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MATHEMATICAL MODEL FOR THE CONTROL OF LYMPHATIC FILARIASIS TRANSMISSION DYNAMICS

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Abstract: In this paper, a mathematical model for the transmission dynamics of lymphatic filariasis is presented by incorporating the infected without symptom, the infected with symptom and treatment compartments. The model is shown to have two equilibrium states: the disease-free equilibrium (DFE) and the endemic equilibrium states. An explicit formula for the effective reproduction number was obtained in terms of the demographic and epidemiological parameters of the model. Using the method of linearization, the disease-free equilibrium state was found to be locally asymptotically stable if the basic reproduction number is less than unity. By constructing a suitable Lyapunov function, the disease-free equilibrium state was found to be globally asymptotically stable. This means that lymphatic filariasis could be put under control in a population when the effective reproduction number is less than one. The endemic equilibrium state was found to be locally asymptotically stable. By constructing yet another Lyapunov function, the endemic equilibrium state was found to be globally asymptotically stable under certain conditions. Sensitivity analysis was carried out on the effective reproduction number, the most sensitive parameters were the treatment rate of human population and the infected rate of human population. Results from the simulation carried out showed that treatment

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level coverage of human population should target a success rate of 75% for LF to be under control in the population.

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1. INTRODUCTION

Lymphatic filariasis, commonly known as elephantiasis, is a neglected tropical disease that has a major social and economic effect in Asia, Africa, the Western Pacific and parts of the United States [1]. It is one of the number one cause of long-term and constant disability [2]. About one billion people are known to be at risk of this disease in 86 countries [3-4]. In 73 endemic countries, the disease affects approximately 120 million people globally and is a weakening disease and one of the most frequent yet most neglected tropical conditions that has severe economic and social impact [5]. Women, men and children of all ages are affected by lymphatic filariasis. It is a moustache disease caused by Brugia Malayi, Brugia Timori and Wuchereria banrofti tissue dwelling nematodes [6-7].

In order to manage and control and LF, many studies have described and studied mathematical models to achieve new insight into the dynamics of the disease. [8], developed a mathematical model for the transmission and application of lymphatic filariasis. The authours built a Wuchereria-induced stochastic transmission model for LF and evaluate its incidence using computational models. The main objective of their research was to evaluate the impact of vector control in the Pondicherry (India) context and, in particular, the constant effect of the post-control period. A mathematical model was formulated for investigation of the long-lasting medical effects in Indonesia of lymph filariasis. [9-10] developed a mathematical model for lymphatic filariases transmission and prevention, with therapy only for those with elephantiasis symptoms. [11-12] formulated lymphatic filariasis models with age structure and transmission. Lymphatic filariosis has also been seen in many mathematics and non-mathematical studies [13-19]. In view of the work of the above-mentioned researchers, the work of the above authors is complemented by incorporating relevant features such as, the classes undergoing treatment, vector control (using bed-net and insecticide) and drug administration to both the infected class with symptoms and without symptoms.

The present paper is organized as follows: section 2 deals with the formulation of the model and definitions of the model parameters and variables, in section 3, the model analysis is carried out

section 4 deals with the major findings, section 5 deals with discussion of results and section 6 deals with some concluding remarks about major findings in the present study.

2. MATERIALS AND METHODS

2.1 Model formulation

We propose a deterministic model to examine lymphatic filariais transmission dynamics with specific features such as treatment classes, vectors (bed-net and insecticide use) and medication in a susceptible class, an infected class with and without symptoms. The total human population represented by $N_h(t)$ is divided into six classes, susceptible human in the absence of treatment $S_1(t)$, treatment of susceptible human beings $S_2(t)$, human infected but without symptoms of elephantiasis $I_1(t)$, infected humans with exhibiting signs of elephantiasis $I_2(t)$, humans receiving care with infected humans (not displaying symptoms of elephantiasis) $T_1(t)$ and humans receiving treatment with infected humans with symptoms of elephantiasis $T_2(t)$. Mosquitoes and human beings are recruited at rates Λ_v and Λ_h respectively into their susceptible corresponding populations. Mosquitoes suffer natural death at a rate μ_v and insecticide death at a rate δ. In the same vein, people are subjected in proportion to each human class, to natural death at a μ_h rate. The mosquito absorbs microfilaria when a person with filariasis (elephantiasis that causes nematodes) is bitten at a rate β_v by infection force $\frac{\beta_v(\theta_h I_1(t)+I_2(t))}{N_h(t)}$ where β_v is the average rate of mosquito bites that cause infected humans to susceptible mosquito transmission of disease and $\theta_h \in (0,1)$ represents less microfilariasis in the bloodstream of people infected but with no symptoms of elephantiasis. When the vector (mosquitos) is infected, it enters $V_2(t)$ carrier-class. Microfilariae transfer into the hemocoel and advance into filariform juveniles through the mosquito gut. Filariform juveniles escape from mosquitoes when an insect feeds and then enter a human's injury system at the rate β_h with an

infection force $\frac{\beta_h V_2(t)}{N_v(t)}$ where β_h is the average amount of mosquito bites that cause disease transmission to humans sensitive per mosquito by the carriers of the parasite (mosquito). Humans are thus infected at a rate β_v to reach exposed class $I_1(t)$ following a mosquito bite. $I_1(t)$ individuals move to the stage where they show signs of filariasis $I_2(t)$ at a rate ρ. However, some migrate to $I_2(t)$ due to secondary infection the rate β_v. Individuals in $S_2(t)$, $I_1(t)$ and $I_2(t)$ received treatment at a rate τ and progress the classes undergoing treatment. The schematic representation of the model is given in figure 1. Based on the model development description above and the schematic diagram in Figure 1, the following model equations are derived:

$$
\begin{aligned}
S_1^{'} &= \Lambda_h - \frac{\beta_h(1-\theta)}{N_v} V_2 S_1 - (\mu_h + \tau) S_1 + (T_1 + T_2)\gamma + \omega_1 S_2 \\
S_2^{'} &= \tau S_1 - (\mu_h + \omega_1) S_2 \\
I_1^{'} &= \frac{\beta_h(1-\theta)}{N_v} V_2 S_1 - \frac{\beta_h(1-\theta)}{N_v} V_2 I_1 - (\mu_h + \tau + \rho) I_1 + \omega_1 T_1 \\
I_2^{'} &= \frac{\beta_h(1-\theta)}{N_v} V_2 I_1 + \rho I_1 - (\mu_h + \tau) I_2 + \omega_2 T_2 \qquad (1) \\
T_1^{'} &= \tau I_1 - (\mu_h + \gamma + \omega_1) T_1 \\
T_2^{'} &= \tau I_2 - (\mu_h + \gamma + \omega_2) T_2 \\
V_1^{'} &= \Lambda_v - \frac{\beta_v(\theta_h I_1 + I_2) V_1}{N_h} - (\mu_v + \delta) V_1 \\
V_2^{'} &= \frac{\beta_v(\theta_h I_1 + I_2) V_1}{N_h} - (\mu_v + \delta) V_2
\end{aligned}
$$

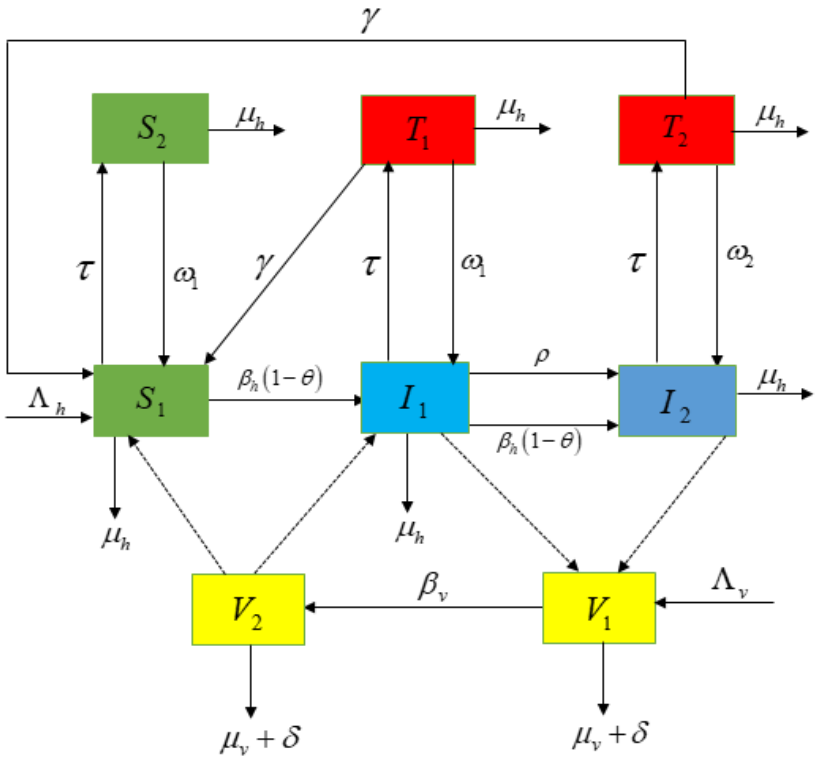

Figure 1: Schematic diagram of the model

Table 1: Notation and definition of variables and parameter

Symbol	Description
$S_1(t)$	Susceptible individuals who do not take drugs
$S_2(t)$	Susceptible individuals subject to therapy
$I_1(t)$	Infected but not exhibiting signs or symptoms of disease
$I_2(t)$	Infected individuals exhibiting signs or symptoms of disease
$T_1(t)$	Individuals undergoing treatment from infected individuals with no symptoms
$T_2(t)$	Individuals undergoing treatment from infected individuals with symptoms
$V_1(t)$	Non-carrier vectors (mosquitoes)
$V_2(t)$	Carrier vectors (mosquitoes)

Symbol	Description
Λ_h	Human population recruitment rate
Λ_v	Mosquito population Human population recruitment rate
β_v	The rate of ingestion by mosquitoes while biting an infected person
β_h	Human population infection rate
μ_h	Mortality rates for population of humans
μ_v	Mortality rates for mosquito population
δ	Vector (mosquitoes) mortality rate with the application of insecticide
ρ	Progression rate of human from $I_1(t)$ to $I_2(t)$
θ	The portion of vulnerable individuals using mosquito nets and insecticide
γ	Rate of recovery or the rate at which individuals treated lose their immunity
τ	Rate of treatment for individuals undergoing therapy
θ_h	Rate of microfilariae in the blood system of infected individuals without signs of disease.
ω_1	Rate at which people in $S_2(t)$ and $T_1(t)$ quit taking medications
ω_2	Rate at which people in $T_2(t)$ quit taking medications

3. MODEL ANALYSIS

The total human population is

$$\frac{dN_h(t)}{dt} = \frac{dS_1(t)}{dt} + \frac{dS_2(t)}{dt} + \frac{dI_1(t)}{dt} + \frac{dI_2(t)}{dt} + \frac{dT_1(t)}{dt} + \frac{dT_2(t)}{dt}$$

Thus,

$$N_h^{'} = \Lambda_h - \mu_h N_h$$

Also, the total mosquito population is

$$\frac{dNv(t)}{dt} = \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt}$$

$$N_v^{'} = \Lambda_v - (\mu_v + \delta)N_v$$

Let, $k_1 = (\mu_h + \tau)$, $k_2 = (\mu_h + \omega_1)$, $k_3 = (\mu_h + \tau + \rho)$, $k_4 = (\mu_h + \gamma + \omega_1)$, $k_5 = (\mu_h + \gamma + \omega_2)$, $k_6 = (\mu_v + \delta)$

Thus, equation (1) becomes,

$$S_1^{'} = \Lambda_h - \frac{\beta_h(1-\theta)}{N_v} V_2 S_1 - k_1 S_1 + (T_1 + T_2)\gamma + \omega_1 S_2$$

$$S_2^{'} = \tau S_1 - k_2 S_2$$

$$I_1^{'} = \frac{\beta_h(1-\theta)}{N_v} V_2 S_1 - \frac{\beta_h(1-\theta)}{N_v} V_2 I_1 - k_3 I_1 + \omega_1 T_1$$

(2) $$I_2^{'} = \frac{\beta_h(1-\theta)}{N_v} V_2 I_1 + \rho I_1 - k_1 I_2 + \omega_2 T_2$$

$$T_1^{'} = \tau I_1 - k_4 T_1$$

$$T_2^{'} = \tau I_2 - k_5 T_2$$

$$V_1^{'} = \Lambda_v - \frac{\beta_v(\theta_h I_1 + I_2)V_1}{N_h} - k_6 V_1$$

$$V_2^{'} = \frac{\beta_v(\theta_h I_1 + I_2)V_1}{N_h} - k_6 V_2$$

Theorem 1:

Let $$\Omega = \left\{ \begin{array}{l} (S_1, S_2, I_1, I_2, T_1, T_2) \in \Re_+^6 : S_1(0) > 0, S_2(0) > 0, I_1(0) > 0, I_2(0) > 0, T_1(0) > 0, T_2(0) > 0 \\ S_1 + S_2 + I_1 + I_2 + T_1 + T_2 \le \frac{R_h}{\mu_h} \\ (V_1, V_2) \in \Re_+^2 : V_1(0) > 0, V_2(0) > 0 \\ V_1 + V_2 \le \frac{R_v}{k_6} \end{array} \right\}$$

then the solutions of $\{S_1(t), S_2(t), I_1(t), I_2(t), T_1(t), T_2(t), V_1(t), V_2(t)\}$ of system (2) are non-negative for $t \ge 0$.

Proof:

$$\frac{dS_1}{dt} = \Lambda_h - \frac{\beta_v(1-\theta)}{N_v} V_2 S_1 - k_1 S_1 + (T_1 + T_2)\gamma + \omega_1 S_2$$

$$\frac{dS_1}{dt} \geq -k_1 S_1$$

$$\frac{dS_1}{S_1} \geq -k_1 dt \tag{3}$$

$$\int \frac{dS_1}{S_1} \geq \int -k_1 dt$$

$$S_1(t) \geq S_1(0) e^{-k_1 t} \geq 0$$

Following the same approach above,

$S_2(t) \geq S_2(0) e^{-k_2 t} \geq 0$, $I_1(t) \geq I_1(0) e^{-k_3 t} \geq 0$, $I_2(t) \geq I_2(0) e^{-k_1 t} \geq 0$, $T_1(t) \geq T_1(0) e^{-k_4 t} \geq 0$,

$T_2(t) \geq T_2(0) e^{-k_5 t} \geq 0$, $V_1(t) \geq V_1(0) e^{-k_6 t} \geq 0$, $V_2(t) \geq V_2(0) e^{-k_6 t} \geq 0$

Hence, $\{S_1(1), S_2(t), I_1(t), I_2(t), T_1(t), T_2(t), V_1(1), V_2(t)\}$ of the system (2) are positive for all $t \geq 0$

3.1 Feasible Region of the Model

Theorem 2: The system (2) has solutions which are contained in the feasible region Γ.

Proof:

Let $\Gamma = (S_1, S_2, I_1, I_2, T_1, T_2) \in \Re_+^6$ and $\Psi = (V_1, V_2) \in \Re_+^2$ be any solution of the system with positive initial conditions, then adding the equations of the system (2) for human population, we have,

$$N_h^{'} = \Lambda_h - \mu_h N_h \tag{4}$$

$$N_h^{'} + \mu_h N_h = \Lambda_h$$

$$\text{IF} = e^{\mu_h t}$$

$$N_h(t) e^{\mu_h t} = \int \Lambda_h e^{\mu_h t} dt + C$$

$$N_h(t) = \frac{\Lambda_h}{\mu_h} + C e^{-\mu_h t}$$

By using the initial condition;

$t = 0,\ N_h(0) = N_{h(0)}$

$$C = N_{h(0)} - \frac{\Lambda_h}{\mu_h}$$

(5) $$N(t) = \frac{\Lambda_h}{\mu_h} + \left(N_{h(0)} - \frac{\Lambda_h}{\mu_h} \right) e^{-\mu_h t}$$

Applying Birkoff and Rota's theorem on differential inequality [20], we obtain $0 \le N_h \le \frac{\Lambda_h}{\mu_h}$ as $t \to \infty$. The total population of human approaches $\frac{\Lambda_h}{\mu_h}$.

also for mosquito population,

$$\frac{dN_v}{dt} = \Lambda_v - k_6 N_v$$

$$\frac{dN_v}{dt} + k_6 N_v = \Lambda_v$$

IF$= e^{k_6 t}$

$$N_v(t) e^{k_6 t} = \int \Lambda_v e^{k_6 t} dt + C$$

$$N_v(t) = \frac{\Lambda_v}{k_6} + C e^{-k_6 t}$$

Using the initial condition;

$t = 0,\ N_v(0) = N_{v(0)}$

$$C = N_{v(0)} - \frac{\Lambda_v}{k_6}$$

(6) $$N_v(t) = \frac{\Lambda_v}{k_6} + \left(N_{v(0)} - \frac{\Lambda_v}{k_6} \right) e^{-k_6 t}$$

Following the same approach in (5) we obtain $0 \le N_v \le \frac{\Lambda_v}{k_6}$ as $t \to \infty$. The total population of

vector approaches $\frac{\Lambda_v}{k_6}$.

Thus, the examine the region for the system (2),

$$(7)\quad \Omega = \Gamma \times \Psi = \left\{ \begin{array}{l} (S_1, S_2, I_1, I_2, T_1, T_2) \in \Re_+^6 : (S_1 + S_2 + I_1 + I_2 + T_1 + T_2) \le \frac{\Lambda_h}{\mu_h}, \\ S_1 \ge 0,\ S_2 \ge 0,\ I_1 \ge 0,\ I_2 \ge 0,\ T_1 \ge 0,\ T_2 \ge 0 \\ (V_1, V_2) \in \Re_+^2 : V_1 + V_2 \le \frac{\Lambda_v}{k_6},\ V_1 \ge 0,\ V_2 \ge 0 \end{array} \right\}$$

The vector field points to the interior of Ω on the part of the boundary when

$(S_1 + S_2 + I_1 + I_2 + T_1 + T_2) = \frac{\Lambda_h}{\mu_h}$ and $(V_1 + V_2) = \frac{\Lambda_v}{k_6}$ then, it is positively invariant.

Hence, it has been shown that all the solutions of equation (2) are in $\Re_+^6$ and $\Re_+^2$ respectively provided that the initial conditions are positive. Our feasible solution set of the model therefore enters the region Γ. Our model is epidemiologically relevant and mathematically well described in this field.

3.2 Disease free equilibrium

Disease free equilibrium states, where there is no infection, are equilibrium solutions. All infected classes will therefore be zero, and susceptible individuals and susceptible vectors will be included in the entire population. Thus, the DFE from (2) is given as,

$$\left(S_1^0, S_2^0, I_1^0, I_2^0, T_1^0, T_2^0, V_1^0, V_2^0\right) = \left(\frac{\Lambda_h k_2}{(k_1 k_2 - \tau\omega_1)}, \frac{\Lambda_h \tau}{(k_1 k_2 - \tau\omega_1)}, 0, 0, 0, 0, \frac{\Lambda_v}{k_6}, 0 \right)$$

3.3. Effective Reproduction Number, R_C

One of the important aspects of mathematical biology is to identify threshold conditions that decide when the disease is introduced into the population, an infection can spread to a susceptible population. The effective number of secondary infections caused by a typical infected person during the entire period of infectiousness is described as basic reproduction number. Following the approach of [21-22], we obtain,

(7) $$F=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\beta_h(1-\theta)S_1^0}{N_v^0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_v\theta_h V_1^0}{N_h^0} & \frac{\beta_v V_1^0}{N_h^0} & 0 & 0 & 0 \end{bmatrix},$$

(8) $$V=\begin{bmatrix} k_3 & 0 & -\omega_1 & 0 & 0 \\ -\rho & k_1 & 0 & -\omega_2 & 0 \\ -\tau & 0 & k_4 & 0 & 0 \\ 0 & -\tau & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & k_6 \end{bmatrix},\ V^{-1}=\begin{bmatrix} b_1 & 0 & b_2 & 0 & 0 \\ b_3 & b_4 & b_5 & b_6 & 0 \\ b_7 & 0 & b_8 & 0 & 0 \\ b_9 & b_{10} & b_{11} & b_{12} & 0 \\ 0 & 0 & 0 & 0 & b_{13} \end{bmatrix}$$

where,

$$b_1=\frac{k_4}{k_3k_4-\tau\omega_1},\ b_2=\frac{\omega_1}{k_3k_4-\tau\omega_1},\ b_3=\frac{\rho k_4k_5}{(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)}$$

$$b_4=\frac{k_5}{(k_1k_5-\tau\omega_2)},\quad b_5=\frac{\rho k_5\omega_1}{(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)},\quad b_6=\frac{\omega_2}{(k_1k_5-\tau\omega_2)}$$

$$b_7=\frac{\tau}{(k_3k_4-\tau\omega_1)},\quad b_8=\frac{k_3}{k_3k_4-\tau\omega_1},\quad b_9=\frac{\rho\tau k_4}{(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)}$$

$$b_{10}=\frac{\tau}{(k_1k_5-\tau\omega_2)},\quad b_{11}=\frac{\rho\omega_1\tau}{(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)},\ b_{12}=\frac{k_1}{(k_1k_5-\tau\omega_2)},\ b_{13}=\frac{1}{k_6}$$

(9) $$FV^{-1}=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\beta_h(1-\theta)b_{13}S_1^0}{N_v^0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\beta_v\theta_h b_1+\beta_v b_3) & \beta_v b_4 & (\beta_v\theta_h b_2+\beta_v b_5) & \beta_v b_6 & 0 \end{bmatrix}$$

From (9), we calculate the eigenvalues to determine the effective reproduction number R_c by taking the spectral radius (dominant eigenvalue) of the matrix FV^{-1}. Computing $|A-\lambda I|=0$, we have

$$R_c = \sqrt{\frac{\beta_h \beta_v b_{13}(1-\theta)(\theta_h b_1 + b_3)S_1^0}{N_v^0}} \tag{10}$$

R_c is the average number of secondary infectious cases that an infectious individual with lymphatic filariasis would produce in a totally susceptible population.

Substituting the values of $b_1, b_3, b_{13}, S_1^0 and\, N_v^0$ into (10) gives

$$R_c = \sqrt{\frac{\beta_h \beta_v k_2 k_4 (1-\theta)((k_1 k_5 - \tau\omega_2)\theta_h + \rho k_5)\mu_h}{k_6(k_3 k_4 - \tau\omega_1)(k_1 k_5 - \tau\omega_2)(k_1 k_2 - \tau\omega_1)}} \tag{11}$$

3.3 Global Stability of Disease-Free Equilibrium (DFE), E_0

Theorem 1: The DFE, E_0 of system (2) is globally asymptotically stable in R_+^8 if $R_C \le 1$

Proof:

Using LaSalle's invariance principle [27], consider the Lyapunov-LaSalle function.

$$L = \beta_v(1-\theta)k_2 k_4 \mu_h[(k_1 k_5 - \tau\omega_2)\theta_h + \rho k_5][I_1 + I_2 + T_1 + T_2] + (1-\theta)(k_1 k_2 - \tau\omega_1)(k_1 k_5 - \tau\omega_2)(k_3 k_4 - \tau\omega_1)V_2 \tag{12}$$

Differentiating (12) with respect to time gives

$$L' = \beta_v(1-\theta)k_2 k_4 \mu_h[(k_1 k_5 - \tau\omega_2)\theta_h + \rho k_5]\left[\frac{d\mathrm{I}_1}{dt} + \frac{d\mathrm{I}_2}{dt} + \frac{dT_1}{dt} + \frac{dT_2}{dt}\right] + (1-\theta)(k_1 k_2 - \tau\omega_1)(k_1 k_5 - \tau\omega_2)(k_3 k_4 - \tau\omega_1)V_2' \tag{13}$$

$$L' = \beta_v(1-\theta)k_2 k_4 \mu_h[(k_1 k_5 - \tau\omega_2)\theta_h + \rho k_5]\left[\frac{\beta_h(1-\theta)}{N_v}V_2 S_1 - \mu_h(\mathrm{I}_1 + \mathrm{I}_2) - (\mu_h + \gamma)(T_1 + T_2)\right] + (1-\theta)(k_1 k_2 - \tau\omega_1)(k_1 k_5 - \tau\omega_2)(k_3 k_4 - \tau\omega_1)\left[\frac{\beta_v(1-\theta)(\theta_h \mathrm{I}_1 + \mathrm{I}_2)}{N_h}V_1 - k_6 V_2\right] \tag{14}$$

$$\begin{aligned} L' = {} & \beta_v(1-\theta)k_2 k_4 \mu_h[(k_1 k_5 - \tau\omega_2)\theta_h + \rho k_5]\frac{\beta_h(1-\theta)V_2 S_1}{N_v} - \beta_v(1-\theta)k_2 k_4 \mu_h^2(k_1 k_5 - \tau\omega_2)\theta_h \mathrm{I}_1 - \\ & \beta_v(1-\theta)k_2 k_4 \mu_h^2(k_1 k_5 - \tau\omega_2)\theta_h \mathrm{I}_2 - \beta_v(1-\theta)k_2 k_4 \mu_h^2 \rho k_5(\mathrm{I}_1 + \mathrm{I}_2) - \\ & \beta_v(1-\theta)k_2 k_4 \mu_h[(k_1 k_5 - \tau\omega_2)\theta_h + \rho k_5](\mu_h + \gamma)(T_1 + T_2) + \\ & (k_1 k_2 - \tau\omega_1)(k_3 k_4 - \tau\omega_1)(k_1 k_5 - \tau\omega_2)\frac{\beta_v(1-\theta)\theta_h \mathrm{I}_1 V_1}{N_h} + \\ & (k_1 k_2 - \tau\omega_1)(k_3 k_4 - \tau\omega_1)(k_1 k_5 - \tau\omega_2)\frac{\beta_v(1-\theta)\mathrm{I}_2 V_1}{N_h} - \\ & (k_1 k_2 - \tau\omega_1)(k_1 k_5 - \tau\omega_2)(k_3 k_4 - \tau\omega_1)k_6 V_2 \end{aligned}$$

Since $S_1 = N_h - (S_2 + \mathrm{I}_1 + \mathrm{I}_2 + T_1 + T_2)$, we have,

$$\begin{aligned} L' = {} & \beta_h\beta_v(1-\theta)^2 k_2k_4\mu_h\left[(k_1k_5-\tau\omega_2)\theta_h+\rho k_5\right]V_2 - \\ & \beta_h\beta_v(1-\theta)^2 k_2k_4\mu_h\left[(k_1k_5-\tau\omega_2)\theta_h+\rho k_5\right](S_2+\mathrm{I}_1+\mathrm{I}_2+T_1+T_2)\frac{V_2}{N_v} - \\ & \beta_v(1-\theta)(k_1k_5-\tau\omega_2)(\theta_h\mathrm{I}_1+\mathrm{I}_2)\left[k_2k_4\mu_h^2-\frac{(k_1k_2-\tau\omega_1)(k_3k_4-\tau\omega_1)}{N_h}\right]- \\ & \beta_v(1-\theta)k_2k_4\mu_h^2\rho k_5(\mathrm{I}_1+\mathrm{I}_2)-\beta_v(1-\theta)k_2k_4\mu\left[(k_1k_5-\tau\omega_2)+\rho k_5\right](\mu_h+\gamma)(T_1+T_2)- \\ & (k_1k_2-\tau\omega_1)(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)k_6V_2 \end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned} L' \le {} & \beta_h\beta_v(1-\theta)^2 k_2k_4\mu_h\left[(k_1k_5-\tau\omega_2)\theta_h+\rho k_5\right]V_2 + \\ & (k_1k_2-\tau\omega_1)(k_1k_5-\tau\omega_2)(k_3k_4-\tau\omega_1)k_6V_2 \end{aligned} \tag{16}$$

That is,

$$L' \le k_6(k_1k_2-\tau\omega_1)(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)V_2\left[\frac{\beta_h\beta_vk_2k_4(1-\theta)\left[(k_1k_5-\tau\omega_2)\theta_h+\rho k_5\right]\mu_h}{(k_1k_2-\tau\omega_1)(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)k_6}-1\right] \tag{17}$$

$$L' \le (k_1k_2-\tau\omega_1)(k_3k_4-\tau\omega_1)(k_1k_5-\tau\omega_2)V_2\left[R_0^2-1\right] \tag{18}$$

Thus, $L' \le 0$ if an only if $R_C \le 1$ with $L' = 0$ if and only if $V_2 = \mathrm{I}_1 = \mathrm{I}_2 = T_1 = T_2 = 0$. It follows from the LaSalle invariance principle [27] that $V_2, \mathrm{I}_1, \mathrm{I}_2, T_1,$ and $T_2 \to 0$ as $t \to \infty$ that is, the disease dies out.

3.4 Endemic Equilibrium

This is the equilibrium state in the presence of infection, to obtain it, we set the LHS of (1) to zero. Thus,

$$S_1^* = \frac{\begin{array}{l}\beta_vk_4\Lambda_h(\Lambda_v-k_6V_2^{**})\left((k_1k_2-\tau\omega_1)\theta_hN_v^{**}+k_5\left(\beta_h(1-\theta)V_2^{**}+\rho N_v^{**}\right)\right)+ \\ k_6^2\tau\gamma\left(k_2(k_1k_5-\tau\omega_2)N_v^{**}+k_4\left(\beta_h(1-\theta)V_2^{**}+\rho N_v^{**}\right)\right)N_h^{**}N_v^{**}V_2^{**}\end{array}}{\beta_vk_4(\Lambda_h-k_6V_2^{**})\left((k_1k_2-\tau\omega_1)N_v^{**}+k_2\beta_hV_2^{**}\right)\left(\begin{array}{l}(k_1k_5-\tau\omega_2)\theta_hN_v^{**}+ \\ k_5(\beta_h(1-\theta)V_2^{*}+\rho N_v^{**}\end{array}\right)}$$

$$S_2^* = \frac{\begin{array}{l}\beta_vk_4\Lambda_h\tau(\Lambda_v-k_6V_2^{**})\left((k_1k_2-\tau\omega_1)\theta_hN_v^{**}+k_5\left(\beta_h(1-\theta)V_2^{**}+\rho N_v^{**}\right)\right)N_v^{**}+ \\ k_6^2\tau^2\gamma\left(k_2(k_1k_5-\tau\omega_2)N_v^{**}+k_4\left(\beta_h(1-\theta)V_2^{**}+\rho N_v^{**}\right)\right)N_h^{**}N_v^{**}V_2^{**}\end{array}}{\beta_vk_4(\Lambda_h-k_6V_2^{**})\left((k_1k_2-\tau\omega_1)N_v^{**}+k_2\beta_hV_2^{**}\right)\left(\begin{array}{l}(k_1k_5-\tau\omega_2)\theta_hN_v^{**}+ \\ k_5(\beta_h(1-\theta)V_2^{*}+\rho N_v^{**}\end{array}\right)}$$

$$I_1^{**} = \frac{k_6^2(k_1k_5 - \tau\omega_2)N_h^{**}N_v^{**}V_2^{**}}{\beta_v(\Lambda_v - k_6V_2^{**})((k_1k_5 - \tau\omega_2)N_v^{**} + k_5(\beta_h(1-\theta)V_2^{**} + \rho N_v^{**}))}$$

$$I_2^{**} = \frac{k_5k_6^2(\beta_h(1-\theta)V_2^{**} + \rho N_v^{**})N_v^{**}V_2^{**}}{\beta_v(\Lambda_v - k_6V_2^{**})((k_1k_5 - \tau\omega_2)N_v^{**} + k_5(\beta_h(1-\theta)V_2^{**} + \rho N_v^{**}))}$$

$$T_1^{**} = \frac{k_6^2\tau(k_1k_5 - \tau\omega_2)N_v^{2**}V_2^{**}}{k_4\beta_v(\Lambda_h - k_6V_2^{**})((k_1k_5 - \tau\omega_2)\theta_hN_v^{**} + k_5(\beta_h(1-\theta)V_2^{**} + \rho N_v^{**}))}$$

$$T_2^{**} = \frac{k_6^2\tau(\beta_h(1-\theta)V_2^{**} + \rho N_v^{**})N_v^{**}V_2^{**}}{\beta_v(\Lambda_h - k_6V_2^{**})((k_1k_5 - \tau\omega_2)\theta_hN_v^{**} + k_5(\beta_h(1-\theta)V_2^{**} + \rho N_v^{**}))}$$

$$V_1^{**} = \frac{\Lambda_v(\Lambda_v - k_6V_2^{**})}{k_6 + (\Lambda_v - k_6V_2^{**})}$$

$$V_2^{**} = \frac{\Lambda_v(\Lambda_v - k_6V_1^{**})}{k_6 + (\Lambda_v - k_6V_1^{**})}$$

3.5. Global Stability of Endemic Equilibrium State

Theorem 2. If $R > 1$, then the endemic equilibrium E^1 of the equation (2) is globally asymptotically stable if

$S_1 = S_1^*, S_2 = S_2^*, \mathrm{I}_1 = \mathrm{I}_1^*, \mathrm{I}_2 = \mathrm{I}_2^*, T_1 = T_1^*, T_2 = T_2^*, V_1 = V_1^*, V_2 = V_2^*$ and $X < Y$ unstable $R_c \le 1$.

Proof:-To establish the global stability of EE (E^1) we use the constructed Lyapunov function by [28-29], the global stability of the EE is proved.

By defining the Lyapunov function as follows:

$$\begin{aligned} L(S_1^*, S_2^*, \mathrm{I}_1^*, \mathrm{I}_2^*, T_1^*, T_2^*, V_1^*, V_2^*) = &\left(S_1 - S_1^* - S_1^*\log\frac{S_1^*}{S_1}\right) + \left(S_2 - S_2^* - S_2^*\log\frac{S_2^*}{S_2}\right) + \\ &\left(\mathrm{I}_1 - \mathrm{I}_1^* - \mathrm{I}_1^*\log\frac{\mathrm{I}_1^*}{\mathrm{I}_1}\right) + \left(\mathrm{I}_2 - \mathrm{I}_2^* - \mathrm{I}_2^*\log\frac{\mathrm{I}_2^*}{\mathrm{I}_2}\right) + \\ &\left(T_1 - T_1^* - T_1^*\log\frac{T_1^*}{T_1}\right) + \left(T_2 - T_2^* - T_2^*\log\frac{T_2^*}{T_2}\right) \\ &\left(V_1 - V_1^* - V_1^*\log\frac{V_1^*}{V_1}\right) + \left(V_2 - V_2^* - V_2^*\log\frac{V_2^*}{V_2}\right) \end{aligned} \tag{19}$$

By direct calculating the derivative of L along the solution of equation (2)

We have;

$$\frac{dL}{dt}=\left(\frac{S_1-S_1^*}{S_1}\right)\frac{dS_1}{dt}+\left(\frac{S_2-S_2^*}{S_2}\right)\frac{dS_2}{dt}+\left(\frac{\mathrm{I}_1-\mathrm{I}_1^*}{\mathrm{I}_1}\right)\frac{d\mathrm{I}_1}{dt}+\left(\frac{\mathrm{I}_2-\mathrm{I}_2^*}{\mathrm{I}_2}\right)\frac{d\mathrm{I}_2}{dt}+ \left(\frac{T_1-T_1^*}{T_1}\right)\frac{dT_1}{dt}+\left(\frac{T_2-T_2^*}{T_2}\right)\frac{dT_2}{dt}+\left(\frac{V_1-V_1^*}{V_1}\right)\frac{dV_1}{dt}+\left(\frac{V_2-V_2^*}{V_2}\right)\frac{dV_2}{dt} \tag{20}$$

$$\frac{dL}{dt}=\left(\frac{S_1-S_1^*}{S_1}\right)\left(\Lambda_h-\frac{\beta_h(1-\theta)}{N_v}V_2S_1-k_1S_1+T_1\gamma+T_2\gamma+\omega_1S_2\right)+\left(\frac{S_2-S_2^*}{S_2}\right)\left(\tau S_1-k_2S_2\right)+ \left(\frac{\mathrm{I}_1-\mathrm{I}_1^*}{\mathrm{I}_1}\right)\left(\frac{\beta_h(1-\theta)}{N_v}V_2S_1-\frac{\beta_h(1-\theta)}{N_v}V_2\mathrm{I}_1-k_3\mathrm{I}_1+\omega_1T_1\right)+\left(\frac{\mathrm{I}_2-\mathrm{I}_2^*}{\mathrm{I}_2}\right)\left(\begin{matrix}\frac{\beta_h(1-\theta)}{N_v}V_2\mathrm{I}_1+\rho\mathrm{I}_1-\\ k_1\mathrm{I}_2+\omega_2T_2\end{matrix}\right)+ \left(\frac{T_1-T_1^*}{T_1}\right)\left(\tau\mathrm{I}_1-k_4T_1\right)+\left(\frac{T_2-T_2^*}{T_2}\right)\left(\tau\mathrm{I}_2-k_5T_2\right)+\left(\frac{V_1-V_1^*}{V_1}\right)\left(\Lambda_v-\frac{\beta_v(\theta_h\mathrm{I}_1+\mathrm{I}_2)}{N_h}V_1-k_6V_1\right)+ \left(\frac{V_2-V_2^*}{V_2}\right)\left(\frac{\beta_v(\theta_h\mathrm{I}_1+\mathrm{I}_2)}{N_h}V_1-k_6V_2\right) \tag{21}$$

Implies that

$$\frac{dL}{dt}=\left(\frac{S_1-S_1^*}{S_1}\right)\left(\begin{matrix}\Lambda_h-\frac{\beta_h(1-\theta)}{N_v}(V_2-V_2^*)(S_1-S_2^*)-k_1(S_1-S_1^*)+(T_1-T_1^*)\gamma+(T_2-T_2^*)\gamma+\\ \omega_1(S_2-S_2^*)\end{matrix}\right)+ \left(\frac{S_2-S_2^*}{S_2}\right)\left(\tau(S_1-S_1^*)-k_2(S_2-S_2^*)\right)+ \left(\frac{\mathrm{I}_1-\mathrm{I}_1^*}{\mathrm{I}_1}\right)\left(\frac{\beta_h(1-\theta)}{N_v}(V_2-V_2^*)(S_1-S_1^*)-\frac{\beta_h(1-\theta)}{N_v}(V_2-V_2^*)(\mathrm{I}_1-\mathrm{I}_1^*)-k_3(\mathrm{I}_1-\mathrm{I}_1^*)+\omega_1(T_1-T_1^*)\right)+ \left(\frac{\mathrm{I}_2-\mathrm{I}_2^*}{\mathrm{I}_2}\right)\left(\frac{\beta_h(1-\theta)}{N_v}(V_2-V_2^*)(\mathrm{I}_1-\mathrm{I}_1^*)+\rho(\mathrm{I}_1-\mathrm{I}_1^*)-k_1(\mathrm{I}_2-\mathrm{I}_2^*)+\omega_2(T_2-T_2^*)\right)+ \left(\frac{T_1-T_1^*}{T_1}\right)\left(\tau(\mathrm{I}_1-\mathrm{I}_1^*)-k_4(T_1-T_1^*)\right)+\left(\frac{T_2-T_2^*}{T_2}\right)\left(\tau(\mathrm{I}_2-\mathrm{I}_2^*)-k_5(T_2-T_2^*)\right)+ \left(\frac{V_1-V_1^*}{V_1}\right)\left(\Lambda_v-\frac{\beta_v(\theta_h(\mathrm{I}_1-\mathrm{I}_1^*)+(\mathrm{I}_2-\mathrm{I}_2^*))}{N_h}(V_1-V_1^*)-k_6(V_1-V_1^*)\right)+ \left(\frac{V_2-V_2^*}{V_2}\right)\left(\frac{\beta_v(\theta_h(\mathrm{I}_1-\mathrm{I}_1^*)+(\mathrm{I}_2-\mathrm{I}_2^*)}{N_h}(V_1-V_1^*)-k_6(V_2-V_2^*)\right)$$

$\Rightarrow$

$$\frac{dL}{dt} = \Lambda_h - \frac{\Lambda_h S_1^*}{S_1} - \left(\frac{(S_1 - S_1^*)^2}{S_1}\right)\left(\frac{\beta_h}{N_v} V_2 + k_1 + \frac{\beta_h \theta}{N_v} V_2^*\right) + \left(\frac{(S_1 - S_1^*)^2}{S_1}\right)\left(\frac{\beta_h}{N_v} V_2^* + \frac{\beta_h \theta}{N_v} V_2\right) +$$
$$\left(\gamma T_1 + \frac{\gamma S_1^* T_1^*}{S_1} + \gamma T_2 + \frac{\gamma S_1^* T_2^*}{S_1} + \omega_1 S_2 + \frac{\omega_1 S_1^* S_2^*}{S_1} + \tau S_1 + \frac{\tau S_1^* S_2^*}{S_2}\right) -$$
$$\left(\gamma T_1^* + \frac{\gamma S_1^* T_1}{S_1} + \frac{\gamma S_1^* T_2}{S_1} + \gamma T_2^* + \omega_1 S_2^* + \tau S_1^* + \frac{\omega_1 S_2 S_2^*}{S_2} + \frac{\tau S_1 S_2^*}{S_2}\right) - k_2 \frac{(S_2 - S_2^*)^2}{S_2} +$$
$$\left(\frac{\beta_h}{N_v} S_1 V_2 + \frac{\beta_h S_1 V_2^* \mathrm{I}_1^*}{N_v \mathrm{I}_1} + \frac{\beta_h S_1^* V_2^*}{N_v} + \frac{\beta_h S_1^* V_2 \mathrm{I}_1^*}{N_v \mathrm{I}_1} + \frac{\beta_h \theta}{N_v} S_1 V_2^* + \frac{\beta_h \theta S_1 V_2 \mathrm{I}_1^*}{N_v \mathrm{I}_1}\right) -$$
$$\left(\frac{\beta_h S_1 V_2^*}{N_v} + \frac{\beta_h S_1^* V_2}{N_v} + \frac{\beta_h S_1^* V_2^* \mathrm{I}_1^*}{N_v \mathrm{I}_1} + \frac{\beta_h \theta S_1 V_2}{N_v} + \frac{\beta_h \theta S_1 V_2^* \mathrm{I}_1^*}{N_v \mathrm{I}_1}\right) + \left(\frac{\beta_h \theta S_1^* V_2}{N_v} + \frac{\beta_h \theta S_1^* V_2^* \mathrm{I}_1^*}{N_v \mathrm{I}_1}\right) -$$
$$\left(\frac{\beta_h S_1^* V_2^*}{N_v} + \frac{\beta_h \theta S_1^* V_2 \mathrm{I}_1^*}{N_v \mathrm{I}_1}\right) + \frac{(\mathrm{I}_1 - \mathrm{I}_1^*)^2}{\mathrm{I}_1}\left(\frac{\beta_h \theta V_2}{N_v} + \frac{\beta_h V_2^*}{N_v}\right) - \frac{(\mathrm{I}_1 - \mathrm{I}_1^*)^2}{\mathrm{I}_1}\left(\frac{\beta_h V_2}{N_v} + \frac{\beta_h \theta V_2^*}{N_v} + k_3\right) +$$
$$\left(\begin{array}{l} \omega_1 T_1 + \frac{\omega_1 T_1 \mathrm{I}_1^*}{\mathrm{I}_1} + \frac{\beta_h \mathrm{I}_2 V_2}{N_v} + \frac{\beta_h \mathrm{I}_1^* \mathrm{I}_2^* V_2^*}{N_v \mathrm{I}_1} + \frac{\beta_h \mathrm{I}_1^* V_2}{N_v} + \frac{\beta_h \mathrm{I}_2^* V_2^*}{N_v} + \frac{\beta_h \mathrm{I}_1^* \mathrm{I}_2^* V_2}{N_v \mathrm{I}_1} + \frac{\beta_h \theta \mathrm{I}_2 V_2^*}{N_v} + \\ \frac{\beta_h \theta \mathrm{I}_1^* \mathrm{I}_2^* V_2}{N_v \mathrm{I}_1} + \frac{\beta_h \theta \mathrm{I}_2 V_2}{N_v} + \frac{\beta_h \theta \mathrm{I}_1^* \mathrm{I}_2^* V_2^*}{N_v \mathrm{I}_1} + \rho \mathrm{I}_1 + \frac{\rho \mathrm{I}_1^* \mathrm{I}_2^*}{\mathrm{I}_2} \end{array}\right) -$$
$$\left(\begin{array}{l} \omega_1 T_1^* + \frac{\omega_1 T_1 \mathrm{I}_1^*}{\mathrm{I}_1} + \frac{\beta_h \mathrm{I}_2 V_2^*}{N_v} + \frac{\beta_h \mathrm{I}_2 V_2 \mathrm{I}_1^*}{\mathrm{I}_1} + \frac{\beta_h V_2^* \mathrm{I}_1^* \mathrm{I}_2^*}{N_v \mathrm{I}_1} + \frac{\beta_h \theta \mathrm{I}_2 V_2}{N_v} + \frac{\beta_h \theta \mathrm{I}_1^* \mathrm{I}_2^* V_2^*}{N_v \mathrm{I}_1} + \frac{\beta_h \theta \mathrm{I}_1^* V_2^*}{N_v} + \\ \frac{\beta_h \theta \mathrm{I}_1^* \mathrm{I}_2^* V_2}{N_v \mathrm{I}_1} + \frac{\rho \mathrm{I}_1 \mathrm{I}_2^*}{\mathrm{I}_2} + \rho \mathrm{I}_1^* \end{array}\right) -$$
$$k_1 \frac{(\mathrm{I}_2 - \mathrm{I}_2^*)^2}{\mathrm{I}_2} + \left(\omega_2 T_2 + \frac{\omega_2 T_2^* \mathrm{I}_2^*}{\mathrm{I}_2} + \tau \mathrm{I}_1 + \frac{\tau \mathrm{I}_1^* T_1^*}{T_1} + \tau \mathrm{I}_2^* + \frac{\tau \mathrm{I}_2^* T_2^*}{T_2} + \Lambda_v\right) -$$
$$\left(\omega_2 T_2^* + \frac{\omega_2 T_2 \mathrm{I}_2^*}{\mathrm{I}_2} + \frac{\tau \mathrm{I}_1 T_1}{T_1} + \tau \mathrm{I}_1^* + \tau \mathrm{I}_2^* + \frac{\tau \mathrm{I}_2 T_2^*}{T_2}\right) - k_4 \frac{(T_1 - T_2^*)^2}{T_1} - k_5 \frac{(T_2 - T_2^*)^2}{T_2} +$$
$$\frac{(V_1 - V_1^*)^2}{V_1}\left(\frac{\beta_v \theta_h \mathrm{I}_1^*}{N_h} + \frac{\beta_v \mathrm{I}_2^*}{N_h}\right) - \frac{(V_1 - V_1^*)^2}{V_1}\left(\frac{\beta_v \theta_h \mathrm{I}_1}{N_h} + \frac{\beta_v \mathrm{I}_2}{N_h} + k_6\right) -$$
$$\left(\begin{array}{l} \Lambda_v \frac{V_1^*}{V_1} + \frac{\beta_v \theta_h \mathrm{I}_1^* V_1}{N_h} + \frac{\beta_v \theta_h \mathrm{I}_1 V_1 V_2^*}{N_h V_2} + \frac{\beta_v \theta_h \mathrm{I}_1 V_1^*}{N_h} + \frac{\beta_v \theta_h \mathrm{I}_1^* V_1^* V_2^*}{N_h V_2} + \frac{\beta_v \mathrm{I}_2^* V_1}{N_h} + \frac{\beta_v \mathrm{I}_2 V_1 V_2^*}{N_h V_2} \\ \frac{\beta_v \mathrm{I}_2 V_1^*}{N_h} + \frac{\beta_v \mathrm{I}_2^* V_1^* V_2^*}{N_h V_2} \end{array}\right) + \tag{22}$$

$$\left(\begin{array}{l}\frac{\beta_v\theta_h I_1 V_1}{N_h}+\frac{\beta_v\theta_h I_1^* V_1 V_2^*}{N_h V_2}+\frac{\beta_v\theta_h I_1^* V_1^*}{N_h}+\frac{\beta_v\theta_h I_1 V_2^*}{N_h V_2}+\frac{\beta_v I_2 V_1}{N_h}+\\ \frac{\beta_v I_2^* V_1 V_2^*}{N_h V_2}+\frac{\beta_v I_2^* V_1^*}{N_h}+\frac{\beta_v I_2 V_1^* V_2^*}{N_h V_2}\end{array}\right)-k_6\frac{(V_2-V_2^*)^2}{V_2}$$

Rearranging the positive and negative terms in (22) leads to

$$\frac{dL}{dt}=X-Y \tag{23}$$

where,

$$X=\left(\frac{(S_1-S_1^*)^2}{S_1}\right)\left(\frac{\beta_h}{N_v}V_2^*+\frac{\beta_h\theta}{N_v}V_2\right)+\left(\begin{array}{l}\Lambda_h+\gamma T_1+\frac{\gamma S_1^* T_1^*}{S_1}+\gamma T_2+\frac{\gamma S_1^* T_2^*}{S_1}+\omega_1 S_2+\\ \frac{\omega_1 S_1^* S_2^*}{S_1}+\tau S_1+\frac{\tau S_1^* S_2^*}{S_2}\end{array}\right)+$$
$$\left(\frac{\beta_h}{N_v}S_1V_2+\frac{\beta_h S_1 V_2^* I_1^*}{N_v I_1}+\frac{\beta_h S_1^* V_2^*}{N_v}+\frac{\beta_h S_1^* V_2 I_1^*}{N_v I_1}+\frac{\beta_h\theta}{N_v}S_1V_2^*+\frac{\beta_h\theta S_1 V_2 I_1^*}{N_v I_1}\right)+$$
$$\left(\frac{\beta_h\theta S_1^* V_2}{N_v}+\frac{\beta_h\theta S_1^* V_2^* I_1^*}{N_v I_1}\right)+\frac{(I_1-I_1^*)^2}{I_1}\left(\frac{\beta_h\theta V_2}{N_v}+\frac{\beta_h V_2^*}{N_v}\right)+$$
$$\left(\begin{array}{l}\omega_1 T_1+\frac{\omega_1 T_1 I_1^*}{I_1}+\frac{\beta_h I_2 V_2}{N_v}+\frac{\beta_h I_1^* I_2^* V_2^*}{N_v I_1}+\frac{\beta_h I_1^* V_2}{N_v}+\frac{\beta_h I_2^* V_2^*}{N_v}+\frac{\beta_h I_1^* I_2^* V_2}{N_v I_1}+\frac{\beta_h\theta I_2 V_2^*}{N_v}+\\ \frac{\beta_h\theta I_1^* I_2^* V_2}{N_v I_1}+\frac{\beta_h\theta I_2 V_2}{N_v}+\frac{\beta_h\theta I_1^* I_2^* V_2^*}{N_v I_1}+\rho I_1+\frac{\rho I_1^* I_2^*}{I_2}\end{array}\right)+$$
$$\left(\omega_2 T_2+\frac{\omega_2 T_2^* I_2^*}{I_2}+\tau I_1+\frac{\tau I_1^* T_1^*}{T_1}+\tau I_2^*+\frac{\tau I_2^* T_2^*}{T_2}+\Lambda_v\right)+\frac{(V_1-V_1^*)^2}{V_1}\left(\frac{\beta_v\theta_h I_1^*}{N_h}+\frac{\beta_v I_2^*}{N_h}\right)+$$
$$\left(\begin{array}{l}\frac{\beta_v\theta_h I_1 V_1}{N_h}+\frac{\beta_v\theta_h I_1^* V_1 V_2^*}{N_h V_2}+\frac{\beta_v\theta_h I_1^* V_1^*}{N_h}+\frac{\beta_v\theta_h I_1 V_2^*}{N_h V_2}+\frac{\beta_v I_2 V_1}{N_h}+\\ \frac{\beta_v I_2^* V_1 V_2^*}{N_h V_2}+\frac{\beta_v I_2^* V_1^*}{N_h}+\frac{\beta_v I_2 V_1^* V_2^*}{N_h V_2}\end{array}\right) \tag{24}$$

$$Y = \frac{\Lambda_h S_1^*}{S_1} + \left(\frac{(S_1 - S_1^*)^2}{S_1}\right)\left(\frac{\beta_h}{N_v}V_2 + k_1 + \frac{\beta_h\theta}{N_v}V_2^*\right) +$$
$$\left(\gamma T_1^* + \frac{\gamma S_1^* T_1}{S_1} + \frac{\gamma S_1^* T_2}{S_1} + \gamma T_2^* + \omega_1 S_2^* + \tau S_1^* + \frac{\omega_1 S_2 S_2^*}{S_2} + \frac{\tau S_1 S_2^*}{S_2}\right) + k_2\frac{(S_2 - S_2^*)^2}{S_2} +$$
$$\left(\frac{\beta_h S_1 V_2^*}{N_h} + \frac{\beta_h S_1^* V_2}{N_v} + \frac{\beta_h S_1^* V_2^* \mathrm{I}_1^*}{N_v \mathrm{I}_1} + \frac{\beta_h\theta S_1 V_2}{N_v} + \frac{\beta_h\theta S_1 V_2^* \mathrm{I}_1^*}{N_v \mathrm{I}_1}\right) +$$
$$\left(\frac{\beta_h S_1^* V_2^*}{N_v} + \frac{\beta_h\theta S_1^* V_2 \mathrm{I}_1^*}{N_v \mathrm{I}_1}\right) + \frac{(\mathrm{I}_1 - \mathrm{I}_1^*)^2}{\mathrm{I}_1}\left(\frac{\beta_h V_2}{N_v} + \frac{\beta_h\theta V_2^*}{N_v} + k_3\right) +$$
$$\left(\begin{array}{l}\omega_1 T_1^* + \frac{\omega_1 T_1 \mathrm{I}_1^*}{\mathrm{I}_1} + \frac{\beta_h \mathrm{I}_2 V_2^*}{N_v} + \frac{\beta_h \mathrm{I}_2 V_2 \mathrm{I}_1^*}{\mathrm{I}_1} + \frac{\beta_h V_2^* \mathrm{I}_1^* \mathrm{I}_2^*}{N_v \mathrm{I}_1} + \frac{\beta_h\theta \mathrm{I}_2 V_2}{N_v} + \frac{\beta_h\theta \mathrm{I}_1^* \mathrm{I}_2^* V_2^*}{N_v \mathrm{I}_1} + \frac{\beta_h\theta \mathrm{I}_1^* V_2^*}{N_v} + \\ \frac{\beta_h\theta \mathrm{I}_1^* \mathrm{I}_2^* V_2}{N_v \mathrm{I}_1} + \frac{\rho \mathrm{I}_1 \mathrm{I}_2^*}{\mathrm{I}_2} + \rho \mathrm{I}_1^*\end{array}\right) +$$
$$k_1\frac{(\mathrm{I}_2 - \mathrm{I}_2^*)^2}{\mathrm{I}_2} + \left(\omega_2 T_2 + \frac{\omega_2 T_2^* \mathrm{I}_2^*}{\mathrm{I}_2} + \tau \mathrm{I}_1 + \frac{\tau \mathrm{I}_1^* T_1^*}{T_1} + \tau \mathrm{I}_2^* + \frac{\tau \mathrm{I}_2^* T_2^*}{T_2} + \Lambda_v\right) +$$
$$\left(\omega_2 T_2^* + \frac{\omega_2 T_2 \mathrm{I}_2^*}{\mathrm{I}_2} + \frac{\tau \mathrm{I}_1 T_1}{T_1} + \tau \mathrm{I}_1^* + \tau \mathrm{I}_2^* + \frac{\tau \mathrm{I}_2 T_2^*}{T_2}\right) + k_4\frac{(T_1 - T_2^*)^2}{T_1} + k_5\frac{(T_2 - T_2^*)^2}{T_2} +$$
$$\frac{(V_1 - V_1^*)^2}{V_1}\left(\frac{\beta_v\theta_h I_1}{N_h} + \frac{\beta_v \mathrm{I}_2}{N_h} + k_6\right) +$$
$$\left(\begin{array}{l}\Lambda_v\frac{V_1^*}{V_1} + \frac{\beta_v\theta_h \mathrm{I}_1^* V_1}{N_h} + \frac{\beta_v\theta_h \mathrm{I}_1 V_1 V_2^*}{N_h V_2} + \frac{\beta_v\theta_h \mathrm{I}_1 V_1^*}{N_h} + \frac{\beta_v\theta_h \mathrm{I}_1^* V_1^* V_2^*}{N_h V_2} + \frac{\beta_v \mathrm{I}_2^* V_1}{N_h} + \frac{\beta_v \mathrm{I}_2 V_1 V_2^*}{N_h V_2} + \\ \frac{\beta_v \mathrm{I}_2 V_1^*}{N_h} + \frac{\beta_v \mathrm{I}_2^* V_1^* V_2^*}{N_h V_2}\end{array}\right) +$$
$$k_6\frac{(V_2 - V_2^*)^2}{V_2}$$

Hence, if $X < Y$, then we obtain $\frac{dL}{dt} \le 0$. Nothing that $\frac{dL}{dt} = 0$ if and only if $S_1 = S_1^*, S_2 = S_2^*, \mathrm{I}_1 = \mathrm{I}_1^*, \mathrm{I}_2 = \mathrm{I}_2^*, T_1 = T_1^*, T_2 = T_2^*, V_1 = V_1^*, V_2 = V_2^*$. Therefore the largest compact invariant set $\left\{\left(S_1^*, S_2^*, \mathrm{I}_1^*, \mathrm{I}_2^*, T_1^*, T_2^*, V_1^*, V_2^*\right) \in \Gamma : \frac{dL}{dt} = 0\right\}$ is the singleton $\left\{E^1\right\}$ where E^1 is the Endemic equilibrium. Hence, by the LaSalle's invariant principle [27], it implies that E^1 is globally asymptotically stable in Γ if $X < Y$.

Table 1: Values for parameters used for the model

Parameter	Value	Source
μ_h	0.0189 yr^{-1}	Estimated
μ_v	0.00013 yr^{-1}	Estimated
Λ_h	3,348,245 yr^{-1}	Estimated
Λ_v	384,800 yr^{-1}	Estimated
N_h	177,155,754 yr^{-1}	Estimated
N_v	2,960,000,000 yr^{-1}	Assumed
β_h	0.009926 yr^{-1}	[30]
β_v	0.000249 yr^{-1}	[30]
ρ	0.00002797 yr^{-1}	[30]
δ	0.0017 yr^{-1}	Assumed
θ_h	0.25 yr^{-1}	Assumed
θ	(0-1)	Varying parameter
γ	0.1667 yr^{-1}	Assumed
ω_1	0.01 yr^{-1}	Assumed
ω_2	0.0001 yr^{-1}	Assumed
τ	0.125 yr^{-1}	Estimated

3.6. Sensitivity Analysis

For infectious disease models in particular, the sensitivity analysis of the effective reproduction number R_c with respect to the model parameters has been performed to determine the importance of the epidemic model parameters [31]. The sensitivity of the magnitude of R_c will be computed with respect to the model parameter values to determine the relative influence of each parameter

on the transmission and control of the disease. Sensitivity analysis determines which parameters and initial conditions (inputs) affect the quantities of interest (outputs) of the models most [32-33]. The normalized sensitivity index of the effective reproduction number R_c with respect to a parameter value X is given by

$$S_X^{R_c} = \frac{\partial R_c}{X} \times \frac{X}{R_c}$$

where

$$X = \{\beta_h, \beta_v, , \delta, \rho, \theta, \gamma, \tau, \omega_1, \omega_2\}$$

Table 2: Sensitivity indices of R_C to the model parameters. The parameters are ordered from the most sensitive to the least.

Parameter	Value	Sensitivity index
τ	0.125	-0.837635142079760
β_v	0.009926	0.7650680900
β_h	0.000249	0.650547300
θ	0.5	-0.50000
δ	0.0017	-0.4644808743
ω_1	0.01	0.1625662160
γ	0.1667	-0.0197997
ρ	0.00002797	0.0002869403718
ω_2	0.0001	0.0000001817748297

Sensitivity index of the effective reproductive number, R_C with respect to each parameter, is computed as shown in table 2 for the model equation (2). The parameters have both positive and negative effects on the effective reproduction number and are ordered from the most sensitive to the least as seen in table 2. The most sensitive parameter is the treatment rate for the population undergoing treatment (τ), the next important parameter is the rate at which the mosquitoes ingests microfilariae when bitting a human (β_h). The parameters with the least SA are the rate of progression of human from $\mathrm{I}_1(t)$ to $\mathrm{I}_2(t)$ (ρ) and the rate at which individuals in $T_2(t)$ stop

taking drugs (ω_2).

4. NUMERICAL SIMULATION

Graphical simulation was performed to see the effects of the controls. The following initial conditions are considered. $S_1(0)$ =114,315,754, $S_2(0)$ =30,000,000, $I_1(0)$ =12,000,000, $I_2(0)$ =6,840,000, $T_1(0)$ =8,000,000, $T_2(0)$ =6,000,000, $V_1(0)$ = 2,000,000,000, $V_2(0)$ =96,000,000

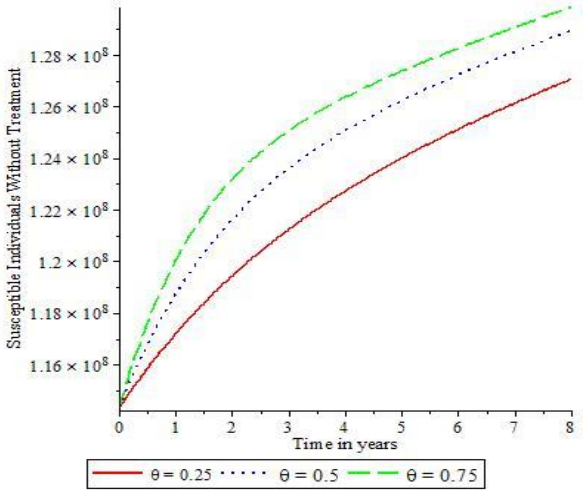

Figure 2: The effect of effective mosquito net on the susceptible individuals not undergoing treatment $S_1(t)$

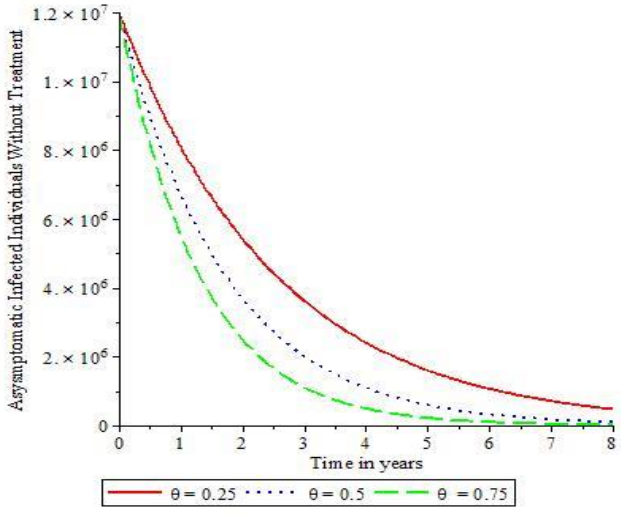

Figure 3: The effects of mosquito net on the asymptomatic infected individuals not undergoing treatment $I_1(t)$

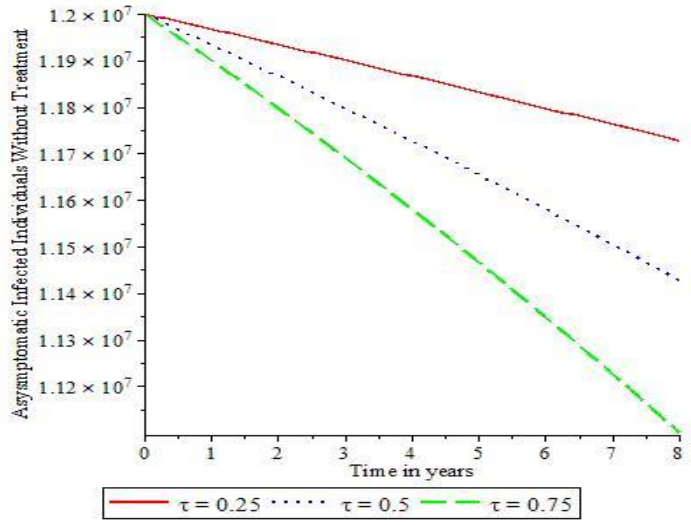

Figure 4: The effects of treatment on the asymptomatic infected individuals $I_1(t)$

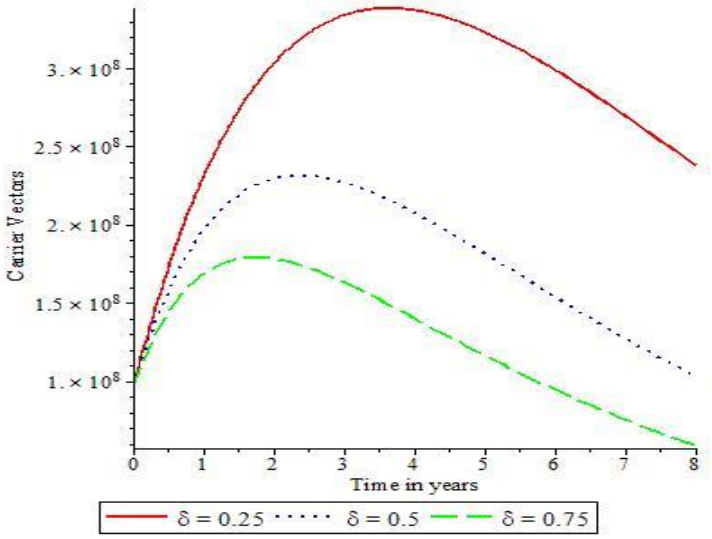

Figure 5: The effects of Insecticide on the Carrier vector δ

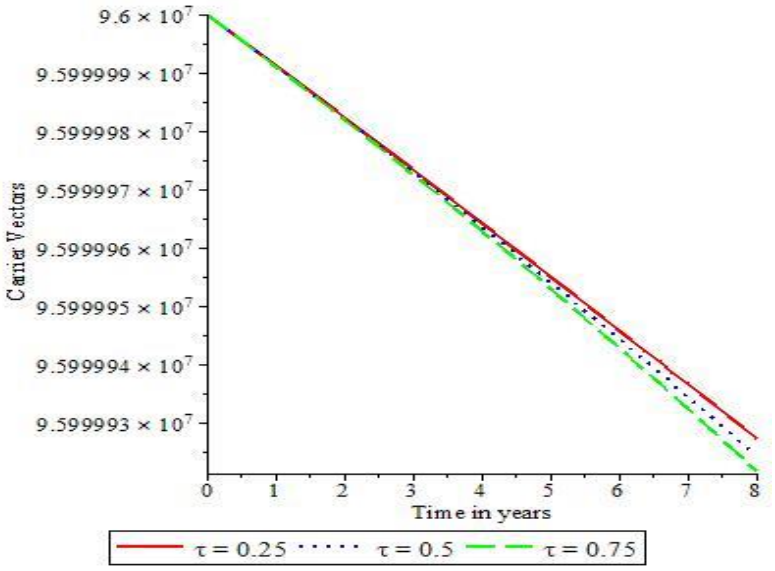

Figure 6: The effects of treatment on infected individuals with and without symptoms

5. DISCUSSION OF RESULTS

Figure 2 shows the comparison between the effects of using effective mosquito net at any coverage rate on the susceptible individuals without treatment $S_1(t)$. This shows that as the rate of using effective mosquito net increases, the susceptible individuals without treatment increases with time.

Figure 3: A comparison between the effects of using effective mosquito net at any coverage rate on the asymptomatic infected individuals without treatment $I_1(t)$. This shows that as the rate of using effective mosquito net increases, the susceptible individuals without treatment increases with time.

Figure 4: A comparison between the effects of treatment at any coverage rate on asymptomatic infected individuals $I_1(t)$. This show that the effect of treatment on asymptomatic infected individuals at any coverage rate will lead to control of lymphatic filariasis but treatment at a high percentage will better control the disease.

Figure 5: The effect of the different death rate of Mosquitoes δ due to Insecticide on Carrier Vector δ. It is observed that as the death rate of Mosquitoes due to Insecticide increases, the carrier vector decreases with time, this shows that the carrier vectors are close to the environment where the insecticide is applied.

Figure 6: A comparison between the effects of treatment τ on the Carrier vector $V(t)$ population.

It reveals that the treatment of infected individuals with symptom $I_2(t)$ and infected individuals without symptoms $I_1(t)$ at any coverage rate decreases the carrier vector population.

6. CONCLUSION

The study presents a mathematical model for the transmission dynamics of lymphatic filariasis incorporating treatment and other control measure parameters. It was shown that the model is mathematically well-posed in a feasible region and realistic from an epidemiological point of view. An explicit threshold value for the effective reproduction number (R_c) was obtained. The threshold value was used to obtain conditions for local and global stabilities of the DFE and EE states. Sensitivity analysis was also conducted on the effective reproduction number and the most sensitive parameters were identified. The simulation results shown that LF can be put under control in the population if treatment level coverage should target a success rate of 75% because the treatment rate is the most sensitive parameter from our analysis.

Some Useful results

Variables and Parameter Estimation

We estimate the parameter value on the basis of available information from the World Health Organization (WHO), the Central Intelligence Agency (CIA), the Center for Disease Control (CDC), and prevention and accurate explanatory literature. It is difficult to obtain reliable data. In the following subsections, the calculations are clearly clarified.

Total human population of Nigeria, N_h

The total population of Nigeria is 177,155,7544, according to WHO (2015).

$N_h = 177{,}155{,}754$

Total number of infected human but not showing signs of elephantiasis symptoms in Nigeria, I_1

Infected people, who have contacted elephantiasis but are not infective, but are asymptomatic are not evidence of elephantiasis. About 6.77 percent of the population of Nigerians is actually infected but asymptomatic (WHO, 2015), so

$$I_1 = \frac{6.77}{100} \times 177{,}155{,}754 = 12{,}000{,}000$$

Total number of infected with symptoms, (I_2)

According to WHO, (WHO,2015). There are currently 6,840,000 cases of elephantiasis with symptoms in Nigeria thus; $I_2 = 6{,}840{,}000$

Total number of susceptible individuals not taking drug in Nigeria, (S_1)

Recall $N_h = S_1 + S_2 + I_1 + I_2 + T_1 + T_2$

$$S_1 = N_h - (S_2 + I_1 + I_2 + T_1 + T_2)$$

$$S_1 = 177{,}155{,}754 - (30{,}000{,}000 + 12{,}000{,}000 + 6{,}840{,}000 + 8{,}000{,}000 + 6{,}000{,}0000)$$

$$S_1 = 114{,}315{,}754 \ \ yr^{-1}$$

Natural death rate of humans in Nigeria, (μ_h)

The death rate is defined as the inverse of life expectancy at birth (Moualen*et al*., 2013).
The life expectancy at the year 2015 is 53.02 years (CIA, 2016). This gives the natural
Death rate for Nigeria to be $\mu_h = \frac{1}{53.02} = 0.0189$

Recruitment number of humans due to birth (Λ_h)

According to CIA (2014), the birth rate for Nigeria at the year 2014 is 38.03 birth per year per 1000 people. This gives the birth rate as $\frac{38.03}{1000} = 0.03803$. However, the recruitment
Number due to birth in Nigeria was gotten from

$$\Lambda_h = N_h \times \mu_h = 177{,}155{,}754 \times 0.0189$$

$$\Lambda_h = 3{,}348{,}245$$

Recruitment number of mosquitoes, (Λ_v)

We assumed a recruitment rate of mosquitoes from (Bhunu and Mushayabasa, 2012) that the mosquitoes were caught for months and we use the months to get population of the mosquitoes for a year.

$$\Lambda_v = \mu_v N_v$$

$\Lambda_v = 0.00013 \times 2{,}960{,}000{,}000$

$\Lambda_v = 384{,}800$

Natural death rate of mosquitoes, (μ_v)

An average mosquito lives for 21 days (WHO, 2015). Thus, we have the natural death rate of mosquitoes as $\frac{1}{21} \times \frac{1}{366}$

Hence $\mu_v = 0.00013\ yr^{-1}$

Treatment rate (τ)

The effective years for treatment for both Infected with symptom and infected without symptom is 8years.

We therefore assume that $\tau = \frac{1}{8} = 0.125\, yr^{-1}$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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