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ON ENTROPY OF PAIRWISE CONTINUOUS MAP IN BITOPOLOGICAL DYNAMICAL SYSTEMS

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Abstract. Topological entropy is an important and widely used measure of complexity in topological dynamical system where only one topology is involved in the entire mathematical process. Bitopological dynamical system is a new area of dynamical system to investigate dynamical properties in terms of a bitopological space which involves two topologies. In this paper we introduce entropy in bitopological dynamical system as a measure of complexity and produce some results related to entropy. Also, we introduce weighted bitopological Shannon entropy as an extension of Shannon entropy in information theory. Recently, Acharjee et al. (S. Acharjee, K. Goswami, HK. Sarmah, on forward iterated Hausdorffness and development of embryo from zygote in bitopological dynamical systems (communicated)) proved that the postgastrulation part of human embryo development is a bitopological dynamical system. As an application of our theory, we find bitopological entropy of the mitosis map in the bitopological space of postgastrulation part of human embryo development. Also, we find the weighted bitopological Shannon entropy in this space.

Keywords: bitopological dynamical system; weakly pairwise compactness; bitopological entropy; weighted bitopological Shannon entropy; embryo; mitosis.

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1. Introduction

Bitopological dynamical system is a new area of dynamical system to investigate dynamical properties in terms of a bitopological space. It overcomes the difficulties faced by topological dynamical system during the study of a system having two parallel states. Recently, Acharjee et al. [1] studied the notion of transitivity in bitopological dynamical systems and produced interesting results related to human embryo development. This motivates us for this paper.

Kelly [2] introduced the concept of bitopological space. Later, bitopological space was extensively studied by many researchers of various branches and the study is still going on. Pervin [3] introduced the concept of pairwise continuity in a bitopological space. For recent theoretical works in bitopological space, one may refer to Acharjee and Tripathy [4], Acharjee et al. [5], Acharjee et al. [6] and many others. Recently, bitopological space has been applied in many areas of science and social science, viz. medical sciences [7], economics ([8], [9]), computer science [10], etc.

The term 'entropy' was formed from two Greek words: 'en' which means 'in' and 'trope' which means 'transformation' [11]. This term was widely used in many different fields with different meanings before it was used in dynamical systems. Clausius [12] used the term entropy in thermodynamics. In 1948, Shannon [13] introduced the concept of entropy in information theory as a measure of the average information content associated with a random outcome. Kolmogorov ([14], [15]) introduced the concept of entropy in dynamical systems as a measure-preserving transformation. Later, Sinai [16] (a student of Kolmogorov) introduced a general version of entropy which is known as Kolmogorov-Sinai entropy.

In 1965, Adler et al. [17] introduced the concept of topological entropy in a compact topological space as an invariant for continuous mappings. Later, Bowen [18] introduced topological entropy for a uniformly continuous mapping on a metric space. Recently, Liu et al. [19] defined topological entropy for continuous mappings on arbitrary topological spaces. In general, topological entropy is used to determine the complexity of a dynamical system [11]. According to ([20], [21]), if the topological entropy of a system is positive, then the system is topologically chaotic.

Bitopological dynamical system is a recently explored area of dynamical systems. Till now no work has been done to measure the complexity of a bitopological dynamical system. In this paper we introduce the notion of entropy in bitopological dynamical system and produce some results related to entropy. Moreover, we introduce weighted bitopological Shannon entropy as an extension of Shannon entropy [13] in bitopological space. Finally, we produce some applications of our theory in biology.

2. Preliminary Definitions

To make the paper self-sufficient to read, we recall some existing definitions of bitopological space and bitopological dynamical system.

Definition 2.1.[2] A quasi-pseudo-metric on a set X is a non-negative real-valued function $p(,)$ on the product $X \times X$ such that

(i) $p(x,x) = 0$, where $x \in X$

(ii) $p(x,z) \leq p(x,y) + p(y,z)$, where $x,y,z \in X$

Definition 2.2.[2] Let $p(,)$ be a quasi-pseudo-metric on X, and let $q(,)$ be defined by $q(x,y) = p(y,x)$, where $x,y \in X$. Then, $q(,)$ is also a quasi-pseudo-metric on X. We say that $p(,)$ and $q(,)$ are conjugate, and denote the set X with the structure by (X, p, q).

If $p(,)$ is a quasi-pseudo-metric on a set X, then the open p-sphere with centre x and radius $\varepsilon > 0$ is the set $S_p(x, \varepsilon) = \{y : p(x,y) < \varepsilon\}$. The collection of all open p-spheres forms a base for a topology. Similarly, $q(,)$ determines a topology for X. We shall denote the topology determined by $p(,)$ by τ_1 and the topology that of $q(,)$ by τ_2.

Definition 2.3.[2] A space X on which are defined two (arbitrary) topologies τ_1 and τ_2 is called a bitopological space and denoted by (X, τ_1, τ_2).

Definition 2.4.[22] A function f from a bitopological space (X, τ_1, τ_2) into a bitopological space (Y, ψ_1, ψ_2) is said to be pairwise continuous (respectively, a pairwise homeomorphism) iff the induced functions $f : (X, \tau_1) \to (Y, \psi_1)$ and $f : (X, \tau_2) \to (Y, \psi_2)$ are continuous (respectively, homeomorphisms).

Pervin [3] called this a continuous map. However, we call this as pairwise continuous map, due to Reilly [22].

Definition 2.5.[23] A cover $\mathscr{U}$ of a bitopological space (X, τ_1, τ_2) is pairwise open if $\mathscr{U} \subset$

$\tau_1 \cup \tau_2$, $\mathscr{U} \cap \tau_1$ contains a non-empty set, and $\mathscr{U} \cap \tau_2$ contains a non-empty set.

Definition 2.6.[23] A bitopological space (X, τ_1, τ_2) is pairwise compact provided every pairwise open cover of X has a finite subcover.

Definition 2.7.[24] A set $A \subset X$ is $+$invariant when $f(A) \subset A$ and A is $-$invariant when $A \subset f(A)$. A is called invariant when $f(A) = A$.

Definition 2.8.[25] A map $f : X \to X$ is called $+$invariant if for all $A \subset X$, $f(A) \subset A$ and $-$invariant when $A \subset f(A)$. The map f is invariant when $f(A) = A$, for all $A \subset X$.

Definition 2.9.[26] Let X be a topological space. A continuous map $f : X \to X$ is said to be a topological dynamical system with discrete time or simply a topological dynamical system. When f is a homeomorphism (that is, a bijective continuous map with continuous inverse), we also say that it is an invertible topological dynamical system.

In [1], we considered $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ as the set of non-negative integers, the set of integers and the set of real numbers, respectively.

Definition 2.10.[1] Let (X, τ_1, τ_2) be a bitopological space. A bitopological dynamical system is a pair (X, f), where (X, τ_1, τ_2) is a bitopological space and $f : X \to X$ is a pairwise continuous map. The dynamics is obtained by iterating the map.

The forward orbit of a point $x \in X$ under f is defined as $O_+(x) = \{f^n(x) : n \in \mathbb{N}\}$, where f^n denotes the n^{th} iteration of the map f. If f is a homeomorphism, then the backward orbit of x is the set $O_-(x) = \{f^{-n}(x) : n \in \mathbb{N}\}$ and the full orbit of x (or simply orbit of x) is the set $O(x) = \{f^n(x) : n \in \mathbb{Z}\}$.

For compact topological space, Adler et al. [17] introduced the concept of topological entropy and studied its various properties. Their definition is as follows:

Definition 2.11.[17] Let X be a compact topological space and $f : X \to X$ a continuous mapping. For any open cover $\mathscr{U}$ of X, let $N(\mathscr{U})$ denotes the number of sets in a subcover of minimal cardinality. Let $H(\mathscr{U}) = logN(\mathscr{U})$. For any two open covers $\mathscr{U}, \mathscr{V}$; $\mathscr{U} \vee \mathscr{V} \equiv \{A \cap B : A \in \mathscr{U}, B \in \mathscr{V}\}$ defines their join. Then, the entropy $ent(f, \mathscr{U})$ of a mapping f with respect to a cover $\mathscr{U}$ is defined as $\lim\limits_{n\to\infty} \frac{H(\mathscr{U} \vee f^{-1}\mathscr{U} \vee \ldots \vee f^{-n+1}\mathscr{U})}{n}$ and the entropy $ent(f)$ of a mapping f is defined as the $sup\, ent(f, \mathscr{U})$, where the supremum is taken over all open covers $\mathscr{U}$.

Recently, Liu et al. [19] proposed a new definition of topological entropy for continuous mappings on arbitrary topological spaces where compactness, metrizability, separation axioms etc. are not necessarily required. Their definition is as follows:

Definition 2.12.[19] Let (X,f) be an arbitrary topological dynamical system, i.e., X is an arbitrary topological space and f is a continuous mapping from X to itself. Let $\mathscr{U}$ be an open cover of X and F be a non-empty compact subset of X invariant under f, i.e. $f(F) \subseteq F$. Let $M_F(\mathscr{U})$ be the smallest cardinality of all subcovers (for F) of $\mathscr{U}$ and $L_F(\mathscr{U}) = logM_F(\mathscr{U})$. For any two open covers $\mathscr{U},\mathscr{V}$; $\mathscr{U} \vee \mathscr{V} = \{A \cap B : A \in \mathscr{U}, B \in \mathscr{V}\}$ defines their join. Also, $\mathscr{K}(X,f)$ denotes the set of all f-invariant non-empty compact subsets of X.

Definition 2.13.[19] Let (X,f) be a topological dynamical system. For $F \in \mathscr{K}(X,f)$ and any open cover $\mathscr{U}$ of X, $ent^*(f,\mathscr{U},F) = \lim\limits_{n\to\infty} \frac{1}{n} L_F(\bigvee_{j=0}^{n-1} f^{-j}(\mathscr{U}))$ is called the topological entropy of f on F relative to $\mathscr{U}$. Further, $ent^*(f,F) = \sup\limits_{\mathscr{U}}\{ent^*(f,\mathscr{U},F)\}$, where the supremum is taken over all open covers $\mathscr{U}$ of X, is called the topological entropy of f on F.

Definition 2.14.[19] Let (X,f) be a topological dynamical system. When $\mathscr{K}(X,f) \neq \emptyset$, define $ent^*(f) = \sup\limits_{F \in \mathscr{K}(X,f)} \{ent^*(f,F)\}$. When $\mathscr{K}(X,f) = \emptyset$, define $ent^*(f) = 0$. Further, $ent^*(f)$ is said to be the entropy of f.

Lemma 2.1.[20] Let $\{a_n\}$ be a sequence of real numbers which is subadditive, i.e. $a_{m+n} \leq a_m + a_n$ for all m,n. Then $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists and has the value $c = \inf \frac{a_n}{n}$.

3. On Entropy of a Pairwise Continuous Map

In this section, our main aim is to introduce entropy in bitopological dynamical systems. For this purpose, we introduce weakly pairwise compactness in a bitopological space.

Definition 3.1. A cover $\mathscr{U}$ of a bitopological space (X,τ_1,τ_2) is weakly pairwise open if $\mathscr{U} \subset \tau_1 \cup \tau_2 \cup \{U \cap V : U(\neq \emptyset, X) \in \tau_1, V(\neq \emptyset, X) \in \tau_2\}$, $\mathscr{U} \cap \tau_1$ contains a non-empty set and $\mathscr{U} \cap \tau_2$ contains a non-empty set.

Definition 3.2. A bitopological space (X,τ_1,τ_2) is weakly pairwise compact provided every weakly pairwise open cover of X has a finite subcover.

It is to be noted that if a bitopological space (X,τ_1,τ_2) is pairwise compact, then it is weakly pairwise compact but the converse may not be true in general. In some particular bitopological spaces where any pair of τ_1-open set ($\neq \phi, X$) and τ_2-open set ($\neq \phi, X$) has empty intersection, pairwise compactness and weakly pairwise compactness are same. We give an example of such type of space in section 6.

Example 3.1. Let us consider the bitopological space $(\mathbb{R}, \tau_1, \tau_2)$, where τ_1 is the left hand topology and τ_2 is the right hand topology. Then, $(\mathbb{R}, \tau_1, \tau_2)$ is weakly pairwise compact since it is pairwise compact according to Cooke and Reilly [27].

Example 3.2. Let X be the unit interval $[0,1]$, $\tau_1 = \{\emptyset, X, \{0\}, [0,a) : a \in X\}$, and $\tau_2 = \{\emptyset, X, \{1\}, (b,1] : b \in X\}$. Then, (X, τ_1, τ_2) is not weakly pairwise compact since the weakly pairwise open cover $\{\{0\}, \{1\}, (b,a) : b(\neq 0), a(\neq 1) \in X, b < a\}$ has no finite subcover.

Also according to Cooke and Reilly [27], (X, τ_1, τ_2) is not pairwise compact since the pairwise open cover $\{\{0\}, (b,1] : b \in X, b \neq 0\}$ of X has no finite subcover.

Now, we define entropy in bitopological dynamical systems which is an extension of the definition of topological entropy [17].

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of non-negative integers, the set of integers and the set of real numbers, respectively.

Definition 3.3. Let (X, f) be a bitopological dynamical system, where (X, τ_1, τ_2) is a bitopological space and $f : X \to X$ is a pairwise continuous map. Let (X, τ_1, τ_2) be weakly pairwise compact. For any weakly pairwise open cover $\mathscr{U}$ of X, let $N(\mathscr{U})$ denotes the number of sets in a subcover of $\mathscr{U}$ with minimal cardinality. Let $L(\mathscr{U}) = \log N(\mathscr{U})$.

For any two weakly pairwise open covers $\mathscr{U}$, $\mathscr{V}$ of X, we define their join by $\mathscr{U} \vee \mathscr{V} = \{U \cap V : U \in \mathscr{U}, V \in \mathscr{V}\}$. Then, $\mathscr{U} \vee \mathscr{V}$ is also a weakly pairwise open cover of X. A weakly pairwise open cover $\mathscr{U}$ is said to be a refinement of a weakly pairwise open cover $\mathscr{V}$, denoted by $\mathscr{V} \prec \mathscr{U}$, if every member of $\mathscr{U}$ is a subset of atleast one member of $\mathscr{V}$.

We have the following properties in a weakly pairwise compact space X. Also, from here we always assume that the bitopological space X is weakly pairwise compact unless otherwise specified.

Property 3.1. Let $\mathscr{U}$ be a weakly pairwise open cover of X. Then, $N(\mathscr{U}) \geq 1$ and therefore $L(\mathscr{U}) \geq 0$.

Property 3.2. Let $\mathscr{U}$ be a weakly pairwise open cover of X. Then for a pairwise continuous map f, $L(f^{-1}(\mathscr{U})) \leq L(\mathscr{U})$.

Proof. Let $\{U_1,...,U_p,V_1,...,V_q,M_1 \cap N_1,...,M_r \cap N_s : U_i,M_j \in \tau_1; V_k,N_l \in \tau_2$ where $i = 1,2,...,p$; $j = 1,2,...r; k = 1,2,...,q$ and $l = 1,2,...,s\}$ be a subcover of $\mathscr{U}$ with minimal cardinality. Then, $\{f^{-1}(U_1),...,f^{-1}(U_p), f^{-1}(V_1),...,f^{-1}(V_q), f^{-1}(M_1 \cap N_1),...,f^{-1}(M_r \cap N_s)\}$ is a weakly pairwise open subcover and it covers X. But, this subcover may not be the minimal subcover which covers X and so, $N(f^{-1}(\mathscr{U})) \leq N(\mathscr{U})$. This gives $L(f^{-1}(\mathscr{U})) \leq L(\mathscr{U})$.

Property 3.3. Let $\mathscr{U}$ and $\mathscr{V}$ be two weakly pairwise open covers of X. Then, $L(\mathscr{U} \vee \mathscr{V}) \leq L(\mathscr{U}) + L(\mathscr{V})$.

Proof. Let $\{M_1,...,M_r\}$ be a subcover of $\mathscr{U}$ with minimal cardinality and $\{N_1,...,N_s\}$ be a subcover of $\mathscr{V}$ with minimal cardinality where M_i's and N_j's are either τ_1-open sets or τ_2-open sets or sets of the form $U \cap V$, $U \in \tau_1, V \in \tau_2$. Then, $\{M_i \cap N_j : i = 1,...,r$ and $j = 1,...,s\}$ is a subcover of $\mathscr{U} \vee \mathscr{V}$ and it is not always minimal. Thus, $N(\mathscr{U} \vee \mathscr{V}) \leq N(\mathscr{U}).N(\mathscr{V})$. This gives $L(\mathscr{U} \vee \mathscr{V}) \leq L(\mathscr{U}) + L(\mathscr{V})$.

Property 3.4. Let $\mathscr{U}$ and $\mathscr{V}$ be two weakly pairwise open covers of X. If f is a pairwise open map, then $f^{-1}(\mathscr{U} \vee \mathscr{V}) = f^{-1}(\mathscr{U}) \vee f^{-1}(\mathscr{V})$.

Proof. It is easy to prove the result by using properties of inverse of a function.

Property 3.5. Let $\mathscr{U}$ and $\mathscr{V}$ be two weakly pairwise open covers of X such that $\mathscr{V} \prec \mathscr{U}$. Then $N(\mathscr{V}) \leq N(\mathscr{U})$ and so $L(\mathscr{V}) \leq L(\mathscr{U})$.

Proof. Let $\{M_1,...,M_r\}$ be a subcover of $\mathscr{U}$ with minimal cardinality where M_i's are either τ_1-open sets or τ_2-open sets or sets of the form $U \cap V$, $U \in \tau_1, V \in \tau_2$. Since $\mathscr{V} \prec \mathscr{U}$, so there exists a subcover $\{N_1,...,N_r\}$ of $\mathscr{V}$ consists of the supersets of the sets M_i's where N_j's are either τ_1-open sets or τ_2-open sets or sets of the form $U \cap V$, $U \in \tau_1, V \in \tau_2$. Clearly, this subcover may not be the minimal subcover of $\mathscr{V}$. Hence, $N(\mathscr{V}) \leq N(\mathscr{U})$ and so $L(\mathscr{V}) \leq L(\mathscr{U})$.

Property 3.6. Let $\mathscr{U}$ be a weakly pairwise open cover of X. Then, the limit $\lim\limits_{n\to\infty} \frac{1}{n} L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}))$ exists and it is finite.

Proof. Let $a_n = L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}))$. Using the lemma 2.1., it is sufficient to show that $a_{m+n} \leq a_m + a_n$ for all m,n.

Now using property 3.2. and property 3.3., we get

$$\begin{aligned}
a_{m+n} &= L(\bigvee_{i=0}^{m+n-1} f^{-i}(\mathscr{U})) \\
&= L((\bigvee_{i=0}^{m-1} f^{-i}(\mathscr{U})) \vee (\bigvee_{i=m}^{m+n-1} f^{-i}(\mathscr{U}))) \\
&= L((\bigvee_{i=0}^{m-1} f^{-i}(\mathscr{U})) \vee (f^{-m} \bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}))) \\
&\leq L((\bigvee_{i=0}^{m-1} f^{-i}(\mathscr{U}))) + L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}))) \\
&= a_m + a_n.
\end{aligned}$$

Thus, $a_{m+n} \leq a_m + a_n$ for all m,n. This completes the proof.

Now, we define entropy of a pairwise continuous map in a bitopological space.

Definition 3.4. Let (X, f) be a bitopological dynamical system, where the bitopological space X is weakly pairwise compact. We define the entropy of the mapping f relative to the weakly pairwise open cover $\mathscr{U}$ as $E(f,\mathscr{U}) = \lim\limits_{n\to\infty} \frac{1}{n} L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}))$.

Property 3.7. Let (X, f) be a bitopological dynamical system, where the bitopological space X is weakly pairwise compact. For any weakly pairwise open cover $\mathscr{U}$ of X, $E(f,\mathscr{U}) \leq L(\mathscr{U})$.

Proof. Using property 3.3. and property 3.2., we get

$$\begin{aligned}
E(f,\mathscr{U}) &= \lim_{n\to\infty} \frac{1}{n} L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U})) \\
&\leq \lim_{n\to\infty} \frac{1}{n} (L(\mathscr{U}) + ... + L(f^{-(n-1)}(\mathscr{U}))) \\
&\leq \lim_{n\to\infty} \frac{1}{n} (L(\mathscr{U}) + ... + L(\mathscr{U})) \\
&\leq L(\mathscr{U}).
\end{aligned}$$

Definition 3.5. Let (X, f) be a bitopological dynamical system, where the bitopological space X is weakly pairwise compact. The entropy of f is defined as $E(f) = \sup\limits_{\mathscr{U}} \{E(f,\mathscr{U})\}$, where the supremum is taken over all weakly pairwise open covers $\mathscr{U}$ of X. We call this entropy as bitopological entropy.

We have the following properties of bitopological entropy.

Property 3.8. The bitopological entropy of the identity mapping is zero.

Proof. Let $I : X \to X$ be the identity mapping from a weakly pairwise compact bitopological space X to itself. For any weakly pairwise open cover $\mathscr{U}$ of X, we have $\bigvee_{i=0}^{n-1} I^{-i}(\mathscr{U}) \prec \mathscr{U}$. By property 3.6., $L(\bigvee_{i=0}^{n-1} I^{-i}(\mathscr{U})) \leq L(\mathscr{U})$. This gives $E(I,\mathscr{U}) = \lim\limits_{n\to\infty} \frac{1}{n} L(\bigvee_{i=0}^{n-1} I^{-i}(\mathscr{U})) = 0$. Since $\mathscr{U}$ is arbitrary, hence the entropy of I is $E(I) = \sup\limits_{\mathscr{U}}\{E(I,\mathscr{U})\} = 0$.

Property 3.9. The bitopological entropy of a $+$invariant mapping is zero.

Proof. Let $f : X \to X$ be a $+$invariant mapping from a weakly pairwise compact bitopological space X to itself. Then for all $A \subset X$, $f(A) \subset A$ and so for any weakly pairwise open cover $\mathscr{U}$ of X, we have $N(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U})) \leq N(\mathscr{U})$. This implies that $L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U})) \leq L(\mathscr{U})$. Hence, $E(f,\mathscr{U}) = \lim\limits_{n\to\infty} \frac{1}{n} L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U})) \leq \lim\limits_{n\to\infty} \frac{1}{n} L(\mathscr{U}) = 0$. Since $\mathscr{U}$ is arbitrary, hence the entropy of f is $E(f) = \sup\limits_{\mathscr{U}}\{E(f,\mathscr{U})\} = 0$.

4. On Bitopological Entropy of Bitopologically Expansive Map and Bitopologically Transitive Map

In this section, we introduce bitopologically expansive map and find the entropy of a map which is not a bitopologically expansive map along with some other results.

Expansive map has been studied by many researchers ([28], [29]) for a long time. Reddy [30] introduced positively expansive map on compact metric spaces as follows:

Definition 4.1.[30] A map f from a compact metrizable space X onto itself is said to be positively expansive if there is a metric d for X and a number $c > 0$ such that $x \neq y$ implies the existence of a positive integer n such that $d(f^n(x), f^n(y)) > c$.

Recently, Richeson and Wiseman [31] generalized the notion of positively expansive map into topological spaces.

Now, we introduce expansive map in bitopological dynamical systems.

Definition 4.2. Let (X, f) be a bitopological dynamical system. The map f is called bitopologically expansive with respect to the τ_1-open set $U(\neq \emptyset)$ and τ_2-open set $V(\neq \emptyset)$ if for all $x,y \in X$, $x \neq y$, $x \in U$, $y \in V$ there exist $m,n \in \mathbb{N}$ (depending on x and y) such that $(x,y) \notin f^{-m}(U) \times f^{-n}(V)$.

The map f (or the system (X, f)) is called bitopologically expansive if for every pair of non-empty open sets $U \in \tau_1$ and $V \in \tau_2$, f is bitopologically expansive.

Acharjee et al.[1] introduced the concept of bitopologically transitive map as follows.

Definition 4.3.[1] Let (X, f) be a bitopological dynamical system. For $U \in \tau_1$ and $V \in \tau_2$, we define the following:

$$N(U,V) = \{(m,n) : m,n \in \mathbb{N}, \quad f^m(U) \cap V \neq \emptyset \quad and \quad U \cap f^n(V) \neq \emptyset\}.$$

The map f is called bitopologically transitive (or (m,n) transitive) if for any pair of non-empty open sets $U \in \tau_1$ and $V \in \tau_2$, the set $N(U,V)$ is non-empty.

The following lemma establishes the relation of $-$invariant map with bitopologically expansive map.

Lemma 4.1. Let (X, f) be a bitopological dynamical system. If the map f is $-$invariant, then f is bitopologically expansive and bitopologically transitive.

Proof. Let f be $-$invariant. Then for all $A \subset X$, $A \subset f(A)$. This implies that $f(A) \subset f^2(A)$, $f^2(A) \subset f^3(A)$ and so on. Also, $A \subset f^n(A), \forall n \in \mathbb{N}$, i.e. $f^{-1}(A) \subset A$, $f^{-2}(A) \subset f^{-1}(A)$ and so on. Let $U(\neq \emptyset) \in \tau_1$ and $V(\neq \emptyset) \in \tau_2$ be arbitrary. Let $x, y \in X$, $x \neq y$, $x \in U$, $y \in V$. Now, $U \times V \supset f^{-1}(U) \times f^{-1}(V) \supset f^{-2}(U) \times f^{-2}(V) \supset$ and so on. Then, there exist $m, n \in \mathbb{N}$ (depending on x and y) such that $(x,y) \notin f^{-m}(U) \times f^{-n}(V)$. Thus, f is bitopologically expansive map.

In the next theorem we find the bitopological entropy of a map which is not bitopologically expansive.

Theorem 4.1. Let (X, f) be a bitopological dynamical system. If the map f is not bitopologically expansive, then the bitopological entropy of the map f is zero.

Proof. By lemma 4.1., if f is not bitopologically expansive then it cannot be $-$invariant. That is for all $A \subset X$, $f(A) \subseteq A$. This implies f is a $+$invariant map or an invariant map.

If f is a $+$invariant map, then by using property 3.9., the bitopological entropy of f is zero.

Also, if f is an invariant map then for any weakly pairwise open cover $\mathscr{U}$ of X, we have $\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}) \prec \mathscr{U}$. By property 3.6., $L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U})) \leq L(\mathscr{U})$. This gives $E(f, \mathscr{U})$ $= \lim\limits_{n\to\infty} \frac{1}{n} L(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U})) = 0$. Since $\mathscr{U}$ is arbitrary, the entropy of f, $E(f) = \sup\limits_{\mathscr{U}}\{E(f, \mathscr{U})\} =$ 0.

The following lemma establishes the relation between bitopologically transitive map and $-$invariant map.

Lemma 4.2. Let (X, f) be a bitopological dynamical system. Then, the map f is bitopologically transitive if and only if it is $-$invariant.

Proof. Assume that f is not a $-$invariant map. Let f be bitopologically transitive. Then, for any pair of non-empty open sets $U \in \tau_1$ and $V \in \tau_2$, we have $f^m(U) \cap V \neq \emptyset$ and $U \cap f^n(V) \neq \emptyset$ for $m, n \in \mathbb{N}$. In particular, let $U \in \tau_1$ and $V \in \tau_2$ be such that $U \cap V = \emptyset$. Now, since f is not a $-$invariant map so $f(U) \subseteq U$. This gives $f^m(U) \subseteq U, \forall m \in \mathbb{N}$. This implies that $f^m(U) \cap V \subseteq U \cap V = \emptyset, \forall m \in \mathbb{N}$ - a contradiction. Thus, f is not a bitopologically transitive map.

Next, let f be $-$invariant. Then for all $A \subset X$, $A \subset f(A)$. This implies that $f(A) \subset f^2(A)$, $f^2(A) \subset f^3(A)$ and so on. Let $U(\neq \emptyset) \in \tau_1$ and $V(\neq \emptyset) \in \tau_2$ be arbitrary. Then, $U \subset f(U) \subset f^2(U) \subset f^3(U) \subset ....$ After some iterations, say m, we must have $f^m(U) = X$. Thus, $f^m(U) \cap V \neq \emptyset, \forall\, V \in \tau_2$. Similarly, there exists $n \in \mathbb{N}$ such that $U \cap f^n(V) \neq \emptyset, \forall\, U \in \tau_1$. Thus, f is bitopologically transitive.

Theorem 4.2. Let (X, f) be a bitopological dynamical system. If the mapping f is not bitopologically transitive, then the bitopological entropy of the map f is zero.

We omit the proof of theorem 4.2. as it can be proved similarly as of the proof of theorem 4.1.

5. Weighted Bitopological Shannon Entropy

In 1948, Shannon [13] introduced the concept of entropy in information theory. We procure it from Addabbo and Blackmore [32].

Definition 5.1. [32] Let $S := \{s_1, ..., s_m\}$ be a nonempty finite set of symbols or messages (sometimes referred to as the alphabet) with a discrete probability p assigned to each, such that $p(s_i) \geq 0$ for all $1 \leq i \leq m$ and $p(s_1) + ... + p(s_m) = 1$. Then, the Shannon entropy of the message ensemble $H(S) := -\sum_{i=1}^{m} p(s_i) \log p(s_i)$ is just the average or expected value of the information content of the message.

In this section, we introduce weighted bitopological Shannon entropy and discuss some results related to it. First we discuss an interesting observation in a pairwise compact bitopological space.

Let us consider the bitopological space $(\mathbb{R}, \tau_1, \tau_2)$, where τ_1 is the left hand topology and τ_2 is the right hand topology. According to Cook and Reilly [27], $(\mathbb{R}, \tau_1, \tau_2)$ is pairwise compact since any pairwise open cover $\mathscr{U}$ of $\mathbb{R}$ has at least one τ_1-open set and one τ_2-open set which are not disjoint. These two sets cover $\mathbb{R}$ and so form the required finite subcover. Let $\mathscr{V}_{min}$ be a subcover of $\mathscr{U}$ with minimal cardinality. Then $\mathscr{V}_{min} = \{U, V\}$, where U is a τ_1-open set and V is a τ_2-open set which are not disjoint. Now, a natural question arises- how important is the set U (or V) in the sense of covering $\mathbb{R}$? The answer is they are equaly important. We cannot omit any one of them. So, if we are asked to give some weights according to the importance of U and V in $\mathscr{V}_{min}$ in the scale of $[0,1]$, then we can say that the weight of U is $\frac{1}{2}$ and also weight of V is $\frac{1}{2}$.

This simple observation from the example of Cook and Reilly[27] motivates us for the following definition.

Definition 5.2. Let (X, τ_1, τ_2) be a pairwise compact bitopological space. Let $\mathscr{U} = \{U_i, V_j : U_i \in \tau_1, V_j \in \tau_2\}$ be a pairwise open cover of X. Let $\mathscr{V}_{min}$ be a subcover of $\mathscr{U}$ with minimal cardinality. Since X is pairwise compact, $\mathscr{V}_{min}$ always has finite number of elements. Then to each $U_i \in \mathscr{V}_{min}$ and to each $V_j \in \mathscr{V}_{min}$, we can assign a value $w(U_i)$ and $w(V_j)$ such that $0 \leq w(U_i) \leq 1, 0 \leq w(V_j) \leq 1$ and $\sum\limits_i w(U_i) + \sum\limits_j w(V_j) = 1$. We call $w(U_i)$ the weight of set U_i. The value of the weight is to be given practically based on different characteristics of an open set such as its cardinality, its importance in covering the space X and so on.

Now, based on weight of an open set in a subcover of minimal cardinality in a pairwise compact bitopological space, we define weighted bitopological Shannon entropy which is an extension of Shannon entropy in information theory.

Definition 5.3. Let (X, τ_1, τ_2) be a pairwise compact bitopological space. Let $\mathscr{U} = \{U_i, V_j : U_i \in \tau_1, V_j \in \tau_2\}$ be a pairwise open cover of X. Let $\mathscr{V}_{min} = \{U_i, V_j : i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}$ be a subcover of $\mathscr{U}$ with minimal cardinality. We define the weighted bitopological Shannon entropy of the bitopological space (X,τ_1,τ_2) with respect to the pairwise open cover $\mathscr{U}$ as

$$S(X,\mathscr{U}) = -(\sum_{i=0}^{m} w(U_i)\log w(U_i) + \sum_{j=0}^{n} w(V_j)\log w(V_j)).$$

Definition 5.4. Let (X,τ_1,τ_2) be a pairwise compact bitopological space. We define the weighted bitopological Shannon entropy of the bitopological space (X,τ_1,τ_2) as $S(X) = \sup_{\mathscr{U}}\{S(X,\mathscr{U})\}$, where the supremum is taken over all pairwise open covers $\mathscr{U}$ of X.

It is to be noted that the weighted bitopological Shannon entropy depends on the definition of weight. So, while calculating weighted bitopological Shannon entropy one must consider a fixed definition of weight. Also, the value of weighted bitopological Shannon entropy differs if one considers different definition of weight. So, all the characteristics are to be measured for weights.

Example 5.1. Let us consider the bitopological space $(\mathbb{R},\tau_1,\tau_2)$, where τ_1 is the left hand topology and τ_2 is the right hand topology. According to Cook and Reilly [27], $(\mathbb{R},\tau_1,\tau_2)$ is pairwise compact since any pairwise open cover of $\mathbb{R}$ has at least one τ_1-open set and one τ_2-open set which are not disjoint. Let $\mathscr{U}$ be a pairwise open cover of $\mathbb{R}$. Then, a subcover of $\mathscr{U}$ with minimal cardinality contains exactly two elements- one is a τ_1-open set and the other is a τ_2-open set. So, $\mathscr{V}_{min} = \{U,V : U \in \tau_1, V \in \tau_2\}$. Now, if we consider the importance of U and V from the perspective of covering the space $(\mathbb{R},\tau_1,\tau_2)$, then they are equally important. So, we can assign weights to U and V as $w(U) = w(V) = \frac{1}{2}$. Then, the weighted bitopological Shannon entropy of the bitopological space $(\mathbb{R},\tau_1,\tau_2)$ with respect to the pairwise open cover $\mathscr{U}$ as

$$\begin{aligned}
S(X,\mathscr{U}) &= -(w(U)\log w(U) + w(V)\log w(V)) \\
&= -(\frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{1}{2}) \\
&= \log 2.
\end{aligned}$$

This is true for any pairwise open cover $\mathscr{U}$ of $\mathbb{R}$. Thus, the weighted bitopological Shannon entropy of the bitopological space $(\mathbb{R},\tau_1,\tau_2)$ is

$$\begin{aligned} S(X) &= \sup_{\mathscr{U}}\{S(X,\mathscr{U})\} \\ &= \log 2. \end{aligned}$$

We produce another example in next section.

6. An Example from Biology

Recently, Acharjee et al. [1] constructed a bitopological space (R,τ_1,τ_2) of human embryo development; where:
$\tau_1 = \{(\phi,\tau_1(t_0)),(U_1,\tau_1(t_1)),(U_2,\tau_1(t_2)),...,(U_m,\tau_1(t_m)),(R,\tau_1(T))\}$ and
$\tau_2 = \{(\phi,\tau_2(t_0)),(V_1,\tau_2(t_1)),(V_2,\tau_2(t_2)),...,(V_n,\tau_2(t_n)),(R,\tau_2(T))\}$.
Here; $\phi = Z = U_0 = V_0$ is the zygote, $U_m = X$ is the brain together with central nervous system of the whole organism and $V_n = Y$ is the other body parts of the whole organism except the brain and the central nervous system. Also, U_1, U_2,... represent different development stages of the brain and the central nervous system; and V_1, V_2,... represent different development stages of the other body parts except the brain and the central nervous system. Here, t_0 is the time of fertilization and T is the time of birth. It is important to note that before gastrulation, we have $U_i = V_i$. At the end; X and Y together form the whole organism, the baby R, i.e. $X\cup Y = R$. Moreover, $Z = U_0 \subset U_1 \subset U_2 \subset ... \subset U_m \subset R$ and $Z = V_0 \subset V_1 \subset V_2 \subset ... \subset V_n \subset R$.

In another paper [25], Acharjee et al. proved that (R,h) is a bitopological dynamical system, where the map $h : R \to R$ (it is the map that represents mitosis process) is defined by

$$h(x_i^j) = \{x_i^{j1}, x_i^{j2}\},$$

where x_i^j is the mother cell, x_i^{j1} and x_i^{j2} are the daughter cells.

Let, R^* be the postgastrulation part of the whole organism R; τ_1^* and τ_2^* are relative topologies on R^*. We also consider that h^* is the restriction of the map h on R^*, i.e. $h^* = h|_{R^*}$. Then according to [25], (R^*,τ_1^*,τ_2^*) is a subspace of (R,τ_1,τ_2).
In this section, we mainly focus on the bitopological dynamical system (R^*,h^*).

Now, we estimate the bitopological entropy of the map h^*.

For any pair of open sets $U^*, V^* \subset R^*$, where $U^* \in \tau_1^*$ and $V^* \in \tau_2^*$, we have

$$\begin{aligned} U^* \cap V^* &= (R^* \cap U) \cap (R^* \cap V) \\ &= R^* \cap (U \cap V) \\ &= \emptyset, \end{aligned}$$

as R^* is the postgastrulation part of the whole organism and $U \cap V$ is the developing stage of the embryo before gastrulation.

Thus, the bitopological space $(R^*, \tau_1^*, \tau_2^*)$ is a special kind of bitopological space where pairwise compactness and weakly pairwise compactness coincides.

Lemma 6.1. The bitopological space $(R^*, \tau_1^*, \tau_2^*)$ is weakly pairwise compact.

Proof. Let $\mathscr{U}^* = \{U_i^*, V_j^* : i, j \in \Delta \text{ and } \Delta \text{ is an index set}\}$ be any weakly pairwise open cover of $(R^*, \tau_1^*, \tau_2^*)$.

Case I: $\mathscr{U}^*$ contains R^*. Then, clearly $\mathscr{U}^*$ has the finite subcover $\{R^*\}$.

Case II: $\mathscr{U}^*$ does not contain R^*. Then $\mathscr{U}^*$ has the finite subcover $\{U^*, V^*\}$, where $U^* =$ largest τ_1^*-open set among the sets $\{U_i^* : i \in \Delta\}$ and $V^* =$ largest τ_2^*-open set among the sets $\{V_j^* : j \in \Delta\}$. We can choose such sets because of the increasing nature of open sets of both the topologies. Thus, every weakly pairwise open cover of $(R^*, \tau_1^*, \tau_2^*)$ has a finite subcover and therefore $(R^*, \tau_1^*, \tau_2^*)$ is weakly pairwise compact.

Theorem 6.1. The maximum value of the bitopological entropy of the map h^* is $\log 2$.

Proof. By Lemma 7.1., for any weakly pairwise open cover $\mathscr{U}^*$ of $(R^*, \tau_1^*, \tau_2^*)$ we have $N(\mathscr{U}^*) \leq 2$ and so $L(\mathscr{U}^*) \leq \log 2$. By property 3.7., $E(f, \mathscr{U}^*) \leq L(\mathscr{U}^*)$. This gives $E(f, \mathscr{U}^*) \leq \log 2$. Thus, $E(f) = \sup\limits_{\mathscr{U}^*}\{E(f, \mathscr{U}^*)\} \leq \log 2$.

Theorem 6.2. The weighted bitopological Shannon entropy of the bitopological space $(R^*, \tau_1^*, \tau_2^*)$ is $\log 2$.

Proof. Let $\mathscr{U}^* = \{U_i^*, V_j^* : i, j \in \Delta \text{ and } \Delta \text{ is an index set}\}$ be any pairwise open cover of $(R^*, \tau_1^*, \tau_2^*)$. Then, a subcover of $\mathscr{U}^*$ with minimal cardinality contains exactly two elements- one is a τ_1^*-open set and the other is a τ_2^*-open set. So, $\mathscr{V}_{min}^* = \{U^*, V^* : U^* \in \tau_1^*, V^* \in \tau_2^*\}$ (we omit the trivial case $\mathscr{U}^*$ contains R^* due to realistic nature of embryo development). Now, if we consider the importance of U^* and V^* from the perspective of covering the space $(R^*, \tau_1^*, \tau_2^*)$, then they are equally important. Moreover, this is medically verified [33]. So, we assign weights to U^* and V^* as $w(U^*) = w(V^*) = \frac{1}{2}$. Then, the weighted bitopological Shannon entropy of the bitopological space $(R^*, \tau_1^*, \tau_2^*)$ with respect to the pairwise open cover $\mathscr{U}^*$ is

$$\begin{aligned} S(R^*, \mathscr{U}^*) &= -(w(U^*)\log w(U^*) + w(V^*)\log w(V^*)) \\ &= -(\frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{1}{2}) \\ &= \log 2. \end{aligned}$$

This is true for any pairwise open cover $\mathscr{U}^*$ of R^*. Thus, the weighted bitopological Shannon entropy of the bitopological space $(R^*, \tau_1^*, \tau_2^*)$ is

$$\begin{aligned} S(R^*) &= \sup_{\mathscr{U}^*}\{S(R^*, \mathscr{U}^*)\} \\ &= \log 2. \end{aligned}$$

7. Conclusion

In this paper we introduced the notion of entropy in bitopological dynamical system where the bitopological space is weakly pairwise compact. We introduced the notion of bitopologically expansive map. Then, we found the bitopological entropy of a map which is not bitopologically expansive. Later, we found the bitopological entropy of a map which is not bitopologically transitive. Bitopological entropy has the potential to measure the complexity of a bitopological dynamical system. Moreover, we introduced weighted bitopological Shannon entropy in a bitopological space as an extension of Shannon entropy [13]. In future, weighted bitopological Shannon entropy may become a foundation of bitopology based information theory and to make this happen, we need a deep research in the area related to weighted bitopological Shannon entropy. Finally, as an application of our theory, we calculated bitopological entropy of the mitosis map in the bitopological space of postgastrulation part of human embryo development, which was introduced by Acharjee et al. [25]. Also, we found the weighted bitopological Shannon entropy of this space.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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