Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2021, 2021:18
https://doi.org/10.28919/cmbn/5382
ISSN: 2052-2541

A PREY-PREDATOR MODEL WITH HOLLING TYPE IV RESPONSE FUNCTION UNDER DETERMINISTIC AND STOCHASTIC ENVIRONMENT

DIPANKAR SADHUKHAN*

Department of Mathematics, Haldia Government College, Haldia-721657, Purba Medinipur, West Bengal, India

Abstract: In this paper, both deterministic and stochastic behaviors of a general prey-predator model have been studied with Holling type-IV response function. For the deterministic model, uniform boundedness and persistence of the system have been discussed under the certain condition of the parameter. For local stability and bifurcation analysis, we arrive at the Hopf bifurcation and derived the symbolic condition for Hopf bifurcation. After that, the model has been illustrated with some numerical examples. In the second phase, the system has been perturbed by independent Gaussian white noises for the stochastic environment and the stability of the system have been studied by statistical linearization technique. Finally, a comparison has been made between the stability conditions in deterministic and stochastic cases.

Keywords: prey-predator; Holling type-IV; boundedness; persistence; white noise; Hopf bifurcation.

2010 AMS Subject Classification: 92B05, 93A30.

1. NOTATIONS

x & y: size of the prey and predator populations respectively at time t.

---

*Corresponding author
E-mail address: dipankar.sadhukhan2@gmail.com
Received January 3, 2021

k: environmental carrying capacity of prey.

r: intrinsic growth rate of prey.

a: half saturation constant.

b: efficiency at which the predator converts the consumed prey in to the new predators.

c: maximum per capita predation (consumption) rate.

i: direct measure of the tolerance of the prey.

d: the food independent death rate of predator.

2. INTRODUCTION

Prey-predator relation is a very complex phenomena, which exists in every ecological system such as pond, sea, forest etc. The pioneering work to model this prey-predator relation has been done first by Lotka[2] and Volterra[3],which is known to us as Lotka-Volterra model. Also there are several different types of prey-predator models such as Gauss type prey-predator model etc. These kinds of deterministic models have been made on the relationship in which one species is a part of the food supply for another species. Therefore a functional response works to change the prey density per unit time per predator as a function of prey or both prey and predator. The curve defined by Lotka-Volterra response function was a straight line through the origin and was unbounded. Abrams and Ginzburg[4] formulated analyzed a prey-predator model using a linear prey-dependant response function called Holling type-I function. For more reasonable response function in 1913, Michaelis and Menten proposed the response function which was of the form.

$g(x) = \frac{\xi x}{\eta + x}$, to study the enzyamatic reactions, where $\xi(> 0)$ denotes the maximal growth rate of the species and $\eta(> 0)$ is the half saturation constant[5]. In 1959, Holling[6] also used this function as one of the predator functional responses. It is now referred to as a Michaelis-Menten function or a Holling type-II function. The function $g(x)$is also called a prey-dependant response function since it depends solely on prey density.

Arditi et al.[7], Arditi and Saiah [8], Gutierrez[9]and Kuang and Beretta[10]are considered the response function based on both prey and predator. In recent time Maiti and Samanta[11,12], Maiti et al.[29] and Maiti and Pathak[30] have studied the pre-predator model under deterministic and stochastic environments with Michaelis-Menten defined ratio dependent response functions (i.e. Holling type-II).

In the literature, there is also a prey-predator relation of the type $g(x) = \frac{\xi x^2}{\eta + x^2}$, called Halling type-III response function which is again prey dependant of second order.

Also many researchers studied the prey-predator model under stochastic environment such as Maiti and Samanta [11-13], Samanta[14-20], Baishya and Chakrabarti[21], as in the present world due to global warming and other causes, environmental fluctuation is a very important random phenomena. But till now no comparison study has been made for the model with Holling type-IV response function in deterministic and stochastic environments.

The model for this section is described by

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \phi(x)y \\ \frac{dy}{dt} &= -dy + \hat{b}\phi(x)y \end{aligned} \qquad (1)$$

which was proposed by Freedman and Wolkowicz [32]. It is similar in appearance to the Rosenzweig- MacArthur system[33].

With Holling type-IV response function $\phi(x) = \frac{\hat{c}x}{\hat{a} + x + \frac{x^2}{i}}$ [31], a prey-predator model is of the form

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \frac{cxy}{1 + ax + bx^2} \\ \frac{dy}{dt} &= y\left(-d + \frac{fx}{1 + ax + bx^2}\right) \end{aligned} \qquad (2)$$

Where, $c = \frac{\hat{c}}{\hat{a}}$, $a = \frac{1}{\hat{a}}$, $b = \frac{1}{i\hat{b}}$ and $f = \frac{\hat{b}\hat{c}}{\hat{a}}$ with $x(0 > 0)$, $y(0) > 0$. It may be noted that the denominator of the above system $(i.e. 1 + ax + bx^2)$ does not vanish for x.

This kind of model with reaction diffusion has been studied earlier by Zhang et al.[22], but they did not make any comparative study of the model in both deterministic and stochastic environments.

This kind of model with reaction diffusion has been studied earlier by Zhang, Wang, Xue and Jin[22],but they did not make any comparative study of the model in both deterministic and stochastic environments.

In the present work, we have considered Holling type-IV response function. Firstly we have studied the boundedness, persistence, Hopf bifurcation (Murrey) [23] and stability analysis in deterministic environment. After that we perturbed the system with independent white noise due to environmental fluctuation and derived the stability condition using statistical linearization method of Valsakumar et al.[1] and Routh-Whoriatz criteria.

3. STEADY STATE AND DYNAMICAL BEHAVIOUR

The steady states of the system (2) in the positive quadrant are at $(0,0)$ (total extinct), $(k,0)$ and interior equilibrium $E(x^*,y^*)$ (cf. Zhang et al[22]).

When $(a,b,c,d,f,r,k) \in E_1$,

Where, $E_1 = \left\{ \begin{matrix} (a,b,c,d,f,r,k): f > ad, (f-ad)^2 > bd^2, \sqrt{(f-ad)^2 - 4bd^2} > \\ \frac{-f^2 - a^2d^2 + 2fad + fbdk - abd^2k + 2bd^2}{bdk + ad - f} > 0, \frac{a}{b} - \frac{f}{bd} + k < 0 \end{matrix} \right\}$, there exists a unique stationary coexistent state at (x_1^*, y_1^*).

With $x_1^* = \frac{(f-ad) - \sqrt{(f-ad)^2 - 4bd^2}}{2bd}$ and $y_1^* = \frac{rf\{(bdk+ad-f)x_1^* + d\}}{bcd^2k}$.

Again when $(a,b,c,d,f,r,k) \in E_2$,

Where, $E_2 = \left\{ \begin{matrix} (a,b,c,d,f,r,k): f > ad, (f-ad)^2 > bd^2, \sqrt{(f-ad)^2 - 4bd^2} > \\ -\frac{-f^2 - a^2d^2 + 2fad + fbdk - abd^2k + 2bd^2}{bdk + ad - f} > 0, \frac{a}{b} - \frac{f}{bd} + k > 0 \end{matrix} \right\}$, there exists another equilibrium point at (x_2^*, y_2^*).

Where $x_2^* = \frac{(f-ad) + \sqrt{(f-ad)^2 - 4bd^2}}{2bd}$ and $y_2^* = \frac{rf\{(bdk+ad-f)x_2^* + d\}}{bcd^2k}$.

In our present investigation, we consider the equilibrium point (x^*, y^*) of the region E_1. The analysis of the equilibrium point (x_2^*, y_2^*) of the region E_2 can be done similarly.

Theorem 1. All the solutions of system (2) which exist in $\Re_+^2$ are uniformly bounded.

Proof: Let $\big(x(t), y(t)\big)$ be any solution of (2) with positive initial conditions.

As $\frac{dx}{dt} \le rx\left(1-\frac{x}{k}\right)$, therefore from standard comparison test, we have

$\lim_{t\to\infty} \sup x(t) \le \mu$, where $\mu = \max\{x(0), k\}$.

Considering $W = fx + cy$ and using (2), we have

$$\frac{dW}{dt} = frx\left(1-\frac{x}{k}\right) - cdy.$$
$$\le frx - cdy = 2frx - frx - cdy$$
$$\le 2rf\mu - \delta W.$$

Where $\delta = min(r, d)$

Therefore, $\frac{dW}{dt} + \delta W \le 2rf\mu$ (3)

Now using the Theorem of Birkhoff and Rota [24] on differential inequalities, we get

$$0 \le W(x,y) \le \frac{W\big(x(0),y(0)\big)}{e^{\delta t}} + \frac{2rf\mu}{\delta}$$

Therefore for $t \to \infty$, $0 \le W(x,y) \le \frac{2rf\mu}{\delta}$

So, $\left\{(x,y): 0 \le fx + cy \le \frac{2rf\mu}{\delta} + \epsilon, for\ \epsilon > 0\right\} = B\ (say)$ (4)

is the region where all solutions of the system (2) which exists in $\mathfrak{R}_+^2$, are uniformly bounded. Hence the theorem.

Theorem 2. If the interior equilibrium point exists then the system (2) is uniformly persistent.

Proof: From the Theorem-1, it is clear that the solution of the system lies in B for sufficiently large t.

Also from the above theorem, we have

$$\lim_{t\to\infty} \sup x(t) \le k$$

So we can take $M = \sup_{x\in B} x(t)$ to be any number larger than k.

Let us now define the average Lyapunov function given by $\rho\,(z) = x^{r_1}y^{r_2}$ where r_1, r_2 are positive constants to be specified later.

Now, $\psi(z) = \frac{\dot\rho(z)}{\rho(z)} = rr_1 - dr_2 + x\left(\frac{r_2 f}{1+ax+bx^2} - \frac{rr_1}{k}\right)$for $z = (x,y) \in H_2$

Where $H_1 = \{(x,y) \in \mathfrak{R}_+^2 : x = 0\}$ and $H_1 = \{(x,y) \in \mathfrak{R}_+^2 : y = 0\}$

So we choose, $r_1 = 1, r_2 = \frac{r_1\left(1+aM+bM^2\right)}{kf}$

guarantee that for $x_1 \in \Omega(\partial\Re^2_+) \cap H_2,\ \psi(x_2) \geq rr_1 - dr_2 = \mu_2\,(say)$, where $\partial\Re^2_+ = \cup_{i=1}^{2} H_i = B$ and $\Omega(\partial\Re^2_+)$ is the $\omega - limit$ set of system (2) in the boundary of the positive cone (Gard[26]).

Since $M - k$ is arbitrarily small, the persistence condition $\mu_2 > 0$ can be written as $f - ad - bdk > \frac{d}{k} > 0$. which is also the condition for existence of the interior equilibrium point (x^*, y^*).

4. Local Stability Analysis

To perform the stability analysis, we use the variational matrix of the system (2) as

$$V(x,y) = \begin{bmatrix} -\frac{rx}{k} + \frac{acxy+2bcx^2y}{(1+ax+bx^2)^2} & -\frac{cx}{(1+ax+bx^2)} \\ \frac{fy(1-bx^2)}{(1+ax+bx^2)^2} & -d + \frac{fx}{(1+ax+bx^2)} \end{bmatrix} \tag{5}$$

Here we mainly focus on the stability of the interior equilibrium point (x^*, y^*). Therefore from (2), for the interior equilibrium point (x^*, y^*) we get.

$$V(x^*,y^*) = \begin{bmatrix} -\frac{rx^*}{k} + \frac{acx^*y^*+2bcx^{*2}y^*}{(1+ax^*+bx^{*2})^2} & -\frac{cx^*}{(1+ax^*+bx^{*2})} \\ \frac{fy^*(1-bx^{*2})}{(1+ax^*+bx^{*2})^2} & 0 \end{bmatrix} \tag{6}$$

Theorem 3. The interior equilibrium point (x^*, y^*) is locally asymptotically stable or unstable according as

$$abd^2k + \left\{(f-ad) - \sqrt{(f-ad)^2 - 4bd^2}\right\}\left(bdk - f + \frac{1}{2}\sqrt{(f-ad)^2 - 4bd^2}\right) < or > 0$$

Proof: From the variational matrix (6) we have

$$tr\,V(x^*,y^*) = -\frac{rx^*}{k} + \frac{acx^*y^*+2bcx^{*2}y^*}{(1+ax^*+bx^{*2})^2}$$

$$= \frac{rf}{2bdx^*k} \times \left[abd^2k + \left\{(f-ad) - \sqrt{(f-ad)^2 - 4bd^2}\right\}\left(bdk - f + \frac{1}{2}\sqrt{(f-ad)^2 - 4bd^2}\right)\right] \tag{7}$$

And the determinant of the variational matrix (6) is

$\det V(x^*,y^*) = \frac{fcx^*y^*(1-bx^{*2})}{(1+ax^*+bx^{*2})^3}$, which is obviously positive as $b = \frac{1}{i\hat{b}}$, so $0 < x^* < \frac{1}{\sqrt{b}}$.

Therefore the interior equilibrium point. (x^*, y^*)will be asymptotically stable if $\det V(x^*, y^*) > 0$ and $tr\, V(x^*, y^*) < 0$ and unstable if $\det V(x^*, y^*) > 0$ and $tr\, V(x^*, y^*) > 0$.

Therefore the interior equilibrium point (x^*, y^*) will be asymptotically stable or unstable according as

$$tr\, V(x^*, y^*) < or > 0. \tag{8}$$

i.e.,

$$abd^2k + \left\{(f - ad) - \sqrt{(f - ad)^2 - 4bd^2}\right\}\left(bdk - f + \frac{1}{2}\sqrt{(f - ad)^2 - 4bd^2}\right) < or > 0 \tag{9}$$

Hence the theorem.

Theorem 4. If the interior equilibrium point (x^*, y^*) exists and

$$k^* = \frac{\left(f - \frac{1}{2}\sqrt{(f - ad)^2 - 4bd^2}\right)\left\{(f - ad) - \sqrt{(f - ad)^2 - 4bd^2}\right\}}{abd^2 + bd\left\{(f - ad) - \sqrt{(f - ad)^2 - 4bd^2}\right\}}$$

Then Hopf bifurcation occurs at $k = k^*$.

Proof: With the help of the previous Theorem-3, it is clear that

(i) $[tr\, V(x^*, y^*)\,]_{k=k^*} = 0$

(ii) $J = [det\, V(x^*, y^*)\,]_{k=k^*} > 0$

(iii) As $J > 0$, the roots of the characteristic equation $\lambda^2 + J = 0$ are purely imaginary at $k = k^*$.

(iv) $\left[\frac{d}{dk} tr\, V(x^*, y^*)\,\right]_{k=k^*} = \frac{rf}{k^2}\left(2f - \sqrt{(f - ad)^2 - 4bd^2}\right) \neq 0.$

Therefore all the conditions of Hopf bifurcation theorem [23] are satisfied and hence the theorem.

5. STOCHASTIC MODEL

As we are only interested on the dynamics of the system (2) about the interior equilibrium point in the first quadrant, so by scaling the system $dt = (1 + ax + bx^2)dT$, the transformed system(2) takes the form

$$\begin{aligned} \frac{dx}{dT} &= \left(rx - \frac{r}{k}x^2\right)(1 + ax + bx^2) - cxy \\ \frac{dy}{dT} &= -dy(1 + ax + bx^2) + fxy \end{aligned} \tag{10}$$

Where a, b, c, d, r, f & $k > 0$.

Now the system (2) and the scaled system (10) have the same equilibrium point (x^*, y^*) in $\Re_+^2$. Let $X = x - x^*$ and $Y = y - y^*$ be the perturbed values, then the system of equations (10) in terms of deviated variables (X, Y) can be written as

$$\frac{dX}{dT} = p_1 X + q_1 X^2 + r_1^/ X^3 + +s_1 X^4 + d_1 Y + f_1 XY \tag{11}$$

Where

$$\begin{aligned} p_1 &= \left(r - \frac{2r}{k}x^*\right)(1 + ax^* + bx^{*2}) + \left(rx^* - \frac{r}{k}x^{*2}\right)(a + 2bx^*) - cy^*, \\ q_1 &= -\frac{r}{k}\,(1 + ax^* + bx^{*2}) + b\left(rx^* - \frac{r}{k}x^{*2}\right) - \left(r - \frac{2r}{k}x^*\right)(a + 2bx^*), \\ r_1^/ &= b\left(r - \frac{2r}{k}x^*\right) - \frac{r}{k}(a + 2bx^*), \\ s_1 &= -\frac{r}{k}b, \\ d_1 &= -cx^*, \\ f_1 &= -c. \end{aligned} \tag{12}$$

And

$$\frac{dY}{dT} = p_2 X + q_2 X^2 + d_2 Y + f_2 XY + g_2 X^2 Y. \tag{13}$$

Where,

$$\begin{aligned} p_2 &= fy^* - dy^*(a + 2bx^*), \\ q_2 &= -bdy^*, \\ d_2 &= fx^* - d(1 + ax^* + bx^{*2}), \\ f_2 &= f - d(a + 2bx^*), \\ g_2 &= -bd. \end{aligned} \tag{14}$$

These are the basic deterministic ordinary differential equations for determining the behaviour of the system about the steady state (x^*, y^*). The solutions $(X(T), Y(T))$ of (11) and (13) together with the initial value $(X(0), Y(0))$ represent the status of the system at time $T > 0$.

To introduce the randomization of the system, we have to make small changes of the initial value system consisting (11) and (13). Here we consider the case when the right hand side of (11) and (13) are given a small perturbation. This has been made by using Gaussian white noise, by which the transformed differential equations are mathematically known as $It\hat{o}$ stochastic differential equations. This is a very useful concept to model rapidly fluctuating phenomenon. Though white noise does not occur in nature, but this noise is a good approximation of the thermal noise in electrical resistance, climate fluctuating, etc. Also, it can be proved that the process $(X(T), Y(T))$, a solution of (11) and (13), is a Markovian if and only if the external noise is white. This is a very beautiful idealisation of white noise (Horsthemke and Lefever [27]).

To analyse the fluctuations of the system about the steady state, the system of equations (11) and (13) can be extended to the corresponding $It\hat{o}$ type differential equations (non-linear coupled bivariate Langevin equations) as:

$$\begin{aligned} \frac{dX}{dT} &= p_1 X + q_1 X^2 + r_1^{/} X^3 + s_1 X^4 + d_1 Y + f_1 XY + \eta_1(T) \\ \frac{dY}{dT} &= p_2 X + q_2 X^2 + d_2 Y + f_2 XY + g_2 X^2 Y + \eta_2(T) \end{aligned} \tag{15}$$

where $\eta_1(T)$ & $\eta_2(T)$ are independent Gaussian white noises with zero mean and correlation:

$$\begin{aligned} &\langle \eta_i(T) \rangle = 0, \\ &\langle \eta_i(T)\eta_i(T^{/}) \rangle = 2\epsilon_i \delta(T - T^{/}),\ i = 1,2. \end{aligned} \tag{16}$$

where ϵ_i is the intensity of the noise and the bracket $\langle . \rangle$ denotes mean value with respect to the noise and δ is the Dirac delta-function.

6. Moment Equations

After the statistical linearization of the system (15), we get the linear system as

$$\begin{aligned} \frac{dX}{dT} &= \alpha_1 X + \beta_1 Y + c_1 + \eta_1(T) \\ \frac{dY}{dT} &= \alpha_2 X + \beta_2 Y + c_2 + \eta_2(T) \end{aligned} \tag{17}$$

Now the error for above linearization are

$$\begin{aligned} E_1 &= p_1 X + q_1 X^2 + r_1^{/} X^3 + +s_1 X^4 + d_1 Y + f_1 XY - \alpha_1 X - \beta_1 Y - c_1 \\ E_2 &= p_2 X + q_2 X^2 + d_2 Y + f_2 XY + g_2 X^2 Y - \alpha_2 X - \beta_2 Y - c_2 \end{aligned} \tag{18}$$

The parameters $\alpha_i, \beta_i,\ c_i\ (i = 1, 2.)$ of the equations (18) are determined from (*cf*. Valsakumar et al. [1], Bandyopadhyay and Chakrabarti [25])

$$\frac{\partial}{\partial \alpha_i}\langle E_i^2 \rangle = \frac{\partial}{\partial \beta_i}\langle E_i^2 \rangle = \frac{\partial}{\partial c_i}\langle E_i^2 \rangle = 0, (i = 1, 2.) \tag{19}$$

Now using the relations (Valsakumar et al. [1]),

$$\begin{aligned}
&\langle X^4\rangle = 3\langle X^2\rangle^2 - 2\langle X\rangle^4,\\
&\langle X^2Y^2\rangle = \langle X^2\rangle\langle Y^2\rangle + 2\langle XY\rangle^2 - 2\langle X\rangle^2\langle Y\rangle^2,\\
&\langle X^3Y\rangle = 3\langle X^2\rangle\langle XY\rangle - 2\langle X\rangle^3\langle Y\rangle,\\
&\langle X^3\rangle = 3\langle X\rangle\langle X^2\rangle - 2\langle X\rangle^3,\\
&\langle Y^3\rangle = 3\langle Y\rangle\langle Y^2\rangle - 2\langle Y\rangle^3,\\
&\langle X^2Y\rangle = 2\langle X\rangle\langle XY\rangle - 2\langle X\rangle^2\langle Y\rangle + \langle X^2\rangle\langle Y\rangle,\\
&\langle Y^2X\rangle = 2\langle Y\rangle\langle XY\rangle - 2\langle Y\rangle^2\langle X\rangle + \langle Y^2\rangle\langle X\rangle.
\end{aligned} \tag{20}$$

$\alpha_i, \beta_i,\ c_i\ (i = 1, 2.)$ can be written as,

$$\begin{aligned}
&\alpha_1 = p_1 + 2q_1\langle X\rangle + 3r_1^{/}\langle X^2\rangle + 4s_1\langle X^3\rangle + f_1\langle Y\rangle,\\
&\beta_1 = d_1 + f_1\langle X\rangle,\\
&c_1 = q_1(\langle X^2\rangle - 2\langle X\rangle^2) + 2r_1^{/}\langle X\rangle^3 + s_1(\langle X^4\rangle - 4\langle X^3\rangle\langle X\rangle).
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_2 = p_2 + 2q_2\langle X\rangle + f_2\langle Y\rangle + g_2\langle XY\rangle,\\
&\beta_2 = d_2 + f_2\langle X\rangle + g_2\langle X^2\rangle,\\
&c_2 = q_2(\langle X^2\rangle - 2\langle X\rangle^2) + 2f_1(\langle XY\rangle - 2\langle X\rangle\langle Y\rangle) - 2g_2\langle X\rangle^2\langle Y\rangle.
\end{aligned}$$

since coefficients are functions of the parameters involved with the model and also with the different moments involving X and Y. Now with the help of (20), we have the system of differential equations of first two moments as:

$$\begin{aligned}
&\frac{d\langle X\rangle}{dT} = p_1\langle X\rangle + q_1\langle X^2\rangle + r_1^{/}\langle X^3\rangle + s_1\langle X^4\rangle + d_1\langle Y\rangle + f_1\langle XY\rangle,\\
&\frac{d\langle Y\rangle}{dT} = p_2\langle X\rangle + q_2\langle X^2\rangle + d_2\langle Y\rangle + f_2\langle XY\rangle + g_2\langle X^2Y\rangle,\\
&\frac{d\langle X^2\rangle}{dT} = 2\left[p_1\langle X^2\rangle + q_1\langle X^3\rangle + r_1^{/}\langle X^4\rangle + s_1\langle X^5\rangle + d_1\langle XY\rangle + f_1\langle X^2Y\rangle\right] + 2\epsilon_1,\\
&\frac{d\langle Y^2\rangle}{dT} = 2[p_2\langle XY\rangle + q_2\langle X^2Y\rangle + d_2\langle Y^2\rangle + f_2\langle XY^2\rangle + g_2\langle X^2Y^2\rangle] + 2\epsilon_2,\\
&\frac{d\langle XY\rangle}{dT} = p_1\langle XY\rangle + q_1\langle X^2Y\rangle + r_1^{/}\langle X^3Y\rangle + s_1\langle X^4Y\rangle + d_1\langle Y^2\rangle + f_1\langle XY^2\rangle +\\
&\qquad\qquad p_2\langle X^2\rangle + q_2\langle X^3\rangle + d_2\langle XY\rangle + f_2\langle X^2Y\rangle + g_2\langle X^3Y\rangle
\end{aligned} \tag{21}$$

where $\epsilon_1 = \langle X\eta_1\rangle$ and $\epsilon_2 = \langle Y\eta_2\rangle$. Also we use $\langle X\eta_1\rangle = 0$ and $\langle Y\eta_2\rangle = 0$.

Here we assume that the system size expansion is valid such that the correlations ϵ_1 and ϵ_2 are assumed to be the order of inverse population size N (*cf.* Valsakumar et al. [1], Baishya and Chakrabarti [21], Bandopadhyay and Chakrabarti [25])

$$\epsilon_i \propto o\left(\frac{1}{N}\right), i = 1, 2. \tag{22}$$

Now using (21), (22) and taking the lowest order term by replacing the averages $\langle X\rangle$ and $\langle Y\rangle$ by their steady state values $\langle X\rangle = \langle Y\rangle = 0$ (Nicolis and Prigogine [28]), we get the following system of equations of second order moments as:

$$\begin{aligned}
\frac{d\langle X^2\rangle}{dT} &= 2p_1\langle X^2\rangle + 2d_1\langle XY\rangle \\
\frac{d\langle Y^2\rangle}{dT} &= 2p_2\langle XY\rangle + 2d_2\langle Y^2\rangle \\
\frac{d\langle XY\rangle}{dT} &= (p_1+d_2)\langle XY\rangle + p_2\langle X^2\rangle + d_1\langle Y^2\rangle
\end{aligned} \tag{23}$$

7. FLUCTUATION AND STABILITY ANALYSIS

Now eliminating $\langle X^2\rangle$ and $\langle Y^2\rangle$ from the system (23), we have the following third order equation of the form:

$$\frac{d^3\langle XY\rangle}{dT^3} + 3a_1\frac{d^2\langle XY\rangle}{dT^2} + 3a_2\frac{d\langle XY\rangle}{dT} + a_3\langle XY\rangle = 0 \tag{24}$$

The auxiliary equation of the above equation (24) is

$$\lambda^3 + 3a_1\lambda^2 + 3a_2\lambda + a_3 = 0 \tag{25}$$

where

$$\begin{aligned}
a_1 &= -(p_1+d_2) = -p_1 = -\frac{fx^*}{d}tr\,V(x^*,y^*), \\
a_2 &= \frac{2}{3}\big((p_1+d_2)^2 + 2(p_1d_2 - p_2d_1)\big) = \frac{2}{3}(p_1^2 - 2p_2d_1), \\
a_3 &= -4(p_1+d_2)(p_1d_2 - p_2d_1) = 4p_1p_2d_1.
\end{aligned} \tag{26}$$

Now let $H = a_1^2 - a_2 = \frac{1}{3}p_1^2 + \frac{4}{3}p_2d_1$.

Therefore the nature of the roots of (25) will dependant on the behaviour of a_1 and H . Also $2a_1^3 + 3a_1a_2 + a_3 = 0$.

Case-1. $\boldsymbol{H \le 0}$.

In this case the roots of the auxiliary equation are $\lambda_1 = -a_1, \lambda_2 = -a_1 + i\sqrt{-3H}$ *and* $\lambda_3 = -a_1 - i\sqrt{-3H}$.

Therefore the solutions of the system (23) are given by

$$\begin{aligned}
\langle XY\rangle &= e^{-a_1t}\{A_{11} + A_{12}\cos t\sqrt{-3H} + A_{13}\sin t\sqrt{-3H}\}, \\
\langle X^2\rangle &= e^{-a_1t}\{A_{21} + A_{22}\cos t\sqrt{-3H} + A_{23}\sin t\sqrt{-3H}\} + B_1e^{2p_1t}, \\
\langle Y^2\rangle &= e^{-a_1t}\{A_{31} + A_{32}\cos t\sqrt{-3H} + A_{33}\sin t\sqrt{-3H}\} + B_2e^{2d_2t}.
\end{aligned} \tag{27}$$

where $A_{ij}\ (i,j\ =\ 1,2,3.), B_1$ & B_2 are constants. So if $p_1 < 0$, then $a_1 > 0$, consequently from the above system each of $\langle X^2 \rangle$ and $\langle XY \rangle$ decreases to zero as time advances and $\langle Y^2 \rangle$ approaches to a constant value since $d_2 = 0$. So from the stability criteria of second order moments, the interior equilibrium point $(x^*,\ y^*)$ is stable for $p_1 < 0$, which is also the stability condition for deterministic case. Again when $p_1 > 0$, i.e. $a_1 < 0$ then in stochastic case $(x^*,\ y^*)$ is unstable as the second order moments diverges with time. So when $p_1 < 0$, the deterministic stability criteria

$$abd^2k + \left\{(f - ad) - \sqrt{(f - ad)^2 - 4bd^2}\right\}\left(bdk - f + \frac{1}{2}\sqrt{(f - ad)^2 - 4bd^2}\right) < 0$$

is satisfied and which also guarantees the stability of the system (2) under stochastic environment, which is also confirmed by Routh-Hurwitz criteria **(Appendix)**. Also if the system is unstable in deterministic case, then the system is unstable too in stochastic arena.

Case-2. $\boldsymbol{H > 0}$.

In this case the roots of the auxiliary equation are $\lambda_1 = -a_1, \lambda_2 = -a_1 + \sqrt{3H}\ and\ \lambda_3 = -a_1 - \sqrt{3H}$. Therefore the solutions of the system (23) are given by

$$\begin{aligned}
\langle XY \rangle &= e^{-a_1 t}\left\{P_{11} + P_{12}e^{t\sqrt{3H}} + P_{13}e^{-t\sqrt{3H}}\right\},\\
\langle X^2 \rangle &= e^{-a_1 t}\left\{P_{21} + P_{22}e^{t\sqrt{3H}} + P_{23}e^{-t\sqrt{3H}}\right\} + Q_1 e^{2p_1 t},\\
\langle Y^2 \rangle &= e^{-a_1 t}\left\{P_{31} + P_{32}e^{t\sqrt{3H}} + P_{33}e^{-t\sqrt{3H}}\right\} + Q_2 e^{2d_2 t}.
\end{aligned} \tag{28}$$

where $P_{ij}(i,j = 1,2,3.), Q_1$ & Q_2 are constants. So if $p_1 < 0$, then $a_1 > 0$, so the deterministic stability criteria

$$abd^2k + \{(f - ad) - \sqrt{(f - ad)^2 - 4bd^2}\}\left(bdk - f + \tfrac{1}{2}\sqrt{(f - ad)^2 - 4bd^2}\right) < 0$$

is satisfied. With the conditions $p_1 < 0$, and $a_1 > \sqrt{3H}$, it is seen that each of the second order moments $\langle X^2 \rangle, \langle Y^2 \rangle$ and $\langle XY \rangle$ converges with the increment of time. So in this scenario, the stochastic system is stable in the sense of second order moment along with the deterministic (in this case only $p_1 < 0$ is required) one.

On the other hand when $p_1 < 0$ with $a_1 > \sqrt{3H}$, all the second order moments $\langle X^2 \rangle, \langle Y^2 \rangle$ and $\langle XY \rangle$ diverge with time. Hence stochastic system becomes unstable in this stochastic environment, though the system is stable in deterministic case. For all other cases, the stochastic system will be unstable.

8. Numerical Experiments

Example-1. Here we investigate the model (4.2) for the parametric values $r = 2.5, k = 3, a = 0.1, b = 0.2, c = 0.9, d = 0.01\ and\ f = 0.5$ in suitable units.

With these values of the parameters, the interior equilibrium point is $(x^*, y^*) = (1.0008, 231.49)$ and in this case $p_1 = -207.3325 < 0, H = 13429 > 0\ and\ a_1 > \sqrt{3H}$.

Also $abd^2k + \left\{(f - ad) - \sqrt{(f - ad)^2 - 4bd^2}\right\}\left(bdk - f + \frac{1}{2}\sqrt{(f - ad)^2 - 4bd^2}\right) = -0.0000136 < 0$.

Corresponding Stable behaviour of the system (2) in the sense of second order moments with respect to the above se of parametric values is

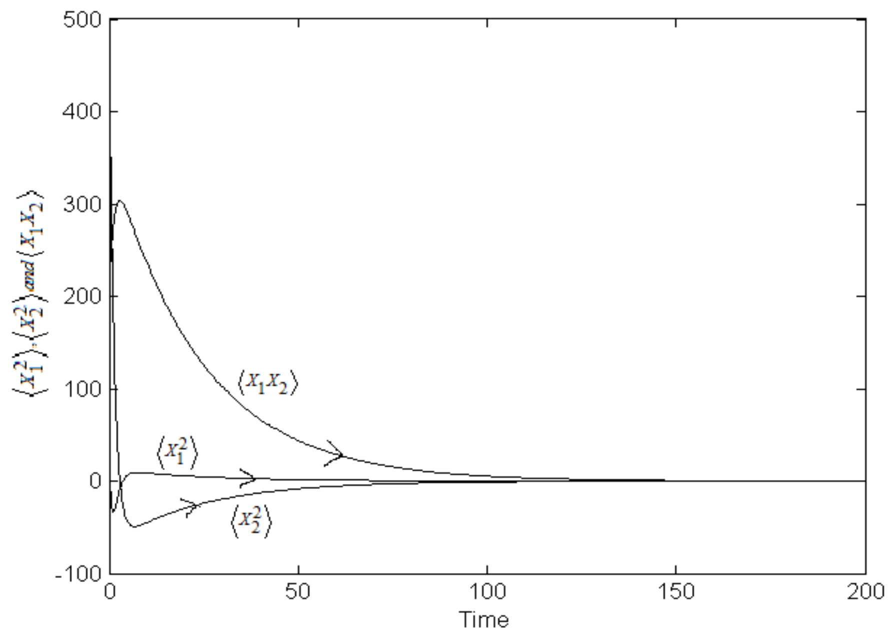

Figure-1: Stable behaviour of the system in the sense of second order moments.

So on the basis of the above inequality, the system is stable in deterministic case. With the same parametric values, in stochastic case, the second-order moments from the expressions (23) are evaluated and plotted in the Figure-1 which also depicts the stable behaviour of the system.

Example-2. Here to find a numerical solution and verification of the analysis of the deterministic model (2), we consider the parametric values of the system as: $r = 1.5, k = 1.5, a = 0.1, b = 1.1, c = 0.5, d = 0.15\ and\ f = 0.4$ in suitable units.

With these values of the parameters, the interior equilibrium point is $(x^*, y^*) = (0.50, 2.65)$ and $abd^2k + \{(f - ad) - \sqrt{(f - ad)^2 - 4bd^2}\}\left(bdk - f + \frac{1}{2}\sqrt{(f - ad)^2 - 4bd^2}\right) = -0.0031 < 0$.

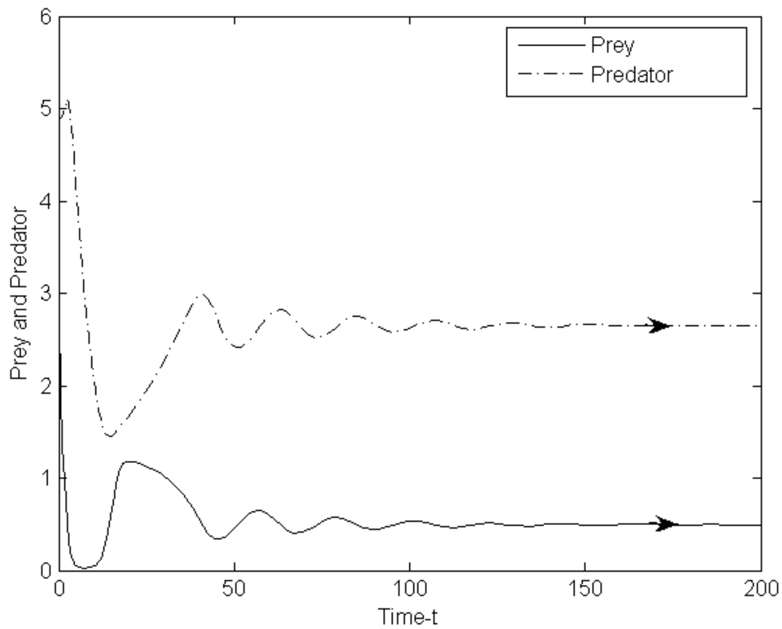

Figure-2: Stability diagram of the system with $x(0) = 2\ and\ y\,(0) = 5$.

Under these conditions, the configurations of $x(t)$ and $y(t)$ are presented in Fig.-2. Therefore the above interior equilibrium point is locally asymptotically stable. This behaviour has been depicted in the Figure-2. With the same parametric values, in stochastic case, the second-order moments from the expressions (23) are evaluated and plotted in the Figure-3 which depicts the unstable behaviour of the system. This is because $a_1 < \sqrt{3H}$, instead $a_1 > \sqrt{3H}$ (as in Case-2. $H > 0$) though in this case $p_1 (= -1.4172) < 0$ and $H\ (= 0.6858) > 0$.

Also for these values of parameters, we get the bifurcation value of k from Theorem-4 as $k^* = 1.6044$.

Now for $k = 1.605\ (> k^*)$, the value of

$abd^2k + \{(f-ad) - \sqrt{(f-ad)^2 - 4bd^2}\}\left(bdk - f + \frac{1}{2}\sqrt{(f-ad)^2 - 4bd^2}\right) =$

$0.000018(> 0)$.

and this makes the system (2) unstable (cf. Theorem-3).

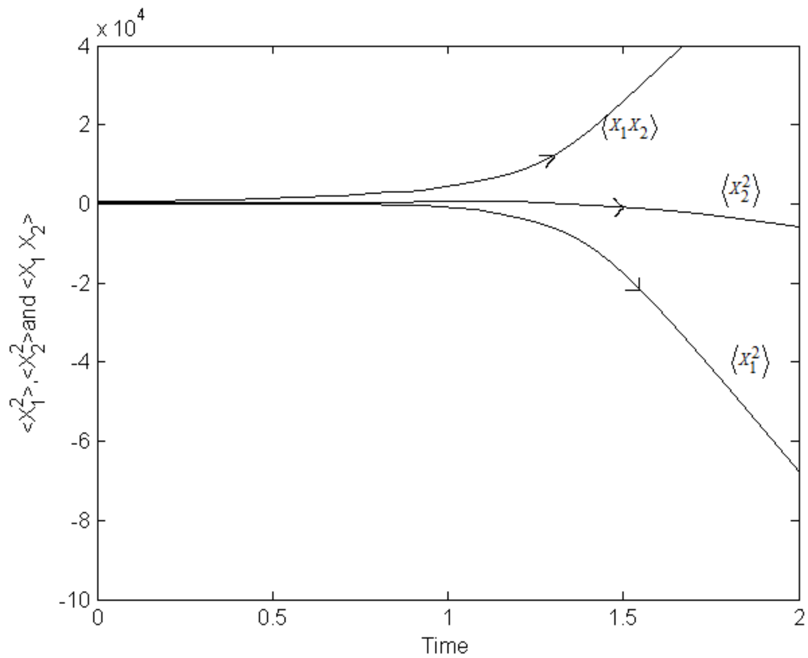

Figure-3: Unstable behaviour of the system in the sense of second order moments.

9. Conclusion and Ecological Implication

In this paper for the first model, we have studied the deterministic and stochastic behaviour of a population system with Holling type-IV response function in a region where unique positive stationary point exists. In deterministic case we have shown the boundedness of the system together with the persistence. Also we have investigated the stability condition of the system, which is necessary for the long term existence of species. Again as the carrying capacity is an important parameter for any biological species, so with the help of parametric values, we determine the critical value of bifurcation parameter k. Since in reality the ecological world is very fluctuating, so by inducing white noise we investigate the behaviour of the same system in random environment and it is very interesting that if a system is stable in deterministic case, then the system will also be stable under stochastic case for Case-1. This is evident as the equation (9) under deterministic environment appears in case-1 of stochastic environment. But under the condition in Case-2, the system which is stable in deterministic case, may not be so in stochastic case. This is because the equation (9) appears in Case-2 along with the some restrictions $\left(a_1 > or < \sqrt{3H}\right)$. It is also quite clear from the examples-1 and-2 with corresponding diagrams (cf. Figs.-1, 2 and 3) that for the same parametric values, the system which is stable under deterministic case, may or may not be stable under stochastic case.

In recent days due to over weight of human biomass and their continuous need for better life there is a tremendous pressure on natural ecological balance, So we face global warming, draught, flood and other uncertain phenomenon of nature which make a direct impact on every ecosystem.

So, to make any prediction on ecological balance through mathematical modelling now a days it is not enough to investigate the behaviour of the system under deterministic analysis only, because it does gives sufficient accuracy that is why present day study needs to include more uncertainty. Mathematically which is possible by incorporating stochasticity in the model and in this study we have checked the relevance of uncertainty by a comparative study of stochastic and deterministic cases.

From these studies, it is conclude that if the system is unstable in deterministic case, it will be automatically unstable in stochastic environment.

ACKNOWLEDGEMENT

This research work is partially supported by University Grants Commission’s Minor Research Project: No.- PSW-136/14-15, (ERO) ID No. WV6-033, Date- 10-Feb-2017 .

Also I thank Prof. M. Maiti (Retired), Dept. of Mathematics, Vidyasagar University and Dr. B. Mondal, Associate Professor, Dept. of Mathematics, Raja N. L. K. women’s College, Midnapore, for their comments and suggestion that greatly improved the manuscript.

APPENDIX

The characteristic equation of the above equation (23) is

$$\begin{vmatrix} 2p_1-\gamma & 0 & 2d_1 \\ 0 & 2d_2-\gamma & 2p_2 \\ p_2 & d_1 & p_1+d_2-\gamma \end{vmatrix} = 0 \tag{29}$$

$$\Rightarrow \gamma^3 + 3a_1\gamma^2 + 3a_2\gamma + a_3 = 0 \tag{30}$$

where

$$\begin{aligned} a_1 &= -(p_1+d_2), \\ a_2 &= \tfrac{2}{3}\left((p_1+d_2)^2 + 2(p_1d_2 - p_2d_1)\right) = \tfrac{2}{3}(p_1^2 - 2p_2d_1) \\ a_3 &= -4(p_1+d_2)(p_1d_2-p_2d_1) = 4p_1p_2d_1 \end{aligned} \tag{31}$$

Now

$$(p_1+d_2) = \frac{rf}{2bdx^*k}\left[abd^2k + \left\{(f-ad) - \sqrt{(f-ad)^2-4bd^2}\right\}\left(bdk - f + \frac{1}{2}\sqrt{(f-ad)^2-4bd^2}\right)\right] \tag{32}$$

$$\text{and } p_1d_2 - p_2d_1 = cx^*y^*\{f - d(a+2bx^*)\} \tag{33}$$

Also the conditions for the system to be stable i.e. $Re\,(\gamma) < 0$, from Routh-Hurwitz criteria, are

$$3a_1 > 0, 9a_1a_2 - a_3 > 0 \text{ and } a_3 > 0 \tag{34}$$

$$\text{i.e. } a_1 > 0, 9a_1a_2 - a_3 > 0 \text{ and } a_3 > 0 \tag{35}$$

In our study, $p_2 > 0, d_1 < 0$ and $d_2 = 0$.

Also $a_3 = 4p_1p_2d_1$ and $9a_1a_2 - a_3 = 2p_1(-3p_1^2 + 4p_2d_1)$.

Since $4p_2d_1 < 0$ and $-3p_1^2 + 4p_2d_1 < 0$ so, these three conditions $a_1 > 0, 9a_1a_2 - a_3 > 0$ and $a_3 > 0$ now depends only on the sign of p_1. If $p_1 < 0$, then all the above conditions hold i.e. the system is stable in stochastic case. Also the condition $p_1 < 0$ implies

$$abd^2k + \left\{(f-ad) - \sqrt{(f-ad)^2-4bd^2}\right\}\left(bdk - f + \frac{1}{2}\sqrt{(f-ad)^2-4bd^2}\right) < 0.$$

which is the stability condition of the system for deterministic case also. Therefore if the system is stable in deterministic case, then system will also be stable under stochastic case for $H < 0$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] M.C. Valsakumar, K.P.N. Murthy, G. Ananthakrishna, On the linearization of nonlinear langevin-type stochastic differential equations, J. Stat. Phys. 30 (1983), 617–631.

[2] A.J. Lotka, Elements of Physical Biology. Williams & Wilkins, Baltimore, 1925.

[3] V. Volterra, Variazioni e fluttuazioni del numero d'individui in specie animali conviventi, Mem. Acad. Lincei. 2 (1926), 31–113

[4] P.A. Abrams, L.R. Ginzburg, The nature of predation: prey dependent, ratio dependent or neither? Trends Ecol. Evol. 15 (2000), 337–341.

[5] M. F. Dimentberg, Statistical Dynamics of Non-linear and Time-varying Systems, John Wiley and Sons, New York, 1988.

[6] C.S. Holling, The Functional Response of Predators to Prey Density and its Role in Mimicry and Population Regulation, Mem. Entomol. Soc. Can. 97 (1965), 5–60.

[7] R. Arditi, L.R. Ginzburg, H.R. Akcakaya, Variation in plankton densities among lakes: a case for ratio dependent models. Amer. Naturalist, 138 (1991), 1287–1296.

[8] R. Arditi, H. Saiah, Empirical Evidence of the Role of Heterogeneity in Ratio-Dependent Consumption, Ecology. 73 (1992), 1544–1551.

[9] A.P. Gutierrez, Physiological Basis of Ratio-Dependent Predator-Prey Theory: The Metabolic Pool Model as a Paradigm, Ecology. 73 (1992), 1552–1563.

[10] Y. Kuang, E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, J. Math. Biol. 36 (1998), 389–406.

[11] A. Maiti, G.P. Samanta, Deterministic and stochastic analysis of a prey-dependent predator–prey system, Int. J. Math. Educ. Sci. Technol. 36 (2005), 65–83.

[12] Maiti, G.P. Samanta, Deterministic and stochastic analysis of a ratio-dependent prey-predator system, Int. J. Syst. Sci. 37 (2006), 817–826.

[13] G.P. Samanta, A. Maiti, Stochastic Gomatam Model of Interacting Species: Non-Equilibrium Fluctuation and Stability, Syst. Anal. Model. Simul. 43 (2003), 683–692.

[14] G.P. Samanta, Fluctuation and stability in a diffusive predator-prey system, Appl. Math. Lett. 3 (1990), 115–118.

[15] G. P. Samanta, Stochastic analysis of a noisy oscillator, Appl. Math. Lett. 4 (1991), 61- 63.

[16] G.P. Samanta, Complex stochastic averaging approach to a two-state predator system, Int. J. Math. Educ. Sci. Technol. 23 (1992), 739–744.

[17] G.P. Samanta, Influence of environmental noise in Gompertzian growth model. J. Math. Phys. Sci, 26 (1992), 503-511.

[18] G.P. Samanta, Logistic growth under colored noise, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. 37 (1993), 115-122.

[19] G.P. Samanta, Stochastic analysis of a prey-predator system, Int. J. Math. Educ. Sci. Technol. 25 (1994), 797–803.

[20] G.P. Samanta, Influence of environmental noises on the Gomatam model of interacting species, Ecol. Model. 91 (1996), 283–291.

[21] M.C. Baishya, C.G. Chakrabarti, Non-equilibrium fluctuation in Lotka-Volterra system, Bull. Math. Biol. 49(1987), 125–131.

[22] L. Zhang, W. Wang, Y. Xue, Z. Jin, Complex dynamics of a Holling-type IV predator-prey model, ArXiv:0801.4365 [q-Bio]. (2008).

[23] J.D. Murray, Mathematical Biology, Springer-Verlag, New York, 1993.

[24] G. Brikhoff, G.C. Rota, Ordinary Differential Equations, Ginn, Boston, 1982.

[25] M. Bandyopadhyay, C.G. Chakrabarti, Deterministic and stochastic analysis of a nonlinear prey-predator system, J. Biol. Syst. 11 (2003), 161–172.

[26] T.C. Gard, Uniform persistence in multispecies population models, Math. Biosci. 85 (1987), 93–104.

[27] W. Horsthemke, Noise Induced Transitions, in: C. Vidal, A. Pacault (Eds.), Non-Equilibrium Dynamics in Chemical Systems, Springer Berlin Heidelberg, Berlin, Heidelberg, 1984: pp. 150–160.

[28] G. Nicolis, L. Prigogine, Self-organisation in non-equilibrium Systems, Wiley, New York, 1977.

[29] A. Maiti, M.M. Jana, G.P. Samanta, Deterministic and Stochastic Analysis of a Ratio-Dependent Predator-Prey System with Delay, Nonlinear Anal., Model. Control. 12 (2007), 383–398.

[30] Maiti, S. Pathak, A Modified Holling-Tanner Model in Stochastic Environment, Nonlinear Anal., Model. Control. 14 (2009), 51–71.

[31] M. Kot, Elements of Mathematical Ecology, Cambridge University Press, Cambridge, U.K. 2001.

[32] H. Freedman, G. Wolkowicz, Predator-prey systems with group defence: The paradox of enrichment revisited, Bull. Math. Biol. 48 (1986), 493–508.

[33] M. Rosenzweig, R. MacArthur, Graphical representation and stability conditions of predator-prey interactions, AmermNaturalist, 97 (1963), 209-223.