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EXTENDED OPTIMAL FEEDBACK CONTROL OF INFORMATION DISSEMINATION IN ONLINE ENVIRONMENTS

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Abstract. In this article, we consider a discrete-time model describing the dissemination of information from one person to another via word-to-mouth transmission or in certain types of online environments such as Facebook, WhatsApp and Twitter. The impact of information sharing is becoming more and more noticeable in the world with the increase of technologies, its potential has become clearer through controlling people's behavior and opinions. In this paper, we consider control strategies depending on the rate of diffusion of the information, and its effect, by considering feedback controls of the populations' numbers. We present the first scenario, where we consider the feedback control of the number of sharers and removed individuals, and the second scenario when we propose a feedback control of the three population functions. Based on a discrete version of Pontryagin's Maximum principle we characterize optimal controls. We provide several numerical examples to compare and discuss the proposed scenarios.

Keywords: dissemination; information; online environment; feedback control; optimal control.

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1. INTRODUCTION

Information spreading is to broadcast or distribute information [1], it is considered as an ubiquitous process in society which describes a wide variety of phenomena [2], however, quantitative analyses of networking usage have shown that people use microblogging as an information-broadcasting platform [3]. Nowadays, the information has been diffused from one person to another contagiously and could be dissiminated through the majority of population in a “viral” way [4]. Day after day, the propagation of data has become more and more accelerated [5]. After the creation of various communication channels, social media have become an important part of people’s everyday lives [6]. Furthermore, a combination of methods is used to access social media, including computer web browsers, mobile apps, tablet apps, and mobile web browsers [7].

The most greater sources of information are social media [8]. Over the past few years, the online messages have undergone huge growth, some of the conversations are usually relevant to a user’s social circle (social status apdates, sharing stories or experiences ..), while a large portion of the messages in social networking are responses triggered by events including natural disasters, political events, protests, marches... [9]. The impact of information sharing is becoming more and more noticeable in the world with the increase of technologies, its potential has become clearer through controlling people’s behavior and opinions [10, 11, 12].

Individuals have changed degrees of agency in initiating, modifying, or reacting to the course of events, and reactions contain observations of occurrence, expressions including emotions, or a call to action [13]. A real example is clearly presented in the Occupy Wall Street (OWS) protest movement against the economic inequality, which was a widely participated movement known to use social media for advertising and dissimination nationwide [9]. A further example is also presented in the Arab spring, which emphasizes the power of social media in spreading democratic ideas across international borders [14]. Civil society leaders in Arab countries emphasized the role of social networking in the protests and exercising freedom of speech [15].

The majority of social sites highlight activities of a person’s social network links which are created between people who are similar, or whose contributions they find interesting, the dynamics of information on a social media could be different from its dynamics within the general population. There are actually some social news sites that provide an opportunity to understand dynamics of information spread on social media. The microblogging service Twitter for instance has become an important source of timely information for people , short text messages are shared including links to retweet mesages of others or news stories and comment on.While the number of fans a user has on each site exhibits a long-tail distribution. Therefore, the information contagion starts with story's submitter and grows as the story accrues fan retweets [16, 17].

Though information is fastly propagated within social networking sites, this fact gives rise to the question of how people assess the source credibility of this information [18], this question's answer has become so important for social media's consumers. For a reason, the gatekeeper function is shifting from producers to consumers of information for new technologies [19]. Despite the existence of debate about the precise factor-structure of source credibility one common factor-structure found contains three dimensions of source credibility ; the first is expertise or competence (the degree to which a perceiver believes a sender to know the truth), second is trustworthiness (the degree to which a perceiver believes a sender will tell the truth as he knows it), and the last one is goodwill (the degree to which a perceiver believes a sender has his best interests at heart) [20]. This change of the gatekeeping function away from producers of content onto consumers of that content has created a shift from the traditional notion of "gatekeeping" to what is referred to as "gatewatching" [21]. Therefore, gatewatchers structurally diffuse or promote information by making sources or stories known to others in the new media environment. Instead of publishing unique information, they make others' information known and add to it. This can be seen in some social networks such as Twitter when a user publishes a link and then comments on it, and similarly in Facebook where the user does the same thing [22].

The widespread of information during the recent years is not controled, since every body could send, share, and diffuse unconfirmed information in a very short period of time. This may lead to disseminating rumors [23]. Misinformation, disinformation, and rumor are three different terms that discribe interchangeably information that lacks truthfulness [24], and due to difficulties in identifying the intention of the source, researchers often adopt the word misinformation to broadly describe false claims. On the other hand, rumor is largely defined as a piece of information that has not been confirmed [25]. Nowadays false information is a part of the contemporary media system where varying degrees of information sources compete for our attention [26]. However, scholar argue that rumor gains its power when it is repeated and shared along from one individual to another [25]. Unsecured information has two main characteristics are dynamic mode and collective process that unfolds over time [26].

The way in which information spreades is similar to an epidemic [37, 38], rumors are contagious due to their fast transmission, all that is needed to infect a person is to transfer the message, once a rumor is started, almost everyone will aknowledge it, thus, the infection of the rumor (virus) has been caused by the person who has started the rumor [27].

Therefore, reseachers look for models in order to understand information dissemination's process. Rumor models are the same as epidemic models such as Susceptible-Infected-Recovered (SIR) and Susceptible-Infected-Susceptible (SIS), a further attention is payed on the rumor diffusion process by means of mathematical analysis [28], particularly, the random analysis methods used by Daley and Kendal within the contributive landmark study (DK model) many years ago [29]. Mainly taken from the susceptible-infected-removed (SIR) model of epidemics, the DK model divided all the homogeneous populations into three groups ; spreaders (they actively spread the missinformation), Ignorants (people who are ignorant of the rumor), and the stiflers (those who have heard the rumor, but no longer are interested in spreading it) [30].

The biological epidemic models Recovery of a spreader to a stifler is spontaneous and independent of others in SIR/SIS models. At some rate spreaders are generated to ignorant-spreader contact-dynamics, same thing as rumor models [31]. For instance when an ignorant comes across an advertisement, the information is started to be spread. Also, when a stifler sees the advertisement, the perception about an information being stale or fashion/product is changed, and information keep spreading again and again [32]. Giving as example nodes in the network which can have three states ignorant, spreader and recovered. Intersting in some information without receiving it makes the node ignorant, while nodes that have recieved a copy of information and are ready to spread the information can be seen as spreaders. If a node has neither interest in the information nor the desire to disseminate it, can be looked upon as a recovered node [33]. In social networking sites, we consider the existance of one node having the information initially, which is the spreading node. At first, all other nodes have interest in the information, and willing to receive the information. However, the majority of the nodes cannot keep the same interest all along. Some ignorant nodes could not be interested any more later on, and refuse to disseminate the information. More clearly, an ignorant node can directly become a recovered node, which is called pre-immunity .Besides this, a spreading node may stop spreading the information when it meets a recovered friend-node, this action is called immunity. Moreover, spreading nodes may stop dissemination without any contacts because of their unwillingness to deliver the information [34].

Human beings are ratiol creatures, yet they are very often guided by emotions and non-ratiol elements. It can frequently be seen that usually people are more influenced by emotions than ratiol elements. On twitter for instance, this research has found that emotionally charged twitter messages tend to be retweeted more often ,and more quickly compared to neutral ones . This is one of the main reasons why companies pay more attention to the analysis of sentiment related to their brands and products in social media [35]. Rumors have been the basis for violent death and destruction throughout history ; When rumor spreads like wild fire, its impact on real social life is enormous . Once the rumor spreads, it easily leaves an impression on the people's minds. Though there is difference between rumor and objective reality, the individuals easily fall prey to rumor, and somtimes they would be shocked if somebody tells them that the rumor is false [25] However, rumors may be spreaded for good publicity of the person concerned or for the promotion of the product, enhance a candidate's chances for voting during elections... But most of rumors are intensely harmful as well as painful. They may even break a person's life [36].

In recent years, social media has played a greater role in disseminating information within large amount of people in a short period. However, this represents a two edged sword due to the possibility of diffusing unsecured information and rumors that could affect negatively various fields of society such as economics, politics, health, art, and even education. Disseminating information within different social networking sites has become a daily activity for the whole world, in some cases the fact of diffusing information becomes a serious addiction. because all individual's communication has transformed from the real world to a virtual social platforms.

The similarities between the spread of the epidemic and the spread of information allowed the researcher to use epidemiological models to model information dissemination [37, 38]. In this article, we divide the population into three groups, which will make it possible to study the development of ignorant people (people who do not know the information), spreaders (people who are interested in this information, who find pleasure in sharing it), removed (people who see that this information lacks relevance and compatibility with their profiles, then they refuse to share it). In this work, we are using feedback control to control the dissemination of information, which means that the control we provide is the feedback from the number of ignorant, spreaders, and removed individuals. We use a discrete version of Pontryagin's Maximum Principle to describe optimal controls, then simulate our results numerically to evaluate the effectiveness of this type of optimal control in reducing the number of spreaders and/or increasing the number of removals and ignorance at a cost optimum.

2. Presentation of the Model

Information is easily spread, by all means, word of mouth, emails, phone calls, social networks, etc. With the help of all the advanced technologies that facilitate human communication, information spreads quickly. One of the most important factors in spreading information is the introduction of the "Share" button that accompanies any status update, link, video, or image posted. Content viewers (for example, friends of the creator and subscribers) are allowed to share the post. For example, on Facebook, if the content was originally posted publicly, anyone can view and share it [37]. Based on the ideas published in [37], we devise here a compartmental model to study the dissemination of information in online environments of N users (Facebook, WhatsApp or Tweeter groups or pages) by posting, sharing and discussing. In these online environments, when a user posts information (Text, image, video...), only his neighbors can see it and determine if that information needs to be shared again or not. If the information is so interesting and some neighbors decide to share it, neighbors of the author's neighbors can then see and re-share them again. The influence of the information then exceeded the local scope of the author and can be widely distributed on the network. On the other hand, if none of the original author's neighbors are attracted by this information, it will disappear soon and very few users will see it. When a user shares a post, the information is displayed on its homepage for a long time even if he does not care about it anymore, all his neighbors can always see the information he has shared. At the same time, if the neighbors have seen the post and do not share it immediately, they may lose interest gradually and ignore that information.

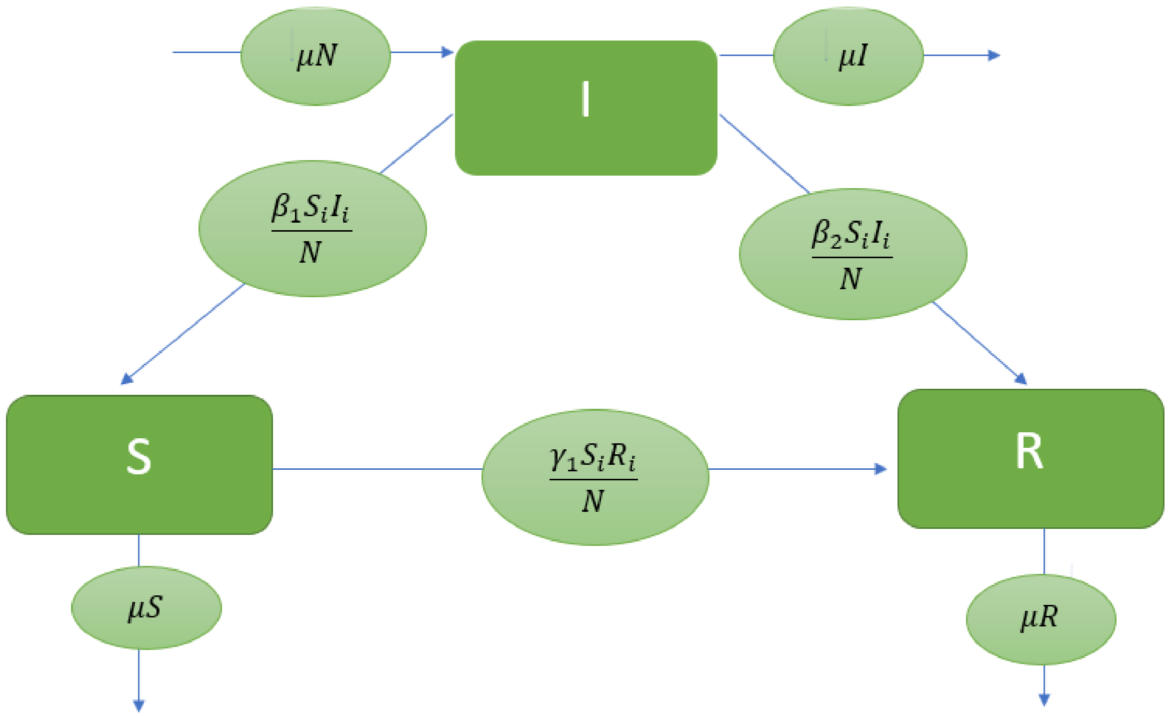
If a user notices that some information is repeated and shared by several of his neighbors, then he will discuss it with his friends through the chat tools or face to face, so he can determine the relevance of this information, and then decide to share it or not.

Our model consists of three compartments: Ignorants (I), Sharers or spreaders (S), and Removed people (R). The term "ignorant" means a person that does not know about the information. The word "Sharer" is used to denote that a person is attracted by the information and/or he finds it funny or interesting, then he decides to share it. The term "Removed" means a person who has seen the post and has decided not to share it. For example, because of irrelevance or for other personal reasons. We kept the term Removed from the classical SIR epidemiological model to denote individuals removed from the sharing system. All transmissions are modeled using the mass action principle, which accounts for the probability of transmission in contact between the different compartments.

Each information has the potential of sharing, but one can find some information not useful or does not fit the user interests, and then there is no need to share it. For example, if the information is about a concern of the public opinion (Raising costs of education, election cheats, public safety...), the probability of shares will be very important. Therefore, the potential relevance of the information will be taken into account and it will be defined based on the proportions of shares. Let's define the potential relevance of the information by the average β_1 , while the potential irrelevance of the information is defined by the average β_2.

We assume that when information is shared, it is relevant. An Ignorant becomes a Sharer just after he shares the information at the rate $\frac{\beta_1 I_i S_i}{N}$. An Ignorant that decides not to share the information becomes Removed at a rate $\frac{\beta_2 I_i S_i}{N}$. A Sharer that contacts a Removed and he decides not to share the information anymore, becomes a Removed at a rate $\gamma\frac{S_i R_i}{N}$. Ignorant, Sharer and Removed individuals can leave the environment at a rate μI_i, μS_i, and μR_i, respectively.

FIGURE 1. Flow chart for the model (1-3)



We assume that all new members are recruited as Ignorants at the constant rate Λ. All these interactions happen at the instant i.

The model resulting from these assumptions is governed by the following equations

$$I_{i+1} = I_i + \Lambda - \mu I_i - \frac{\beta I_i S_i}{N} \tag{1}$$

$$S_{i+1} = S_i - \mu S_i + \frac{\beta_1 I_i S_i}{N} - \gamma \frac{S_i R_i}{N} \tag{2}$$

$$R_{i+1} = R_i - \mu R_i + \frac{\beta_2 I_i S_i}{N} + \gamma \frac{S_i R_i}{N} \tag{3}$$

Where $S_i > 0$, $I_i > 0$ and $R_i > 0$ for all i. For simplicity, we have put $\beta_1 + \beta_2 = \beta$, and without loss of generality, we assume that the number of new users is equal to the number of outgoing users, that is $\Lambda = \mu N$.

Note that $N = N_i = I_i + S_i + R_i$, in fact

$$N_{i+1} = N_i + \Lambda - \mu N_i = N_i$$

Parameter	description
Λ	Recruitment rate
β_1	Potential relevance rate of information
β_2	Potential irrelevance rate of information
γ	Sharer to Removed transition rate
μ	Exit rate

TABLE 1. Parameters description

A flow chart for the model is shown in Fig. 1, and parameters description can be found in Table 1.

3. THE OPTIMAL CONTROL PROBLEM

3.1. Presentation of the controls. To eradicate the dissemination of information, some governments prefer to block all communications and can ban social media platforms [39]. But this strategy of control can lead to some protests or upsets. In seeking good interventions, governments began using social media to control the spread of some annoying information, by commenting on false information to correct it, or on real ones to confirm it.

By following this direction in the seek of control strategies, we propose a control strategy using a new feedback optimal control that will be interact in function of the number of ignorant, spreaders, and removed individuals. This control represents the effectiveness of comments and clarifications from official institutions such as government or any credible source. For example, the publication of the press release of the Moroccan Ministry of Health on its official Facebook page [40] and on its official web site [41] regarding the clarifications of the Influenza's situation and to ensure the population. This control can be also documents or some videos released in WhatsApp or on YouTube to aware people and/or to reveal the truth. As in the case of the companies in the Egyptian Market that reacted to rumors by publishing short videos explaining manufacturing processes and distributing documents confirming the quality and safety of their products [42].

The originality of this control is evident in that it depends on the number of populations, as the greater the number of posts and shares, the greater the percentage of control over the publication of the necessary explanations and documents, to limit the spread of this information by posts' delete and stopping shares. Where in the situation of rumors, after the truth was revealed, people will not only refrain from sharing that rumor but will also comment on others' posts by pointing to links to the truth, resulting in a lack of re-sharing and deleting posts. One thing that discourages re-sharing is when others comment on it by referring to external sources where the validity of the rumor is discussed. People who spread that rumor may try not to continue sharing it or stay away from these rumors if they know they are wrong [43]. Sometimes, Governments criminalize the re-sharing of some information on social networks due to the seriousness of the situation. In such situations, almost all group administrators delete all publications related to this information and block the publication or sharing of these contents on their pages anymore. Based on these facts, the controlled model is given by

$$I_{i+1} = I_i + \Lambda - \mu I_i - \frac{\beta I_i S_i}{N} \tag{4}$$

$$S_{i+1} = S_i - \mu S_i + \frac{\beta_1 I_i S_i}{N} - \gamma \frac{S_i R_i}{N} - k_i S_i \tag{5}$$

$$R_{i+1} = R_i - \mu R_i + \frac{\beta_2 I_i S_i}{N} + \gamma \frac{S_i R_i}{N} + k_i S_i \tag{6}$$

Where $S_i > 0$, $I_i > 0$ and $R_i > 0$ for all i, $\beta_1 + \beta_2 = \beta$, and $\Lambda = \mu N$.

3.2. First Scenario. In this scenario, as an extension of the work done in [44] we consider the reaction of the control as a function of the number of sharers and removed individuals, therefore we chose the feedback control variable to be as follow $k_i = u_i S_i + v_i R_i$, then the model (4)-(6) takes the following form:

$$I_{i+1} = I_i + \Lambda - \mu I_i - \frac{\beta I_i S_i}{N} \tag{7}$$

$$S_{i+1} = S_i - \mu S_i + \frac{\beta_1 I_i S_i}{N} - \gamma \frac{S_i R_i}{N} - u_i S_i^2 - v_i R_i S_i \tag{8}$$

$$R_{i+1} = R_i - \mu R_i + \frac{\beta_2 I_i S_i}{N} + \gamma \frac{S_i R_i}{N} + u_i S_i^2 + v_i R_i S_i \tag{9}$$

Where $S_i > 0$, $I_i > 0$ and $R_i > 0$ for all i. For simplicity, we put $\beta_1 + \beta_2 = \beta$, $\Lambda = \mu N$.

3.2.1. *Objective functional.* The main objective here is to use a feedback control function, depending on output of the system. We use optimal control strategy to reduce the number of Sharers and increase the number of Removals, and that with optimal costs of applying the control. Then, the problem is to minimize the objective functional given by

$$\begin{aligned} J(u,v) &= (\alpha_S S_{\mathcal{N}} - \alpha_R R_{\mathcal{N}}) \\ &+ \sum_{i=0}^{\mathcal{N}-1} \left(\alpha_S S_i - \alpha_R R_i + \frac{A}{2}(u_i)^2 + \frac{B}{2}(v_i)^2 \right) \end{aligned} \tag{10}$$

Where $A > 0, B > 0, \alpha_S > 0, \alpha_R > 0$ are the weight constants of controls, the sharers and removed, respectively, $u = (u_0, ..., u_{\mathcal{N}-1})$ and $v = (v_0, ..., v_{\mathcal{N}-1})$, and $\mathcal{N}$ is the final time of the control strategy.

Our goal is to minimize Sharers, minimize the cost of applying controls and increase the number of Removed individuals. In other words, we are seeking optimal controls u^* and v^* such that

$$J(u^*, v^*) = min\{J(u,v)/u \in \mathcal{U}, v \in \mathcal{V}\}$$

where $\mathcal{U}$ and $\mathcal{V}$ are the control sets defined by

$$\mathcal{U} = \{u / u_{min} \le u_i \le u_{max}, i = 0, ..., \mathcal{N}-1\} \tag{11}$$

$$\mathcal{V} = \{v / v_{min} \le v_i \le v_{max}, i = 0, ..., \mathcal{N}-1\} \tag{12}$$

such that

$$0 < u_{min} < u_{max} < 1$$

and

$$0 < v_{min} < v_{max} < 1$$

3.2.2. *Sufficient conditions.*

Theorem 1. *There exists an optimal control $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ such that*

$$J(u^*, v^*) = min\{J(u,v)/u \in \mathcal{U}, v \in \mathcal{V}\}$$

subject to the control system (7)-(9) and initial conditions.

Proof. Since the parameters of the system are bounded and there are a finite number of time steps, that is I, S, and R are uniformly bounded for all (u,v) in the control set $\mathscr{U}\times\mathscr{V}$, thus $J(u,v)$ is also bounded for all $(u,v)\in\mathscr{U}\times\mathscr{V}$. Which implies that $\inf_{(u,v)\in\mathscr{U}\times\mathscr{V}} J(u,v)$ is finite, and there exists a sequence $(u^n,v^n)\in\mathscr{U}\times\mathscr{V}$ such that

$$\lim_{n\to+\infty} J(u^n,v^n) = \inf_{(u,v)\in\mathscr{U}\times\mathscr{V}} J(u,v)$$

and corresponding sequences of states I^n, S^n, and R^n.

Since there is a finite number of uniformly bounded sequences, there exists $(u^*,v^*)\in\mathscr{U}\times\mathscr{V}$ and I^*, S^* and R^* such that, on a sequence,

$$(u^n,v^n)\to(u^*,v^*)$$

$$I^n\to I^*$$

$$S^n\to S^*$$

$$R^n\to R^*$$

Finally, due to the finite dimensional structure of the system (7)-(9) and the objective function $J(u,v)$, (u^*,v^*) is an optimal control with corresponding states I^*, S^*, and R^*. Which completes the proof. □

3.2.3. *Necessary conditions.* By using a discrete version of the Pontryagin's maximum principle [45, 46, 47], we derive necessary conditions for our optimal controls. For this purpose, we define the Hamiltonian as:

$$\begin{aligned}
\mathscr{H}(i) &= \alpha_S S_i - \alpha_R R_i + \frac{A}{2}(u_i)^2 + \frac{B}{2}(v_i)^2 \\
&+ \zeta_{1,i+1}\left[I_i + \Lambda - \mu I_i - \frac{\beta I_i S_i}{N}\right] \\
&+ \zeta_{2,i+1}\left[S_i - \mu S_i + \frac{\beta_1 I_i S_i}{N} - \gamma\frac{S_i R_i}{N} - u_i S_i^2 - v_i S_i R_i\right] \\
&+ \zeta_{3,i+1}\left[R_i - \mu R_i + \frac{\beta_2 I_i S_i}{N} + \gamma\frac{S_i R_i}{N} + u_i S_i^2 + v_i S_i R_i\right]
\end{aligned} \tag{13}$$

Theorem 2. *Given optimal controls u^*, v^* and solutions I^*, S^*, and R^*, there exists $\zeta_{k,i}, i = 0 \ldots \mathscr{N}-1, k = 1,2,3$, the adjoint variables satisfying the following equations:*

$$
\begin{aligned}
\Delta\zeta_{1,i} &= -\Bigg[\zeta_{2,i+1}\left(\frac{S_i\beta_1}{N}\right)+\zeta_{3,i+1}\left(\frac{S_i\beta_2}{N}\right) \\
&\quad -\zeta_{1,i+1}\left(\mu+\frac{S_i\,(\beta_1+\beta_2)}{N}-1\right)\Bigg] \\
\Delta\zeta_{2,i} &= -\Bigg[\alpha_S+\zeta_{3,i+1}\left(2u_iS_i+\frac{I_i\beta_2}{N}+\frac{R_i\gamma}{N}+v_iR_i\right) \\
&\quad -\zeta_{2,i+1}\left(\mu+2S_iu_i-\frac{I_i\beta_1}{N}+\frac{R_i\gamma}{N}-1+v_iR_i\right) \\
&\quad -\frac{I_i\zeta_{1,i+1}\,(\beta_1+\beta_2)}{N}\Bigg] \qquad (14) \\
\Delta\zeta_{3,i} &= -\left[\zeta_{3,i+1}\left(\frac{S_i\gamma}{N}-\mu+1+v_iS_i\right)-\alpha_R+\zeta_{2,i+1}\left(-v_iS_i-\frac{S_i\gamma}{N}\right)\right] \qquad (15)
\end{aligned}
$$

where $\zeta_{1,N}=0$, $\zeta_{2,N}=\alpha_S$, $\zeta_{3,N}=-\alpha_R$ are the transversality conditions. In addition

$$u_i^* = \min\left\{\max\left\{u_{min},\frac{S_i^2\,(\zeta_{2,i+1}-\zeta_{3,i+1})}{A}\right\},u_{max}\right\},\; i=0,\ldots,\mathscr{N}-1$$

$$v_i^* = \min\left\{\max\left\{v_{min},\frac{S_iR_i\,(\zeta_{2,i+1}-\zeta_{3,i+1})}{B}\right\},v_{max}\right\},\; i=0,\ldots,\mathscr{N}-1$$

Proof. Using the discrete version of the Pontryagin's maximum principle [45, 46], we obtain the following adjoint equations:

$$
\begin{aligned}
\Delta\zeta_{1,i} &= -\frac{\partial\mathscr{H}}{\partial I_i} \\
&= -\Bigg[\zeta_{2,i+1}\left(\frac{S_i\beta_1}{N}\right)+\zeta_{3,i+1}\left(\frac{S_i\beta_2}{N}\right) \\
&\quad -\zeta_{1,i+1}\left(\mu+\frac{S_i\,(\beta_1+\beta_2)}{N}-1\right)\Bigg] \\
\Delta\zeta_{2,i} &= -\frac{\partial\mathscr{H}}{\partial S_i} \\
&= -\Bigg[\alpha_S+\zeta_{3,i+1}\left(2u_iS_i+\frac{I_i\beta_2}{N}+\frac{R_i\gamma}{N}+v_iR_i\right) \\
&\quad -\zeta_{2,i+1}\left(\mu+2S_iu_i-\frac{I_i\beta_1}{N}+\frac{R_i\gamma}{N}-1+v_iR_i\right) \\
&\quad -\frac{I_i\zeta_{1,i+1}\,(\beta_1+\beta_2)}{N}\Bigg] \\
\Delta\zeta_{3,i} &= -\frac{\partial\mathscr{H}}{\partial R_i} \\
&= -\left[\zeta_{3,i+1}\left(\frac{S_i\gamma}{N}-\mu+1+v_iS_i\right)-\alpha_R+\zeta_{2,i+1}\left(-v_iS_i-\frac{S_i\gamma}{N}\right)\right]
\end{aligned}
$$

With $\zeta_{1,N}=0$, $\zeta_{2,N}=\alpha_S$, $\zeta_{3,N}=-\alpha_R$. To obtain the optimality conditions we take the variation with respect to controls (u_i and v_i) and set it equal to zero

$$\frac{\partial \mathscr{H}}{\partial u_i}=A\,u_i-S_i^2\,\zeta_{2,i+1}+S_i^2\,\zeta_{3,i+1}=0$$

$$\frac{\partial \mathscr{H}}{\partial v_i}=B\,v_i-S_iR_i\,\zeta_{2,i+1}+S_iR_i\,\zeta_{3,i+1}=0$$

Then we obtain the optimal control pair

$$u_i=\frac{S_i^2\,(\zeta_{2,i+1}-\zeta_{3,i+1})}{A}$$

$$v_i=\frac{S_iR_i\,(\zeta_{2,i+1}-\zeta_{3,i+1})}{B}$$

By the bounds in $\mathscr{U}$ and $\mathscr{V}$ of the controls in the definitions (11) and (12), it is easy to obtain u_i^* and v_i^* in the following form

$$u_i^* = min\left\{max\left\{u_{min},\frac{S_i^2\,(\zeta_{2,i+1}-\zeta_{3,i+1})}{A}\right\},u_{max}\right\}, i=0,...,\mathscr{N}-1$$

$$v_i^* = min\left\{max\left\{v_{min},\frac{S_iR_i\,(\zeta_{2,i+1}-\zeta_{3,i+1})}{B}\right\},v_{max}\right\}, i=0,...,\mathscr{N}-1$$

□

3.3. Second Scenario.

In this scenario, we consider the reaction of the control as a function of the number of ignorants, sharers and removed individuals, therefore we chose the feedback control variable to be as follow $k_i=u_iS_i+v_iR_i+w_iI_i$, then the model (4)-(6) takes the following form:

$$I_{i+1} = I_i+\Lambda-\mu I_i-\frac{\beta I_iS_i}{N} \tag{16}$$

$$S_{i+1} = S_i-\mu S_i+\frac{\beta_1 I_iS_i}{N}-\gamma\frac{S_iR_i}{N}-u_iS_i^2-v_iR_iS_i-w_iI_iS_i \tag{17}$$

$$R_{i+1} = R_i-\mu R_i+\frac{\beta_2 I_iS_i}{N}+\gamma\frac{S_iR_i}{N}+u_iS_i^2+v_iR_iS_i+w_iI_iS_i \tag{18}$$

Where $S_i>0$, $I_i>0$ and $R_i>0$ for all i, $\beta_1+\beta_2=\beta$, and $\Lambda=\mu N$.

3.3.1. *Objective functional.* In this scenario, the objective function takes the following form:

$$J(u,v,w) = (\alpha_S S_{\mathcal{N}} - \alpha_R R_{\mathcal{N}}) + \sum_{i=0}^{\mathcal{N}-1} \left(\alpha_S S_i - \alpha_R R_i + \frac{A}{2}(u_i)^2 + \frac{B}{2}(v_i)^2 + \frac{C}{2}(w_i)^2 \right) \tag{19}$$

Where $A > 0, B > 0, C > 0, \alpha_S > 0, \alpha_R > 0$ are the weight constants of controls, the sharers and removed, respectively, $u = (u_0, ..., u_{\mathcal{N}-1})$, $v = (v_0, ..., v_{\mathcal{N}-1})$and $w = (w_0, ..., w_{\mathcal{N}-1})$, and $\mathcal{N}$ is the final time of our strategy of control.

Our goal is to minimize Sharers, minimize the cost of applying controls and increase the number of Removals. In other words, we are seeking optimal controls u^* , $v^*$and w^* such that

$$J(u^*, v^*, w^*) = min\{J(u,v,w)/u \in \mathcal{U}, v \in \mathcal{V}, w \in \mathcal{W}\}$$

where $\mathcal{U}$ and $\mathcal{V}$ are the control sets defined by (11) and (12) respectively, and

$$\mathcal{W} = \{w/w_{min} \leq w_i \leq w_{max}, i = 0, ..., \mathcal{N}-1\} \tag{20}$$

such that

$$0 < w_{min} < w_{max} < 1$$

3.3.2. *Sufficient conditions.*

Theorem 3. *There exists an optimal control* $(u^*, v^*, w^*) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ *such that*

$$J(u^*, v^*, w^*) = min\{J(u,v,w)/u \in \mathcal{U}, v \in \mathcal{V}, w \in \mathcal{W}\}$$

subject to the control system (16)-(18) and initial conditions.

3.3.3. *Necessary conditions.* The Hamiltonian is defined as follows:

$$
\begin{aligned}
\mathscr{H}(i) &= \alpha_S S_i - \alpha_R R_i + \frac{A}{2}(u_i)^2 + \frac{B}{2}(v_i)^2 + \frac{C}{2}(w_i)^2 \\
&+ \zeta_{1,i+1}\left[I_i + \Lambda - \mu I_i - \frac{\beta I_i S_i}{N}\right] \\
&+ \zeta_{2,i+1}\left[S_i - \mu S_i + \frac{\beta_1 I_i S_i}{N} - \gamma\frac{S_i R_i}{N} - u_i S_i^2 - v_i S_i R_i - w_i S_i I_i\right] \\
&+ \zeta_{3,i+1}\left[R_i - \mu R_i + \frac{\beta_2 I_i S_i}{N} + \gamma\frac{S_i R_i}{N} + u_i S_i^2 + v_i S_i R_i + w_i S_i I_i\right]
\end{aligned} \tag{21}
$$

Theorem 4. *Given optimal controls u^*, v^*, w^* and solutions I^*, S^*, and R^*, there exists $\zeta_{k,i}$, $i = 0...\mathscr{N}-1$, $k = 1,2,3$, the adjoint variables satisfying the following equations:*

$$
\begin{aligned}
\Delta\zeta_{1,i} &= -\left[\zeta_{2,i+1}\left(\frac{S_i\,\beta_1}{N} - w_i S_i\right) + \zeta_{3,i+1}\left(\frac{S_i\,\beta_2}{N} + w_i S_i\right)\right. \\
&\quad -\zeta_{1,i+1}\left(\mu + \frac{S_i\,(\beta_1+\beta_2)}{N} - 1\right)] \\
\Delta\zeta_{2,i} &= -\left[\alpha_S + \zeta_{3,i+1}\left(2u_i S_i + \frac{I_i\,\beta_2}{N} + \frac{R_i\,\gamma}{N} + v_i R_i + w_i I_i\right)\right. \\
&\quad -\zeta_{2,i+1}\left(\mu + 2S_i u_i - \frac{I_i\,\beta_1}{N} + \frac{R_i\,\gamma}{N} - 1 + v_i R_i + w_i I_i\right) \\
&\quad \left. -\frac{I_i\,\zeta_{1,i+1}\,(\beta_1+\beta_2)}{N}\right]
\end{aligned} \tag{22}
$$

$$
\Delta\zeta_{3,i} = -\left[\zeta_{3,i+1}\left(\frac{S_i\,\gamma}{N} - \mu + 1 + v_i S_i\right) - \alpha_R + \zeta_{2,i+1}\left(-v_i S_i - \frac{S_i\,\gamma}{N}\right)\right] \tag{23}
$$

where $\zeta_{1,N} = 0$, $\zeta_{2,N} = \alpha_I$, $\zeta_{3,N} = -\alpha_R$ are the transversality conditions. In addition

$$
u_i^* = min\left\{max\left\{u_{min}, \frac{S_i^2\,(\zeta_{2,i+1} - \zeta_{3,i+1})}{A}\right\}, u_{max}\right\}, i = 0,...,\mathscr{N}-1 \tag{24}
$$

$$
v_i^* = min\left\{max\left\{v_{min}, \frac{S_i R_i\,(\zeta_{2,i+1} - \zeta_{3,i+1})}{B}\right\}, v_{max}\right\}, i = 0,...,\mathscr{N}-1 \tag{25}
$$

$$
w_i^* = min\left\{max\left\{w_{min}, \frac{S_i I_i\,(\zeta_{2,i+1} - \zeta_{3,i+1})}{C}\right\}, w_{max}\right\}, i = 0,...,\mathscr{N}-1 \tag{26}
$$

I_0	S_0	R_0	Λ	β_1	β_2	γ	μ
1000	10	10	μN	0.081	0.0310	0.002	2.6×10^{-4}

TABLE 2. Parameters values utilized for the resolution of the discrete systems (1-3) and (16-18), and then leading to simulations obtained from Figure 2 to Figure 7, with the initial conditions I_0, S_0, R_0.

4. NUMERICAL SIMULATIONS

We now present numerical simulations associated with the above-mentioned optimal control problems. We write code in MATLAB^{TM} and simulated our results using data from Table 2. The optimality systems are solved based on an iterative discrete scheme that converges following an appropriate test similar to the one related to the Forward-Backward Sweep Method (FBSM). The state system with an initial guess is solved forward in time and then the adjoint system is solved backward in time because of the transversality conditions. Afterward, we update the optimal control values using the values of state and co-state variables obtained at the previous steps. Finally, we execute the previous steps until a tolerance criterion is reached.

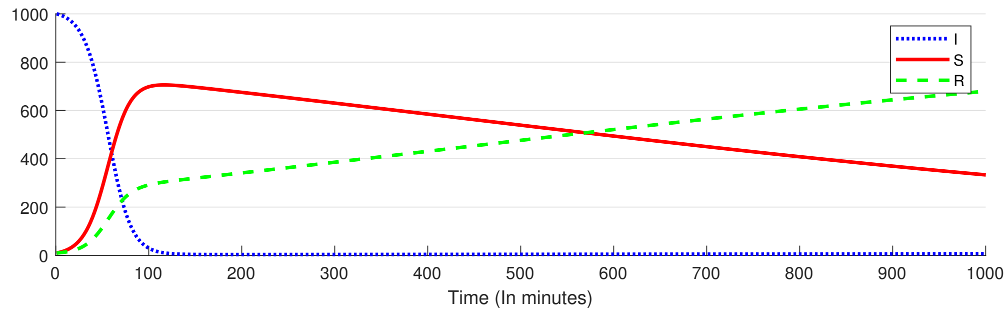
In all the simulations below, the minutes were used as a time unit because the spread of information occurs faster in time. We focus here on information that is more appealing and has the potential to be shared.

Without loss of generality, and as an example, we have chosen as a studied population, a group (In Facebook, Tweeter, WhatsApp ...) with 1000 members, that can be considered as the ignorant group.

At the initial time $i = 0$, ten sharers, and ten removed individuals are introduced into this group with information under investigation which can be a posted as a video or image or text.

All parameters of the table 2 are chosen to get a situation in which the number of sharers rises above 700 individuals of the population and the removed group remains small. In this situation, it can be shown that our proposed strategy of optimal control is very efficient to reduce the number of sharers and thus the amount of the information while it increases the number of the removed and/or Ignorant population and that with an optimal cost.

FIGURE 2. Dynamic of the system (1-3) without the controls: Ignorants (I), Sharers (S) and Removeds (R).



In Fig.2 It can be seen that after about 100 hours of information injection, no one is ignorant of it. Which means that the information reaches almost all the members of the group. We talk then about an explosion of the information. In the case of false information, this situation can lead to serious economic and/or political damages. Because it can be seen from this figure that the more the number of sharers is big the more of the amount of the information is huge. Because it can be seen from this figure that the greater the number of shares, the greater the huge volume of information. Thus, the information cannot be stopped from spreading, and thus it can spread to outside groups and reach other publishers elsewhere. The reason for the sharers' decrease is that the information has already reached all members of the group and may have reached other groups resulting in a loss of interest in sharing this information.

4.1. First scenario (Feedback of S and R). Fig. 3 shows the dynamic of I, S, R in the model (16-18) with the control function $k_i = u_i S_i + v_i R_i$. Where it can be seen that the number of sharers remains very small, while the number of the removed people rise to about 650 individuals before it starts to decrease by about 550 individuals at the end of this simulation. Making comments and explanations on social networks could be another way to ensure people's safety in an emergency. More people reading official comments, namely correct information or interpretations, will probably deal with the subject rationally.

To achieve this optimal result, we suggest using our control strategy in the first 8 hours of information appearing, to rapidly lower participants' climax. When the people involved don't provide more explanatory information quickly, people can be left feeling something is wrong,

FIGURE 3. Dynamic of the system (16-18) with the control $k_i = u_i S_i + v_i R_i$: Ignorants (I), Sharers (S) and Removeds (R).

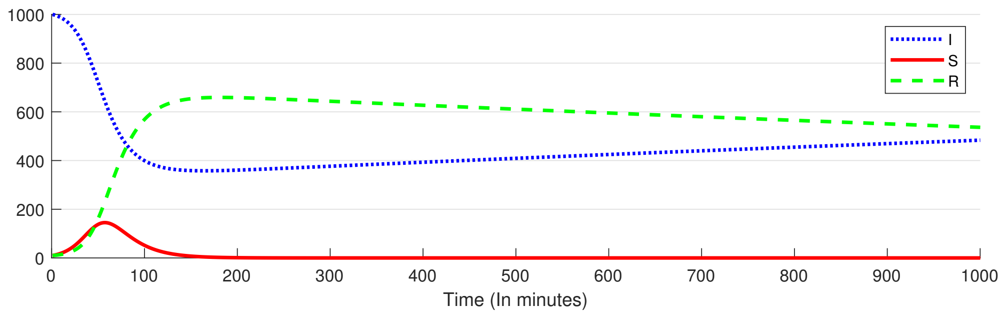
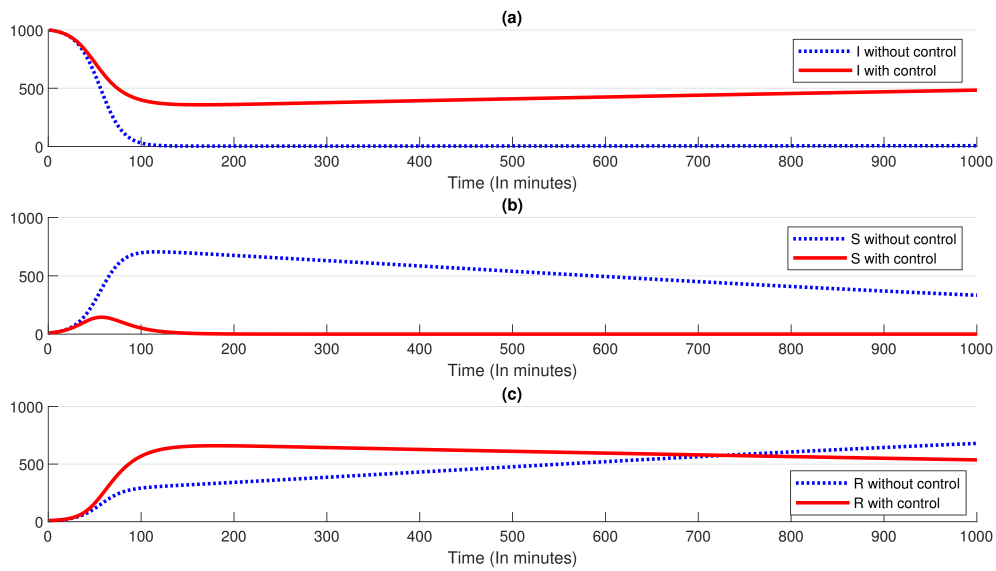


FIGURE 4. Comparison between the dynamic of the system (1-3) and the system (16-18) with the control $k_i = u_i S_i + v_i R_i$: (a) Ignorants, (b) Sharers, and (c) Removeds.



leading to a lot of gossip. In the case of government rumors, some information can build trust and boost social stability.

Fig.4 shows the comparison of the different states of the proposed model with and without the control function. It can be seen from that figure in (a) that before using the optimal control, the number of ignorant people decreases very rapidly compared to the case when there is control. This fact can be explained by the efficiency of the control approach we used here to block information to reach more people. Where we can see that after about 100 minutes the number of ignorant people begins to increase slightly again. Which means that this information starts to wear out. Sub-figure (b) shows the comparison between the number of sharers with and without the control, where it can be seen that when using this control we can bring forward the peak of sharing and avoid the spread of the information. The number of shares does not exceed 200 members when the control is used, compared to the case when there is no control where this number reaches about 700 individuals to begin decreasing slowly up to the end of the simulation with a value exceeds 400 members. In sub-figure (c) we can see the comparison between the number of removal with and without the use of the optimal control, as it can be seen that when using the control, the number of removal increases rapidly from the first 100 minutes to exceeding 600 individuals and then begins to decrease slightly to reach 500 individuals at the end of these Simulation, compared to a non-controlling state, grows slightly since the start of simulation. From these figures, we can see the efficiency of the optimal control that we propose in this paper, in reducing the number of shares and increase the number of removed and ignorant individuals with an optimal cost.

4.2. Second scenario (Feedback of I, S, and R). Fig. 5 shows the dynamic of I, S, and R in the model (16-18) with the control function $k_i = u_iS_i + v_iR_i + w_iI_i$. Where it can be seen that the number of sharers remains almost zero, while the number of the removed people rise to about 100 individuals, and the ignorant population decreases by about 100 and then starts to rise slightly. This result shows the strong control over information in making almost all group members ignore it.

FIGURE 5. Dynamic of the system (16-18) with the control $k_i = u_iS_i + v_iR_i + w_iI_i$: Ignorants (I), Sharers (S) and Removeds (R).

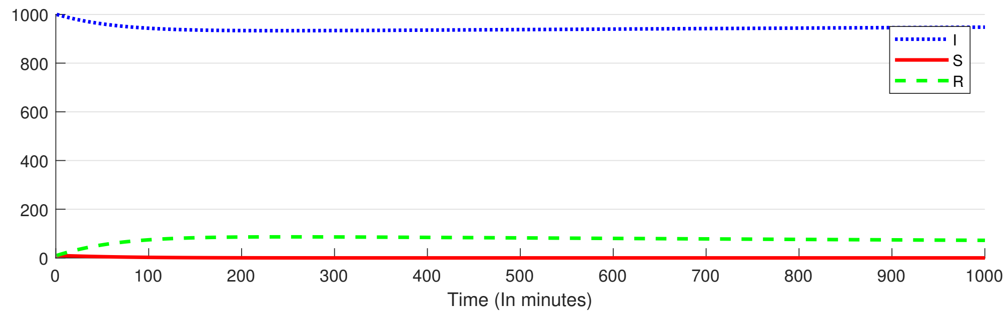


FIGURE 6. Comparison between the dynamic of the system (1-3) and the system (16-18) with the control $k_i = u_iS_i + v_iR_i + w_iI_i$: (a) Ignorants, (b) Sharers, and (c) Removeds.

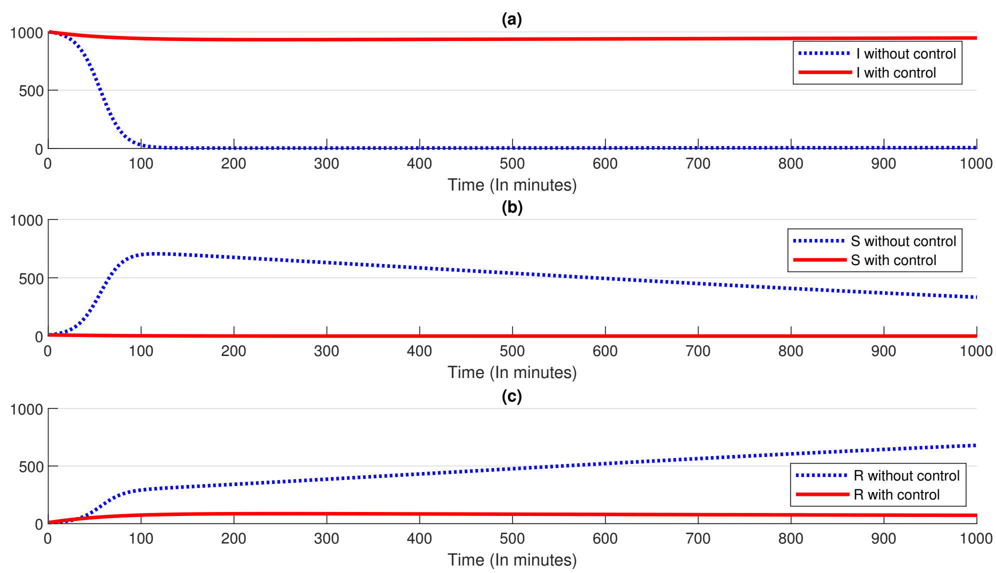


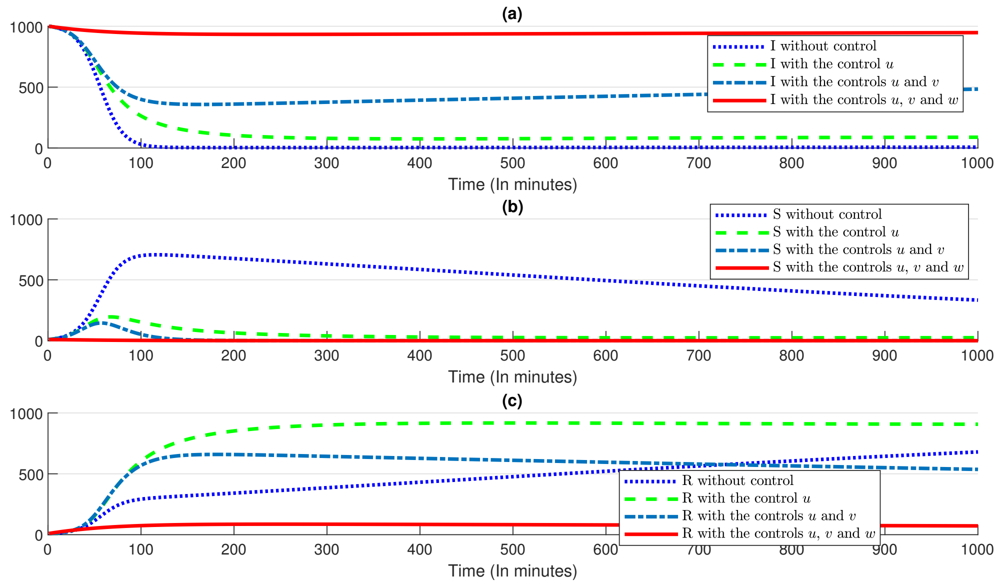
Fig.6 shows the comparison of the different states of the proposed model with and without the control function in the second scenario. In the sub-figure (a), after using the optimal control, we can see that the number of ignorant people remains almost insensitive to the effect of this information, where it decreases by a small value and begins increasing again compared to the case when there is no control where it decreases quickly towards zero by about 100 minutes. This fact can be explained by the efficiency of the control approach we utilized in blocking the proliferation of the information and making all members ignore it.

Sub-figure (b) shows the comparison between the number of sharers with and without control, as it can be seen that when using this control we can stop the multiplication of information by making the number of shares almost null, compared to the case when there is no control where this number reaches about 700 individuals to begin to decline Slowly until the end of the simulation with a value greater than 400 members. In the sub-figure (c) we can see the comparison between the number of removed with and without the use of the optimal control, where it can be seen that when the control is utilized the number of removed rises quickly by about the first 10 minutes to reach about 100 individuals then to continue decreasing slightly, compared to the case when there is no control it grows slightly from the beginning of the simulation.

From these figures, we can see the efficiency of the optimal control that we propose in this paper, in reducing the number of spreaders and removed individuals while increasing the number of ignorant individuals with an optimal cost.

In Fig.7 we can see the comparison between the different scenarios, without the optimal control, with the control $k_i = u_i S_i$, with the control $k_i = u_i S_i + v_i R_i$ and with the control $k_i = u_i S_i + v_i R_i + w_i I_i$. Sub-figure (a) shows the comparison between the ignorant populations in the different scenarios, where we can see clearly that when we use the control presented in the second scenario defined by the feedback control of the three states I, S and R, the number of ignorant individuals is insensitive to the influence of information, in other words, we can say that in this scenario, optimal control can efficiently prevent information from reaching the ignorant and thus eliminate its spread. Sub-figure (b) shows the comparison between the number of shares in the different scenarios, where we can see that when we use the feedback of the three states we can eliminate the shares of the information radically, while in the other scenarios, it can be seen that the number of sharers rise a while and then begins to decrease towards zero. Sub-figure (c) shows the comparison between the numbers of removed individuals in these scenarios, where it can be seen that the number of removal in the second scenario takes the small values compared to other scenarios, and the number of removals takes the big values in the case when we use the feedback of S only.

FIGURE 7. Comparison between the dynamic of the system (1-3) and the system (16-18) with the different controls $y_i = u_i S_i$, $y_i = u_i S_i + v_i R_i$, and $y_i = u_i S_i + v_i R_i + w_i I_i$: (a) Ignorants, (b) Sharers, and (c) Removeds.



The use of each scenario depends on the objective of the intervention. If the goal is only to get rid of the information, then the best scenario would be the second scenario when we use the feedback of the three functions. But if the goal of the intervention is to control the spread of information and make the largest number of people conscious as possible, then only function S feedback control would be the best scenario.

5. CONCLUSION

In this article, we considered a discrete-time model for describing the dissemination of information from one person to another, it can be shared word-to-mouth or in certain types of online environments such as Facebook, WhatsApp and Twitter. The impact of information sharing is becoming more and more noticeable in the world with the increase of technologies, its potential has become clearer through controlling people's behavior and opinions. In this paper, we considered control strategies depending on the rate of diffusion of the information, and its effect, by considering feedback controls of the populations' numbers. We presented the first scenario, where we considered the feedback control of the number of sharers and removed individuals, and the second scenario when we proposed a feedback control of the three population functions. Based on a discrete version of Pontryagin's Maximum principle we characterized optimal controls and simulated and discussed our results numerically. We found that if the goal of the intervention is only to eliminate the information, then the best scenario would be the second scenario when we use the feedback of the three functions. But if the goal is to control the spread of information and make the largest number of people conscious as possible, then only function S feedback control would be the best scenario.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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