Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2021, 2021:15
https://doi.org/10.28919/cmbn/5302
ISSN: 2052-2541

DYNAMICS IN A RATIO-DEPENDENT ECO-EPIDEMIOLOGICAL PREDATOR-PREY MODEL HAVING CROSS SPECIES DISEASE TRANSMISSION

DEBASISH BHATTACHARJEE, ANKUR JYOTI KASHYAP*, HEMANTA KUMAR SARMAH, RANU PAUL

Department of Mathematics, Gauhati University, Guwahati, Assam 781014, India

Abstract. We have proposed and analyzed an eco-epidemiological predator-prey interaction model having disease in both the population with ratio-dependent functional response. The total population has been classified into susceptible prey, infected prey, susceptible predator, and infected predator. The infection propagation is considered directly proportional to the number of individuals come in to contact with one infected individual. The predation ability of infected predator is neglected during the disease. The infection is considered to be weak infection in case of predator. Predator can recover themselves due to their natural immune system or application of external curative stimulants. Though such weak infections are not generally considered for study but these cannot be completely ignored. The positivity of the solutions, the existence of various biologically feasible equilibrium points, their stability are investigated. The numerical analysis is carried out with hypothetical set of parameters to substantiate the analytical findings that our model exhibits. The oscillatory coexistence of the species which is very common in nature is observed for disease free as well as coexisting system. The stability nature of the Hopf-bifurcating periodic solutions of the disease free as well as coexisting equilibrium are determined by computing the Lyapunov coefficients. Further, the system undergoes the Bogdanov-Takens bifurcation in two-parameter space around the disease free as well as coexisting equilibrium. It is also observed that the system will be disease free through proper predational strategies.

Keywords: epidemiological model; ratio-dependent functional response; local and global stability; Lyapunov coefficient; generalised Hopf; Bogdanov-Taken bifurcation.

*Corresponding author
E-mail address: ankurjyoti99@gmail.com
Received December 05, 2020

2020 AMS Subject Classification: 92B05, 34A34, 37N25, 37G15, 37C29, 37M05, 37M10, 37M25.

1. INTRODUCTION

The study of the interaction between predator and prey population was first introduced by Alfred J Lotka [1]. One of the important parts of predator-prey population modeling is the mathematical formulation of predator-prey interaction, termed as a functional response. There are several types of functional responses : Holling I-III type [2, 3]; Hassell–Varley type [4]; Beddington–DeAngelis type, introduced independently by Beddington [5] and DeAngelis et al. [6] ; the Crowley–Martin type [7]; and the recent well-known ratio-dependence type by Arditi and Ginzburg [8]. The ratio-dependent functional response is a subtype of predator-dependent functional response. It is assumed that the prey eaten per predator per unit time is a function of prey to predator ratio. For predator-prey interaction where predation involves serious searching, the ratio-dependent predator-prey models become more appropriate compared to the other types, for example, *see* [9]. The ratio-dependent models are, in fact, more flexible and versatile for which we are interested in studying the dynamics of the predator-prey system with this functional response.

The importance of the study of diseases in predator-prey species is essential because of its effect on both the populations towards extinction. The interaction of the susceptible– infected–recovered population has become an important subject of research after the work of Kermack, and Mc Kendrick [10]. Anderson and May(1986)[11]; Hadeler and Freedman(1989)[12]; Venturino(1995)[13]; Chattopadhyay and Arino(1999)[14] have merged the scenario of epidemiology in prey-predator system. After those, diverse predator-prey ecological systems having the disease in one or both types of species have been reported. Among them, the authors find two types of systems.

In the first case, the prey-predator species, one is susceptible or vector to a disease that crosses the species barrier and the other species become infected. Due to lack of immunity, the disease becomes fatal to the other species [16, 15]. In the Pelican-Tilapia ecology of Salton sea [16], it has been reported that tilapia, while dying of Vibrio infections, provides fatal doses of botulism when Pelicans predate tilapia, which causes the death of pelicans, showing the infection crossing the predator-prey barrier. However, in [16], the dynamic property of the infected predator (pelican) has been discussed implicitly. "Cross border" scenario has been considered explicitly by Hadeler and Freedman[12], Anderson and May(1986)[11], Venturino(1995)[13], Latif et al.[17], Hsieh and Hsiao[18]. In the article[18], Hsieh and Hsiao considered disease transmission while predating citing the example of transmission of avian influenza (H5N1) from aves to human and then the pandemic spread of the disease among humans. In all these cases, the disease crossing the barrier of species has been fatal for other species.

In the other case, the disease can not cross the species barrier, or one species is not responsible for the disease spread to the other while predating. One cause may be the predator, say, is not susceptible to the disease. Some natural examples are a marine planktonic system where both phytoplankton (such as Cryptophyte) and zooplankton (such as Rotifers) are infected by some viral disease [19]. Still the disease is not transmitted from one species to another. Many fish related viral or bacterial disease does not cause any infection in human being. There have been some initiatives in this category, in literature, having a disease in the prey but not in predator [14, 20, 21, 22], both in the prey and predator but not "crossing the species-barrier" [19, 24, 23] and having a disease in the predator species[25, 26] but not in prey. However, there may be a third scenario in between. The disease is transmitted from one species to another but the disease is curable in the latter. This kind of system is not discussed in the literature to the best of the author's knowledge. Some immunocompromised predators may develop a weak infection due to high pathogen load, added by the stressed environment (in the case of fish) in some cases, but ultimately recovered, showing that the predator is not apparently affected by the disease of prey. This type of scenario is difficult to detect but cannot be ruled out. The recovery may be due to natural immunity or the application of external curative stimulants. For example, Herring (prey) can be infected with and carry the kidney disease bacterium *Renibacterium Salmoninarum*, which can cause a threat to Salmon (predator)[27]. The infection of this bacteria can be cured through proper antibiotic therapy [28] in cultured aquatics or fish farm or laboratory. Transmission of VHS virus Genotype IV is horizontal from fish to fish and likely by ingestion of infected fish [29]. In the Japanese flounder (Paralichthys olivaceus: predator)-Japanese sand lance (Ammodytes personatus: prey) ecosystem [30], both are susceptible to VHS virus [31]. Even fish has variable morbidity due to this deadly virus. Experimentally it is shown in [32] that shifting of water temperature to 20^0C immediately after VHS infection or maintaining the water temperature at 20^0C can be considered as a control measure reducing the mortality to 0%. However, the virus can grow at that temperature. At 20^0C, the infection shows less clinical sign [32] showing higher and quicker immunity response of Japanese flounder. We have considered a prey-predator system as having a disease in prey, and the disease is transmitted horizontally both in prey and predator but not vertically. The disease is also transmitted during prey-predator interaction. Ratio dependent functional response has been considered in prey-predator interaction. Although it creates a weak infection, the disease is not fatal, creating a temporal reduction of survival ability in the case of a predator. We also assume that predator, after recovery, does not become immune to the disease. This scenario may be seen in cultured aquatics, fish farms, etc., where proper care of infected fish is done. The model is also applicable to the scenario where an infected prey is a vector to the predator's infectious disease. The model is only comparable with the other eco epidemiological models having disease transmission while predating. Only a few models have been discussed in this case as per the author's knowledge [12, 13, 17, 18, 33]. Our model is different from the comparable models in terms of functional response and disease-related mortality in predators.

The paper is organized as follows: In section 2, the model formulation has been discussed. The proposed system's boundedness, condition for the existence of equilibrium has been discussed in section 3. In section 4, the vanishing equilibrium's nature and the stability condition of other equilibrium have been discussed. In section 5, numerical simulation verifying the analytical results so obtained has been discussed. Besides these, some numerical bifurcation scenario has been put forward in this section. A few comparisons of the considered system have been made with the numerically simulated trajectory of its corresponding counterpart having time delay (section 6). Finally, in section 7, a brief description of the outcomes obtained from the current study is provided. For analytical computation, numerical simulation, etc., advanced software like Mathematica, Matlab, MatCont have been used.

2. Model Formulation

To formulate the model we make the following assumptions:

(1) Both the populations are divided into two categories, susceptible and infected. Total prey population density is denoted by N, total predator population density is denoted by P, the susceptible prey X, infected prey Y, the susceptible predator P_1 and infected predator P_2. $N = X + Y$ and $P = P_1 + P_2$.

(2) The susceptible prey population can reproduce in a logistic fashion while the infected ones cannot reproduce. But the infected prey growth is due to the law of mass action. Both the populations coexist with a carrying capacity K.

(3) Susceptible prey becomes infected when they interact with the infected one. The infection propagation is directly proportional to the number of individuals coming in to contact one infected individual due to horizontal transmission of the disease or predation. The susceptible population of both predator and prey is sufficiently large. The time span for the existence of the model is relatively small, the disease transmission rate is constant and assumed to be λ_1, λ_2, respectively. The infected predator population does not breed, but this population's growth factor is from the law of mass action.

(4) Both prey and predator population follow ratio-dependent functional response. This functional response considers the fact that if $P_1 + P_2 \to \infty$ then the functional response value tends to $\frac{pX}{m}(p = p'$ case), i.e. if the predators become very large then the consumption rate of the susceptible prey is $\frac{pX}{m}$, i.e. the prey will not vanish. This is because there will be competition among predators to consume prey. The infected prey is unable to recover while the infected predators are capable of recover or become immune with a rate γ'.

(5) The mortality due to infection in the prey population has been considered. The natural mortality of the prey and predator has been viewed as dependent on respective carrying capacity. The growth rate of susceptible predators P_1 is proportional to their predation of susceptible prey.

(6) Due to the self-recovery of the predators, the amount of susceptible species increases by the law of mass action.

(7) Predation rates of both susceptible and infected species are different towards both the prey species.

(8) Both the susceptible and infected predators have different growth rates, say, δ and δ'. The disease transmission is not vertical.

Under the above assumptions, we have the following model equations:

$$
\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X+Y}{K}\right)-\lambda XY-\frac{(pP_1+p'P_2)X}{m(P_1+P_2)+X+Y}\\
\frac{dY}{dT} &= \lambda XY-\gamma Y-\frac{(cP_1+c'P_2)Y}{m(P_1+P_2)+X+Y}\\
\frac{dP_1}{dT} &= (\delta P_1+\delta' P_2)\left(1-\frac{P_1+P_2}{X}\right)-\lambda' P_1P_2-\frac{c\alpha P_1 Y}{m(P_1+P_2)+X+Y}+\gamma' P_2\\
\frac{dP_2}{dT} &= \lambda' P_1P_2+\frac{c\alpha P_1 Y}{m(P_1+P_2)+X+Y}-\gamma' P_2
\end{aligned}
\tag{1}
$$

r = species growth rate of prey in the susceptible population.

K = carrying capacity of the ecosystem.

$\lambda =$ disease transmission coefficient of prey.

$p, p' =$ searching rate of susceptible and infected predator towards susceptible prey.

$c, c' =$ searching rate of susceptible and infected predator towards infected prey.

m = a positive constant. $\gamma =$ per capita death rate of infected prey(e.g say 10%)

$\delta =$species growth rate of susceptible predator .

$\delta' =$species growth rate of infected predator .

$\lambda' =$ disease transmission coefficient of predator.

γ'=conversion rate of infected to susceptible predator. (if one predator takes f days to recover, so in one day $1/f = \gamma'$ of the pradator recovers.)

α = Infection transmission proportionality due to predation.

The system (1) is to be analyzed with the following conditions: $X(0) > 0; Y(0) > 0; P_1(0) > 0; P_2(0) > 0$

We further simplify the model by assuming that the disease affects both the predation and the reproduction rate of infected predators temporarily till recovery. Hence for our model $p' = 0$, $c' = 0$ and $\delta' = 0$.

Under the above assumptions our model becomes,

$$(2)\qquad \begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X+Y}{K}\right)-\lambda XY-\frac{pP_1X}{m(P_1+P_2)+X+Y}\\
\frac{dY}{dT} &= \lambda XY-\gamma Y-\frac{cP_1Y}{m(P_1+P_2)+X+Y}\\
\frac{dP_1}{dT} &= \delta P_1\left(1-\frac{P_1+P_2}{X}\right)-\lambda' P_1P_2-\frac{c\alpha P_1Y}{m(P_1+P_2)+X+Y}+\gamma' P_2\\
\frac{dP_2}{dT} &= \lambda' P_1P_2+\frac{c\alpha P_1Y}{m(P_1+P_2)+X+Y}-\gamma' P_2
\end{aligned}$$

with conditions: $X(0)>0; Y(0)>0; P_1(0)>0; P_2(0)>0$

In order to simplify the system we use the following variables,

$$t=rT, x=X/K, y=Y/K, z=P_1/K, w=P_2/K$$

Thus we obtain,

$$\begin{aligned}
\frac{dx}{dt} &= x(-x-y+1)-a_1xy-\frac{a_2xz}{m(w+z)+x+y}\\
\frac{dy}{dt} &= a_1xy-b_1y-\frac{b_2yz}{m(w+z)+x+y}\\
\frac{dz}{dt} &= u_1z\left(1-\frac{z}{x}\right)-u_3wz-\frac{u_4zy}{m(w+z)+x+y}+u_5w\\
\frac{dw}{dt} &= u_3wz+\frac{u_4zy}{m(w+z)+x+y}-u_5w
\end{aligned}$$

where, $a_1=\lambda K/r; a_2=p/r; b_1=\gamma/r; b_2=c/r$

$$u_1=\delta/r; u_3=\lambda' k/r; u_4=c\alpha/r; u_5=\gamma'/r$$

with conditions: $x(0)>0; y(0)>0; z(0)>0; w(0)>0.$

3. Model Analysis

Theorem 1. *All the solutions of the system* (2)*,which initiate in* $\mathbf{R}_+{}^4$ *are uniformly bounded.*

Proof. From the first equation of system (2) we obtain, $\frac{dx}{dt}\le x(1-x)$. By solving the differential inequality we get, $\limsup_{t\to\infty} x(t)\le 1$ or $x(t)\le 1$.

Now define a function, $G(t)=x(t)+y(t)+z(t)+w(t)$ and then by taking the derivative along the solution of system (2),we get

$$\frac{dG}{dt} \leq x(1-x) - b_1 y + u_1 z - u_5 w \leq x - (b_1 y - u_1 + u_5 w)$$

where $\eta = \min\{b_1, u_1, u_5\}$ then we get

$$\frac{dG}{dt} = (1+\eta)x - \eta G \leq (1+\eta) - \eta G$$

Now, by using Gronwall lemma,

$$0 < G(t) \leq \frac{1+\eta}{\eta}\left(e^{-\eta t} - 1\right) - G(0)e^{-\eta t}$$

Thus $G(t) \leq \frac{1+\eta}{\eta}$ as $t \to \infty$ that is independent of initial conditions and hence the system (2) is bounded. Thus all the solutions of the system (2) are confined in the region.

$$\Omega = \left\{(x,y,z,w) : 0 \leq x(t) + y(t) + z(t) + w(t) \leq \frac{1+\eta}{\eta} + \varepsilon, \forall \varepsilon > 0\right\}$$

□

3.1. EXISTENCE OF EQUILIBRIUM POINTS.

The system (2) has equilibrium points namely $E_i, i = 0, 1, 2, 3, 4, 5, 6.....$

(a) The vanishing equilibrium $E_0(0,0,0,0)$

(b) The axial equilibrium point $E_1(1,0,0,0)$, which always exist

(c) The predator free equilibrium $E_2 = \left(\frac{b_1}{a_1}, -\frac{b_1 - a_1}{a_1(a_1+1)}, 0, 0\right)$.

which exists under the sufficient condition: $a_1 > b_1$

(d) The disease free equilibrium $E_3(x_3, 0, z_3, 0)$, where, $x_3 = \frac{m+1-a_2}{m+1}; z_3 = \frac{m+1-a_2}{m+1}$.

E_3 exist under the sufficient condition: $m+1 > a_2$.

(e)The Infected prey free equilibrium: $E_4(x_4, 0, z_4, w_4)$,where

$$x_4 = \frac{u_5}{u_3}$$
$$z_4 = \frac{u_5}{u_3} \tag{3}$$
$$w_4 = \frac{u_5\left(u_3\left(a_2 - m - 1\right) + (m+1)u_5\right)}{mu_3\left(u_3 - u_5\right)}$$

3.1.1. ***Existence of the coexisting equilibrium point*.** Solving infected prey nullcline for w and applying it to solve infected predator nulcline for y, one can obtain

$$w = \frac{-a_1 mxz - a_1 x^2 + a_1 x(-y) + b_1 mz + b_1 x + b_1 y + b_2 z}{m\,(a_1 x - b_1)} \tag{4}$$

$$y = \frac{b_2\,(u_3 z - u_5)\left(a_1(-m)xz - a_1 x^2 + b_1 mz + b_1 x + b_2 z\right)}{(a_1 x - b_1)\,(-a_1 m u_4 x + b_1 m u_4 + b_2 u_3 z - b_2 u_5)} \tag{5}$$

Again using (4) and (5) and solving steady state equation of susceptible predator, for z, one may obtain

$$z = x \tag{6}$$

x can be obtained applying (4),(5),(6) from the equation

$$A_0 x^3 + A_1 x^2 + A_2 x + A_3 = 0 \tag{7}$$

where, $A_0 = a_1\left(-a_1 b_2\,(-a_2 u_3 + b_2(m+1)u_3 + m u_4) + a_2 a_1^2(-m)u_4 - b_2^2 m u_3\right)$
$A_1 = a_1^2\,(a_2\,(3b_1 m u_4 - b_2 u_5) + b_2\,(b_2(m+1)u_5 + m u_4))$
$+a_1 b_2\left(-2a_2 b_1 u_3 + b_2\,(u_3\,(b_1(m+1)-1) + m u_5) + 2b_1 m u_4 + b_2^2 u_3\right) + b_2^2 u_3\,(b_1 m + b_2)$
$A_2 = a_2 b_1\,(b_1\,(b_2 u_3 - 3a_1 m u_4) + 2a_1 b_2 u_5)$
$-b_2\left(b_1\,(b_2\,(u_5\,(a_1(m+1)+m) - u_3) + 2a_1 m u_4) + b_2 u_5\,(a_1\,(b_2-1) + b_2) + b_1^2 m u_4\right)$
$A_3 = b_1\,(a_2 b_1 + b_2)\,(b_1 m u_4 - b_2 u_5)$

Theorem 2. *The coexisting equilibrium point exists if*

(a) $0 < x < 1, 0 < b_1 < a_1 x, 0 < u_3 < \frac{u_5}{x}, \frac{b_2(x-1)}{a_1 x - b_1} + a_2 < 0, b_2 \geq \frac{(a_1 x+1)(a_1 x - b_1)}{(a_1+1)x}$

Or,

(b) $0 < x < 1, 0 < b_1 < a_1 x, 0 < u_3 < \frac{u_5}{x}, \frac{b_2(x-1)}{a_1 x - b_1} + a_2 < 0, a_1 x < b_1 + b_2,$
$b_2 < \frac{(a_1 x+1)(a_1 x - b_1)}{(a_1+1)x}, \frac{b_2\left(a_1(b_1+b_2-1)x + a_1^2\left(-x^2\right) + b_2 x + b_1\right)}{(b_1 - a_1 x)^2} + a_2 > 0$

Proof. We show the existence of the tuple $\bar{x} = (x, y, z, w, a_1, a_2, b_1, b_2, m, u_1, u_3, u_4, u_5) > 0$, satisfying $\psi(\bar{x}) = 0$, $\psi(x) = (-\frac{a_2 xz}{m(w+z)+x+y} - a_1 xy + x(-x-y+1), a_1 xy - \frac{b_2 yz}{m(w+z)+x+y} - b_1 y, -\frac{u_4 yz}{m(w+z)+x+y}$
$- u_3 wz + u_5 w + u_1 z\left(1 - \frac{z}{x}\right), \frac{u_4 yz}{m(w+z)+x+y} + u_3 wz - u_5)$

x is made free to choose, shifting the dependency to m, we have, from (7), for m,

(8)
$$m = \frac{b_2\left(u_3x-u_5\right)\left(a_1^2x^2\left(a_2-b_2\right)+a_1x\left(b_2\left(b_1+b_2-1\right)-2a_2b_1\right)+a_2b_1^2+b_2\left(b_2x+b_1\right)\right)}{\left(a_1x-b_1\right)\left(a_2u_4\left(b_1-a_1x\right)^2+b_2u_4(x-1)\left(a_1x-b_1\right)+\left(a_1+1\right)b_2^2x\left(u_3x-u_5\right)\right)}$$

Applying (8) and (6) on (4) and (5), we have,

$$y = \frac{a_2b_1+a_1a_2(-x)-b_2(x-1)}{\left(a_1+1\right)b_2} \tag{9}$$

$$w = \frac{u_4\left(a_1x-b_1\right)\left(-a_2b_1+a_1a_2x+b_2(x-1)\right)}{\left(a_1+1\right)b_2^2\left(u_3x-u_5\right)} \tag{10}$$

It is sufficient to show the positiveness of m, y, w so obtained, when x and other parameters are chosen positive so that $\bar{x} > 0$ satisfying $\psi(\bar{x}) = 0$.

We consider,

$\Delta_1 = b_2\left(a_1^2x^2\left(a_2-b_2\right)+a_1x\left(b_2\left(b_1+b_2-1\right)-2a_2b_1\right)+a_2b_1^2+b_2\left(b_2x+b_1\right)\right)$

$\Delta_2 = a_2u_4\left(b_1-a_1x\right)^2+b_2u_4(x-1)\left(a_1x-b_1\right)+\left(a_1+1\right)b_2^2x\left(u_3x-u_5\right)$

$\phi_1 = a_2b_1+a_1a_2(-x)-b_2(x-1)$, $\Delta = \frac{\Delta_1}{\Delta_2}$, $\phi_2 = \frac{a_1x-b_1}{u_3x-u_5}$

then $y = \frac{\phi_1}{(1+a_1)b_2}$, $w = \frac{-\phi_2 y}{b_2}$, $m = \frac{\Delta}{\phi_2}$

It is sufficient to show that $\phi_1 > 0; \phi_2, \Delta < 0$

$\phi_2 < 0$ implies,

Case 1: $u_3 < \frac{u_5}{x}, x > \frac{b_1}{a_1}$

$a_2 > 0, \phi_1 > 0$ implies $\frac{b_2(x-1)}{a_1x-b_1} + a_2 < 0$ and $0 < x < 1$.

Now $\frac{\Delta_2}{(xa_1-b_1)^2u_4} = \frac{b_2(x-1)}{a_1x-b_1} + a_2 + \frac{(a_1+1)b_2^2(u_3x-u_5)}{(xa_1-b_1)^2u_4} < 0 \Rightarrow \Delta_2 < 0$. So, for a feasible region, we must have $\Delta_1 > 0$.

$$\Rightarrow a_2 > \frac{a_1^2b_2x^2-a_1b_2^2x+a_1b_2x-a_1b_1b_2x-b_2^2x-b_1b_2}{-2a_1b_1x+a_1^2x^2+b_1^2}$$

if

$$a_1^2b_2x^2-a_1b_2^2x+a_1b_2x-a_1b_1b_2x-b_2^2x-b_1b_2 > 0$$

then for feasibility, one must have,

$$\frac{a_1^2b_2x^2-a_1b_2^2x+a_1b_2x-a_1b_1b_2x-b_2^2x-b_1b_2}{-2a_1b_1x+a_1^2x^2+b_1^2} < a_2 < \frac{b_2-b_2x}{a_1x-b_1}$$

$$a_1^2b_2x^2-a_1b_2^2x+a_1b_2x-a_1b_1b_2x-b_2^2x-b_1b_2 > 0$$

$$\iff b_2 < \frac{-a_1b_1x + a_1^2x^2 + a_1x - b_1}{a_1x + x}$$

Further,

$$\frac{a_1^2b_2x^2 - a_1b_2^2x + a_1b_2x - a_1b_1b_2x - b_2^2x - b_1b_2}{-2a_1b_1x + a_1^2x^2 + b_1^2} < \frac{b_2 - b_2x}{a_1x - b_1}$$

$\iff a_1x - b_1 < b_2$

(a) is obtained.

If

$$a_1^2b_2x^2 - a_1b_2^2x + a_1b_2x - a_1b_1b_2x - b_2^2x - b_1b_2 < 0$$

then

$$b_2 \geq \frac{-a_1b_1x + a_1^2x^2 + a_1x - b_1}{a_1x + x}, 0 < a_2 < \frac{b_2 - b_2x}{a_1x - b_1}$$

Hence (b).

Case 2: $u_3 > \frac{u_5}{x}, x < \frac{b_1}{a_1}$

$\Delta_1 = a_1b_2^2x\varepsilon + a_1b_2^3x + a_2b_2\varepsilon^2 + b_2^3x + b_2^2\varepsilon > 0$ considering $b_1 = a_1x + \varepsilon$ for some $\varepsilon > 0$

as well as,

$$\frac{\Delta_2}{u_4\left(a_1x - b_1\right)^2} = \frac{b_2(x-1)}{a_1x - b_1} + a_2\frac{\left(a_1+1\right)b_2^2x\left(u_3x - u_5\right)}{u_4\left(a_1x - b_1\right)^2} > 0,$$

□

4. Local Stability Analysis of Equilibrium Points

4.1. Nature of system around $E_0(0,0,0,0)$: As the community matrix is not defined at the equilibrium point $(0,0,0,0)$, we apply technique developed by Arino et al [34].

Let,

$$\frac{dX}{dt} = H\left(X\right) + Q\left(X\right), X = (x,y,z,w) \tag{11}$$

where,

$$
\begin{aligned}
&H(X) = (H_1(X), H_2(X), H_3(X), H_4(X)); Q(X) = (Q_1(X), Q_2(X), Q_3(X), Q_4(X))\\
&H_1(X) = x - \frac{x(a_2z)}{m(w+z)+x+y}; H_2(X) = -b_1y - \frac{y(b_2z)}{m(w+z)+x+y}\\
&H_3(X) = -\frac{u_4yz}{m(w+z)+x+y} + u_5w + (u_1z)\left(1-\frac{z}{x}\right)\\
&H_4(X) = \frac{u_4yz}{m(w+z)+x+y} - u_5w; Q_1(X) = -x(x+y) - a_1xy\\
&Q_2(X) = a_1xy; Q_3(X) = -u_3wz; Q_4(X) = u_3wz
\end{aligned}
$$

Then $H(X)$ is C^1 function and continuous outside the origin and $H(sX) = sH(X)\ \forall s \geq 0$ and Q is a C^1 function such that $Q(X) = o(X)$ in the neighbourhood of the origin. If $X(t)$ is a solution of (11) and $\lim_{t\to+\infty} \inf \|X(t)\| = 0$ and X is bounded then $\exists\ X(t_n + .), t_n \to \infty$ such that $X(t_n + .) \to 0$ locally uniformly on $s \in \mathbb{R}$

Let

$$y_n(s) = \frac{x(t_n+s)}{\|x(t_n+s)\|}$$

It can be checked as shown in [34] that y_n converges to some function y locally uniformly on $\mathbb{R}$ such that

$$\frac{dy}{dt} = H(y(t)) - (y(t), H(y(t)))y(t), |y(t)| = 1 \quad \forall\ t \tag{12}$$

where $\|.\|$ is the Euclidean norm and $\langle . \rangle$ is the Euclidean inner product.

The steady states of H are vectors v satisfying

$$H(v) = (v, H(v))\, v$$

Alternatively,

$$H(v) = \mu v \tag{13}$$

where $\|v\| = 1$ and $\mu = (v, H(v))$. The solutions of (13) correspond to fixed directions through which the trajectory may meet the origin asymptotically. In our case,we have where,

Direction	**Eigenvalue**
$(0,0,0,w)$	$-u_5$
$(0,y,0,0)$	$-b_1$
$(x,0,0,0)$	1
$(wx_{11},0,wz_{11},w)$	$-u_5$
$(wx_{12},0,wz_{12},w)$	$-u_5$
$(x_{21}z,0,z,0)$	$\frac{-a_2+l_1+mu_1+m+u_1}{2m}$
$(x_{22}z,0,z,0)$	$\frac{-a_2-l_1+mu_1+m+u_1}{2m}$
$(x_{31}z,0,z,w_{31}z)$	$-u_5$
$(x_{32}z,0,z,w_{32}z)$	$-u_5$
$(x,0,xz_{31},0)$	$\frac{-a_2+l_1+mu_1+m+u_1}{2m}$
$(x,0,xz_{32},0)$	$\frac{-a_2-l_1+mu_1+m+u_1}{2m}$
$(x,0,xx_{32},w_{41}x)$	$-u_5$
$(x,0,xz_{33},xw_{42}$	$-u_5$

$$l_1=\sqrt{(a_2+(m-1)u_1-m)^2+4m(u_1-1)u_1};l_2=\sqrt{(a_2u_5+mu_1(u_5+1))^2-4mu_1u_5(u_5+1)^2}$$

$$\theta=(m+2)(u_5+1)-a_2;x_{11}=\frac{\theta(-m)u_1(u_5+1)-(a_2u_5-l_2)(-a_2+mu_5+m)}{2(u_5+1)(u_5(-a_2+mu_5+m)+u_1((m+1)(u_5+1)-a_2))}$$

$$x_{12}=-\frac{(a_2u_5+12)(-a_2+mu_5+m)+\theta mu_1(u_5+1)}{2(u_5+1)(u_5(-a_2+mu_5+m)+u_1((m+1)(u_5+1)-a_2))}$$

$$z_{11}=-\frac{-a_2u_5+12+m(u_5+1)(u_1+2u_5)}{2(u_5(-a_2+mu_5+m)+u_1((m+1)(u_5+1)-a_2))}$$

$$z_{12}=\frac{a_2u_5+12-m(u_5+1)(u_1+2u_5)}{2(u_5(-a_2+mu_5+m)+u_1((m+1)(u_5+1)-a_2))};x_{21}=-\frac{a_2+l_1+m(u_1-1)-u_1}{2(u_1-1)}$$

$$x_{22}=\frac{-a_2+l_1-mu_1+m+u_1}{2(u_1-1)};x_{31}=\frac{a_2u_5-l_2+mu_1(u_5+1)}{2u_5(u_5+1)};x_{32}=\frac{a_2u_5+l_2+mu_1(u_5+1)}{2u_5(u_5+1)}$$

$$w_{31}=\frac{a_2u_5+l_2-m(u_5+1)(u_1+2u_5)}{2mu_5(u_5+1)};w_{32}=-\frac{-a_2u_5+l_2+m(u_5+1)(u_1+2u_5)}{2mu_5(u_5+1)}$$

$$z_{31}=-\frac{-a_2+l_1-mu_1+m+u_1}{2mu_1};z_{32}=\frac{a_2+l_1+m(u_1-1)-u_1}{2mu_1}$$

$$z_{33}=\frac{a_2u_5-l_2+mu_1(u_5+1)}{2mu_1(u_5+1)}$$

$$w_{41} = -\frac{(a_2u_5+l_2)(-a_2+mu_5+m)+mu_1(u_5+1)((m+2)(u_5+1)-a_2)}{2m^2u_1(u_5+1)^2}$$

$$w_{42} = \frac{((a_2u_5-l_2)(-(-a_2+mu_5+m))-mu_1(u_5+1)((m+2)(u_5+1)-a_2))}{2m^2u_1(u_5+1)^2})$$

Therefore, the origin can be reached along the directions where the eigen values become negative.

4.2. Stability of the equilibrium points E_1, E_2 and E_3:

Theorem 3. *The equillibrium points*

(i) $E_1(1,0,0,0), E_2\left(\frac{b_1}{a_1}, -\frac{b_1-a_1}{a_1(a_1+1)}, 0, 0\right)$ *are saddle.*

(ii) $E_3(x_3,0,z_3,0)$ *is locally asymptotically stable in* $\mathbf{R}_+{}^4$ *under the conditions* $s_1 < 0$; $b_2 + (1+m)b_1 > a_1(1+m-a_2)$ *and* $u_5 > \frac{(-a_2+m+1)u_3}{1+m}$

Proof. (i) Clearly, the Jacobian at E_1 has two negative values and one positive eigenvalues, $-1, a_1-b_1, u_1, -u_5$. Threrefore, the equilibrium point E_1 is a saddle point. In fact, E_1 may be achieved when there is no predator and infected prey. Also, E_1 is achieved when $a_1 < b_1$. But once predator comes in to existence, the axial equilibrium can not be achieved.

Clearly, Jacobian matrix of the system (2) at E_2 has the following eigenvalues,

$$\Lambda_1 = -\frac{b_1+\sqrt{\phi_1}}{2a_1}$$

$$\Lambda_2 = \frac{\sqrt{\phi_1}-b_1}{2a_1}$$

$$\Lambda_{3,4} = \frac{1}{2a_1(b_1+1)}\left\{a_1((b_1+1)u_1-(b_1+1)u_5-u_4)+b_1u_4 \pm \sqrt{\Delta_1}\right\}$$

where, $\phi_1 = b_1\left(4a_1b_1-4a_1^2+b_1\right)$ and

$$\Delta_1 = (a_1(b_1+1)u_1+u_4(b_1-a_1))^2+a_1^2(b_1+1)^2u_5^2+2a_1(b_1+1)u_5(a_1(b_1+1)u_1+u_4(a_1-b_1))$$

As $a_1 > b_1$, $\phi_1 < b_1$, so , $\Lambda_{1,2} < 0$. Since

$$a_1^2(b_1+1)^2u_5^2+2a_1(b_1+1)u_5(a_1(b_1+1)u_1+u_4(a_1-b_1)) > 0$$

so, $\Delta_1 > a_1((b_1+1)u_1-(b_1+1)u_5-u_4)+b_1u_4$ which results $\Lambda_{1,2,3} < 0$. But $\Lambda_4 > 0$. Hence E_2 is saddle. It may be observed that predation free equillibrium point is a stable fixed point when there is no predator, but when predator comes in to existence due to their density dependent death, the species can not be extinct unless their food is extinct, so, the predator free equillibrium can not be reached from a coexisting situation.

(ii) Jacobian at E_3 has the following eigenvalues,

$$\Lambda_1 = -\frac{a_1(a_2-m-1)+b_1(m+1)+b_2}{m+1}$$
$$\Lambda_{2,3} = s_1 \pm -\frac{(m+1)^2}{2(-a_2+m+1)^4}\sqrt{s_2}$$

where,

$$s_1 = \frac{a_2(m+2)-(m+1)^2(u_1+1)}{2(m+1)^2}$$
$$s_2 = \frac{(-a_2+m+1)^8\left(2a_2(m+1)^2(mu_1-m-2)+a_2^2(m+2)^2+(m+1)^4(u_1-1)^2\right)}{(m+1)^8}$$
$$\Lambda_4 = \frac{u_3(-a_2+m+1)}{m+1}-u_5$$

All the real parts of characteristic roots of E_3 are negative in $\mathbf{R}_+{}^4$ under the conditions $s_1<0$; $b_2+(1+m)b_1 > a_1(1+m-a_2)$ and $u_5 > \frac{(-a_2+m+1)u_3}{1+m}$. Hence E_3 is asymptotically stable under the said parametric region.

□

4.3. Direction of Hopf-bifurcation at disease free equilibrium (DFE) E_3:

Theorem 4. *The system undergoes a hopf bifurcation along the parametric surface* $(m+1)^2(u_1+1)-a_2(m+2)=0$ *at the equilibrium point* E_3

Proof. The characteristic roots are

$$\Lambda_1 = -\frac{a_1(a_2-m-1)+b_1(m+1)+b_2}{m+1}$$
$$\Lambda_{2,3} = s_1 \pm -\frac{(m+1)^2}{2(-a_2+m+1)^4}\sqrt{s_2}$$

where,

$$s_1 = \frac{a_2(m+2)-(m+1)^2(u_1+1)}{2(m+1)^2}$$

$$s_2 = \frac{(-a_2+m+1)^8\left(2a_2(m+1)^2(mu_1-m-2)+a_2^2(m+2)^2+(m+1)^4(u_1-1)^2\right)}{(m+1)^8}$$

$$\Lambda_4 = \frac{u_3(-a_2+m+1)}{m+1} - u_5$$

The given parametric surface is obtained for $s_1 = 0$.

If u_1 is the control parameter then,

$$u_1^*(say) = \frac{a_2(m+2)-(m+1)^2}{(m+1)^2}$$

$$\therefore s_2 = 4(-a_2+m+1)^9\left((m+1)^2 - a_2(m+2)\right) < 0 \text{ at } u_1^*$$

provided,

$$(m+1)^2 - a_2(m+2) < 0$$

The region where $\Lambda_{1,4} < 0$, $(m+1)^2 - a_2(m+2) < 0$ is

$$\frac{(m+1)^2}{(m+2)} < a_2 < m+1$$

$$u_5 > \frac{(-a_2+m+1)u_3}{1+m}$$

$$b_2 > a_1(1+m-a_2) - (1+m)b_1$$

The transversality condition $\frac{\partial s_1}{\partial u_1}|_{u_1^*} = -\frac{1}{2} < 0$. Hence hopf bifurcation occurs.

One may set the control parameter as a_2, then,

$a_2 = \frac{(m+1)^2(u_1+1)}{m+2}$ along with $u_1 < \frac{1}{1+m}$ will make the characteristic roots $\Lambda_{3,4}$ purely imaginary.

The other conditions for hopf bifurcation are

$$m > a_2 - 1$$

$$b_2 > a_1(1+m-a_2) - (1+m)b_1$$

$$u_5 > \frac{(-a_2+m+1)u_3}{1+m}$$

□

Theorem 5. *There exists a locally defined smooth two dimensional parameter dependent attracting center manifold $W^c_{loc}(0)$ of system which is locally tangent to T_c at the Disease free equilibrium $(x_3, 0, z_3, 0)$. The restriction to $W^c_{loc}(0)$ exhibits Hopf bifurcation with negative first Lyapunov coefficient.*

Proof. The characteristic polynomial of the Jacobian matrix of the system at the DFE has a pair of purely imaginary roots $\lambda_{1,2} = \pm i\omega$, $\omega > 0$ for

$$u_1 = 1 + \frac{(2+m)a_2}{(1+m)^2}; a_2, m > 0$$

Now we investigate the asymptotic dynamics on the existing center manifold of the system as well as the stability of the resulting Hopf bifurcation by computing the first lyapunov coefficient of the restricted dynamics on the center manifold using the method outlined in [39].

Let q be a complex eigenvalue corresponding to $\lambda_1$which satisfies $Aq = i\omega q$ (A=Jacobian matrix of the system at the DFE). $A\bar{q} = -i\omega\bar{q}$. We choose adjoint eigen vector p which satisfies $A^T\bar{p} = -i\omega p, A^T\bar{p} = iwp$ and $\langle p, q\rangle = 1$ where $\langle p, q\rangle = \sum_{i=1}^{n} \bar{p}_i q_i$ is the standard scaler product in C^n. We choose,

$$q = \begin{pmatrix} \frac{-a_2(m+2)+(m+1)^2(1-i\omega)}{(m+1)^2-a_2(m+2)} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{p} = \begin{pmatrix} p_{11} \\ p_{21} \\ p_{31} \\ 1 \end{pmatrix}$$

where

$$p_{11} = -\frac{\left(a_2(m+2)+i\rho^2\omega-\rho^2\right)\left(-\rho u_3\left(\rho-a_2\right)+i\rho^2\omega+\rho^2 u_5\right)}{a_2\left(a_2\left(\rho u_3-m(m+2)\right)\right)-im\rho^2\omega+m\rho^2-\rho^2 u_3+\rho^2 u_5)}$$

$$p_{21} = \frac{a_2^2 u_3\left((m+1)^2+\rho\right)-a_1\rho\left(\rho-a_2\right)\left(-\rho u_3\left(\rho-a_2\right)+i\rho^2\omega+\rho^2 u_5\right)+\rho^2\left(-\left(i\rho^2\omega-\rho^2 u_3+\rho^2 u_5\right)\right)}{a_2 m\left(a_1(-m-1)\left(\rho-a_2\right)+b_2 m+b_1\rho^2+b_2+i\rho^2\omega\right)}$$
$$+\frac{a_2\left((m+2)\left(\rho^2 u_5+i\rho^2\omega\right)-(2m+3)\rho^2 u_3+m\rho u_4\right)}{a_2 m\left(a_1(-m-1)\left(\rho-a_2\right)+b_2 m+b_1\rho^2+b_2+i\rho^2\omega\right)}$$
$$-\frac{\rho\left(-\rho u_3\left(\rho-a_2\right)+i\rho^2\omega+\rho^2 u_5\right)}{a_2\left(\rho u_3-m(m+2)\right)+m^3+2m^2-im\rho^2\omega+m-\rho^2 u_3+\rho^2 u_5}$$
$$\times\frac{u_3\left(\rho-a_2\right)\left(-a_2(m+2)+a_1\rho\left(\rho-a_2\right)+\rho^2\right)+a_2\left(\rho u_5\left(a_1\rho+m+2\right)+m u_4\right)-\left(a_1+1\right)\rho^3 u_5}{a_2 m\left(a_1(-m-1)\left(\rho-a_2\right)+b_2 m+b_1\rho^2+b_2+i\rho^2\omega\right)}$$

$$p_{31} = -\frac{\rho u_3\left(\rho-a_2\right)-i\rho^2\omega-\rho^2 u_5}{a_2\left(\rho u_3-m(m+2)\right)-im\rho^2\omega+m\rho^2-\rho^2 u_3+\rho^2 u_5}$$

$$\omega = \sqrt{\frac{a_2\left(2m^2+5m+3\right)-a_2^2(m+2)-(m+1)^3}{(m+1)^3}}$$

$$\rho = (1+m)$$

In order to satisfy normalization condition. We choose $v_1 = \langle \bar{p}, q\rangle$ such that $\langle p, q\rangle = 1$.

Hence we take $p = \frac{1}{v_1}\bar{p}$. System at the equilibrium point E_3 can be written as

$$\dot{x} = Ax + F(x, u_1) + O(H)^4$$

where $x = (x, y, z, w)^T$,

$$F(x, u_1) = \frac{B(x, x)}{2} + \frac{C(x, x, x)}{2}$$

B and C are multilinear functions where,

$$B_i = \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial \xi_i \partial \xi_k}\bigg|_{\xi=0} x_j y_k$$

$$C_i(x, y, z) = \sum_{j,k,l=1}^{3} \frac{\partial^3 F_i(\xi)}{\partial xi_j \partial xi_k \partial xi_l}\bigg| x_j y_k z_l$$

Solving the corresponding linear system gives $s = A^{-1}B(q, \bar{q})$ and $r = (2i\omega E - A)^{-1}B(q, q)$.

Finally we get the Lyapunov coefficient

$$l_1(0) = \frac{1}{2\omega} Re[\langle p, c(q, q, \bar{q})\rangle - 2\langle p, B(q, s)\rangle + \langle p, B(\bar{q}, r)\rangle] \tag{14}$$

□

Theorem 6. *The system undergoes a saddle-node (fold) bifurcation along the parametric surface*

$$a_1\left(a_2-m-1\right)-b_1(m+1)-b_2=0$$

and

$$\frac{u_3\left(-a_2+m+1\right)}{m+1}-u_5=0$$

respectively at the equilibrium point E_3.

Proof. The first parametric surface is obtained when the characteristic root λ_1 at E_3 is equal to zero. If the control parameter is considered as a_1. Let $\bar{a_1}=\frac{b_1m+b_1+b_2}{-a_2+m+1}$. The transversality condition is satisfied and if $\frac{a_2(m+2)-(m+1)^2}{(m+1)^2}<u_1, u_5>\frac{u_3(-a_2+m+1)}{m+1}$, then the other characteristic roots are negative. Hence there is a saddle node bifurcation in this region.

Similarly, the second parametric surface is obtained by $\lambda_4=0$ and considering the other parametric region as

$$a_1<\frac{b_1(m+1)+b_2}{-a_2+m+1}, \frac{a_2(m+2)-(m+1)^2}{(m+1)^2}<u_1$$

and setting the control parameter as u_5, we obtain a saddle node bifurcation in this region . □

Theorem 7. *The equilibrium point*

$$E_4\left(\frac{u_5}{u_3}, 0, \frac{u_5}{u_3}, \frac{u_5\left(u_3\left(a_2-m-1\right)+(m+1)u_5\right)}{mu_3\left(u_3-u_5\right)}\right)$$

is not locally asymptotically stable

Proof. One of the eigen values of the system (2) at the equilibrium point E_4 is $-\frac{a_2(b_1mu_3-a_1mu_5)+b_2m(u_3-u_5)}{a_2mu_3}$. We consider the remaining part of the characteristic polynomial

$$\lambda^3-c_1\lambda^2+c_2\lambda-c_3=0$$

$c_1=\frac{u_3^3\left(a_2(mu_1-(m+1)u_5)+a_2^2u_5-m\right)+u_5u_3^2(a_2(m(-u_1)+(m+1)u_5+m)+3m)-(a_2+3)mu_5^2u_3+mu_5^3}{a_2mu_3^2(u_3-u_5)}$

$c_2=-\frac{u_5^2\left(a_2mu_3+m^2u_1+mu_1+2mu_3+2u_3\right)}{a_2mu_3^2}-\frac{u_5\left(a_2^2u_3+a_2u_3-2m^2u_1-2mu_1-mu_3-u_3\right)}{a_2mu_3}-\frac{-a_2mu_1+a_2^2u_3+m^2u_1+mu_1}{a_2m}$
$+\frac{(m+1)u_5^3}{a_2mu_3^2}-\frac{a_2u_3^2}{m(u_5-u_3)}$

$c_3=\frac{(m+1)u_1u_5^3}{a_2u_3^2}+\frac{u_1u_5^2(a_2-2m-2)}{a_2u_3}-\frac{u_1u_5(a_2-m-1)}{a_2}.$

For stability, all the eigen values should have negative real parts, which implies, $c_3 < 0$. Existence condition of the equilibrium implies that,

$a_2 = \frac{u_3-u_5}{u_3} + \phi_1, m = \frac{\phi_2(-a_2u_3+u_3-u_5)}{u_5-u_3}, u_5 = u_3 - \phi_3, \phi_1 > 0, 0 < \phi_2 < 1, 0 < \phi_3 < u_3,$

$$c_3 = -\frac{u_1\phi_1(\phi_2 - 1)\phi_3(u_3 - \phi_3)}{u_3\phi_1 + \phi_3} > 0$$

which implies at least one of the real part of eigen values non negative. Hence the result. □

4.4. Stability of the coexisting equilibrium.

Jacobian matrix of the system (2) at E^* is

$$J^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

where

$$A_{11} = -\frac{a_2z(m(w+z)+y)}{(m(w+z)+x+y)^2} - a_1y - 2x - y + 1; A_{12} = x\left(\frac{a_2z}{(m(w+z)+x+y)^2} - a_1 - 1\right)$$

$$A_{13} = -\frac{a_2x(mw+x+y)}{(m(w+z)+x+y)^2}; A_{14} = \frac{a_2mxz}{(m(w+z)+x+y)^2}$$

$$A_{21} = y\left(a_1 + \frac{b_2z}{(m(w+z)+x+y)^2}\right); A_{22} = a_1x - \frac{b_2z(m(w+z)+x)}{(m(w+z)+x+y)^2} - b_1$$

$$A_{23} = -\frac{b_2y(mw+x+y)}{(m(w+z)+x+y)^2}; A_{24} = \frac{b_2myz}{(m(w+z)+x+y)^2}$$

$$A_{31} = z\left(\frac{u_4y}{(m(w+z)+x+y)^2} + \frac{u_1z}{x^2}\right); A_{32} = -\frac{u_4z(m(w+z)+x)}{(m(w+z)+x+y)^2}$$

$$A_{33} = -\frac{u_4y(mw+x+y)}{(m(w+z)+x+y)^2} - u_3w + \frac{u_1(x-2z)}{x}; A_{34} = \frac{mu_4yz}{(m(w+z)+x+y)^2} + u_3(-z) + u_5$$

$$A_{41} = -\frac{u_4yz}{(m(w+z)+x+y)^2}; A_{42} = \frac{u_4z(m(w+z)+x)}{(m(w+z)+x+y)^2}$$

$$A_{43} = \frac{u_4y(mw+x+y)}{(m(w+z)+x+y)^2} + u_3w; A_{44} = -\frac{mu_4yz}{(m(w+z)+x+y)^2} + u_3z - u_5$$

The characteristic equation of J^* can be written as,

$$\lambda^4 + L_1\lambda^3 + L_2\lambda^2 + L_3\lambda + L_4 = 0$$

Theorem 8. *The coexisting equilibrium E^* is stable if $L_4, L_3, L_2L_3 - L_4, \Delta_2 = L_1L_2L_3 - L_3^2 - L_1^2L_4 > 0$ and the system undergoes Hopf bifurcation at this fixed point in a certain parametric region.*

The stability condition is due to Routh–Hurwitz criteria. The control parameter has been set as u_1 as the so obtained coexistence condition is independent of u_1. The numerical evidence of Hopf bifurcation scenario has been obtained applying the following criteria:

Theorem 9 (Liu[35])**.** *The Hopf bifurcation criteria is equivalent to the following condition*

(1) $L_4, L_3, L_2L_3 - L_4 > 0, \Delta_2 = 0$

(2) $\frac{\partial \Delta_2}{\partial \mu} \neq 0$*, where* μ *is the bifurcation point.*

4.5. Direction of Hopf-bifurcation at the coexisting equilibrium. In order to determine the direction and stability criterion of the bifurcating periodic solution, we reduce the set of differential equations in the system (2) into it's normal form using the procedure described by Hassard et al.[37]. For the sake, introducing the new variables $x = s_1 + x^*, y = s_2 + y^*, z = s_3 + z^*, w = s_4 + w^*$, the system (2) can be written in matrix form as,

$$\dot{X} = AX + B \tag{15}$$

where A is the Jacobian matrix of the modified system, AX is the linear part of the system and B represents the nonlinear part. Moreover,

$$X = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, B = \begin{pmatrix} B_1(s_1,s_2,s_3,s_4) \\ B_2(s_1,s_2,s_3,s_4) \\ B_3(s_1,s_2,s_3,s_4) \\ B_4(s_1,s_2,s_3,s_4) \end{pmatrix}$$

where $a_{ij}(i, j = 1,2,3,4)$, $B_i(s_1,s_2,s_3,s_4)(i = 1,2,3,4)$ are in Appendix. We consider two conjugate imaginary eigenvalues $\lambda_{1,2} = \pm\beta$ and two other eigenvalues $\lambda_{3,4} = \nu_{1,2}$ the characteristic equation of the system (2). Next, we seek a transformation matrix T which reduces the matrix

A to the form,

$$T^{-1}AT = \begin{pmatrix} 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & v_1 & 0 \\ 0 & 0 & 0 & v_2 \end{pmatrix}$$

where the nonsingular matrix T is given as

$$T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}$$

where $c_{ij}(i.j = 1,2,3,4)$ can be calculated using basic matrix operations.

To obtain the normal form of the Equation (15), we make another change of variable i.e. $X = TY$, where $Y = (y_1, y_2, y_3, y_4)'$

Then the normal form of system (15) can be given by,

$$\dot{Y} = T^{-1}ATY + F \tag{16}$$

where

$$F = T^{-1}B = \begin{pmatrix} F_1(y_1, y_2, y_3, y_4) \\ F_2(y_1, y_2, y_3, y_4) \\ F_3(y_1, y_2, y_3, y_4) \\ F_4(y_1, y_2, y_3, y_4) \end{pmatrix}$$

where $F_i(y_1, y_2, y_3, y_4)(i = 1,2,3,4)$ can be obtained by transforming B_i's using the variables $s_1 = y_1 + y_3 + y_4, s_2 = c_{21}y_1 + c_{22}y_2 + c_{23}y_3 + c_{24}y_4; s_3 = c_{31}y_1 + c_{32}y_2 + c_{33}y_3 + c_{34}y_4; s_4 = c_{41}y_1 + c_{42}y_2 + c_{43}y_3 + c_{44}y_4$.

Equation (16) is the normal form of Equation (15) from which The stability and direction of the Hopf bifurcation can be computed. In Equation (16), on the right hand side of the first term is linear and the second is non-linear in y_i's. For evaluating the direction of periodic solution, we can evaluated the following quantities at $u_1 = u_1(0)$ and origin.

$$g_{11} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2} + i\left(\frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right)\right]$$

$$g_{02} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2} + i\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right]$$

$$g_{20} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2} + i\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right]$$

$$G_{21} = \frac{1}{8}\left[\frac{\partial^3 F_1}{\partial y_1^3} + \frac{\partial^3 F_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 F_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F_2}{\partial y_2^3} + i\left(\frac{\partial^3 F_2}{\partial y_1^3} + \frac{\partial^3 F_2}{\partial y_1 \partial y_2^2} - \frac{\partial^3 F_1}{\partial y_1^2 \partial y_2} - \frac{\partial^3 F_1}{\partial y_2^3}\right)\right]$$

$$G_{110}^j = \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1 \partial y_j} + \frac{\partial^2 F_2}{\partial y_2 \partial y_j} + i\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_j} - \frac{\partial^2 F_1}{\partial y_2 \partial y_j}\right)\right]$$

$$G_{101}^j = \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1 \partial y_j} - \frac{\partial^2 F_2}{\partial y_2 \partial y_j} + i\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_j} + \frac{\partial^2 F_1}{\partial y_2 \partial y_j}\right)\right]$$

$$h_{11}^j = \frac{1}{4}\left[\frac{\partial^2 F^j}{\partial y_1^2} + \frac{\partial^2 F^j}{\partial y_2^2}\right]$$

$$h_{20}^j = \frac{1}{4}\left[\frac{\partial^2 F^j}{\partial y_1^2} - \frac{\partial^2 F^j}{\partial y_2^2} - 2i\frac{\partial^2 F^j}{\partial y_1 \partial y_2}\right]$$

$$w_{11}^j = \frac{h_{11}^j}{\nu_j}$$

$$w_{20}^j = \frac{h_{20}^j}{(\nu_j + 2i\beta)}, j = 1,2$$

and

$$g_{21} = G_{21} + \sum_{j=1}^{2}\left(2G_{110}^j w_{11}^j + G_{101}^j w_{20}^j\right)$$

$$C_1(0) = \frac{i}{2\beta}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}$$

(17) $$\mu_2 = -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\Delta'(u_1(0))\}}$$

where Δ' can be calculated using (CH2) in [35]. The system (2) undergoes supercritical (subcritical) Hopf bifurcation if $\mu_2 > 0$ ($\mu_2 < 0$). Furthermore, the bifurcating periodic solutions are asymptotically stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

4.6. Global stability analysis.

Theorem 10. The DFE equilibrium E_3 is globally asymptotically stable under the sufficient conditions: $a_1\theta_2 < b_1$, $\frac{L_3 z_3 u_4 \theta_2}{\theta_1} - L_2\theta_1 b_2 < 0$, $(w\theta_2 u_3 - z_3 u_5) < 0$

Proof. At the equilibrium point E_3 system (2) reduces to,

$$x^*(1-x^*-y^*) - a_1 x^* y^* - \frac{a_2 x^* z^*}{m(w^*+z^*)+x^*+y^*} = 0$$

$$u_1 z^*\left(1-\frac{z^*}{x^*}\right) - u_3 w^* z^* - \frac{u_4 z^* y^*}{m(w^*+z^*)+x^*+y^*} + u_5 w^* = 0$$

To study the globally asymptotically stability of the DFE E_3 the following positive definite Lyapunov function is considered:

$$V(x,y,z,w) = L_1\left(x - x^* - x^*\ln\frac{x}{x^*}\right) + L_2 y + L_3\left(z - z^* - z^*\ln\frac{z}{z^*}\right) + L_4 w$$

Now taking the time derivative of $V(x,y,z,w)$ along the solutions of system (2), $\dot{V}(x,y,z,w)$ is given by,

$$\frac{dV(x,y,z,w)}{dt} = L_1\frac{(x-x^*)}{x}.\frac{dx}{dt} + L_2.\frac{dy}{dt} + L_3\frac{(z-z^*)}{z}.\frac{dz}{dt} + L_4.\frac{dw}{dt}$$

Suppose $L_1 = \frac{L_2 a_1}{1+a_1}$; $L_3 = L_4$ and $\theta_1 < x,y,z,w < \theta_2$. Since $x_3 = z_3$ therefore,

$$\frac{dV(x,y,z,w)}{dt} = -L_1(x-x_3)^2 - (x-x_3)\left(L_1 y + L_1 y a_1 + \frac{L_1 z a_2}{\phi} + \frac{L_1 z_3 a_2}{x_3 + m z_3}\right) + L_2 x y a_1 - L_2 y b_1$$
$$- \frac{L_2 y z b_2}{\phi} + \left(L_3 u_1 - \frac{L_3 z u_1}{x}\right)(z - z_3) + L_3 w z_3 u_3 + \frac{L_3 y z_3 u_4}{\phi} - \frac{L_3 w z_3 u_5}{z}$$

where $m(w+z)+x+y = \phi$. When $x \to x_3$ and $z \to z_3$,

$$\begin{aligned}\frac{dV}{dt} &= L_2 x y a_1 - L_2 y b_1 - \frac{L_2 y z b_2}{\phi} + L_3 w z_3 u_3 + \frac{L_3 y z_3 u_4}{\phi} - \frac{L_3 w z_3 u_5}{z} \\ &= L_2 y(x a_1 - b_1) + \frac{z}{\phi}\left(\frac{L_3 z_3 y u_4}{z} - L_2 y b_2\right) + L_3\left(w z_3 u_3 - \frac{w z_3 u_5}{z}\right) \\ &= L_2\theta_1 g_1(\theta_1,\theta_2) + \frac{\theta_1}{2\theta_2(1+m)} g_2(\theta_1,\theta_2) + L_3 g_3(\theta_1,\theta_2)\end{aligned}$$

where $g_1(\theta_1,\theta_2) = a_1\theta_2 - b_1$, $g_2(\theta_1,\theta_2) = \frac{L_3 z_3 u_4 \theta_2}{\theta_1} - L_2\theta_1 b_2$, $g_3(\theta_1,\theta_2) = (w\theta_2 u_3 - z_3 u_5)$

Therefore $\frac{dV}{dt}$ is negative definite under the sufficient conditions $g_1(\theta_1,\theta_2) < 0$, $g_2(\theta_1,\theta_2) < 0$

and $g_3(\theta_1, \theta_2) < 0$

□

Theorem 11. The coexisting equilibrium E^* is globally asymptotically stable under the sufficient conditions (18).

Proof. At the equilibrium point E^* system (2) reduces to,

$$x^*(1-x^*-y^*) - a_1x^*y^* - \frac{a_2x^*z^*}{m(w^*+z^*)+x^*+y^*} = 0$$

$$a_1x^*y^* - b_1y^* - \frac{b_2y^*z^*}{m(w^*+z^*)+x^*+y^*} = 0$$

$$u_1z^*\left(1-\frac{z^*}{x^*}\right) - u_3w^*z^* - \frac{u_4z^*y^*}{m(w^*+z^*)+x^*+y^*} + u_5w^* = 0$$

$$u_3w^*z^* + \frac{u_4z^*y^*}{m(w^*+z^*)+x^*+y^*} - u_5w^* = 0$$

To study the globally asymptotically stability of E^* the following positive definite Lyapunov function is considered:

$$V(x,y,z,w) = L_1\left(x-x^*-x^*\ln\frac{x}{x^*}\right) + L_2\left(y-y^*-y^*\ln\frac{y}{y^*}\right) + L_3\left(z-z^*-z^*\ln\frac{z}{z^*}\right) + L_4\left(w-w^*-w^*\ln\frac{w}{w^*}\right)$$

Now taking the time derivative of $V(x,y,z,w)$ along the solutions of system (2), $\dot{V}(x,y,z,w)$ is given by,

$$\frac{dV(x,y,z,w)}{dt} = L_1\frac{(x-x^*)}{x}.\frac{dx}{dt} + L_2\frac{(y-y^*)}{y}.\frac{dy}{dt} + L_3\frac{(z-z^*)}{z}.\frac{dz}{dt} + L_4\frac{(y-y^*)}{y}.\frac{dw}{dt}$$

Suppose $L_1 = \frac{L_2a_1}{1+a_1}$; $L_3 = L_4$ and $\theta_1 < x,y,z,w < \theta_2$. Since $x^* = z^*$ therefore,

$$\begin{aligned}\frac{dV(x,y,z,w)}{dt} &= -L_1(x-x^*)^2 - \frac{L_1x^*a_2}{\phi_1} - \frac{L_2y^*b_2}{\phi_1} + \frac{z}{\phi}(\xi_{11}+\chi_{11}) + L_4z^*(\xi_{21}+\chi_{22}) + L_4z(\xi_{31}+\chi_{32}) \\ &\le -L_1(x-x^*)^2 - \frac{L_1x^*a_2}{\phi_1} - \frac{L_2y^*b_2}{\phi_1} + \frac{z}{\phi}\Psi + nz^*g_1(\theta_1,\theta_2) + nzg_2(\theta_1,\theta_2) \\ &+ L_4\left(-\frac{z^2u_1}{x} + \frac{zz^*u_1}{x}\right) + L_4\left(wu_5 - \frac{wz^*u_5}{z}\right) + L_4w^*u_5\end{aligned}$$

where $m(w+z)+x+y = \phi$ and $m(w^*+z^*)+x^*+y^* = \phi_1$ and $\xi_{11}, \chi_{11}, \xi_{21}, \chi_{21}, \xi_{31}, \chi_{31}, \Psi, g_1(\theta_1,\theta_2), g_2(\theta_1,\theta_2)$ are given in Appendix.

In the neighbourhood of the equilibrium point E^*, $\lim_{(x,y,z)\to(x^*,y^*,z^*)}\left(-\frac{z^2u_1}{x} + \frac{zz^*u_1}{x}\right) = 0$ and $\lim_{(x,y,z)\to(x^*,y^*,z^*)}\left(wu_5 - \frac{wz^*u_5}{z}\right) = 0$

$\therefore |-\frac{z^2u_1}{x}+\frac{zz^*u_1}{x}|<\delta_1$ and $|\left(wu_5-\frac{wz^*u_5}{z}\right)|<\delta_2$

Therefore

$$\frac{dV(x,y,z,w)}{dt}\leq -L_1(x-x^*)^2-\frac{L_1x^*a_2}{\phi_1}-\frac{L_2y^*b_2}{\phi_1}+\frac{\theta_1}{2\theta_2(1+m)}\Psi+L_4z^*g_1(\theta_1,\theta_2)+L_4\theta_1g_2(\theta_1,\theta_2)$$
$$+L_4|\delta_1|+L_4|\delta_2|+L_4w^*u_5$$

Then the coexisting equilibrium point E^* is Globally asymptotically stable under the sufficient condition,

$$-L_1(x-x^*)^2-\frac{L_1x^*a_2}{\phi_1}-\frac{L_2y^*b_2}{\phi_1}+\frac{\theta_1}{2\theta_2(1+m)}\Psi+L_4z^*g_1(\theta_1,\theta_2)+L_4\theta_1g_2(\theta_1,\theta_2)$$
$$+L_4|\delta_1|+L_4|\delta_2|+L_4w^*u_5<0 \tag{18}$$

provided $\Psi<0$,$g_1(\theta_1,\theta_2)<0$ and $g_2(\theta_1,\theta_2)<0$. Further, we can always find a neighbourhood of the equilibrium point E^* such that $L_4|\delta_1|+L_4|\delta_2|<|\varepsilon|$ which satisfies (18). □

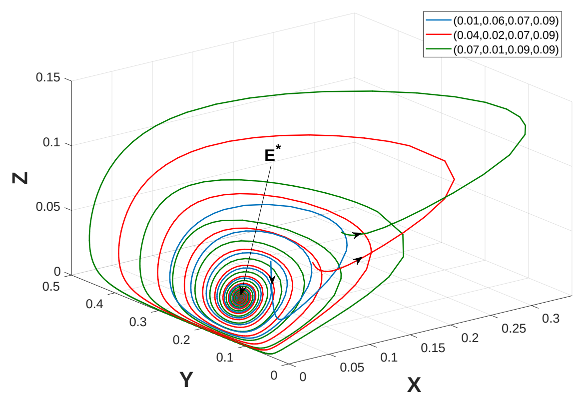
5. Numerical Simulations

TABLE 1. Paramteter Values

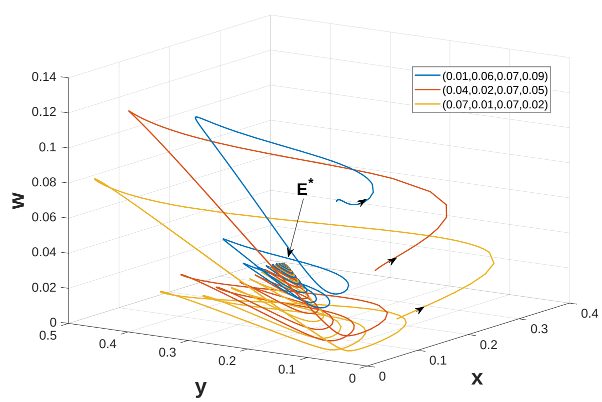
Parameters	Definition	Default value
r	Growth rate of susceptible prey	3
K	Carrying capacity	250
λ	Disease transmission coefficient of prey	0.06
λ'	Disease transmission coefficient of predator	0.03
p	searching rate of susceptible predator towards susceptible prey	0.6
c	searching rate of susceptible predator towards infected prey	0.7
m	a positive constant	0.4
γ	death rate of infected prey	0.3
δ	species growth rate of susceptible predator	2
γ'	conversion rate of infected to susceptible predator	$1/3$
α	Infection transmission proportionality due to predation	0.5

In this section we illustrate some of the key findings of the system (2) numerically around the coexisting equilibrium for a wide range of parameter values given in Table (1). For visualization

of the dynamical behavior of the system we use the software MatCont 6[36]. For the parameters in Table (1) $a_1 = 5, a_2 = 0.2, b_1 = 0.1, b_2 = 0.23, u_1 = 2/3, u_3 = 2.5, u_4 = 0.11667, u_5 = 1/9, m = 0.4$. Our study focuses on the occurrence and termination of the disease. The trajectory of the system (2) is drawn in the Figure(1) for different initial points which shows that the solution of system (2) approaches the coexisting equilibrium $E^* = \{0.0255, 0.1584, 0.0255, 0.0468\}$ which is globally asymptotically stable with the eigen values
$-0.845741, -0.0188191 \pm 0.325998i, -0.037742$.



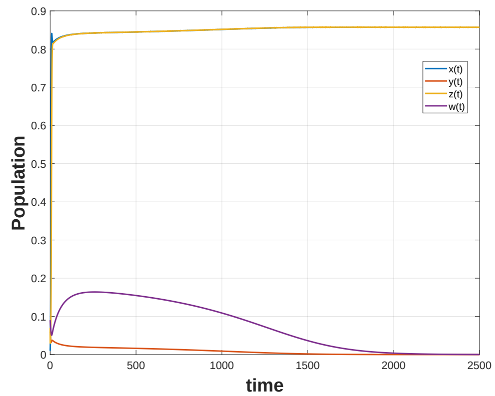
(A)



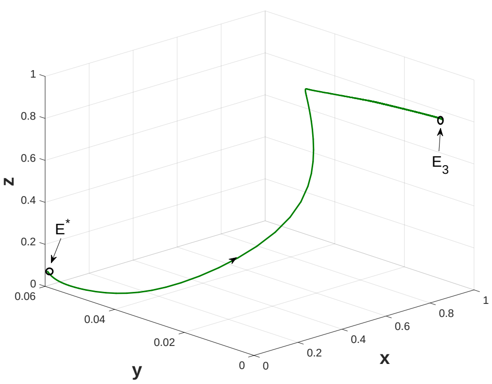
(B)

FIGURE 1. Depicts global stability of the positive equilibrium E^* of the system (2)

5.1. Dynamics of the system for variation of the parameters a_1 and u_3. For the parameter $a_1 = 0.3$ and $u_3 = 0.12$ (keeping the other parameters fixed) the coexisting equilibrium of the system (2) changes to disease free equillibrium E_3 (Figure 2).



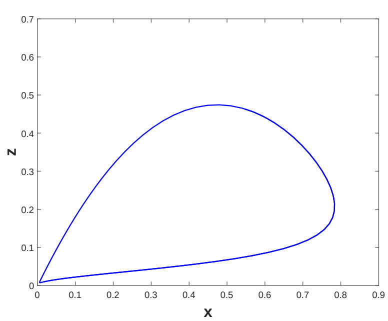
(A)



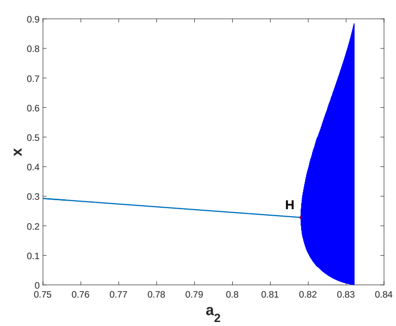
(B)

FIGURE 2. (a)Time series of the solution of system (2) for $a_1 = 0.3$, $u_3 = 0.12$ (keeping other parameters fixed in Table:(1)). (b)Depicts E^* approaches disease free equilibrium E_3

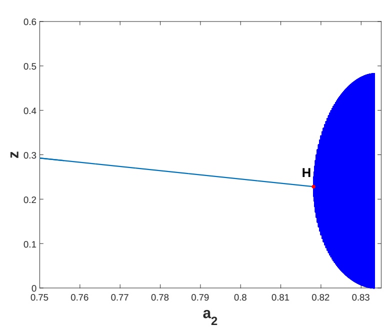
5.2. Hopf bifurcation at the disease free equilibrium. We fix the parameter values as $a_1 = 2, b_1 = 0.75, b_2 = 0.5, u_1 = 0.50, u_3 = 1, u_4 = 0.25, u_5 = 0.62, m = 0.06$ keeping a_2 as bifurcation parameter. For $a_2 = 0.75$, all the trajectories of the system (2) approaches to $(0.2925, 0, 0.2925, 0)$. Increasing the parameter a_2 it is observed that the equilibrium E_3 loses it's stability and the susceptible population oscilate due to supercritical Hopf bifurcation above the threshold value $a_2^* = 0.818155$ (Figure 3) where the first Lyapunov coefficient $l_1 = -0.2080247$ (from equation (14)). Again we fix $a_2 = 0.75$, $m = 0.06$ keeping u_1 as the bifurcation parameter. On increasing the parameter value u_1 it is observed that the equilibrium E_3 loses it's stability and the susceptible population oscilates due to supercritical Hopf bifurcation above the threshold value $u_1^* = 0.375044$ (Figure 4) where the first Lyapunov coefficient is $l_1 = -0.1753540$. We draw Hopf-bifurcation curve of the system (2) at the DFE E_3 in two parametric plane. On further continuation of the Hopf-bifurcation curve shows generalised Hopf-bifurcation (labeled as GH) in $a_2 - m$ parametric space. The GH bifurcation occurs at $(a_2, m) = (0.750031, 0.000028)$, where the first Lyapunov coefficient l_1 becomes zero (Figure 5). On further continuation of the system (2) at the DFE E_3 undergoes Bogdanov-Takens bifurcation (labeled as BT) at $a_2 = 2.000628, m = 1.000440$.



(A)

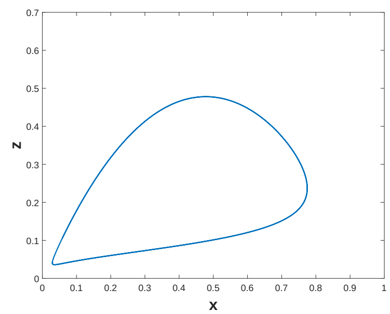


(B)

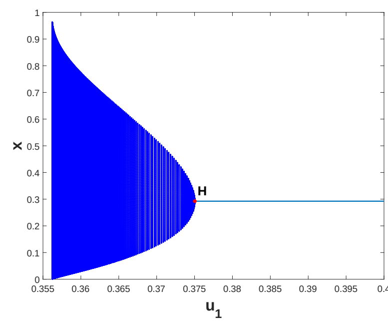


(C)

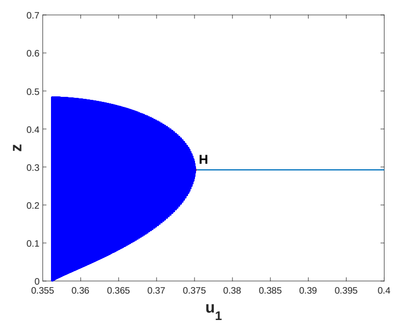
FIGURE 3. (a) depicts limit cycle. (b), (c) Bifurcation diagram at $a_2 = 0.818156$.



(A)



(B)



(C)

FIGURE 4. (a)) depicts limit cycle.(b),(c)Bifurcation diagram at $u_1 = 0.375042$

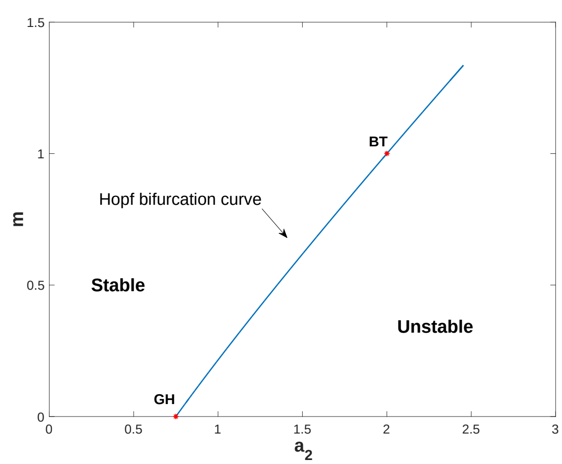


FIGURE 5. Depicts generalised Hopf-bifurcation of the system (2) at the DFE E_3, where $a_2 = 0.750031, m = 0.000028$

5.3. Analysis of bifurcation at the coexisting equilibrium.

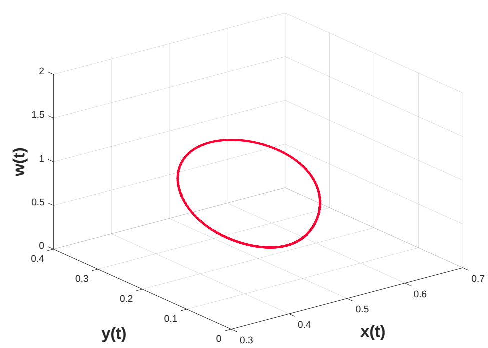
5.3.1. *Existence of Hopf Bifurcation:* We fix $a_1 = 0.303; a_2 = 0.647; b_1 = 0.004; b_2 = 0.257; u_1 = 0.55; u_3 = 2.106; u_4 = 0.261; u_5 = 0.992; m = 0.210$ and initial values $x = 0.02; y = 0.15; z = 0.02; w = 0.04$. It is observed that all the species coexists with populations $x^* = 0.4581; y^* = 0.1555; z^* = 0.4581; w^* = 0.779$. Now we set u_1 as the bifurcation parameter. On increasing u_1 the system (2) loses it's stability and shows oscillatory behavior above the threshold parameter value $u_1^* = 0.579726$ via supercritical Hopf bifurcation where the first Lyapunov coefficient $l_1 = -1.893888 \times 10^-2$. (Figure 6). Though we are using MatCont to determine the nature of the Hopf bifurcation as well as the Lyapunov coefficient, the nature of the Hopf bifurcation can also be determined using equation (17). For the above parameter values $\mu_2 > 0$ which confirms that the system undergoes supercritical Hopf bifurcation.

$$g_{11} = -0.378465 - 0.778659i \qquad g_{20} = 3.96973 - 1.52318i$$

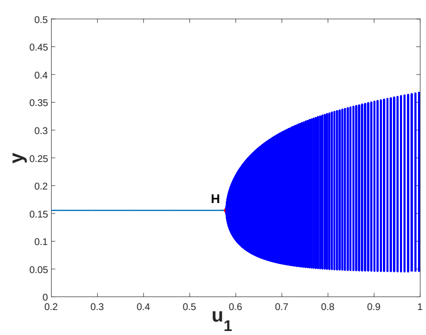
$$g_{02} = -3.47332 + 1.74093i \qquad g_{21} = 6.22262 + 210.947i$$

$$C_1(0) = 3.18282 + 105.211i \qquad \Delta'(u_1(0)) = -0.118163$$

While drawing the Hopf-bifurcation curve of the system (2) in two parametric plane we observe that the system becomes unstable with an increase of the parameter u_1. On continuation, the Hopf bifurcation curve shows generalised Hopf-bifurcation (labeled as GH) in (u_1, a_1), (u_1, a_2), (u_1, b_2), (u_1, u_3) and (u_1, u_4) parametric spaces. In (u_1, a_1) parametric space GH bifurcation occurs at $u_1 = 1.085892$, $a_1 = 0.221289$ where the first lypunov coefficient becomes zero. Similarly in (u_1, a_2) parametric space GH bifurcation occurs at $u_1 = 0.379809$, $a_2 = 0.705966$, in (u_1, b_2) parametric space at $u_1 = 1.486966$, $b_2 = 0.363905$, in (u_1, u_3) parametric space at $u_1 = 5.616859$, $u_3 = 16.339412$ and in (u_1, u_5) parametric space at $u_1 = 0.297226$, $u_5 = 1.075390$ (Figure 7). Further, it is observed that for $u_1 = 5.867$, and $u_3 = 17.55$ in the parametric space $u_1 - u_3$, the system shows multiple limit cycles around coexisting equilibrium point $E^* = (0.0565, 0.07, 0.0565, 1.918)$. Here, all the populations approaches to a fixed value (stable equilibrium), if the initial populations start inside the unstable limit cycle otherwise the the solution trajectories approaches to the stable limit cycle (Figure 8). Similar scenarios can also be obtained for (u_1, a_2),(u_1, b_2) and (u_1, a_1) parameter space. Some other similar scenarios can be found in [38].



(A)



(B)

FIGURE 6. (a) Depicts limit cycle at $u_1 = 0.62$ (b) Depicts appearance of limit cycle at $u_1 = 0.579726$

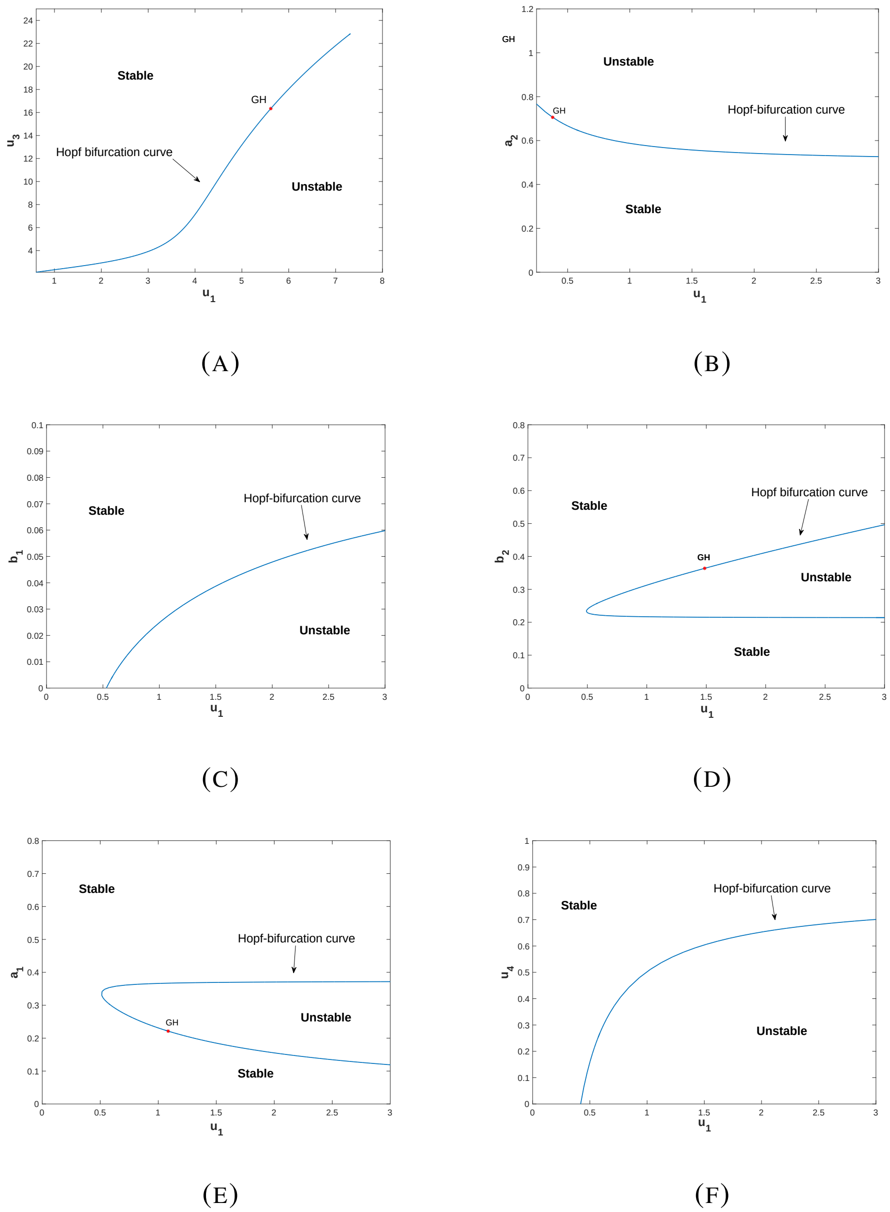


FIGURE 7. Two dimensional projection of Hopf-bifurcation curves of the system (2)

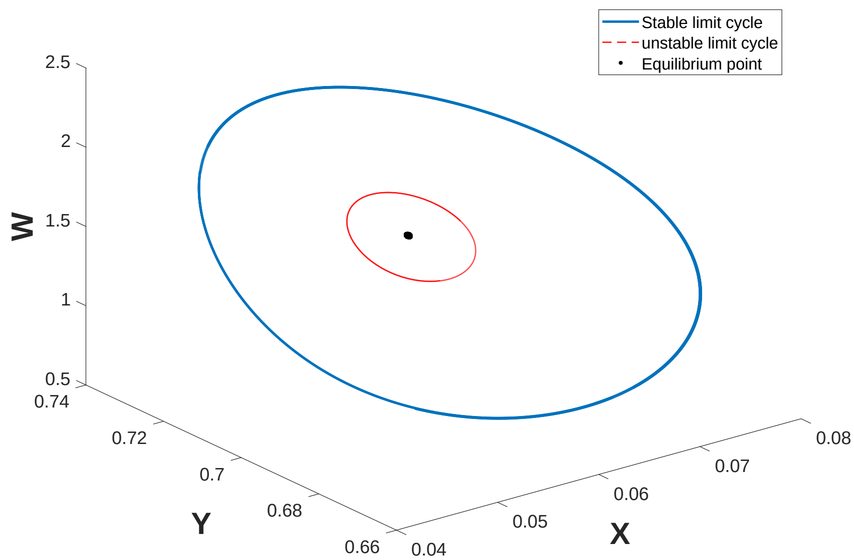


FIGURE 8. Depicts multiple limit cycles of the system where $u_1 = 5.867, u_3 = 17.55$ (keeping other parameters as in subsection 5.3.1)

5.4. Further analysis of the system:

5.4.1. *Saddle-Node and Global Bifurcations.* The coexisting equilibrium E^* (obtained for the parameter values in Table:(1)) undergoes a saddle-node (fold) bifurcation at the parameter value $a_2 = 2.443240$ (keeping the other parameter fixed as in Table:(1)). Figure(9) depicts a fold bifurcation point LP at which $x = 0.031321, y = 0.061225, z = 0.031321, w = 0.055466$ having eigen values $(-0.370998 \pm 0.371244i, -0.00724178, 0)$ and H1,H2,H3 are neutral saddles.

On further analysis for global bifurcations it is seen that the truncated E^* undergoes Bogdanov-Taken and Cusp bifurcation for the parameter space (a_2, b_2), which can be obtained using MatCont continuation of the fold bifurcation point LP (Figure 9). The system (2) undergoes Bogdanov-Taken bifurcations at the point BT1 for the parameter values $a_2 = 2.430790$, $b_2 = 0.244934$ (keeping other parameters fixed as Table:(1)).

At the point BT1 in the parameter space $x = 0.032326, y = 0.059344, z = 0.032326, w = 0.059688$ with eigen values $(-0.374472 \pm 0.371057i, 0, 0)$. The trajectories approaches cusp bifurcation at the points CP1 and CP2 at which the parameter values are $a_2 = 2.483263, b_2 = 0.313008$ and $a_2 = 2.426031, b_2 = 0.304523$ respectively. At the point CP1 the system has eigen values $(-0.401618 \pm 0.39349i, 0.0883421, 0)$.

Moreover, the trajectories undergoes Bogdenov-Takens bifurcations at another points BT2 $(a_2 = 2.142068, b_2 = 0.278798)$ and BT3 $(a_2 = 1.437625, b_2 = 0.849279)$(Figure 10). In the parameter space (a_2, b_2) the Zero-Hopf(ZH) point is a neutral saddle, Cusp point CP2 and Bogdanov-Taken point BT2 exists with infected prey free equilibrium which is unstable. The only bifurcation points with stable equilibrium are BT1 and CP1. Further, we show the Hopf bifurcation curve of the system in the parametric space (a_2, b_2) which meets at the point BT1 (Figure 11).

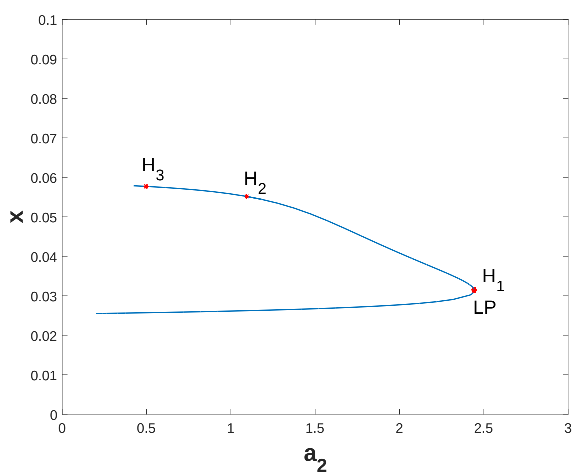


FIGURE 9. Depicts Fold (saddle-node) bifurcation point.

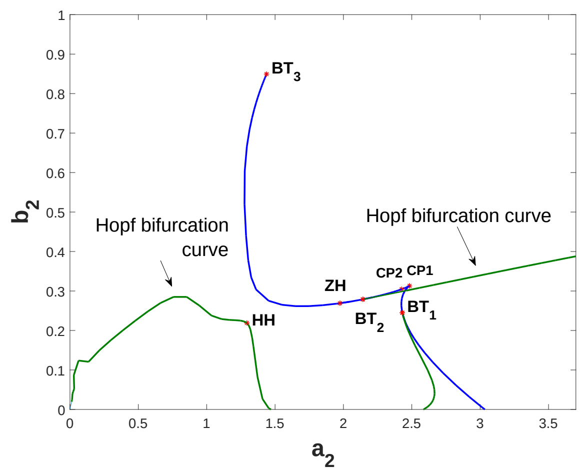


FIGURE 10. Depicts Hopf curves and LP curves in (a_2, b_2) parameter space.

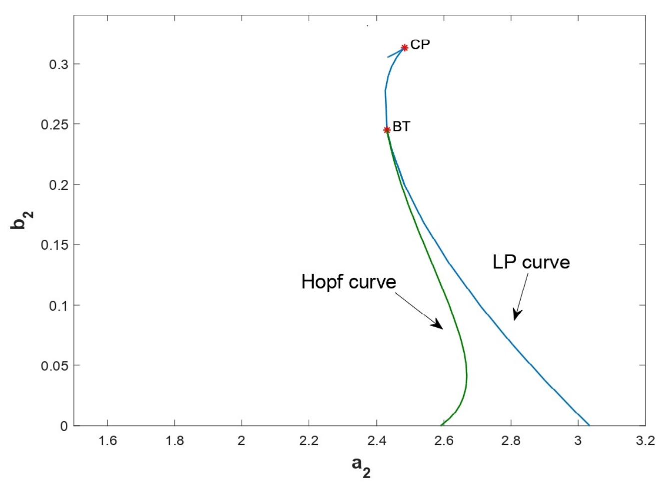
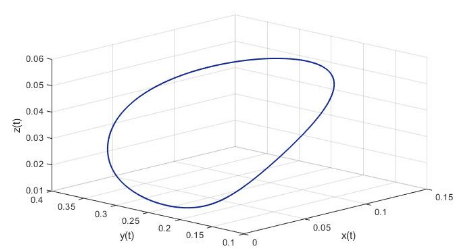


FIGURE 11. Depicts Hopf curve and LP curve in (a_2, b_2) parameter space.

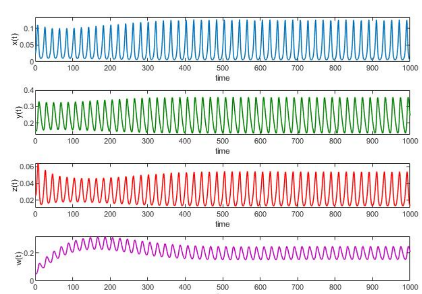
6. Effect of Time Delay

6.1. Effect of time delay in disease transmission. For the parameter values $a_1 = 3, a_2 = 0.5, b_1 = 0.1, b_2 = 0.23, u_1 = \frac{2}{3}, u_3 = 3.5, u_4 = 0.11667, u_5 = 0.11, m = 0.4$ the coexisting equilibrium E^* is unstable. But for a small time delay $\tau = 0.2$ in the disease transmission (a_1) in prey population with the same parameter values the coexisting equilibrium of the delay system shows oscillatory behaviour of all the population (Figure 12).

6.2. Effect of time delay in disease recovery. For the parameter values $a_1 = 5, a_2 = 0.5, b_1 = 0.1, b_2 = 0.23, u_1 = \frac{2}{3}, u_3 = 2.5, u_4 = 0.11667, u_5 = 0.11, m = 0.4$ the coexisting equilibrium E^* is stable. With a three different time delays $\tau = 0.5, 3.0, 5.5$ in the recovery rate (u_5) it is seen that time delay have no effect on stability of the coexisting equilibrium E^*(Figure 13).

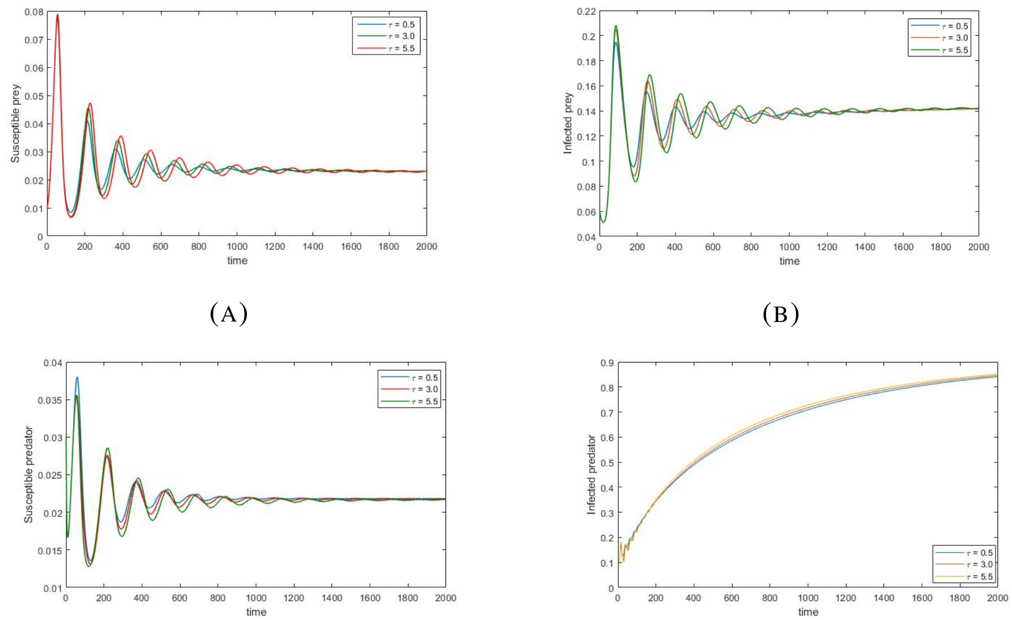


(A)



(B)

FIGURE 12. (a)Limit cycle for $\tau = 0.2$ (b)Time series of the solution of system with delay for $\tau = 0.2$



(A) (B)

(C) (D)

FIGURE 13. Figures (a), (b), (c) and (d) depicts populations for various initial time delays

7. Discussion and Conclusion

In this paper,an ecological system consisting of prey–predator food web model with ratio-dependent type of function response is proposed and analyzed. It is assumed that both the populations are affected by a disease. The disease transmission is assumed to be horizontal with the consideration that some of the infected predators can recover due to their natural immune system. The existence and boundedness of the solution of the proposed model together with the conditions of existence and local stability of all possible equilibrium points are obtained. Finally for the suitable hypothetical set of parameter values, the proposed system is solved numerically in order to verify the obtained analytical results. The obtained results can be summarized as follows:

(1) The disease free equilibrium and the coexisting equilibrium of the system are globally asymptotically stable.

(2) The system can achieve a stable population containing both susceptible and infected one,by maintaining proper parameter value (Figure 1) and hence the infection is controlled to a certain extent.

(3) When the disease transmission rate related parameter is decreased further in both prey and predator, the population achieves infection free equilibrium and thus the disease is eradicated. So, decreasing the disease transmission related parameter (a_1, u_3) of both the population, one crosses the parametric surface barrier related to saddle node bifurcation (Theorem 4) , the system becomes disease free.

(4) If the predation rate related parameter exceeds the threshold value, a coexisting population which was to go to a stable infection free state enters in to a stable oscillatory state resulting in a limit cycle (Figure 3) among the susceptible predator and prey maintaining still infection free system. Similar stable oscillatory state of the infection free state appears if the growth rate related parameter exceeds the threshold value (Figure 4).

(5) It is observed that the reproduction rate in predator species as well as catching rate of susceptible prey may destabilize the system and produce periodic oscillations via supercritical Hopf-bifurcation.

(6) Further, we have calculated the Lyapunov coefficient and showed that the Hopf-bifurcation in the system around the disease free equilibrium can be both supercritical and subcritical depending on the parameter associated with predation rate as well as reproduction rate in predators. We have also performed two-parameter bifurcation analysis and showed that the system undergoes the Bogdanov-Takens bifurcation at the DFE (Figure 5).

(7) If the growth rate related parameter (u_1) is increased beyond the threshold value, the coexisting population which was to go to a stable coexisting fixed point achieves a stable oscillatory state (Figure 6) resulting in a stable limit cycle. One may observe the corresponding time series, the growth rate of the susceptible and the infected prey population is inversely related whereas the growth rate of the susceptible predator and prey are directly

related. A relatively large oscillation is found in the susceptible prey compared to the susceptible predator. Similarly, a relatively large oscillation is generated in the infected sub population compared to the other sub population.

(8) We have calculated the Lyapunov coefficient and showed that the Hopf-bifurcation in the system around the coexisting equilibrium can be both supercritical and subcritical depending on the parameter associated with the reproduction rate in predators and disease transmission rate in them (Figure 7(a)). Moreover, we have shown the supercritical and subcritical nature of the Hopf bifurcation that may appear in the coexisting equilibrium due to the reproduction rate in predators and other parameters (Figure 7(b)-(f)).

(9) We have also performed two-parameter bifurcation analysis at the coexisting equilibrium. It is observed that the system undergoes the Bogdanov-Takens bifurcation together with a cusp bifurcation. We draw Hopf curve and LP curve in parameter space containing the parameters associated to predation rate of both the prey population (Figure 10-11).

(10) In general recovery from a disease is not an instantaneous process. Usually it is also a delay process in nature. System (2) has no effect on delay of the recovery factor. Figure (13) shows effect three different time delays in recovery factor on the system (2).

APPENDIX

Expressions of Theorem (11)

$$\xi_{11} = \frac{a_2 L_1 x^* z}{\phi} + \frac{b_2 L_2 y^* z}{\phi}$$
$$\chi_{11} = -\frac{L_4 u_4 w^* yz}{w\phi}$$
$$\xi_{21} = \frac{a_2 L_1 x z^*}{\phi_1} + \frac{b_2 L_2 y z^*}{\phi_1}$$
$$\chi_{21} = -\frac{L_4 u_4 w y^* z^*}{w^* \phi_1} + \frac{L_4 u_4 y z^*}{\phi} - L_4 u_1 z^*$$
$$\xi_{31} = -\frac{a_2 L_1 x z}{\phi} - \frac{b_2 L_2 y z}{\phi}$$
$$\chi_{31} = -\frac{L_4 u_5 w^* z}{z^*} + \frac{L_4 u_4 y^* z}{\phi_1} + L_4 u_1 z$$
$$\Psi = a_2 L_1 x^* + b_2 L_2 y^* - L_4 u_4 w^*$$
$$g_1(\theta_1, \theta_2) = -\frac{a_2 L_1 + b_2 L_2}{2(m+1)L_4} + \frac{\theta_1 u_4 (-y^*}{2\theta_1 (m+1) w^*} - u_1 + u_4$$
$$g_2(\theta_1, \theta_2) = -\frac{a_2 L_1 + b_2 L_2}{(2(m+1))L_4} + \frac{1}{2}\theta_1 (m+1) u_4 y^* - \frac{u_5 w^*}{z^*} + u_1$$

Expressions from equation (16)

(19)

$$a_{11} = -\frac{a_2 (s_3 + z^*)}{m(s_3 + s_4 + w^* + z^*) + s_1 + s_2 + x^* + y^*} + \frac{a_2 (s_1 + x^*)(s_3 + z^*)}{(m(s_3 + s_4 + w^* + z^*) + s_1 + s_2 + x^* + y^*)^2}$$
$$- a_1 (s_2 + y^*) - 2s_1 - s_2 - 2x^* - y^* + 1$$
$$a_{12} = \frac{a_2 (s_1 + x^*)(s_3 + z^*)}{(m(s_3 + s_4 + w^* + z^*) + s_1 + s_2 + x^* + y^*)^2} - a_1 (s_1 + x^*) - s_1 - x^*$$
$$a_{13} = -\frac{a_2 (s_1 + x^*)(ms_4 + mw^* + s_1 + s_2 + x^* + y^*)}{(ms_3 + ms_4 + mw^* + mz^* + s_1 + s_2 + x^* + y^*)^2}$$
$$a_{14} = \frac{a_2 m (s_1 + x^*)(s_3 + z^*)}{(m(s_3 + s_4 + w^* + z^*) + s_1 + s_2 + x^* + y^*)^2}$$
$$a_{21} = (s_2 + y^*)\left(a_1 + \frac{b_2 (s_3 + z^*)}{(m(s_3 + s_4 + w^* + z^*) + s_1 + s_2 + x^* + y^*)^2}\right)$$

(20)

$$a_{22} = a_1(s_1+x^*) - \frac{b_2(s_3+z^*)}{m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*} + \frac{b_2(s_2+y^*)(s_3+z^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2} - b_1$$

$$a_{23} = -\frac{b_2(s_2+y^*)(ms_4+mw^*+s_1+s_2+x^*+y^*)}{(ms_3+ms_4+mw^*+mz^*+s_1+s_2+x^*+y^*)^2}$$

$$a_{24} = \frac{b_2 m(s_2+y^*)(s_3+z^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2}$$

$$a_{31} = (s_3+z^*)\left(\frac{u_4(s_2+y^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2} + \frac{u_1(s_3+z^*)}{(s_1+x^*)^2}\right)$$

$$a_{32} = -\frac{u_4(s_3+z^*)(ms_3+ms_4+mw^*+mz^*+s_1+x^*)}{(ms_3+ms_4+mw^*+mz^*+s_1+s_2+x^*+y^*)^2}$$

$$a_{33} = -\frac{u_4(s_2+y^*)}{m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*} + \frac{mu_4(s_2+y^*)(s_3+z^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2} - u_3(s_4+w^*) - \frac{2u_1(s_3+z^*)}{s_1+x^*} + u_1$$

$$a_{34} = \frac{mu_4(s_2+y^*)(s_3+z^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2} - u_3(s_3+z^*) + u_5$$

$$a_{41} = -\frac{u_4(s_2+y^*)(s_3+z^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2}$$

$$a_{42} = \frac{u_4(s_3+z^*)(ms_3+ms_4+mw^*+mz^*+s_1+x^*)}{(ms_3+ms_4+mw^*+mz^*+s_1+s_2+x^*+y^*)^2}$$

$$a_{43} = \frac{u_4(s_2+y^*)}{m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*} - \frac{mu_4(s_2+y^*)(s_3+z^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2} + u_3(s_4+w^*)$$

$$a_{44} = -\frac{mu_4(s_2+y^*)(s_3+z^*)}{(m(s_3+s_4+w^*+z^*)+s_1+s_2+x^*+y^*)^2} + u_3(s_3+z^*) - u_5$$

$$B_1 = -s_1((a_1+1)s_2+s_1)(m(w^*+z^*)+x^*+y^*)^4 - a_2(m(s_3+s_4-w^*-z^*)+s_1+s_2-x^*-y^*)$$
$$(z^*(ms_4+s_1+s_2)-s_3(mw^*+x^*+y^*))(s_1(m(w^*+z^*)+y^*)-x^*(m(s_3+s_4)+s_2))$$
$$(m(w^*+z^*)+x^*+y^*)^{-4}$$

$$B_2 = a_1 s_1 s_2(m(w^*+z^*)+x^*+y^*)^4 + b_2(m(s_3+s_4-w^*-z^*)+s_1+s_2-x^*-y^*)(z^*(ms_4+s_1+s_2)$$
$$-s_3(mw^*+x^*+y^*)(y^*(m(s_3+s_4)+s_1)-s_2(m(w^*+z^*)+x^*))(m(w^*+z^*)+x^*+y^*)^{-4}$$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A.J. Lotka, Elements of Physical Biology. Williams and Wilkins Company, Baltimore, (1925).

[2] C.S. Holling, The components of predation as revealed by a study of small-mammal predation of the European pine sawfly. Can. Entomol. 91 (1959), 293–320

[3] Holling, C. S.: Some characteristics of simple types of predation and parasitism. Can. Entomol. 91 (1959), 385–395.

[4] M.P. Hassell, G.C. Varley, New Inductive Population Model for Insect Parasites and its Bearing on Biological Control, Nature. 223 (1969), 1133–1137.

[5] J.R. Beddington, Mutual Interference Between Parasites or Predators and its Effect on Searching Efficiency, J. Animal Ecol. 44 (1975), 331-340.

[6] D.L. DeAngelis, R.A. Goldstein, R.V. O'Neill, A Model for Tropic Interaction, Ecology. 56 (1975), 881–892.

[7] P.H. Crowley, E.K. Martin, Functional responses and interference within and between year classes of a dragonfly population. J. North Amer. Benthol. Soc. 8 (1989), 211–221.

[8] R. Arditi, L.R. Ginzburg, Coupling in predator-prey dynamics: Ratio-Dependence, J. Theor. Biol. 139 (1989), 311–326.

[9] Y. Kuang, Rich dynamics of Gause-type ratio-dependent predator-prey system. Fields Inst. Commun. 21 (1999), 325–337.

[10] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proc. R. Soc. Lond. A. 115 (1927), 700–721.

[11] R.M. Anderson, R.M. May, The invasion, persistence and spread of infectious diseases within animal and plant communities, Phil. Trans. R. Soc. Lond. B. 314 (1986), 533–570.

[12] K.P. Hadeler, H.I. Freedman, Predator-prey populations with parasitic infection, J. Math. Biol. 27 (1989), 609–631.

[13] E. Venturino, Epidemics in predator–prey models: diseases in the prey. In: Arino, O., Axelrod, D., Kimmel, M., Langlais, M. (eds.) Mathematical Population Dynamics: Analysis of Heterogeneity, vol. 1: Theory of Epidemics, pp. 381–393. Wuerz, Winnipeg (1995).

[14] J. Chattopadhyay, O. Arino, A predator-prey model with disease in the prey, Nonlinear Anal., Theory Meth. Appl. 36 (1999), 747–766.

[15] N. Bairagi, S. Chaudhuri, J. Chattopadhyay, Harvesting as a disease control measure in an eco-epidemiological system – A theoretical study, Math. Biosci. 217 (2009), 134–144.

[16] J. Chattopadhyay, N. Bairagi, Pelicans at risk in Salton sea — an eco-epidemiological model, Ecol. Model. 136 (2001), 103–112.

[17] R.L. Tayeh, R.K. Naji, Mathematical study of eco–epidemiological system, Math. Theory Model. 4(14) (2014), 2225-0522.

[18] Y.-H. Hsieh, C.-K. Hsiao, Predator-prey model with disease infection in both populations, Math. Med. Biol. 25 (2008), 247–266.

[19] K. pada Das, K. Kundu, J. Chattopadhyay, A predator–prey mathematical model with both the populations affected by diseases, Ecol. Complex. 8 (2011), 68–80.

[20] D. Greenhalgh, Q.J.A. Khan, J.S. Pettigrew, An eco-epidemiological predator-prey model where predators distinguish between susceptible and infected prey, Math. Meth. Appl. Sci. 40 (2017), 146–166.

[21] S. Chakraborty, S. Pal, N. Bairagi, Dynamics of a ratio-dependent eco-epidemiological system with prey harvesting, Nonlinear Anal., Real World Appl. 11 (2010), 1862–1877.

[22] H.W. Hethcote, W. Wang, L. Han, Z. Ma, A predator–prey model with infected prey, Theor. Popul. Biol. 66 (2004), 259–268.

[23] K. pada Das, A study of harvesting in a predator-prey model with disease in both populations, Math. Meth. Appl. Sci. 39 (2016), 2853–2870.

[24] X. Gao, Q. Pan, M. He, Y. Kang, A predator–prey model with diseases in both prey and predator, Physica A: Stat. Mech. Appl. 392 (2013), 5898–5906.

[25] K.P. Das, S. Chaudhuri, Role of harvesting in controlling chaotic dynamics in the predator–prey model with disease in the predator, Int. J. Biomath. 06 (2013), 1350005.

[26] C. Huang, H. Zhang, J. Cao, H. Hu, Stability and Hopf Bifurcation of a Delayed Prey–Predator Model with Disease in the Predator, Int. J. Bifurcation Chaos. 29 (2019), 1950091.

[27] G.S. Traxler, G.R. Bell, Pathogens associated with impounded Pacific herring Clupea harengus pallasi, with emphasis on viral erythrocytic necrosis (VEN) and atypical Aeromonas salmonicida, Dis. Aquatic Organ. 5(2) (1988), 93-100.

[28] A.E. Eissa, E.E. Elsayed, An overview on bacterial kidney disease, Life Sci. J. 3(3) (2006), 58-76.

[29] T.R. Meyers, et al. Diseases of wild and cultured fishes in Alaska. Alaska Department of Fish and Game, Fish Pathology Laboratories, 2019.

[30] T. Tomiyama, Y. Kurita, Seasonal and spatial variations in prey utilization and condition of a piscivorous flatfish Paralichthys olivaceus, Aquatic Biol. 11(3) (2011), 279-288.

[31] H.F. Skall, N.J. Olesen, S. Mellergaard, Viral haemorrhagic septicaemia virus in marine fish and its implications for fish farming - a review, J Fish Dis. 28 (2005), 509–529.

[32] M. Sano, T. Ito, T. Matsuyama, et al. Effect of water temperature shifting on mortality of Japanese flounder Paralichthys olivaceus experimentally infected with viral hemorrhagic septicemia virus, Aquaculture. 286 (2009), 254–258.

[33] D. Greenhalgh, Q.J.A. Khan, F.A. Al-Kharousi, Eco-epidemiological model with fatal disease in the prey, Nonlinear Anal., Real World Appl. 53 (2020), 103072.

[34] O. Arino, A.E. abdllaoui, J. Mikram, J. Chattopadhyay, Infection in prey population may act as a biological control in ratio-dependent predator–prey models, Nonlinearity. 17 (2004), 1101–1116.

[35] W.M. Liu, Criterion of Hopf Bifurcations without Using Eigenvalues, J. Math. Anal. Appl. 182 (1994), 250–256.

[36] A. Dhooge, W. Govaerts, Yu.A. Kuznetsov, H.G.E. Meijer, B. Sautois, New features of the software M at C ont for bifurcation analysis of dynamical systems, Math. Computer Model. Dyn. Syst. 14 (2008), 147–175.

[37] B.D. Hassard, N.D. Kazarinoff, Y. Wan, Theory and applications of Hopf bifurcation. London Math Soc Lecture Note Ser, vol 41. Cambridge University Press, Cambridge, (1981).

[38] S. Pal, N. Pal, S. Samanta, J. Chattopadhyay, Effect of hunting cooperation and fear in a predator-prey model, Ecol. Complex. 39 (2019), 100770.

[39] Y.A. Kuznetsov, Elements of Applied Bifurcation Theory, vol. 112. Springer, New York, (2013).