OPTIMAL CONTROL OF A COMMUNITY VIOLENCE MODEL: COMMUNITY VIOLENCE TREATED AS A CONTAGIOUS DISEASE

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Abstract. Violence was, for a long time, misunderstood and misdiagnosed. This misdiagnosis led to ineffective and counterproductive treatments and control strategies. Recent advances in neuroscience and epidemiology show that violence is a contagious disease. In this paper, community violence is treated as an infection that spreads from person to person through victimization or through witnessing violence. A compartmental model is used for formulating the spread of cummunity violence as a system of differential equations. The distribution of treated individuals is considered as a control variable. Our objective is to characterize an optimal control (treatment) that minimizes the number of individuals who use violence and the cost associated with this treatment. A numerical simulation analysis is used to confirm the effectiveness of our control.

Keywords: SHV model; SIR model; community violence; infectious disease; epidemic disease; contagious disease.

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1. INTRODUCTION

Violence is the utilization of actual power to harm, misuse, harm, or destroy [1]. Community violence covers viciousness among colleagues and outsiders, and incorporates youth brutality; attack by outsiders; violence related with vandalism related misdemeanors; and violence in working environments and different institutions [2]. There is a solid connection between degrees of violence and modifiable factors in a society, for example, concentrated poverty, earning and gender inequality, the destructive utilization of alcohol, and the absence of safe, stable, and supporting family climate. Moreover, violence frequently has deep rooted ramifications for physical and emotional wellness and social working and can slow economic and social development [1]. Violence meets the word reference meaning of a disease, and many studies currently affirm that violence is contagious [3]. The specific contagion of violence is started by victimization or visual exposure and intervened by the brain, similarly as the lungs intercede replication of tuberculosis or the intestines cholera. The brain processes violence exposure into scripts, or replicated practices, and oblivious social assumptions. This processing can likewise prompt a few situationally versatile reactions including aggression, impulsivity, depression, stress, excessive startle reactions, and changes in neurochemistry [4]. [5] used compartmental model to describe the spread of criminal gang membership, and [6] used compartmental model for the analysis of domestic violence. But none of them incorporated the spatial diffusion which plays an important role into the spread of violence. In this paper, we treat community violence as an infection that multiplies through exposure (victimization or witnessing violence). The violence is controlled by identifying and helping individuals at the highest risk factors (drugs, alcohol, poverty, poor education, family structure, ... [4]) to make them more averse to submit viciousness by talking in their terms, examining the expenses of utilizing violence, and assisting them with getting the help and social services (e.g., education, job training, drug treatment) towards behavior change and changes in life course [7]. The rest of this paper is structured as follows: In section 2, we show the mathematical model. Section 3 is about the associated optimal control problem. In section 4 we prove the existence of an optimal solution. Then we formulate the necessary optimality conditions in section 5. The numerical results are showed in section 6. Finally, we give the conclusion of the paper in the 7th section.

2. MATHEMATICAL MODEL

In this work, the total population (T) is divided into three compartments: (S) susceptible individuals, (H) susceptible individuals with high risk factors to use violence and (V) individuals who use violence. Let $T(t)$ be the total population at an instant $t \in [0, \tau]$.

We assume that:

• Violence is contageous,

• An individual can only be infected by violence through contact with violent individuals,

• The population is uniformly mixed,

• The population tend to move to regions,

• The densities depend on time and position in space since the population tend to move to regions (leading to the notations $S(t,x)$, $H(t,x)$ and $V(t,x)$).

Let us define some parameters used in this model:

μ: Natural birth rate

d : Natural death rate

α_1: Rate of becoming (H) individual (some (S) individuals move to the (H) compartment)

α_2: Transmission rate for (S) individuals

α_3: Transmission rate for (H) individuals ($\alpha_3 > \alpha_2$)

r : Recovery rate (recovering from violence)

β : Rehabilitation rate (from (V) to (S)) (some individuals who recover from violence return to the (S) compartment)

c : Rehabilitation rate (from (H) to (S)) (some (H) individuals return to the (S) compartment)

The susceptible individuals can become violent at a rate $\alpha_2 \frac{VS}{T}$. The rate at which susceptible individuals with high risk factors become violent is $\alpha_3 \frac{VH}{T}$. Some violent individuals return to the previous states at a rate rV and the proportion of those who become susceptible with high risk factors is $(1 - \beta)$. Some susceptible individuals with high risk factors return to the susceptible state at a rate cH (without the intervention seen above) and in the other direction the rate at which a susceptible individual move to the second state is $\alpha_1 S$.

We obtain the following system of reaction-diffusion equations as a spatiotemporal SHV model for the spread of community violence:

(1)
$$\begin{cases} \frac{\partial S}{\partial t} = \lambda_1 \Delta S + \mu R - \alpha_1 S - \alpha_2 \frac{VS}{T} + cH + \beta rV - dS \\\\ \frac{\partial H}{\partial t} = \lambda_2 \Delta H - cH + \alpha_1 S - \alpha_3 \frac{VH}{T} + (1-\beta)rV - dH \\\\ \frac{\partial V}{\partial t} = \lambda_3 \Delta V + \alpha_2 \frac{VS}{T} + \alpha_3 \frac{VH}{T} - rV - dV, \ (t,x) \in Q = [0,\tau] \times \Omega \end{cases}$$

with the homogeneous Neumann boundary conditions

(2)
$$\frac{\partial S}{\partial \eta} = \frac{\partial H}{\partial \eta} = \frac{\partial V}{\partial \eta} = 0, \quad (t,x) \in [0,\tau] \times \partial\Omega$$

where Ω is a fixed and bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, the time t belongs to a finite interval $[0,\tau]$, while x varies in Ω.

The initial distribution of the three populations is supposed to be:

(3)
$$S(0,x) = S_0 > 0, \quad H(0,x) = H_0 > 0 \quad \text{and} \quad V(0,x) = V_0 > 0, \quad x \in \Omega$$

3. OPTIMAL CONTROL PROBLEM

In this paper, the community violence is controlled by identifying and helping susceptible individuals with high risk factors to obtain the support and social services, so we include a control u in model (1) where $u(t,x)$ represents the density of beneficiaries per time unit and space, and we assume that they are transfered directly and immediatly to the susceptible class. The controlled system is given by:

(4)
$$\begin{cases} \frac{\partial S}{\partial t} = \lambda_1 \Delta S + \mu R - \alpha_1 S - \alpha_2 \frac{VS}{T} + cH + \beta rV - dS + uH \\\\ \frac{\partial H}{\partial t} = \lambda_2 \Delta H - cH + \alpha_1 S - \alpha_3 \frac{VH}{T} + (1-\beta)rV - dH - uH \\\\ \frac{\partial V}{\partial t} = \lambda_3 \Delta V + \alpha_2 \frac{VS}{T} + \alpha_3 \frac{VH}{T} - rV - dV, \quad (t,x) \in Q = [0,\tau] \times \Omega \end{cases}$$

(5)
$$\frac{\partial S}{\partial \eta} = \frac{\partial H}{\partial \eta} = \frac{\partial V}{\partial \eta} = 0, \quad (t,x) \in [0,\tau] \times \partial\Omega$$

(6) $S(0,x) = S_0 > 0, \quad H(0,x) = H_0 > 0 \quad \text{and} \quad V(0,x) = V_0 > 0, \quad x \in \Omega$

In this current work, we want to minimize the density of the violent individuals and the cost of treating susceptible individuals. The objective functional can be given by:

(7) $J(S,H,V,u) = \frac{1}{2}\|V\|_{L^2(Q)}^2 + \frac{1}{2}\|V(\tau,.)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(Q)}^2$

where u belongs to the set U_{ad} of admissible controls

(8) $U_{ad} = \left\{ u \in U; \ \|u\|_{L^\infty(Q)} \le u_{\max} \right\}$

with $U_{ad} = \left\{ u \in U; \ \|u\|_{L^\infty(Q)} \le u_{max} \right\}$ is a given positive constant and

$$U = \left\{ u \in L^\infty(Q); \ \|u\|_{L^\infty(Q)} < 1 \ and \ u \ge 0 \right\}$$

4. EXISTENCE OF GLOBAL SOLUTION

In this section, we will prove the existence of a global strong solution of the problem (4)–(6). We denote :

$$H(\Omega) = \left(L^2(\Omega)\right)^3 \quad and \quad L(\tau,\Omega) = L^2\left(0,\tau;H^2(\Omega)\right) \cap L^\infty\left(0,\tau;H^1(\Omega)\right)$$
$$y = (y_1,y_2,y_3) = (S,H,V) \quad with \quad y^0 = (y_1^0,y_2^0,y_3^0) = (S_0,H_0,V_0)$$

and let A be the linear operator defined as follows:

$$A : D(A) \subset H(\Omega) \to H(\Omega)$$

(9) $Ay = (\lambda_1 \Delta y_1, \lambda_2 \Delta y_2, \lambda_3 \Delta y_3) \in D(A)$

$$\forall y = (y_1,y_2,y_3) \in D(A)$$

(10) $D(A) = \left\{ y = (y_1,y_2,y_3) \in \left(H^2(\Omega)\right)^3, \frac{\partial y_1}{\partial \eta} = \frac{\partial y_2}{\partial \eta} = \frac{\partial y_3}{\partial \eta} = 0, \text{a.e } x \in \partial \Omega \right\}$

If we consider the function f defined by :

$$f(y(t)) = (f_1(y(t)), \ f_2(y(t)), \ f_3(y(t)))$$

with

(11)
$$
\begin{cases}
f_1(y(t)) = \mu(y_1 + y_2 + y_3) - \alpha_2 \frac{y_1 y_3}{T} + \beta r y_3 - (\alpha_1 + d) y_1 + (c + u) y_2 \\[3em]
f_2(y(t)) = \alpha_1 y_1 - \alpha_3 \frac{y_2 y_3}{T} + (1 - \beta) r y_3 - (d + c + u) y_2 \\[3em]
f_3(y(t)) = \alpha_2 \frac{y_1 y_3}{T} + \alpha_3 \frac{y_2 y_3}{T} - (r + d) y_3
\end{cases}
$$

then the problem (4)–(6) can be written in the space $H(\Omega)$ under the form:

(12)
$$
\begin{cases}
\frac{\partial y}{\partial t} = Ay(t) + f(y(t)) \\[1em]
\qquad\qquad\qquad\qquad , t \in [0, \tau] \\[1em]
y(0) = y^0
\end{cases}
$$

Theorem 1. *Let Ω be a bounded domain from $\mathbb{R}^2$, with the boundary of class C^λ, $\lambda > 2$. If $\alpha_1, \alpha_2, \alpha_3, r, c, \mu, d > 0$, $0 < \beta < 1$, $u \in U$, $y^0 \in D(A)$ and $y_i^0 \geq 0$ on Ω for $i = 1, 2, 3$, then the problem (4)–(6) has a unique (global) strong solution such that: $y \in W^{1,2}(0, \tau; H(\Omega))$, $y_i \in L(\tau, \Omega) \cap L^\infty(Q)$ and $y_i \geq 0$ on Q for $i = 1, 2, 3$.*

In addition , there exists $C > 0$ independant of u such that for all $t \in [0, \tau]$:

(13)
$$
\left\| \frac{\partial y_i}{\partial t} \right\|_{L^2(Q)} + \| y_i \|_{L^2(0, \tau; H^2(\Omega))} + \| y_i \|_{H^1(\Omega)} + \| y_i \|_{L^\infty(Q)} \leq C \quad for \quad i = 1, 2, 3
$$

Proof. The function f is Lipschitz continuous in $y = (y_1, y_2, y_3)$ uniformly with respect to $t \in [0, \tau]$. Since the operator A is self-adjoint and dissipative on $H(\Omega)$, it follows that the problem (4)–(6) admits a unique strong solution $y \in W^{1,2}(0, \tau; H(\Omega))$ with $y_i \in L^2(0, \tau; H^2(\Omega))$ for $i = 1, 2, 3$.

Let us prove that $y_i \in L^\infty(Q)$ for $i = 1, 2, 3$.

We denote $M_i = \max \left\{ \| f_i \|_{L^\infty(\Omega)}, \| y_i^0 \|_{L^\infty(\Omega)} \right\}$ for $i = 1, 2, 3$.

Let $\{ S_i(t), t \geq 0 \}$ be the C_0−semi-group generated by the operator B_i defined as follows:

$B_i : D(B_i) \subset L^2(\Omega) \to L^2(\Omega)$

$B_i z = \lambda_i \Delta z$

$D(B_i) = \left\{ z \in H^2(\Omega), \frac{\partial z}{\partial \eta} = 0, \text{ a.e } in \ \partial\Omega \right\}$

It is easy to see that, for $x \in \Omega$, the function $g_i(t,x) = y_i - M_i t - \left\| y_i^0 \right\|_{L^\infty(\Omega)}$ satisfies the Cauchy problem :

(14)
$$
\begin{cases}
\frac{\partial g_i}{\partial t}(t,x) = \lambda_i \Delta g_i + f_i(y(t)) - M_i \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad , t \in [0,\tau] \\
g_i(0,x) = y_i^0 - \left\| y_i^0 \right\|_{L^\infty(\Omega)}
\end{cases}
$$

and the function defined by $h_i(t,x) = y_i + M_i t + \left\| y_i^0 \right\|_{L^\infty(\Omega)}$ satisfies the Cauchy problem :

(15)
$$
\begin{cases}
\frac{\partial h_i}{\partial t}(t,x) = \lambda_i \Delta h_i + f_i(y(t)) + M_i \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad , t \in [0,\tau] \\
h_i(0,x) = y_i^0 + \left\| y_i^0 \right\|_{L^\infty(\Omega)}
\end{cases}
$$

then $g_i(t,x) = S_i(t)\left(y_i^0 - \left\| y_i^0 \right\|_{L^\infty(\Omega)} \right) + \int_0^t S_i(t-s)(f_i(y(t)) - M_i)ds$ and
$h_i(t,x) = S_i(t)\left(y_i^0 + \left\| y_i^0 \right\|_{L^\infty(\Omega)} \right) + \int_0^t S_i(t-s)(f_i(y(t)) + M_i)ds$

Since $y_i^0 - \left\| y_i^0 \right\|_{L^\infty(\Omega)} \leq 0$, $f_i(y(t)) - M_i \leq 0$, $y + \left\| y_i^0 \right\|_{L^\infty(\Omega)} \geq 0$, $f_i(y(t)) + M_i \geq 0$, it follows that: $(\forall (t,x) \in Q)$, $g_i(t,x) \leq 0$ and $(\forall (t,x) \in Q)$, $h_i(t,x) \geq 0$. Then: $(\forall (t,x) \in Q)$, $|y_i(t,x)| \leq Mt + \left\| y_i^0 \right\|_{L^\infty(\Omega)}$. Thus we have proved that, for $i = 1,2,3$, $y_i \in L^\infty(Q)$.

By the system (4), we know that :

(16)
$$
\begin{cases}
\frac{\partial y_1}{\partial t} = \lambda_1 \Delta y_1 + f_1(y_1,y_2,y_3) \\
\\
\frac{\partial y_2}{\partial t} = \lambda_2 \Delta y_2 + f_2(y_1,y_2,y_3), \ (t,x) \in Q \\
\\
\frac{\partial y_3}{\partial t} = \lambda_3 \Delta y_3 + f_3(y_1,y_2,y_3)
\end{cases}
$$

Since the functions $f_1(y_1,y_2,y_3)$, $f_2(y_1,y_2,y_3)$ and $f_3(y_1,y_2,y_3)$ are continuously differentiable satisfying, for all $y_1,y_2,y_3 \geq 0$, $f_1(0,y_2,y_3) = \mu(y_2 + y_3) + \beta r y_3 + (c+u)y_2 \geq 0$, $f_2(y_1,0,y_3) = \alpha_1 y_1 + (1-\beta)ry_3 \geq 0$ and $f_3(y_1,y_2,0) = 0$, we deduce (see [[8]]) that $y_1(t,x) \geq 0$, $y_2(t,x) \geq 0$ and $y_3(t,x) \geq 0$.

Finally, let us prove that $y_1 \in L^\infty(0,\tau;H^1(\Omega))$. By the first equation of (4), we obtain :

$$
\int_0^t \int_\Omega \left| \frac{\partial y_1}{\partial s} \right|^2 dsdx + \lambda_1^2 \int_0^t \int_\Omega |\Delta y_1|^2 dsdx - 2\lambda_1 \int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \Delta y_1 dsdx
$$
$$
= \int_0^t \int_\Omega \left(\mu(y_1 + y_2 + y_3) - \alpha_1 y_1 - \alpha_2 \frac{y_1 y_3}{T} + \beta r y_3 - dy_1 + (c+u)y_2 \right)^2 dsdx
$$

Using the Green's formula and knowing that , we have :

$$\int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \Delta y_1 \, ds dx = -\frac{1}{2} \int_\Omega \left(|\nabla y_1|^2 - |\nabla y_1^0|^2 \right) dx$$

then

$$\int_0^t \int_\Omega \left| \frac{\partial y_1}{\partial s} \right|^2 ds dx + \lambda_1^2 \int_0^t \int_\Omega |\Delta y_1|^2 ds dx + \lambda_1 \int_\Omega |\nabla y_1|^2 dx - \lambda_1 \int_\Omega |\nabla y_1^0|^2 dx$$
$$= \int_0^t \int_\Omega \left(\mu(y_1 + y_2 + y_3) - \alpha_1 y_1 - \alpha_2 \frac{y_1 y_3}{R} + \beta r y_3 - d y_1 + (c+u) y_2 \right)^2 ds dx$$

It follows that :

$$\lambda_1 \int_\Omega |\nabla y_1|^2 dx \leq \int_0^t \int_\Omega \left(\mu R - \alpha_1 y_1 - \alpha_2 \frac{y_1 y_3}{R} + \beta r y_3 - d y_1 + (c+u) y_2 \right)^2 ds dx$$
$$+ \lambda_1 \int_\Omega |\nabla y_1^0|^2 dx$$

and since $y_1^0 \in H^2(\Omega)$ and $y_1, y_2, y_3 \in L^\infty(Q)$ (with y_1, y_2, y_3 bounded independently of u), we deduce that $y_1 \in L^\infty(0, \tau; H^1(\Omega))$. We can prove similarly that $y_2, y_3 \in L^\infty(0, \tau; H^1(\Omega))$. Thus the inequality (13) holds for $i = 1, 2, 3$.

$\square$

5. EXISTENCE OF OPTIMAL SOLUTION

In this section, we will prove the existence of an optimal solution of the problem (4)-(8).

Theorem 2. *Let Ω be a bounded domain from $\mathbb{R}^2$, with the boundary of class C^λ, $\lambda > 2$. If $\alpha_1, \alpha_2, \alpha_3, r, c, \mu, d > 0$, $0 < \beta < 1$, $y^0 \in D(A)$ and $y_i^0 \geq 0$ on Ω for $i = 1, 2, 3$, then the problem (4)–(8) has an optimal solution (y^*, u^*).*

Proof. For every $u \in U_{ad}$, there exists a unique solution y to the problem (4)–(6) (see Theorem 1).

Let

$$(17) \qquad\qquad\qquad J^* = \inf_{u \in U_{ad}} \{ J(y, u) \}$$

Since J^* is finite, there exists a sequence (y^n, u^n) such that, for all $n \geq 1$:

$$(18) \qquad\qquad u^n \in U_{ad}, \; y^n \in W^{1,2}(0, \tau; H(\Omega)), \; J^* \leq J(y^n, u^n) \leq J^* + \frac{1}{n}$$

and

(19)
$$
\begin{cases}
\dfrac{\partial y_1^n}{\partial t} = \lambda_1 \Delta y_1^n + \mu(y_1^n + y_2^n + y_3^n) - \alpha_2 \dfrac{y_1^n y_3^n}{y_1^n + y_2^n + y_3^n} + \beta r y_3^n - (\alpha_1 + d) y_1^n \\
\qquad\qquad + (c + u^n) y_2^n \\[2em]
\dfrac{\partial y_2^n}{\partial t} = \lambda_2 \Delta y_2^n + \alpha_1 y_1^n - \alpha_3 \dfrac{y_2^n y_3^n}{y_1^n + y_2^n + y_3^n} + (1 - \beta) r y_3^n - (d + c + u^n) y_2^n \\[2em]
\dfrac{\partial y_3^n}{\partial t} = \lambda_3 \Delta y_3^n + \alpha_2 \dfrac{y_1^n y_3^n}{y_1^n + y_2^n + y_3^n} + \alpha_3 \dfrac{y_2^n y_3^n}{y_1^n + y_2^n + y_3^n} - (r + d) y_3^n \quad , (t,x) \in Q
\end{cases}
$$

(20)
$$
\frac{\partial y_1^n}{\partial \eta} = \frac{\partial y_2^n}{\partial \eta} = \frac{\partial y_3^n}{\partial \eta} = 0 \quad , (t,x) \in [0,\tau] \times \partial\Omega
$$

(21)
$$
y_1^n(0,x) = y_1^0 > 0, y_2^n(0,x) = y_2^0 > 0 \text{ and } y_3^n(0,x) = y_3^0 > 0 \quad , x \in \Omega
$$

We know that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, then, for all $t \in [0,\tau]$, $\{y_1^n(t), n \geq 1\}$ is relatively compact in $L^2(\Omega)$ for all $t \in [0,\tau]$. Let us prove that $\{y_1^n, n \geq 1\}$ is equicontinuous at each $t \in [0,\tau]$. From the first equation of (19), we obtain, for all $t \in [0,\tau]$:

(22)
$$
\int_0^t \int_\Omega \frac{\partial y_1^n}{\partial t} y_1^n dx ds = \int_0^t \int_\Omega \Big[\lambda_1 (\Delta y_1^n) y_1^n + \mu(y_1^n + y_2^n + y_3^n) y_1^n
$$
$$
- \alpha_2 \frac{(y_1^n)^2 y_3^n}{y_1^n + y_2^n + y_3^n} + \beta r y_1^n y_3^n - (\alpha_1 + d)(y_1^n)^2 + (c + u^n) y_1^n y_2^n \Big] dx ds
$$

then

(23)
$$
\int_\Omega (y_1^n)^2(t,x) dx = \int_\Omega (y_1^0)^2(x) dx + 2 \int_0^t \int_\Omega \Big[\lambda_1 (\Delta y_1^n) y_1^n + \mu(y_1^n + y_2^n + y_3^n) y_1^n
$$
$$
- \alpha_2 \frac{(y_1^n)^2 y_3^n}{y_1^n + y_2^n + y_3^n} + \beta r y_1^n y_3^n - (\alpha_1 + d)(y_1^n)^2 + (c + u^n) y_1^n y_2^n \Big] dx ds, \forall t \in [0,\tau]
$$

and by Theorem 1, there exists a constant $C > 0$ independant of u such that for all $n \geq 1$ and $t \in [0,\tau]$:

(24)
$$
\left\| \frac{\partial y_i^n}{\partial t} \right\|_{L^2(Q)} \leq C, \ \|y_i^n\|_{L^2(0,\tau;H^2(\Omega))} \leq C, \ \|y_i^n\|_{H^1(\Omega)} \leq C \quad for \quad i = 1,2,3
$$

It follows that, for all $n \geq 1$ and $t \in [0, \tau]$:

$$(25) \qquad \left| \int_\Omega (y_1^n)^2(t,x)dx - \int_\Omega (y_1^n)^2(s,x)dx \right| \leq K|t-s|$$

We deduce that $\{y_1^n, n \geq 1\}$ is equicontinuous at each $t \in [0, \tau]$. The Ascoli-Arzela Theorem (see [[9]]) implies that $\{y_1^n, n \geq 1\}$ is relatively compact in $C\left([0, \tau]; L^2(\Omega)\right)$. Hence, selecting further sequences, if necessary, we have :

$y_1^n \to y_1^*$ in $L^2(\Omega)$ uniformly with respect to t, and analogously:

$y_i^n \to y_i^*$ in $L^2(\Omega)$ uniformly with respect to t for $i = 2, 3$.

Since the sequence Δy_i^n is bounded in $L^2(Q)$ then it has a weakly convergent subsequence, denoted again Δy_i^n, in $L^2(Q)$. For all distribution φ,

$$(26) \qquad \int_Q \varphi \Delta y_i^n = \int_Q y_i^n \Delta \varphi \to \int_Q y_i^* \Delta \varphi = \int_Q \varphi \Delta y_i^*$$

then

$\Delta y_i^n \to \Delta y_i^*$ weakly in $L^2(Q)$ for $i = 1, 2, 3$.

From (24) and Theorem 1, we see that :

$\frac{\partial y_i^n}{\partial t} \to \frac{\partial y_i^*}{\partial t}$ weakly in $L^2(Q)$ for $i = 1, 2, 3$

$y_i^n \to y_i^*$ weakly in $L^2(0, \tau; H^2(\Omega))$ for $i = 1, 2, 3$

$y_i^n \to y_i^*$ weakly in $L^\infty(0, \tau; H^1(\Omega))$ for $i = 1, 2, 3$.

Since u^n is bounded in $L^2(Q)$ then it has a weakly convergent subsequence, denoted again u^n, so $u^n \to u^*$ weakly in $L^2(Q)$ and $u^n y_2^n \to u^* y_2^*$ weakly in $L^2(Q)$.

We also know that U_{ad} is a closed and convex set in $L^2(Q)$. It follows that U_{ad} is weakly closed, so $u^* \in U_{ad}$.

By passing to the limit in $L^2(Q)$ as $n \to \infty$ in (18)-(21), we deduce that y^* is the solution of (1) –(3) corresponding to u^*. And since $J(y^*, u^*) \leq \inf_{u \in U_{ad}} J(y, u)$, then (y^*, u^*) minimizes (7).

$\square$

6. NECESSARY OPTIMALITY CONDITIONS

Let us prove first, that the mapping $u \to y(u)$ is Gateaux sifferentiable with respect to u^* (where $y(u)$ is the corresponding solution of (1) –(3) corresponding to u).

Let (y^*, u^*) be an optimal pair, $u \in U$ and $u^{\mathcal{E}} = u^* + \varepsilon u \in U$ $(\varepsilon > 0)$. From Theorem 1, the problem (1)–(3) admits a unique solution $y^{\mathcal{E}} = y(u^{\mathcal{E}})$ corresponding to $u^{\mathcal{E}}$. Let $z^{\mathcal{E}} = \frac{1}{\varepsilon}(y^{\mathcal{E}} - y^*)$ and put, for $i, j \in \{1, 2, 3\}$:

$$\Lambda(y_1, y_2, y_3) = \frac{y_1 y_2}{y_1 + y_2 + y_3}$$

(27)
$$\begin{cases} M_{ij}^{\mathcal{E}} = \frac{\Lambda(y_i^{\mathcal{E}}, y_j^{\mathcal{E}}, y_k^{\mathcal{E}}) - \Lambda(y_i^*, y_j^{\mathcal{E}}, y_k^{\mathcal{E}})}{y_i^{\mathcal{E}} - y_i^*} & ; i < j \text{ and } k \neq i, j \\[3mm] M_{ij}^{\mathcal{E}} = \frac{\Lambda(y_i^{\mathcal{E}}, y_j^*, y_k^{\mathcal{E}}) - \Lambda(y_i^*, y_j^*, y_k^*)}{y_i^{\mathcal{E}} - y_i^*} & ; i > j \text{ and } k \neq i, j \end{cases}$$

We get, by subtracting the corresponding system (1) – (3) to u^* from the corresponding system (1) – (3) to $u^{\mathcal{E}}$:

(28)
$$\begin{cases} \frac{\partial z_1^{\mathcal{E}}}{\partial t} = \lambda_1 \Delta z_1^{\mathcal{E}} + \left(\mu - \alpha_1 - \alpha_2 M_{13}^{\mathcal{E}} - d\right) z_1^{\mathcal{E}} + \left(\mu + c + u^{\mathcal{E}}\right) z_2^{\mathcal{E}} \\ \qquad + \left(\mu - \alpha_2 M_{31}^{\mathcal{E}} + \beta r\right) z_3^{\mathcal{E}} + u y_2^* \\[3mm] \frac{\partial z_2^{\mathcal{E}}}{\partial t} = \lambda_2 \Delta z_2^{\mathcal{E}} + \alpha_1 z_1^{\mathcal{E}} - \left(\alpha_3 M_{23}^{\mathcal{E}} + c + d + u^{\mathcal{E}}\right) z_2^{\mathcal{E}} \quad ; (t, x) \in Q \\ \qquad + \left((1 - \beta)r - \alpha_3 M_{32}^{\mathcal{E}}\right) z_3^{\mathcal{E}} - u y_2^* \\[3mm] \frac{\partial z_3^{\mathcal{E}}}{\partial t} = \lambda_3 \Delta z_3^{\mathcal{E}} + \alpha_2 M_{13}^{\mathcal{E}} z_1^{\mathcal{E}} + \alpha_3 M_{23}^{\mathcal{E}} z_2^{\mathcal{E}} + \left(\alpha_2 M_{31}^{\mathcal{E}} + \alpha_3 M_{32}^{\mathcal{E}} - r - d\right) z_3^{\mathcal{E}} \end{cases}$$

(29)
$$\frac{\partial z_1^{\mathcal{E}}}{\partial \eta} = \frac{\partial z_2^{\mathcal{E}}}{\partial \eta} = \frac{\partial z_3^{\mathcal{E}}}{\partial \eta} = 0 \quad , (t, x) \in [0, \tau] \times \partial \Omega$$

(30)
$$z_1^{\mathcal{E}}(0, x) = 0 \quad , z_2^{\mathcal{E}}(0, x) = 0 \quad \text{and} \quad z_3^{\mathcal{E}}(0, x) = 0 \quad , \quad x \in \Omega$$

To show that $z_i^{\mathcal{E}}$ are bounded in $L^2(Q)$ uniformly with respect to ε, we denote:

$$E^{\mathcal{E}} = \begin{pmatrix} \mu - \alpha_1 - \alpha_2 M_{13}^{\mathcal{E}} - d & \mu + c + u^{\mathcal{E}} & \mu - \alpha_2 M_{31}^{\mathcal{E}} + \beta r \\[2mm] \alpha_1 & -\alpha_3 M_{23}^{\mathcal{E}} - c - d - u^{\mathcal{E}} & (1 - \beta)r - \alpha_3 M_{32}^{\mathcal{E}} \\[2mm] \alpha_2 M_{13}^{\mathcal{E}} & \alpha_3 M_{23}^{\mathcal{E}} & \alpha_2 M_{31}^{\mathcal{E}} + \alpha_3 M_{32}^{\mathcal{E}} - r - d \end{pmatrix}$$

and $F = \begin{pmatrix} y_2^* \\ -y_2^* \\ 0 \end{pmatrix}$

then (28) – (30) can be written as:

$$
(31) \qquad \begin{cases} \frac{\partial z^\varepsilon}{\partial t} = Az^\varepsilon + E^\varepsilon z^\varepsilon + Fu \\ \\ z^\varepsilon(0) = 0 \end{cases} , \ t \in [0, \tau]
$$

The solution of this problem is given by:

$$
(32) \qquad z^\varepsilon(t) = \int_0^t S(t-s)E^\varepsilon(s)z^\varepsilon(s)ds + \int_0^t S(t-s)Fu(s)ds
$$

Since the coefficients of the matrix E^ε are bounded uniformly with respect to ε and using Gronwall's inequality, we deduce that there exists a constant $C_1 > 0$ such that:

$$
(33) \qquad \|z_i^\varepsilon\|_{L^2(Q)} \leq C_1 \qquad for \quad i = 1, 2, 3
$$

Thus, z_i^ε are bounded in $L^2(Q)$ uniformly with respect to ε.

And since we have:

$$
(34) \qquad \|y_i^\varepsilon - y_i^*\|_{L^2(Q)} = \varepsilon \|z_i^\varepsilon\|_{L^2(Q)}
$$

then $y_i^\varepsilon \underset{\varepsilon \to 0}{\to} y_i^*$ in $L^2(Q)$ for $i = 1, 2, 3$.

By denoting :

$$
(35) \qquad M_{ij}^* = \frac{\partial \Lambda}{\partial y_1}\left(y_i^*, y_j^*, y_k^*\right) \quad for\ i \neq j\ ,\ k \neq i\ and\ k \neq j
$$

$$(36) \quad E = \begin{pmatrix} \mu - \alpha_1 - \alpha_2 M_{13}^* - d & \mu + c + u^* & \mu - \alpha_2 M_{31}^* + \beta r \\[2em] \alpha_1 & -\alpha_3 M_{23}^* - c - d - u^* & (1-\beta)r - \alpha_3 M_{32}^* \\[2em] \alpha_2 M_{13}^* & \alpha_3 M_{23}^* & \alpha_2 M_{31}^* + \alpha_3 M_{32}^* - r - d \end{pmatrix}$$

the system

$$(37) \qquad \begin{cases} \frac{\partial z}{\partial t} = Az + Ez + Fu \\[1em] \qquad\qquad\qquad\qquad , t \in [0, \tau] \\[1em] z(0) = 0 \end{cases}$$

admits a unique solution given by :

$$(38) \qquad z(t) = \int_0^t S(t-s)E(s)z(s)ds + \int_0^t S(t-s)Fu(s)ds$$

By subtracting (38) from (32) we obtain :

$$(39) \qquad z^\varepsilon(t) - z(t) = \int_0^t S(t-s)\left[E^\varepsilon(s)\left(z^\varepsilon - z\right) + \left(E^\varepsilon(s) - E(s)\right)z(s)\right]ds$$

Since all the the elements of the matrix E^ε tend to the corresponding elements of the matrix E in $L^2(Q)$ and by using Gronwall's inequality, it follows that:

$$(40) \qquad z_i^\varepsilon \underset{\varepsilon \to 0}{\to} z_i \quad in \quad L^2(Q) \quad for \quad i = 1, 2, 3$$

Thus, we have proved the following result:

Proposition 3. *The mapping* $y : U \to W^{1,2}(0, \tau; H(\Omega))$, *with the conditions of the theorem 1, is Gateaux differentiable with respect to* u^*. *For* $u \in U$, $z = y'(u^*)u$ *is the unique solution in* $W^{1,2}(0, \tau; H(\Omega))$ *of the following problem :*

$$(41) \qquad \begin{cases} \frac{\partial z}{\partial t} = Az + Ez + Fu \\[1em] \qquad\qquad\qquad\qquad , t \in [0, \tau] \\[1em] z(0, x) = 0 \end{cases}$$

Moreover, the dual system associated to the system (4)-(8) is :

(42)
$$\begin{cases} -\frac{\partial p}{\partial t} - Ap - E^*p = N^*Ny^* \\ \\ p(\tau,x) = N^*Ny^*(\tau,x) \end{cases} \quad , t \in [0,\tau]$$

where $p = (p_1, p_2, p_3)$ is the adjoint variable, (y^*, u^*) is the optimal pair, and the matrix is

defined by : $N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Theorem 4. *Under hypotheses of Theorem 1, if (y^*, u^*) is an optimal pair, then the system (38) admits a unique solution $p \in W^{1,2}(0, \tau; H(\Omega))$ with $p_i \in L(\tau, \Omega)$ for $i = 1, 2, 3$. Moreover,*

(43)
$$u^* = \min\left(u_{\max}, \max\left(0, \frac{y_2^*}{\alpha}(p_2 - p_1)\right)\right)$$

Proof. By making the change of variable $s = \tau - t$ and the change of functions $q_i(s,x) = p_i(\tau - s, x) = p_i(t, x)$ for $(t, x) \in Q$, we can prove, with the same method used in the proof of Theorem 1, that the system (38) admits a unique solution $p \in W^{1,2}(0, \tau; H(\Omega))$ with $p_i \in L(\tau, \Omega)$ for $i = 1, 2, 3$.

Let us now prove the second part of the theorem. Let (y^*, u^*) be an optimal pair, $u^{\varepsilon} = u^* + \varepsilon h \in U$ $(\varepsilon > 0)$ and $y^{\varepsilon} = (y_1^{\varepsilon}, y_2^{\varepsilon}, y_3^{\varepsilon})$ the state solution corresponding to u^{ε}. Then :

$J'(u^*)(h) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J(u^{\varepsilon}) - J(u^*)\right)$

$= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}\left(\int_0^{\tau}\int_{\Omega}\left[(y_3^{\varepsilon})^2 - (y_3^*)^2\right]dxdt + \int_{\Omega}\left[(y_3^{\varepsilon}(\tau,x))^2 - (y_3^*(\tau,x))^2\right]dx\right.$
$\left. + \alpha\int_0^{\tau}\int_{\Omega}\left[(u^{\varepsilon})^2 - (u^*)^2\right]dxdt\right)$

$= \lim_{\varepsilon \to 0} \frac{1}{2}\left(\int_0^{\tau}\int_{\Omega}\left(\frac{y_3^{\varepsilon} - y_3^*}{\varepsilon}\right)(y_3^{\varepsilon} + y_3^*)dxdt + \int_{\Omega}\left(\frac{y_3^{\varepsilon} - y_3^*}{\varepsilon}\right)(y_3^{\varepsilon} + y_3^*)(\tau,x)dx\right.$
$\left. + \alpha\int_0^{\tau}\int_{\Omega}\left(\frac{u_3^{\varepsilon} - u_3^*}{\varepsilon}\right)(u^{\varepsilon} + u^*)dxdt\right)$

$= \lim_{\varepsilon \to 0} \frac{1}{2}\left(\int_0^{\tau}\int_{\Omega}(z_3^{\varepsilon})(y_3^{\varepsilon} + y_3^*)dxdt + \int_{\Omega}(z_3^{\varepsilon})(y_3^{\varepsilon} + y_3^*)(\tau,x)dx\right.$
$\left. + \alpha\int_0^{\tau}\int_{\Omega}h(u^{\varepsilon} + u^*)dxdt\right)$

$= \int_0^{\tau}\langle Ny^*, Nz\rangle_{L^2(\Omega)}dt + \langle Ny^*(\tau,x), Nz(\tau,x)\rangle_{L^2(\Omega)} + \int_0^{\tau}\langle\alpha u^*, h\rangle_{L^2(\Omega)}dt$

$= \int_0^{\tau}\langle N^*Ny^*, z\rangle_{L^2(\Omega)}dt + \langle Ny^*(\tau,x), Nz(\tau,x)\rangle_{L^2(\Omega)} + \int_0^{\tau}\langle\alpha u^*, h\rangle_{L^2(\Omega)}dt$

$= \int_0^{\tau}\langle N^*Ny^*, z\rangle_{L^2(\Omega)}dt + \langle N^*Ny^*(\tau,x), z(\tau,x)\rangle_{L^2(\Omega)} + \int_0^{\tau}\langle\alpha u^*, h\rangle_{L^2(\Omega)}dt$

$$= \int_0^\tau \langle -\frac{\partial p}{\partial t} - Ap - E^* p, z \rangle_{L^2(\Omega)} dt + \langle p(\tau, x), z(\tau, x) \rangle_{L^2(\Omega)} + \int_0^\tau \langle \alpha u^*, h \rangle_{L^2(\Omega)} dt$$

$$= \int_0^\tau \langle p, \frac{\partial z}{\partial t} - Az - Ez \rangle_{L^2(\Omega)} dt + \int_0^\tau \langle \alpha u^*, h \rangle_{L^2(\Omega)} dt$$

$$= \int_0^\tau \langle p, Fh \rangle_{L^2(\Omega)} dt + \int_0^\tau \langle \alpha u^*, h \rangle_{L^2(\Omega)} dt$$

$$= \int_0^\tau \langle F^* p, h \rangle_{L^2(\Omega)} dt + \int_0^\tau \langle \alpha u^*, h \rangle_{L^2(\Omega)} dt$$

$$= \int_0^\tau \langle F^* p + \alpha u^*, h \rangle_{L^2(\Omega)} dt$$

where $z^\varepsilon = (z_1^\varepsilon, z_2^\varepsilon, z_3^\varepsilon) = \frac{1}{\varepsilon}(y^\varepsilon - y^*)$ and $z = y'(u^*)h$

For $h = u - u^*$, we obtain:

(44)
$$J'(u^*)(u - u^*) = \int_0^\tau \langle F^* p + \alpha u^*, u - u^* \rangle_{L^2(\Omega)} dt$$

Since J is Gateaux differentiable at u^*, U_{ad} is convex and the minimum of the objective functional is attained at u^*, we conclude that:

(45)
$$\int_0^\tau \langle F^* p + \alpha u^*, u - u^* \rangle_{L^2(\Omega)} dt \geq 0 \qquad \forall u \in U_{ad}$$

By standard arguments varying u, wo obtain:

(46)
$$u^* = -\frac{1}{\alpha} F^* p = \frac{y_2^*}{\alpha}(p_2 - p_1)$$

then $u^* = \min\left(u_{\max}, \max\left(0, \frac{y_2^*}{\alpha}(p_2 - p_1)\right)\right)$ $\qquad \square$

7. NUMERICAL RESULTS

We used forward-backward sweep method $(FBSM)$ to simulate the state system $(4) - (6)$, the dual system (42) and the characterization of the control (43). We wrote a code in $MATLAB$ where, using a finite difference method, we solved the system $(4) - (6)$ forward in time and the system (43) backward in time. A $40km \times 30km$ grid Ω represents the population's habitat. We start by considering that the population density is 80 per square kilometer at $t = 1$. We suppose that violence spread starts from the middle Ω_1. We consider two situations: the presence of intervention and the absence of the intervention where, in both cases, the spread of violence is displayed over a period of 24 months. $Table 1$, resume the values of the initial conditions and parameters used in our numerical simulation.

In *Figures* $1-3$, the numerical results show that, without any control, community violence spreads throughout Ω and the number of violent individuals increases quickly to attain its peak by the day 720.

Notations	Values	Description
$S_0(x,y)$	$35\ for\ (x,y) \in \Omega_1$ $40\ for\ (x,y) \notin \Omega_1$	Initial susceptible population
$H_0(x,y)$	$35\ for\ (x,y) \in \Omega_1$ $40\ for\ (x,y) \notin \Omega_1$	Initial susceptible population with high risk factors
$V_0(x,y)$	$10\ for\ (x,y) \in \Omega_1$ $0\ for\ (x,y) \notin \Omega_1$	Initial violent population
$\lambda_i\ ,\ i = 1,2,3$	$0.5, 0.5, 0.6$	Diffusion coefficients
μ	0.01	Birth rate
d	0.01	Natural death rate
α_1	0.03	Rate of becoming H
α_2	0.03	Transmission rate (for S)
α_3	0.3	Transmission rate (for H)
r	0.01	Recovery rate
β	0.5	Rehabilitation rate (from V to S)
c	0.01	Rehabilitation rate (from H to S)
u_{max}	0.7	Maximal value of admissible controls

TABLE 1. Initial conditions and parameters values
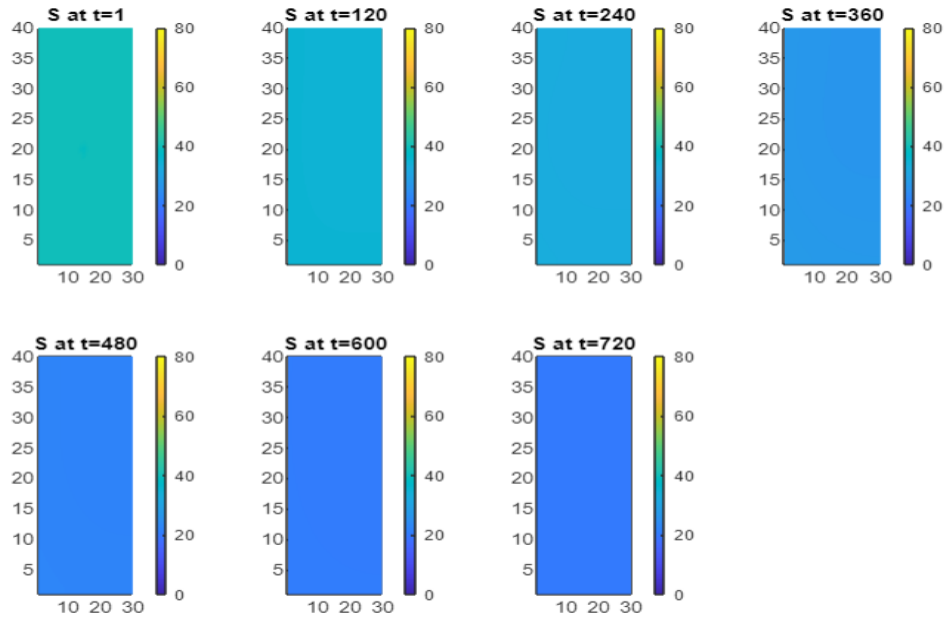
FIGURE 1. Susceptble behavior within Ω without control
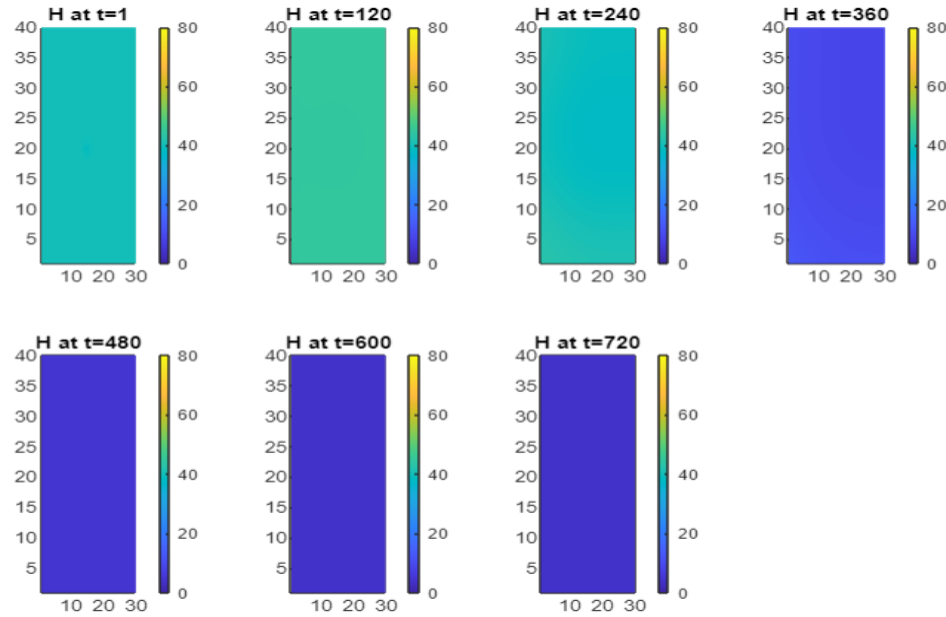
FIGURE 2. Susceptble with high risk factors behavior within Ω without control
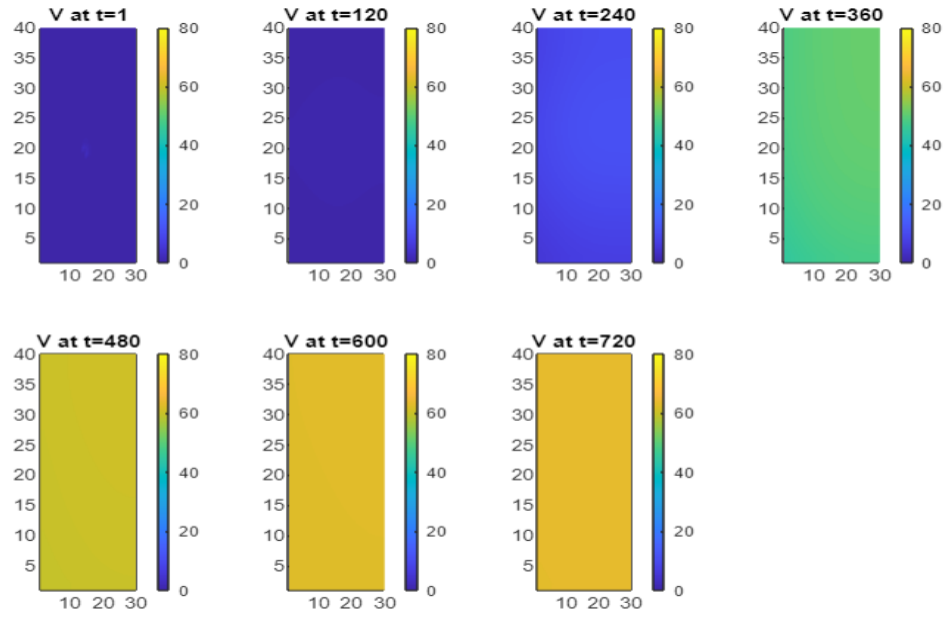
FIGURE 3. Violent behavior within Ω without control

Figures 4 − 6, shows the effect of starting the intervention from the first day. We can see that violence spreads throughout Ω but the number of violent inviduals remains very low during all the period. After 24 months the densitiy of violent indivuduals is around 7 per square kilometer instead of 64 per square kilometer without the intervention. This is mostly due to the high decrease of the number of susceptible individuals with high risk factors from the 4th month to the 8th month as we can see in *Figure* 5, which prove the effectiveness of our intervention strategy.
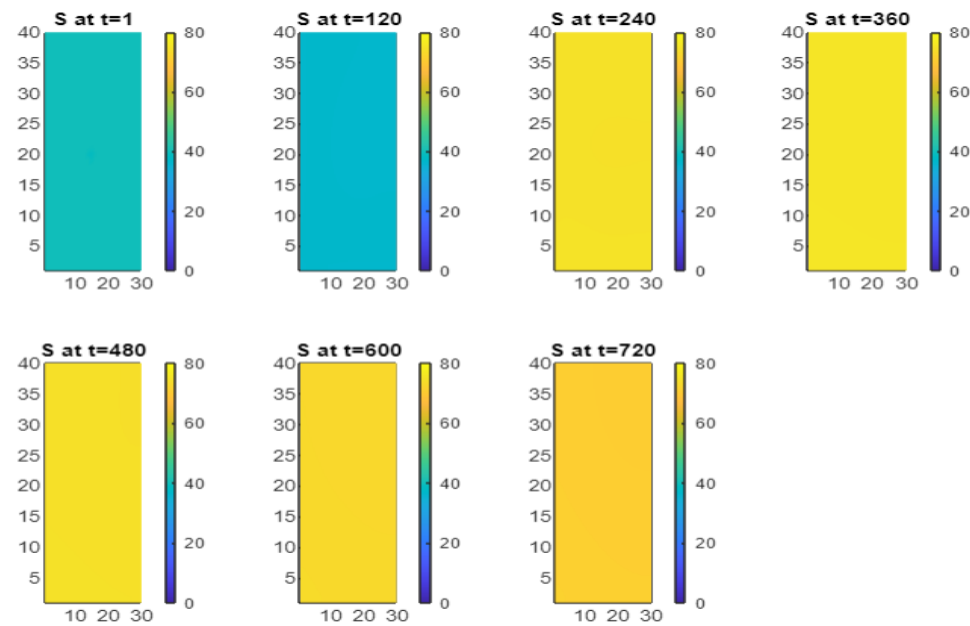

FIGURE 4. Susceptble behavior within Ω without control. The intervention starts from the first day
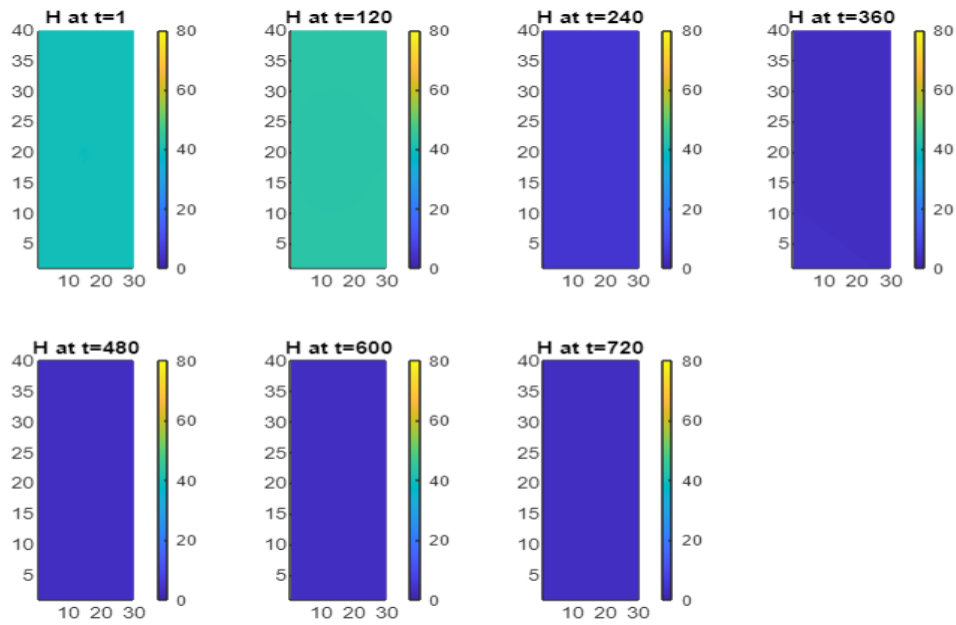
FIGURE 5. Susceptble with high risk factors behavior within Ω without control.

The intervention starts from the first day
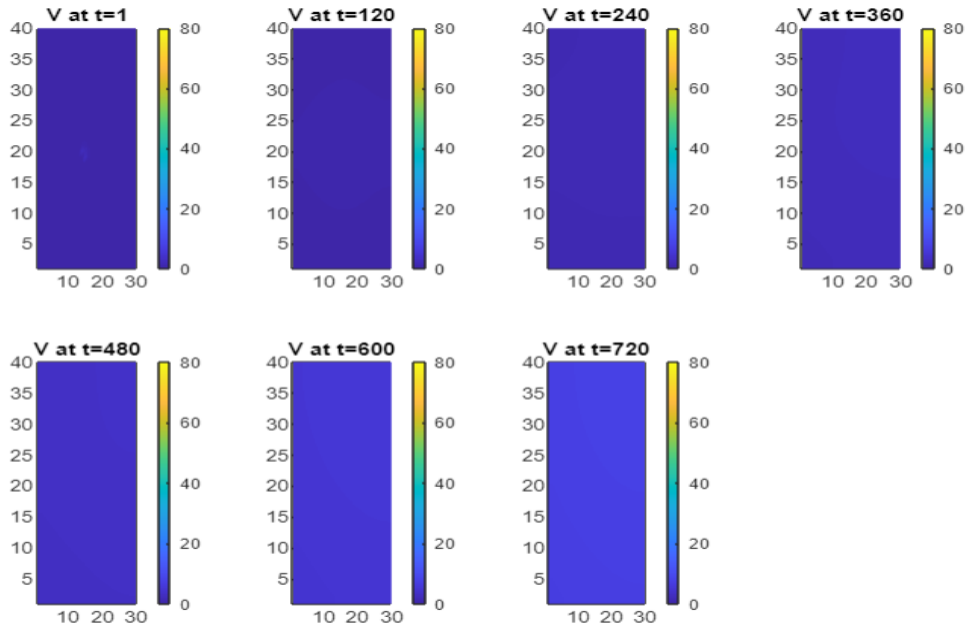
FIGURE 6. Violent behavior within Ω with control . The intervention starts

from the first day

To confirm this effectiveness, we simulate the spread of community violence in the second case where the intervention starts after 6 months. In *Figures* $7-9$, we see clearly that again, the number of violent inviduals remains low, and the density of violent inviduals is around 12 per square kilometer.
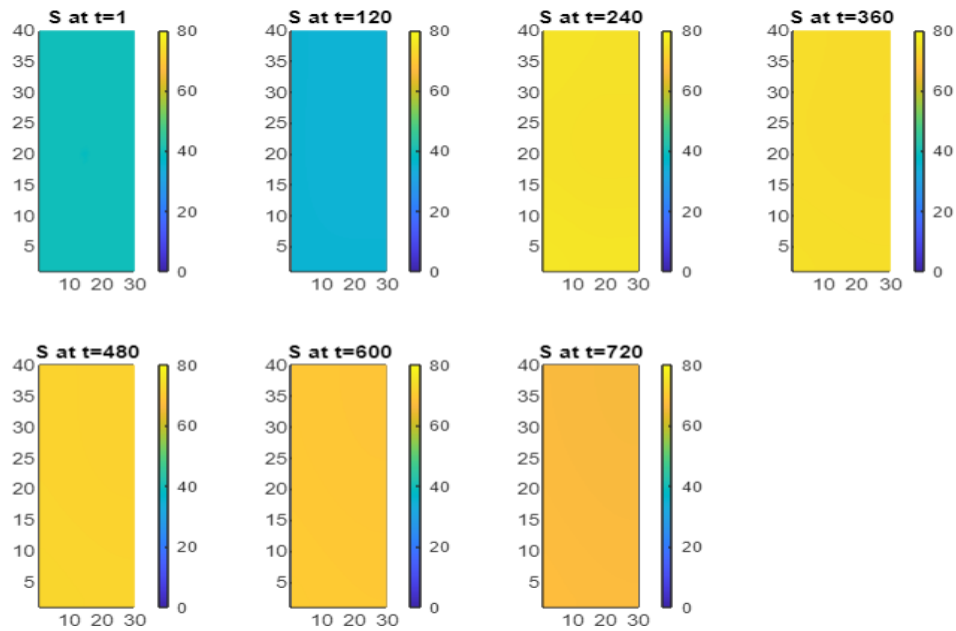
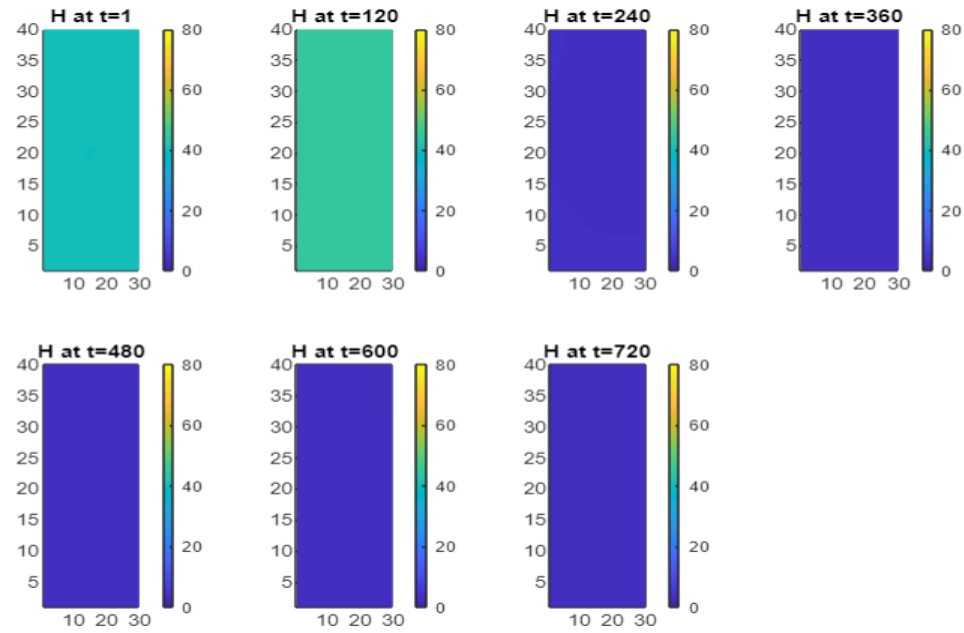FIGURE 7. Susceptible behavior within Ω with control. The intervention starts after 180 days

FIGURE 8. Susceptible with high risk factors behavior within Ω with control.
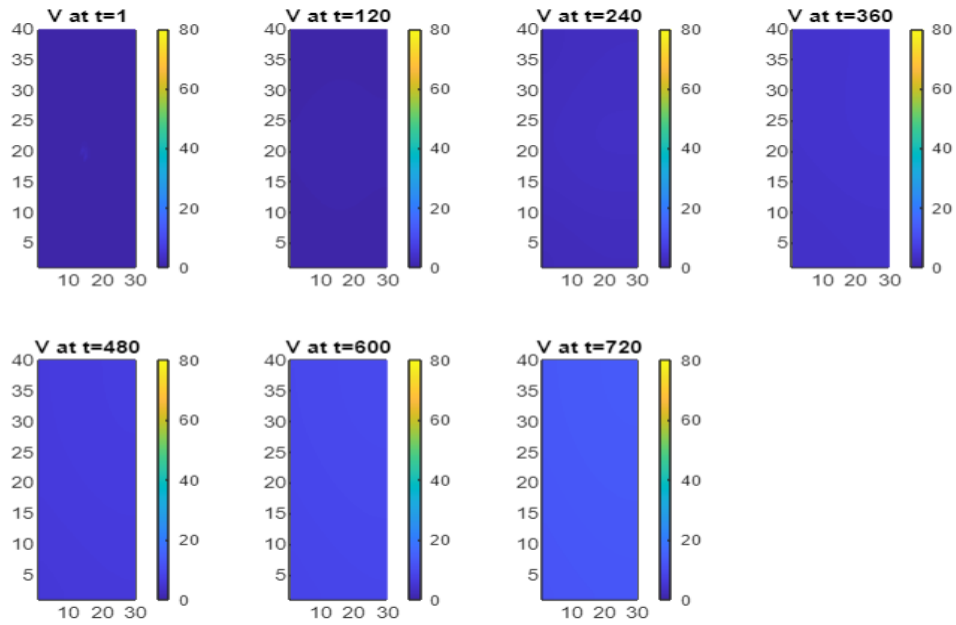The intervention starts after 180 days

FIGURE 9. Violent behavior within Ω with control. The intervention starts after
180 days

Finally, let us consider a third case where the intervention starts after 16 months. *Figure* 12 shows that the number number of violent individuals decreases significantly and immediately after the the intervention and the density goes down from 60 per square kilometer at the beginning of the intervention (16th month) to 49 per square kilometer by the end of the simulation which is still high. This relative high number of violent individuals can be explained by *Figure* 11 where we see that the number of susceptible individuals with high risk factors reaches its peak by the 4th month and that at the start of the intervention (16th month) the vast majority of those individuals beacame already violent. We conclude that, to eliminate violence by using only this intervention method, we need to intervene in the first months. In the other case other intervention methods are needed (social norms, prison, ...).
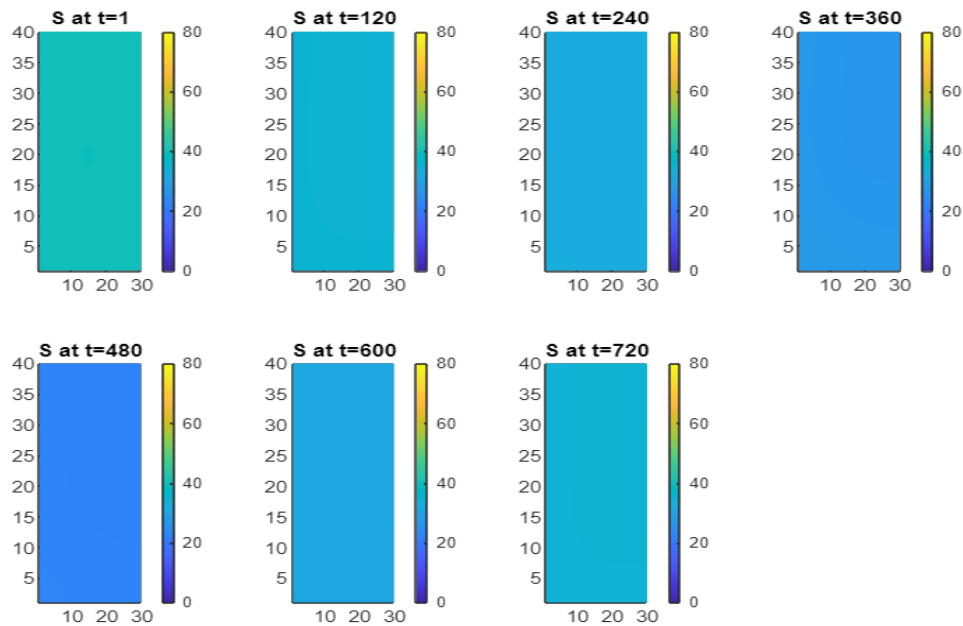

FIGURE 10. Susceptible behavior within Ω with control. The intervention starts after 480 days
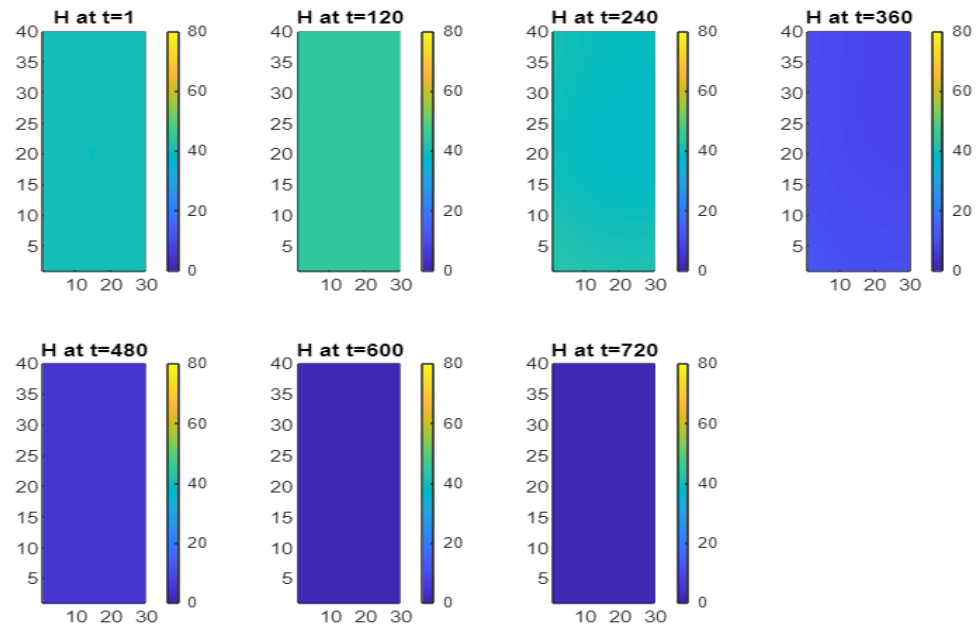
FIGURE 11. Susceptible with high risk factors behavior within Ω with control. The intervention starts after 480 days
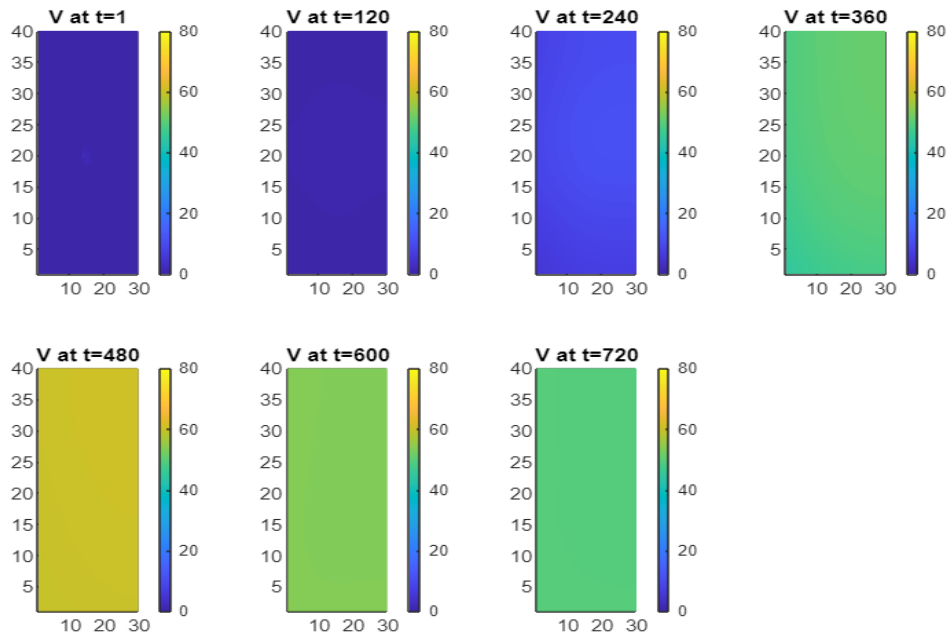

FIGURE 12. Violent behavior within Ω with control. The intervention starts after 480 days

8. CONCLUSION

In this paper, we presented a model (SHV) for treating community violence described by a system of partial differential equations. We treated community violence as an epidemic disease controlled by identifying and helping individuals at the highest risk. To this purpose, we used optimal control theory and we proved the existence and characterization of the optimal control. The numerical results show that the spread of community violence is quick in absence of any intervention. Our simulation proved that the control strategy used in this work is highly effective at stopping community violence from spreading and that we can eliminate violence if we start the intervention in the first months.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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