factors with PRCC values 0.8790 and -0.7919, respectively. After that, the parameters  $\beta_r$  and  $\beta_m$  are greatly affecting the outcomes of basic reproduction number with PRCC values 0.6727 and 0.5433, respectively. It implies that if we decrease the value  $b$  and increase the value  $d_m$  through control policies, we can highly control Rift Valley fever. To be more specific, by controlling these factors, we can greatly decrease and prevent the level of new Rift Valley fever cases. In Figure 2 and Figure 3, the effect of input factors on the threshold parameter  $\mathcal{R}_0$  of the proposed system is demonstrated numerically. We have shown the transmission probabilities  $\beta_r$ ,  $\beta_m$  and the biting rate of mosquitoes are critical factors which increase the level of infection while the increase of input factor  $v$  decrease the threshold parameter which means that infection can controlled through vaccination.

**4.2. Numerical scheme for Caputo operator.** Here in this part of the article, we will introduce a numerical technique to illustrate the solution of the system in the Caputo sense (7). This numerical technique has been presented in [47] and is given as follows

$$(17) \quad {}_0^C D_t^\rho y(t) = g(t, y(t)),$$

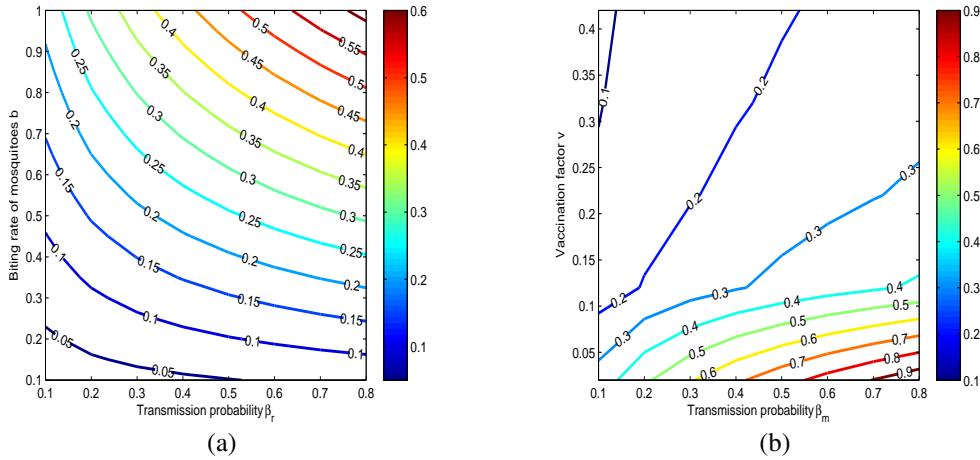
the above system (17) can be written in the following by applying the basic theory

$$(18) \quad y(t) - y(0) = \frac{1}{\Gamma(\rho)} \int_0^t g(\zeta, y(\zeta))(t - \zeta)^{\rho-1} d\zeta,$$

which can be further converted into

$$(19) \quad y(t_{n+1}) - y(0) = \frac{1}{\Gamma(\rho)} \int_0^{t_{n+1}} (t_{n+1} - t)^{\rho-1} g(t, y(t)) dt,$$

Parameter	Interpretation	PRCC values	p values
$\beta_r$	Transmission from mosquitoes to susceptible ruminants	+0.6727	0.0000
$\beta_m$	Transmission from ruminants to susceptible mosquitoes	+0.5433	0.0000
$d_r$	Host ruminants natural death rate	-0.4841	0.0000
$\gamma$	Recovery rate of host individuals	-0.2101	0.0000
$\delta$	Disease induced death rate	-0.2444	0.0000
$\alpha$	Efficacy of vaccine or strength of vaccine	-0.1119	0.0004
$\nu$	Vaccinated fraction of susceptible ruminants	-0.2541	0.0000
$\rho$	Recovery rate through vaccination	+0.0726	0.0223
$d_m$	Vector mosquitoes natural death rate	-0.7919	0.0000
$b$	Biting rate of vector mosquitoes	+0.8790	0.0000

TABLE 1. Sensitivity results of  $\mathcal{R}_0$  with PRCC and corresponding p values.FIGURE 2. Illustration of the reproduction parameter  $\mathcal{R}_0$  (a) with the variation of biting rate  $b$  and transmission probability  $\beta_r$ , (b) and with the variation of vaccination rate  $\nu$  and transmission probability  $\beta_m$ .

where  $t = t_{n+1}$ ,  $n \in N$ , and

$$(20) \quad y(t_n) - y(0) = \frac{1}{\Gamma(\rho)} \int_0^{t_n} (t_n - t)^{\rho-1} g(t, y(t)) dt.$$

Equation (20) and (19) further implies that

$$y(t_{n+1}) = y(t_n) + \frac{1}{\Gamma(\rho)} \int_0^{t_{n+1}} (t_{n+1} - t)^{\rho-1} g(t, y(t)) dt$$



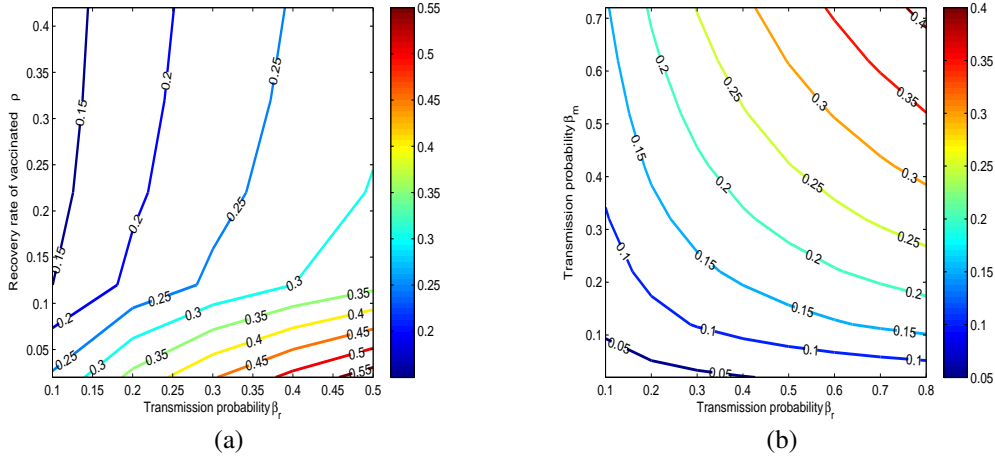


FIGURE 3. Illustration of basic reproduction number  $\mathcal{R}_0$  (a) with the variation transmission probability  $\beta_r$  and  $\rho$ , (b) and with the variation of transmission probability  $\beta_r$  and  $\beta_m$ .

$$(21) \quad -\frac{1}{\Gamma(\rho)} \int_0^{t_n} (t_n - t)^{\rho-1} g(t, y(t)) dt.$$

Let us assume the following

$$(22) \quad \mathcal{B}_{\rho,1} = \frac{1}{\Gamma(\rho)} \int_0^{t_{n+1}} (t_{n+1} - t)^{\rho-1} g(t, y(t)) dt,$$

and

$$(23) \quad \mathcal{B}_{\rho,2} = \frac{1}{\Gamma(\rho)} \int_0^{t_n} (t_n - t)^{\rho-1} g(t, y(t)) dt.$$

Here, we utilized lagrange approximation on the  $g(t, y(t))$ , the following is obtained

$$(24) \quad \begin{aligned} \mathcal{P}(t) &\simeq g(t_n, y_n) \frac{t - t_{n-1}}{t_n - t_{n-1}} + g(t_{n-1}, y_{n-1}) \frac{t - t_n}{t_{n-1} - t_n} \\ &= \frac{g(t_n, y_n)}{h} (t - t_{n-1}) - \frac{g(t_{n-1}, y_{n-1})}{h} (t - t_n). \end{aligned}$$

By applying the above, we get the following

$$(25) \quad \begin{aligned} \mathcal{B}_{\rho,1} &= \frac{g(t_n, y_n)}{h\Gamma(\rho)} \int_0^{t_{n+1}} (t - t_{n-1})(t_{n+1} - t)^{\rho-1} dt \\ &\quad - \frac{g(t_{n-1}, y_{n-1})}{h\Gamma(\rho)} \int_0^{t_{n+1}} (t - t_n)(t_{n+1} - t)^{\rho-1} dt. \end{aligned}$$

After simplification, we obtain

$$\begin{aligned} \mathcal{B}_{\rho,1} &= \left( \frac{2h}{\alpha} t_{n+1}^\alpha - \frac{t_{n+1}^{\rho+1}}{\rho+1} \right) \frac{g(t_n, y_n)}{h\Gamma(\rho)} \\ (26) \quad &- \left( \frac{h}{\rho} t_{n+1}^\rho - \frac{1}{\rho+1} t_{n+1}^{\rho+1} \right) \frac{g(t_{n-1}, y_{n-1})}{h\Gamma(\rho)}. \end{aligned}$$

In the same way

$$\begin{aligned} \mathcal{B}_{\rho,2} &= \frac{1}{\Gamma(\rho)} \int_0^{t_n} (t_n - t)^{\rho-1} \left( \frac{g(t_n, y_n)}{h} (t - t_{n-1}) \right. \\ (27) \quad &\left. - \frac{g(t_{n-1}, y_{n-1})}{h} (t - t_n) \right) dt. \end{aligned}$$

This further implies that

$$\begin{aligned} \mathcal{B}_{\rho,2} &= \left( \frac{h}{\rho} t_n^\rho - \frac{t_n^{\rho+1}}{\rho+1} \right) \frac{g(t_n, y_n)}{h\Gamma(\rho)} \\ (28) \quad &+ \left( \frac{1}{\rho+1} t_n^{\rho+1} \right) \frac{g(t_{n-1}, y_{n-1})}{h\Gamma(\rho)}. \end{aligned}$$

To obtain the required approximate solution of the proposed Rift Valley fever fractional model in Caputo framework, we substitute (27) and (28) in (21), and get the following

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + \frac{g(t_n, y_n)}{h\Gamma(\rho)} \left( \frac{2ht_{n+1}^\rho}{\rho} - \frac{t_{n+1}^{\rho+1}}{\rho+1} + \frac{h}{\rho} t_n^\rho - \frac{t_{n+1}^{\rho+1}}{\rho+1} \right) \\ (29) \quad &+ \frac{g(t_{n-1}, y_{n-1})}{h\Gamma(\rho)} \left( -\frac{h}{\rho} t_{n+1}^\rho + \frac{t_{n+1}^{\rho+1}}{\rho+1} + \frac{t_n^{\rho+1}}{\rho+1} \right). \end{aligned}$$

For the required approximate solution of the proposed fractional model, we use the above technique for the solution of our Rift Valley fever model in Caputo sense (7). In our numerical simulations, we represent the dynamics of Rift Valley fever with the variation of fractional-order  $\vartheta$  in Figure. 4 and Figure. 5 in the Caputo framework. We observed that the infected individuals can be controlled by controlling the fractional-order of the system. This means that the fractional-order is a significant parameter for the control of Rift Valley fever and the infection can be prevented through this parameter.

Symbol	Interpretation	Values	Reference
$d_r$	Ruminants natural death rate	0.000481	[37]
$v$	Vaccination factor of susceptible ruminants to $V_r$	0.7	Assumed
$\gamma$	Recovery rate of infected ruminants	0.0875	[38]
$\vartheta$	Fractional order or index of memory	variable	Assumed
$b$	Biting rate of vectors	0.701	[39]
$\beta_r$	Transmission from mosquitoes to susceptible ruminants	0.14	[40]
$\alpha$	Efficacy of vaccine or strength of vaccine	0.6	Assumed
$\rho$	Recovery through vaccination of ruminants host	0.5	Assumed
$\beta_m$	Transmission from ruminants to susceptible mosquitoes	0.35	[40]
$d_m$	Mosquitoes natural death rate	0.0166	[41]
$\Pi_r$	Recruitment rate of ruminant hosts	Variable	Assumed
$\Pi_r$	Recruitment rate of mosquitoes vectors	Variable	Assumed

TABLE 2. Description with values of input factors for Rift Valley fever

## 5. ATANGANA-BALEANU STRUCTURE OF THE MODEL

Here in this part of the paper, we will investigate the proposed dynamical model of Rift Valley fever in Atangana-Baleanu fractional framework. The system of equation (6) in Atangana-Baleanu framework can be represented as

$$(30) \quad \begin{cases} {}_0^{ABC}D_t^\vartheta S_r &= \Pi_r - b\beta_r S_r I_m - vS_r - d_r S_r, \\ {}_0^{ABC}D_t^\vartheta V_r &= vS_r - (1 - \alpha)b\beta_r V_r I_m - \rho V_r - d_r V_r, \\ {}_0^{ABC}D_t^\vartheta I_r &= b\beta_r S_r I_m + (1 - \alpha)b\beta_r V_r I_m - (d_r + \gamma + \delta)I_r, \\ {}_0^{ABC}D_t^\vartheta R_r &= \rho V_r + \gamma I_r - d_r R_r, \\ {}_0^{ABC}D_t^\vartheta S_m &= \Pi_m - b\beta_m S_m I_r - d_m S_m, \\ {}_0^{ABC}D_t^\vartheta I_m &= b\beta_m S_m I_r - d_m I_m. \end{cases}$$

After this, fixed point theory is used to present the existence and uniqueness of the RVF model (30) solution. Then the fractional model (30) can be written in the following way

$$(31) \quad \begin{aligned} {}_0^{ABC}D_t^\vartheta r(t) &= \mathcal{U}(t, r(t)), \\ r(0) &= r_0, \quad 0 < t < T < \infty. \end{aligned}$$

In the above system (31), the vector function  $\mathcal{U}$  is continuous and assume that  $z_1 = S_r, z_2 = V_r, z_3 = I_r, z_4 = R_r, z_5 = S_m$  and  $z_6 = I_m$ , moreover, the state variables are denoted by  $r(t) = (z_1, z_2, z_3, z_4, z_5, z_6)$  and the vector function  $\mathcal{U}$  is given in the following form

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \\ \mathcal{U}_3 \\ \mathcal{U}_4 \\ \mathcal{U}_5 \\ \mathcal{U}_6 \end{pmatrix} = \begin{pmatrix} \Pi_r - b\beta_r S_r I_m - vS_r - d_r S_r \\ vS_r - (1 - \alpha)b\beta_r V_r I_m - \rho V_r - d_r V_r \\ b\beta_r S_r I_m + (1 - \alpha)b\beta_r V_r I_m - (d_r + \gamma + \delta)I_r \\ \rho V_r + \gamma I_r - d_r R_r \\ \Pi_m - b\beta_m S_m I_r - d_m S_m \\ b\beta_m (\widehat{N}_m - I_m)I_r - d_m I_m \end{pmatrix},$$

with suitable conditions are given by  $r_0(t) = (S_r(0), V_r(0), I_r(0), I_v(0))$ . In addition, the Lipschitz condition is fulfilled by  $\mathcal{U}$ , and is given as

$$(32) \quad \|\mathcal{U}(t, r_1(t)) - \mathcal{U}(t, r_2(t))\| \leq \mathcal{H} \|r_1(t) - r_2(t)\|.$$

In the next step, the existence and uniqueness of the proposed Rift Valley fever system (30) will be presented.

**Theorem 12.** *The solution of Rift Valley fever system (30) in Atangana Baleanu(AB) framework will be unique if the following condition holds true*

$$(33) \quad \frac{(1 - \vartheta)\mathcal{H}}{ABC(\vartheta)} + \frac{\vartheta T_{\max}^{\vartheta}\mathcal{H}}{ABC(\vartheta)\Gamma(\vartheta)} < 1.$$

**Proof 12** To prove the target, we use Definition (4) of fractional integral to the system (31), and obtain the integral equation of the structure

$$(34) \quad r(t) = r_0 + \frac{(1 - \vartheta)\mathcal{U}(t, r(t))}{ABC(\vartheta)} + \frac{\vartheta}{ABC(\vartheta)\Gamma(\vartheta)} \int_0^t (t - \kappa)^{\vartheta-1} \mathcal{U}(\kappa, r(\kappa)) d\kappa.$$

Further, we take  $I = (0, T)$ , and  $\Theta : C(I, R^4) \rightarrow C(I, R^4)$  is given by

$$(35) \quad \Theta[r(t)] = r_0 + \frac{(1 - \vartheta)\mathcal{U}(t, r(t))}{ABC(\vartheta)} + \frac{\vartheta}{ABC(\vartheta)\Gamma(\vartheta)} \int_0^t (t - \kappa)^{\vartheta-1} \mathcal{U}(\kappa, r(\kappa)) d\kappa,$$

then (34) can be expressed in the following form

$$(36) \quad r(t) = \Theta[r(t)].$$

Here, the norm  $\|\cdot\|_I$  on  $I$  indicates the supremum norm and is given as

$$(37) \quad \|r(t)\|_I = \sup_{t \in I} \|r(t)\|, \quad r(t) \in \mathbb{C}.$$

It is evident that the  $\|\cdot\|_I$  on  $C(I, R^4)$  make a Banach space, moreover, the inequality mentioned below can be smoothly realized that

$$(38) \quad \left\| \int_0^t \mathcal{K}(t, \kappa) r(\kappa) d\kappa \right\| \leq T \|\mathcal{K}(t, \kappa)\|_I \|r(t)\|_I,$$

in which  $\mathcal{K}(t, \kappa) \in C(I^2, R)$ ,  $r(t) \in C(I, R^4)$  in order to enable

$$(39) \quad \|\mathcal{K}(t, \kappa)\|_I = \sup_{t, \kappa \in I} |\mathcal{K}(t, \kappa)|.$$

Applying the operator  $\Theta$  defined in (36), we get the following

$$(40) \quad \begin{aligned} \|\Theta[r_1(t)] - \Theta[r_2(t)]\|_I &\leq \left\| \frac{(1-\vartheta)}{ABC(\vartheta)} (\mathcal{U}(t, r_1(t)) - \mathcal{U}(t, r_2(t))) + \frac{\vartheta}{ABC(\vartheta)\Gamma(\vartheta)} \right. \\ &\quad \left. \times \int_0^t (t-\kappa)^{\vartheta-1} (\mathcal{U}(\kappa, r_1(\kappa)) - \mathcal{U}(\kappa, r_2(\kappa))) d\kappa \right\|_I. \end{aligned}$$

Furthermore, using Lipschitz condition (32) with the result in (38), the below is obtained after calculations

$$(41) \quad \|\Theta[r_1(t)] - \Theta[r_2(t)]\|_I \leq \left[ \frac{(1-\vartheta)\mathcal{H}}{ABC(\vartheta)} + \frac{\vartheta\mathcal{H}}{ABC(\vartheta)\Gamma(\vartheta)} T_{max}^{\vartheta} \right] \|r_1(t) - r_2(t)\|_I.$$

As a consequence of this, we get the following

$$(42) \quad \|\Theta[r_1(t)] - \Theta[r_2(t)]\|_I \leq \mathcal{D} \|r_1(t) - r_2(t)\|_I,$$

in which

$$\mathcal{D} = \frac{(1-\vartheta)\mathcal{H}}{ABC(\vartheta)} + \frac{\vartheta\mathcal{H}}{ABC(\vartheta)\Gamma(\vartheta)} T_{max}^{\vartheta}.$$

It is clear that if the condition in (33) holds true then the operator  $\Theta$  will be contraction. This implies that the fractional system (30) of Rift Valley fever has a unique solution.

**5.1. Numerical scheme for AB operator.** In this section, we will propose a numerical scheme for the numerical solution of the Rift Valley fever system (30). To do this, we first present a numerical method for the proposed fractional model with the non-local and non-singular kernel to illustrate the solution pathway. The recent developed iterative scheme [48] is used for the fractional Rift Valley fever system (30). This iterative method is applied to our system in the following way to obtain the approximate solution of AB operator. The proposed fractional system (31) can be converted into the below integral form

$$(43) \quad r(t) - r(0) = \frac{(1 - \vartheta)}{ABC(\vartheta)} \mathcal{U}(t, r(t)) + \frac{\vartheta}{ABC(\vartheta) \times \Gamma(\vartheta)} \int_0^t \mathcal{U}(\kappa, r(\kappa)) (t - \kappa)^{\vartheta-1} d\kappa,$$

where the time  $t = t_{l+1}$ ,  $l = 0, 1, 2, \dots$ , we get the following

$$(44) \quad \begin{aligned} r(t_{l+1}) - r(0) &= \frac{1 - \vartheta}{ABC(\vartheta)} \mathcal{U}(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta) \times \Gamma(\vartheta)} \\ &\quad \times \int_0^{t_{l+1}} \mathcal{U}(\kappa, r(\kappa)) (t_{l+1} - \kappa)^{\vartheta-1} d\kappa, \\ &= \frac{1 - \vartheta}{ABC(\vartheta)} \mathcal{U}(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta) \times \Gamma(\vartheta)} \\ &\quad \times \sum_{j=0}^l \int_{t_j}^{t_{j+1}} \mathcal{U}(\kappa, r(\kappa)) (t_{l+1} - \kappa)^{\vartheta-1} d\kappa. \end{aligned}$$

In the next step, we estimate  $\mathcal{U}(\kappa, r(\kappa))$  over the time interval  $[t_j, t_{j+1}]$ , and use the following interpolation

$$(45) \quad \mathcal{U}(\kappa, r(\kappa)) \cong \mathcal{U}(t_j, r(t_j)) \frac{(t - t_{j-1})}{h} - \mathcal{U}(t_{j-1}, r(t_{j-1})) \frac{(t - t_j)}{h}.$$

putting in equation (44), we obtain the following

$$(46) \quad \begin{aligned} r(t_{l+1}) &= r(0) + \frac{1 - \vartheta}{ABC(\vartheta)} \mathcal{U}(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta) \times \Gamma(\vartheta)} \\ &\quad \times \sum_{j=0}^l \left( \frac{\mathcal{U}(t_j, r(t_j))}{h} \int_{t_j}^{t_{j+1}} (t - t_{j-1})(t_{l+1} - t)^{\vartheta-1} dt \right. \\ &\quad \left. - \frac{\mathcal{U}(t_{j-1}, r(t_{j-1}))}{h} \int_{t_j}^{t_{j+1}} (t - t_j)(t_{l+1} - t)^{\vartheta-1} dt \right). \end{aligned}$$

Finally, we obtain the approximate solution after simplification of the integral as

$$\begin{aligned}
 r(t_{l+1}) &= r(t_0) + \frac{1-\vartheta}{ABC(\vartheta)} \mathcal{U}(t_l, e(t_l)) + \frac{\vartheta}{ABC(\vartheta)} \sum_{j=0}^l \\
 &\quad \left[ \frac{h^\vartheta \mathcal{U}(t_j, r(t_j))}{\Gamma(\vartheta+2)} ((l+1-j)^\vartheta (l-j+2+\vartheta) - (l-j)^\vartheta (l-j+2+2\vartheta)) \right. \\
 (47) \quad &\quad \left. - \frac{h^\vartheta \mathcal{U}(t_{j-1}, r(t_{j-1}))}{\Gamma(\vartheta+2)} ((l+1-j)^{\vartheta+1} - (\vartheta-j)^\vartheta (l-j+1+\vartheta)) \right].
 \end{aligned}$$

As a result, we get the following approximate solution for our proposed fractional system Atangana-Beleanu framework

$$\begin{aligned}
 S_r(t_{l+1}) &= S_r(t_0) + \frac{1-\vartheta}{ABC(\vartheta)} \mathcal{U}_1(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta)} \sum_{j=0}^l \\
 &\quad \left[ \frac{h^\vartheta \mathcal{U}_1(t_j, r(t_j))}{\Gamma(\vartheta+2)} ((l+1-j)^\vartheta (l-j+2+\vartheta) - (l-j)^\vartheta (l-j+2+2\vartheta)) \right. \\
 &\quad \left. - \frac{h^\vartheta \mathcal{U}_1(t_{j-1}, r(t_{j-1}))}{\Gamma(\vartheta+2)} ((l+1-j)^{\vartheta+1} - (l-j)^\vartheta (l-j+1+\vartheta)) \right],
 \end{aligned}$$

$$\begin{aligned}
 V_r(t_{l+1}) &= V_r(t_0) + \frac{1-\vartheta}{ABC(\vartheta)} \mathcal{U}_2(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta)} \sum_{j=0}^l \\
 &\quad \left[ \frac{f^\vartheta \mathcal{U}_2(t_j, r(t_j))}{\Gamma(\vartheta+2)} ((l+1-j)^\vartheta (l-j+2+\vartheta) - (l-j)^\vartheta (l-j+2+2\vartheta)) \right. \\
 &\quad \left. - \frac{h^\vartheta \mathcal{U}_2(t_{j-1}, r(t_{j-1}))}{\Gamma(\vartheta+2)} ((l+1-j)^{\vartheta+1} - (l-j)^\vartheta (l-j+1+\vartheta)) \right],
 \end{aligned}$$

$$\begin{aligned}
 I_r(t_{\kappa+1}) &= I_r(t_0) + \frac{1-\vartheta}{ABC(\vartheta)} \mathcal{U}_3(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta)} \sum_{j=0}^l \\
 &\quad \left[ \frac{h^\vartheta \mathcal{U}_3(t_j, r(t_j))}{\Gamma(\vartheta+2)} ((l+1-j)^\vartheta (l-j+2+\vartheta) - (l-j)^\vartheta (l-j+2+2\vartheta)) \right. \\
 &\quad \left. - \frac{h^\vartheta \mathcal{U}_3(t_{j-1}, r(t_{j-1}))}{\Gamma(\vartheta+2)} ((l+1-j)^{\vartheta+1} - (l-j)^\vartheta (l-j+1+\vartheta)) \right],
 \end{aligned}$$

$$R_r(t_{l+1}) = R_r(t_0) + \frac{1-\vartheta}{ABC(\vartheta)} \mathcal{U}_4(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta)} \sum_{j=0}^l$$

$$\begin{aligned}
& \left[ \frac{f^\vartheta \mathcal{U}_4(t_j, r(t_j))}{\Gamma(\vartheta + 2)} ((l + 1 - j)^\vartheta (l - j + 2 + \vartheta) - (l - j)^\vartheta (l - j + 2 + 2\vartheta)) \right. \\
& \quad \left. - \frac{f^\vartheta \mathcal{U}_4(t_{j-1}, r(t_{j-1}))}{\Gamma(\vartheta + 2)} ((l + 1 - j)^{\vartheta+1} - (l - j)^\vartheta (l - j + 1 + \vartheta)) \right], \\
S_m(t_{l+1}) &= S_m(t_0) + \frac{1 - \vartheta}{ABC(\vartheta)} \mathcal{U}_5(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta)} \sum_{j=0}^l \\
& \left[ \frac{f^\vartheta \mathcal{U}_5(t_j, r(t_j))}{\Gamma(\vartheta + 2)} ((l + 1 - j)^\vartheta (l - j + 2 + \vartheta) - (l - j)^\vartheta (l - j + 2 + 2\vartheta)) \right. \\
& \quad \left. - \frac{f^\vartheta \mathcal{U}_5(t_{j-1}, r(t_{j-1}))}{\Gamma(\vartheta + 2)} ((l + 1 - j)^{\vartheta+1} - (l - j)^\vartheta (l - j + 1 + \vartheta)) \right], \\
I_m(t_{l+1}) &= I_m(t_0) + \frac{1 - \vartheta}{ABC(\vartheta)} \mathcal{U}_6(t_l, r(t_l)) + \frac{\vartheta}{ABC(\vartheta)} \sum_{j=0}^l \\
& \left[ \frac{f^\vartheta \mathcal{U}_6(t_j, r(t_j))}{\Gamma(\vartheta + 2)} ((l + 1 - j)^\vartheta (l - j + 2 + \vartheta) - (l - j)^\vartheta (l - j + 2 + 2\vartheta)) \right. \\
(48) \quad & \left. - \frac{f^\vartheta \mathcal{U}_6(t_{j-1}, r(t_{j-1}))}{\Gamma(\vartheta + 2)} ((l + 1 - j)^{\vartheta+1} - (l - j)^\vartheta (l - j + 1 + \vartheta)) \right].
\end{aligned}$$

We will use the above numerical scheme (48) to investigate the results of the proposed AB fractional model to illustrate the transmission pathway of Rift Valley fever. First, we demonstrate the influence of fractional-order  $\vartheta$  on the dynamics of dengue in the Figure. 6 and then compare the results of fractional operators in the Figure. 7. In Figure. 6, we observed the influence of  $\vartheta$  on the system in Atangana-Baleanu framework, furthermore, we illustrated that the numerical results of Atangana-Baleanu operator more accurate as compared to the other operator.

## 6. CONCLUSION

In this article, we structured an epidemic model for Rift Valley fever with vaccination through the fractional derivative. We determined the essential properties of the proposed fractional-order Rift Valley fever model through the basic knowledge of fractional calculus, which includes positivity, uniqueness, and boundedness of the solution. We investigate the our model of RVF for equilibria and determined the basic reproduction number of the system through next-generation



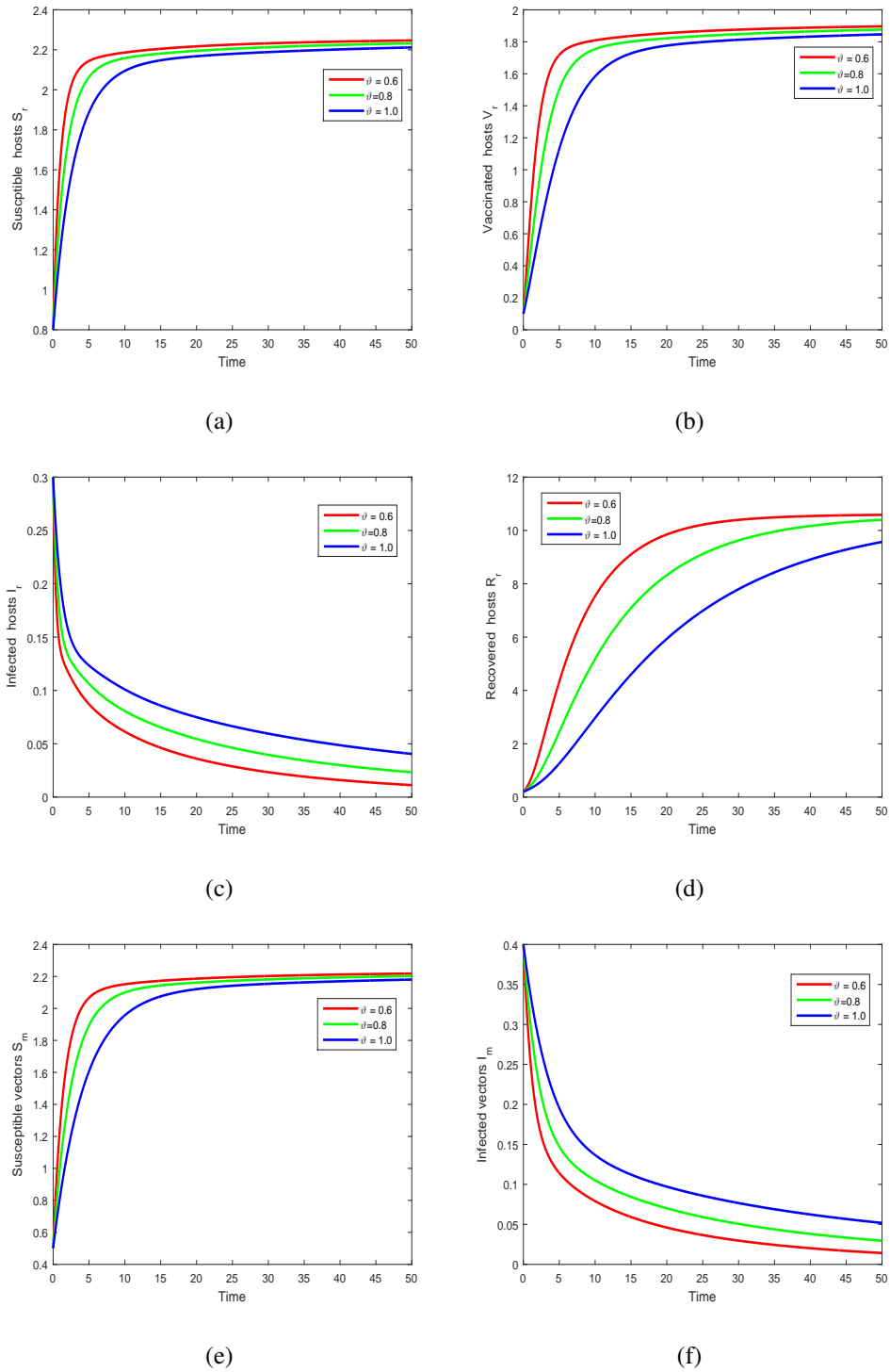
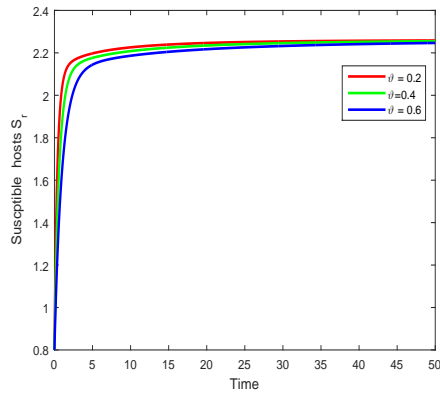
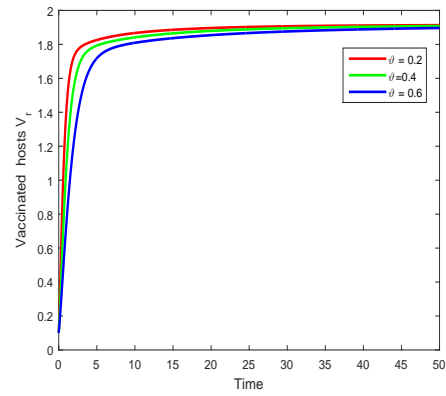


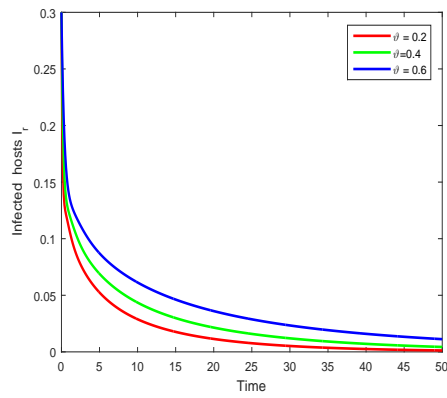
FIGURE 4. Illustration of the transmission pathway of hosts and vectors population in Caputo's framework with fractional-order  $\vartheta = 1.0, 0.8, 0.6$ .



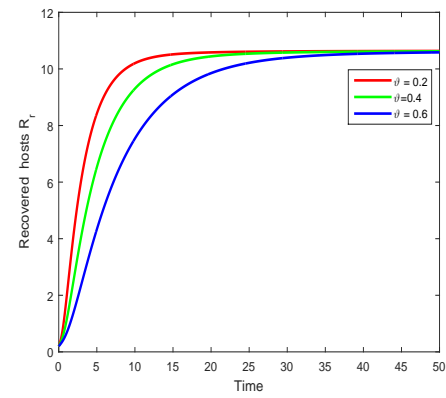
(a)



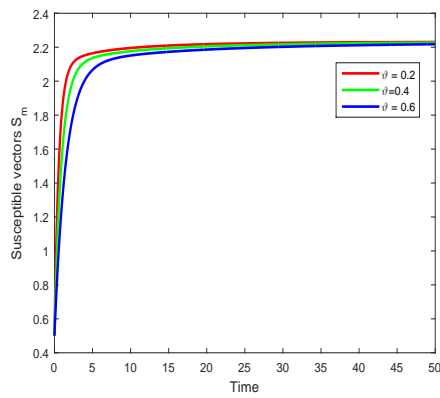
(b)



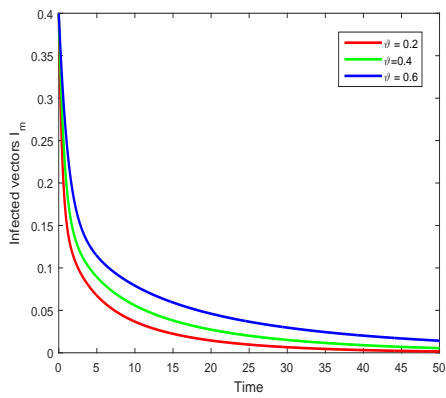
(c)



(d)



(e)



(f)

FIGURE 5. Illustration of the transmission pathway of hosts and vectors population in Caputo's framework with fractional-order  $\vartheta = 0.6, 0.4, 0.2$ .

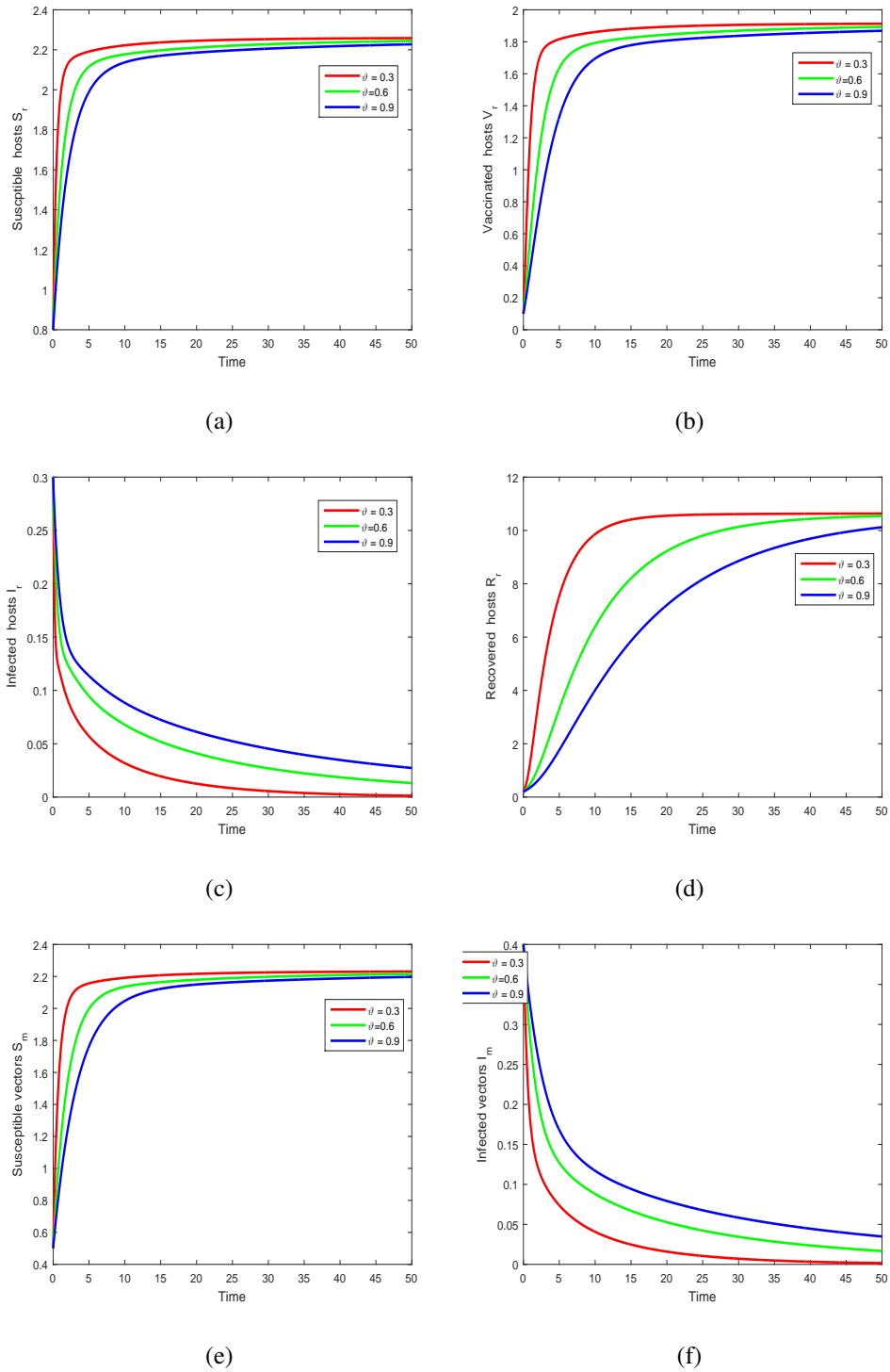


FIGURE 6. Illustration of the transmission pathway of hosts and vectors population in Atangana-Baleanu framework with fractional-order  $\vartheta = 0.3, 0.6, 0.9$ .

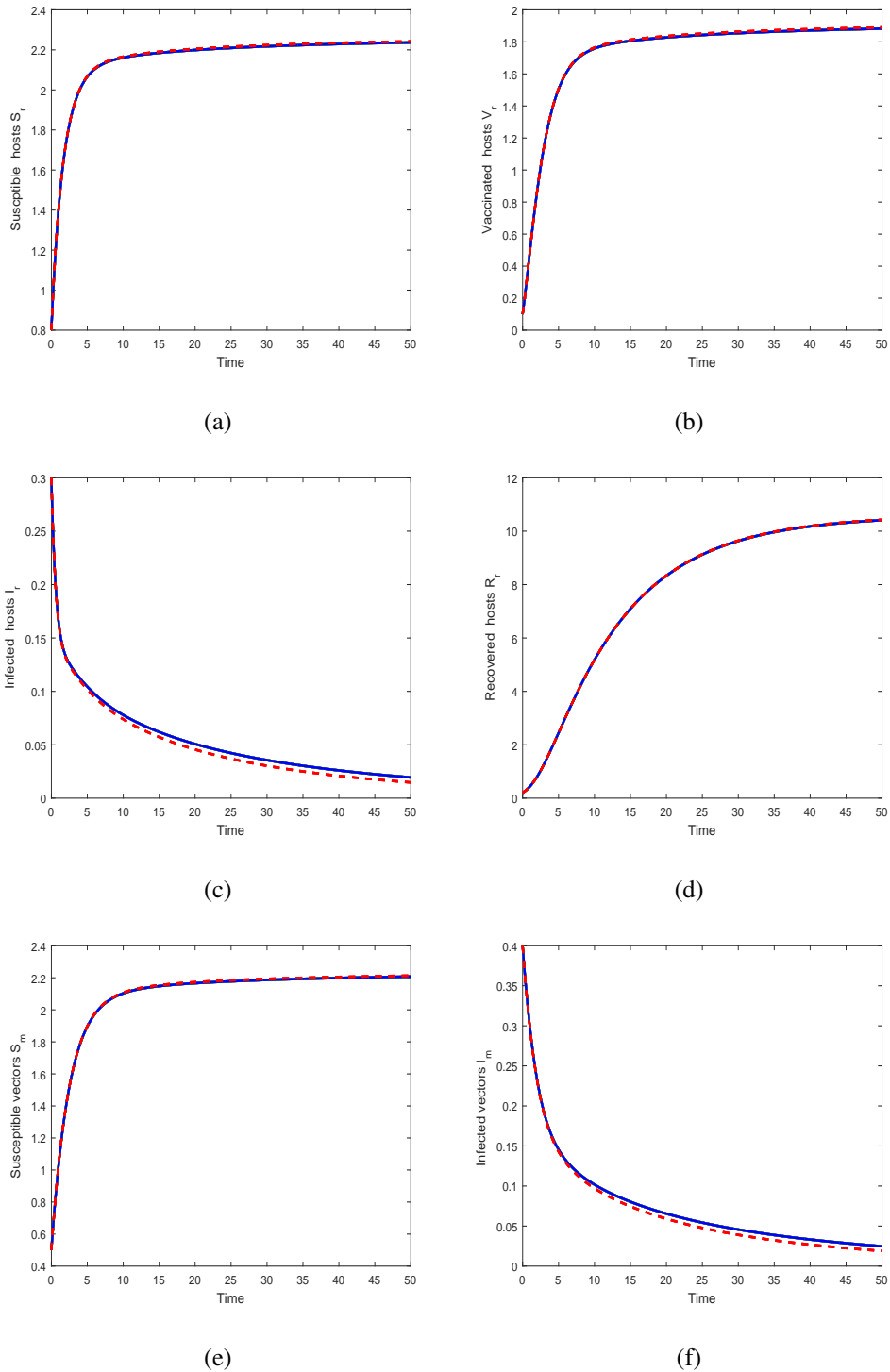


FIGURE 7. Illustration of the time series of hosts and vectors populations with Caputo and Atangana-Baleanu fractional derivatives, where the blue lines represents the dynamics of Rift Valley fever with Caputo and the red lines represents with Atangana-Baleanu fractional derivatives.

technique, indicated by  $\mathcal{R}_0$ . The stability results are established for the infection-free steady-state of the system. Moreover, the sensitivity of  $\mathcal{R}_0$  is analyzed through the partial rank correlation coefficient (PRCC) technique to show the importance of different input factors in  $\mathcal{R}_0$ . Then the Rift Valley fever model is investigated in Atangana-Baleanu sense, in addition, a numerical scheme for the mentioned fractional operators presented to demonstrate the solution pathway of the model. We noticed that the fractional-order dynamics can explain the complex system of Rift Valley fever infection more precisely and accurately rather than the integer-order dynamics. It is also observed that the Atangana-Baleanu operator provides more accurate results than the Caputo fractional operator.

### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

### REFERENCES

- [1] R. Daubney, J.R. Hudson, P.C. Granham, Enzootic hepatitis or Rift Valley Fever: an underscribed virus disease of sheep cattle and man from East Africa, *J. Pathol. Bacteriol.* 34 (1931), 545-579.
- [2] S. Abdo-Salem, A. Tran, V. Grosbois, G. Gerbier, M. Al-Qadasi, K. Saeed, E. Etter, E. Thiry, F. Roger, V. Chevalier, Can Environmental and Socioeconomic Factors Explain the Recent Emergence of Rift Valley Fever in Yemen, 2000–2001? *Vector Borne Zoonotic Dis.* 11 (2011), 773–779.
- [3] Mebus Charles, A: Rift Valley Fever. *AccessScience*. <http://www.accessscience.com/content/rift-valley-fever/757367> (2014). Accessed 12 April 2016
- [4] B. Li, C. Qin, X. Wang, Analysis of an SIRS epidemic model with nonlinear incidence and vaccination, *Commun. Math. Biol. Neurosci.* 2020 (2020), Article ID 2.
- [5] R. Jan, M.A. Khan, J.F. Gomez-Aguilar, Asymptomatic carriers in transmission dynamics of dengue with control interventions, *Optim. Control Appl. Meth.* 41 (2020), 430-447.
- [6] S. Bakht, K. Hattaf, N.I. Fatmi, Dynamics of a class of viral infection models with diffusion, *Commun. Math. Biol. Neurosci.*, 2020 (2020), Article ID 5.
- [7] A. Anyamba, J.P. Chretien, J. Small, C.J. Tucker, P.B. Formenty, J.H. Richardson, S.C. Britch, D.C. Schnabel, R.L. Erickson, Prediction of a Rift Valley fever outbreak, *Proc. Natl. Acad. Sci. USA*, 106 (2009), 955-959.
- [8] B.J. Barnard, Rift Valley fever vaccine-antibody and immune response in cattle to a live and an inactivated vaccine, *J. South Afr. Vet. Assoc.* 50 (1979), 155-157.

- [9] N. Chitnis, J.M. Hyman, C.A. Manore, Modelling vertical transmission in vector-borne diseases with applications to Rift Valley fever, *J. Biol. Dyn.* 7 (2013), 11-40.
- [10] I.H.A. El-Rahim, U. Abd El-Hakim, M. Hussein, An epizootic of Rift Valley fever in Egypt in 1997. *Rev. Sci. Tech.* 18 (1999), 741–748
- [11] Y. Xiao, J.C. Beier, R.S. Cantrell, C. Cosner, D.L. DeAngelis, S.G. Ruan, Modelling the effects of seasonality and socioeconomic impact on the transmission of Rift Valley fever virus, *PLoS Negl. Trop. Dis.* 9 (2015), e3388.
- [12] H.D. Gaff, D.M. Hartley, N.P. Leahy, An epidemiological model of Rift Valley fever, *Electron. J. Differ. Equ.*, 2007 (2007), 115.
- [13] S.C. Mpeshe, H. Haario, J.M. Tchuente, A mathematical model of Rift Valley fever with human host, *Acta Biotheor.* 59 (2011), 231-250.
- [14] L. Xue, H.M. Scott, L.W. Cohnstaedt, C. Scoglio, A network-based meta-population approach to model Rift Valley fever epidemics, *J. Theor. Biol.* 306 (2012), 129-144.
- [15] D.Z. Gao, C. Cosner, R.S. Cantrell, J.C. Beier, S.G. Ruan, Modeling the spatial spread of Rift Valley fever in Egypt, *Bull. Math. Biol.* 75 (2013), 523-542.
- [16] S.I. Oke, M.M. Ojo, M.O. Adeniyi, M.B. Matadi, Mathematical modeling of malaria disease with control strategy, *Commun. Math. Biol. Neurosci.* 2020 (2020), Article ID 43.
- [17] R. Artiono, B.P. Prawoto, The effects of vaccination to the dynamics of rubella virus with seasonality, *Commun. Math. Biol. Neurosci.* 2020 (2020), Article ID 9.
- [18] T. Ikegami, S. Makino, Rift valley fever vaccines, *Vaccine.* 27 (2009), D69-D72.
- [19] F. Chamchod, R. Stephen, C. Cosner, A.N. Hassan, J.C. Beie, S.G. Ruan, A modeling approach to investigate epizootic outbreaks and enzootic maintenance of Rift Valley fever virus, *Bull. Math. Biol.* 76 (2014), 2052-2072.
- [20] R. Jan, Y. Xiao, Effect of partial immunity on transmission dynamics of dengue disease with optimal control, *Math. Meth. Appl. Sci.* 42 (2019), 1967-1983.
- [21] R. Jan, Y. Xiao, Effect of pulse vaccination on dynamics of dengue with periodic transmission functions. *Adv. Differ. Equ.* 2019 (2019), 368.
- [22] S.K. Elagan, M. Sayed, M. Higazy, An Analytical Study on Fractional Partial Differential Equations by Laplace Transform Operator Method, *Int. J. Appl. Eng. Res.* 13 (2018), 545-549.
- [23] M.C. Tripathy, Application of Fractional Capacitor and Fractional Inductor in Automatic Voltage Regulator, *Int. J. Appl. Eng. Res.* 14 (2019), 2812-2816.
- [24] A.M.S. Mahdy, G.M.A. Marai, Numerical Solution for the time-fractional Fokker-Planck equation Using Fractional Power Series Method and The shifted Chebyshev polynomials of the third kind, *Int. J. Appl. Eng. Res.* 13 (2018), 366-374.

- [25] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.
- [26] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* 1 (2015), 73-85.
- [27] I. Ameen, The effect of vaccination and treatment of measles disease described by a fractional order model, *World J. Model. Simul.* 14 (2018), 30-38.
- [28] R. Jan, M.A. Khan, P. Kumam, P. Thounthong, Modeling the transmission of dengue infection through fractional derivatives, *Chaos Solitons Fractals*, 127 (2019), 189-216.
- [29] A. Atangana, E.F.D. Goufo, Computational Analysis of the Model Describing HIV Infection of CD4+T Cells, *Biomed. Res. Int.* 2014 (2014), 618404.
- [30] M. Abdulhameed, D. Vieru, R. Roslan, Magneto-hydrodynamic electroosmotic flow of maxwell fluids with Caputo-Fabrizio derivatives through circular tubes, *Comput. Math. Appl.* 74 (2017), 2503-2519.
- [31] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using operators of Caputo type*, Springer, (2004).
- [32] S. Mao, R. Xu, and Y. Li, A fractional order SIRS model with standard incidence rate, *J. Beihua Univ.* 12 (2012), 379-382.
- [33] Z.M. Odibat, N.T. Shawagfeh, Generalized Taylor's formula, *Appl. Math. Comput.* 186 (2007), 286-293.
- [34] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations* (Vol. 204), Elsevier, (2006).
- [35] J.A.P. Heesterbeek, A brief history of  $R_0$  and a recipe for its calculation, *Acta Biotheor.* 50 (2002), 189-204.
- [36] J. M. Heffernan, R. J. Smith, L. M. Wahl, Perspectives on basic reproduction ratio, *J. R. Soc. Interface.* 2 (2005), 281-293.
- [37] A.A. Majok, K.H. Zessin, M.P.O. Baumann, T.B. Farver, Analyses of baseline survey data on rinderpest in Bahr el Ghazal Province, with proposal of an improved vaccination strategy against rinderpest for southern Sudan, *Trop. Anim. Health Prod.* 23 (1991), 186-196.
- [38] F. Chamchod, R.S. Cantrell, C. Cosner, A.N. Hassan, J.C. Beier, S. Ruan, A Modeling Approach to Investigate Epizootic Outbreaks and enzootic Maintenance of Rift Valley Fever Virus, *Bull. Math. Biol.* 76 (2014), 2052-2072.
- [39] D.V. Canyon, J.L.K. Hii, R. Muller, The frequency of host biting and its effect on oviposition and survival in *Aedes aegypti* (Diptera: Culicidae), *Bull. Entomol. Res.* 89 (1999), 35-39.
- [40] M.J. Turell, K.J. Linthicum, L.A. Patrican, F.G. Davies, A. Kairo, C.I. Bailey, Vector competence of selected African mosquito (Diptera: Culicidae) species for Rift Valley fever virus, *J. Med. Entomol.* 45 (2008), 102-108.

- [41] M.H. Reiskind, C.J. Westbrook, L.P. Lounibos, Exposure to Chikungunya virus and adult longevity in *Aedes aegypti* (L.) and *Aedes albopictus* (Skuse), *J. Vector. Ecol.* 37 (1987), 640-647.
- [42] D. Matignon, Stability results for fractional differential equations with applications to control processing, *Comput. Eng. Syst. Appl.* 2 (1996), 963-968.
- [43] C. Li, Y. Ma. Fractional dynamical system and its linearization theorem, *Nonlinear Dyn.* 71 (2013), 621-633.
- [44] Z. Wang, D. Yang, T. Ma, N. Sun, Stability analysis for nonlinear fractional-order systems based on comparison principle, *Nonlinear Dyn.* 75 (2014), 387-402.
- [45] B. Sara, Z. Omar, T. Abdessamad, R. Mostafa, F. Hanane, Parameters' estimation, sensitivity analysis and model uncertainty for an influenza a mathematical model: case of Morocco, *Commun. Math. Biol. Neurosci.* 2020 (2020), Article ID 57.
- [46] S. Marino, I.B. Hogue, C.J. Ray, D. Kirschner, A methodology for performing global uncertainty and sensitivity analysis in systems biology, *J. Theor. Biol.* 254 (2008), 178-196.
- [47] A. Atangana, K.M. Owolabi, New numerical approach for fractional differential equations, *Math. Model. Nat. Phenom.*, 13 (2018), 3.
- [48] M. Toufik, A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: application to chaotic models, *Eur. Phys. J. Plus.* 132 (2017), 444.