Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2021, 2021:52
https://doi.org/10.28919/cmbn/5807
ISSN: 2052-2541

ON THE CONTROLLABILITY AND OBSERVABILITY OF POSITIVE NONLINEAR CONTINUOUS SYSTEMS

ILYAS SITLI[1,*], MOUHCINE NAIM[1], FOUAD LAHMIDI[1], ABDELWAHED NAMIR[2]

[1]Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'sik, Hassan II University, P.O. Box 7955, Sidi Othman, Casablanca, Morocco

[2]Laboratory of Information Technologies and Modeling (LITM), Faculty of Sciences Ben M'sik, Hassan II University, P.O. Box 7955, Sidi Othman, Casablanca, Morocco

Abstract. In this paper, we study the controllability and observability of continuous nonlinear positive systems. We solve these problems by a technique based on the fixed point theory.

Keywords: positive nonlinear systems; continuous systems; controllability; observability; fixed point theory.

2010 AMS Subject Classification: 93B05, 93B07, 93C10.

1. Introduction

Controllability and observability are two fundamental concepts in the control theory. Systematic studies on these topics in the linear case were started at the beginning of 1960s [1, 2], in nonlinear one in 1970s [3]. Controllability continually appears as a necessary condition for the existence of solutions to many control problems, for example: stabilization of unstable system by feedback, optimal control [4]. Observability plays a crucial role in study of canonical forms of dynamical systems or observer synthesis [5]. Basically a system is controllable if it is

*Corresponding author
E-mail address: sitli.ilyass@gmail.com
Received April 2, 2021

possible to transfer it from an arbitrary initial state to an arbitrary final state using only certain admissible controls; it is observable if the initial state can be determined using the information given by an output over a finite time. There exist many papers in which these two properties for classical discrete and continues systems are studied. A meaningful fact in practise, also in classical systems, is to investigate both properties in their local formulations for nonlinear systems through global notions of controllability and observability by linearisation of the considered systems.

Positive systems are a wide class of systems in which state variables and outputs are constrained to be positive, or at least nonnegative for all time whenever the initial state and inputs are nonnegative [6]. Since the state variables and outputs of many real-world processes represent quantities that may not have meaning unless they are nonnegative because they measure concentrations, temperatures, cell birth or losses, ..., positive systems arise frequently in mathematical modeling of engineering problems, management sciences, economics, social sciences, chemistry, biology, ecology, medicine, and other areas.

The mathematical theory of positive linear systems is based on the theory of nonnegative matrices developed by Perron and Frobenius, see for example, [7, 8]. An excellent survey of positive systems with an emphasis on their applications in the areas of management and social sciences is given by Luenberger in [7]. The more recent monographs by Farina and Rinaldi in [9] and Kaczorek in [6] are devoted entirely to positive linear systems and some of their applications.

The reachability, controllability and observability of positive linear systems is largely studied by several authors since late 1980s for both discrete and continuous systems ([10], [11], [12], [13], [14], [15], [16], ...). The reachability of positive nonlinear systems for continuous and discrete systems has been studied respectively in [17] and [18].

In this work, we solve the problem of controllability and observability for nonlinear positive continuous systems, by two different methods that are mainly based on fixed point techniques. We characterize the set of nonnegative controls which steer the state of a positive system from a nonnegative initial state to a nonnegative desired final state. The set of all nonnegative states which correspond to the given nonnegative output is also characterized. The following notations will be used. $\mathbb{R}_+$ is the set of nonnegative real numbers, $\mathbb{R}^n$ the set of real vectors with n components, $\mathbb{R}^n_+$ the set of all vectors in $\mathbb{R}^n$ with nonnegative components, i.e.,

$$\mathbb{R}^n_+ = \left\{ x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n : x_i \in \mathbb{R}_+, i \in \{1, .., n\} \right\}$$

where T denotes the transpose, $\mathbb{R}^{n\times m}$ the set of real matrices of size $n \times m$ $\left(\mathbb{R}^n = \mathbb{R}^{n\times 1}\right)$, I_n the identity matrix in $\mathbb{R}^{n\times n}$, $L^2([0,T],X)$ the set of square integrable function defined in the time interval $[0,T]$ with values in $X \subset \mathbb{R}^n$, S_F the set of all fixed points of a function F.

2. Preliminaries

Consider the nonlinear continuous system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + Bu(t), \; t \in \mathbb{R}_+, \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

with linear observation

$$y(t) = Cx(t), \; t \in \mathbb{R}_+, \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are respectively the state and control input of system $(1)-(2)$ at time t, $y(t) \in \mathbb{R}^r$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear function, and x_0 represents the initial state.

Definition 1. *The system $(1)-(2)$ is said to be positive if $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^r_+$, $t \in \mathbb{R}_+$, for every nonnegative initial state $x_0 \in \mathbb{R}^n_+$ and all nonnegative inputs $u(t) \in \mathbb{R}^m_+$,$t \in \mathbb{R}_+$.*

Definition 2. *A matrix $A = \left(a_{ij}\right)$ in $\mathbb{R}^{n\times m}$ is said to be nonnegative, and denoted by $A \in \mathbb{R}^{n\times m}_+$, if all of its elements are nonnegative, i.e.,$a_{i,j} \in \mathbb{R}_+$ for all i, j.*

Now, we introduce the notion of Metzler matrix, proposed by L.A. Metzler in [19]. This matrix plays an important role in the mathematical theory of positive continuous systems.

Definition 3. *A square matrix $A = \left(a_{ij}\right)_{1\leq i,j\leq n}$ is said to be a Metzler matrix if its non-diagonal elements are nonnegative, i.e., $a_{ij} \in \mathbb{R}_+$ for all $i \neq j$.*

Remark 4. *Clearly, the nonnegative matrices are Metzler matrices. Moreover, there is a very strong link between the nonnegative matrices and the Metzler matrices. Indeed,* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *is a Metzler matrix if and only if there exists* $\lambda \in \mathbb{R}_+$ *such that*
$(A+\lambda I_n) \in \mathbb{R}_+^{n\times n}$ *, for example,*

$$\lambda = \max\left\{0, -\min_{0\leq i\leq n} a_{ii}\right\}.$$

An important property of Metzler matrices is given by the following result.

Lemma 5. *[6].* $A \in \mathbb{R}^{n\times n}$ *is a Metzler matrix if and only if the associated exponential matrix* $e^{At} \in \mathbb{R}_+^{n\times n}$ *for all* $t \in \mathbb{R}_+$.

We assume that system (1)-(2) satisfies the following assumptions
$(H1)$: $u \in L^2([0,T],\mathbb{R}^m)$ and $x \in L^2([0,T],\mathbb{R}^n)$ with $T > 0$.
$(H2)$: f is a lipschitzian function, i.e., there exists a constant $K > 0$ such that

$$\|f(x_1) - f(x_2)\| \leq K\|x_1 - x_2\| \text{ for all } x_1, x_2 \in \mathbb{R}^n,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

The following sufficient condition for the positivity of system (1)-(2) holds.

Proposition 6. *The system (1)-(2) is positive if*

$$\begin{cases} A \text{ is a Metzler matrix,} \\ f(\mathbb{R}_+^n) \subset \mathbb{R}_+^n \\ B \in \mathbb{R}_+^{n\times m} \\ C \in \mathbb{R}_+^{r\times n} \end{cases} \qquad (3)$$

Proof. Since the conditions $(H1)$ and $(H2)$ satisfied, system (1) has a unique solution $x(t)$ given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} f(x(\tau))\,d\tau + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau,\ t \in [0,T]. \quad (4)$$

Using Picard method [20], the following functional sequence converges to the
solution (4)

$$\begin{cases} \widetilde{x}_{k+1}(t) = e^{At}\widetilde{x}_0 + \int_0^t e^{A(t-\tau)} f(\widetilde{x}_k(\tau))\,d\tau + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau = \Phi_u(\widetilde{x}_k)(t),\ k \in \mathbb{N}, \\ \widetilde{x}_0 = x_0 \in \mathbb{R}_+^n \end{cases}$$

For all $t \in [0,T]$, we have

$$\widetilde{x}_k(t) = \Big[\underbrace{\Phi_u \circ \Phi_u \circ \ldots \circ \Phi_u}_{k-times}\Big](x_0)(t),\ k \in \mathbb{N}.$$

Since A is a Metzler matrix, then by Lemma 5, $e^{At} \in \mathbb{R}_+^{n\times n}$. On the other hand, the conditions $B \in \mathbb{R}_+^{n\times m}$and $f\left(\mathbb{R}_+^n\right) \subset \mathbb{R}_+^n$ ensures that $e^{A(t-\tau)}f(x_0) \in \mathbb{R}_+^n$ and $e^{A(t-\tau)}Bu(\tau) \in \mathbb{R}_+^n$ since $x_0 \in \mathbb{R}_+^n$ and $u(\tau) \in \mathbb{R}_+^m$ for every $\tau \in [0,t)$. Hence $\Phi_u(x_0)(t) \in \mathbb{R}_+^n$ for any $x_0 \in \mathbb{R}_+^n$, and then $\widetilde{x}_k(t) \in \mathbb{R}_+^n$ for all $k \in \mathbb{N}$, which implies that $x(t) \in \mathbb{R}_+^n$. Consequently, if $C \in \mathbb{R}_+^{r\times n}$, then $y(t) \in \mathbb{R}_+^r$. This completes the proof. □

Remark 7. *The linear system obtained from* $(1)-(2)$ *for* $f=0$ *is positive if and only if A is a Metzler matrix,* $B \in \mathbb{R}_+^{n\times m}$ *and* $C \in \mathbb{R}_+^{r\times n}$ *[6].*

In the rest of this paper, we assume that system (1)-(2) is positive, with the conditions (3) satisfied.

3. CONTROLLABILITY

In the next, we shall formulate the definition for controllability of system (1) as follows.

Definition 8. *System* (1) *is said to be controllable in a finite time* $t_f > 0$ *if for any initial state* $x_0 \in \mathbb{R}_+^n$ *and any desired final state* $x_f \in \mathbb{R}_+^n$*, there exists a nonnegative input* $u \in L^2([0,t_f],\mathbb{R}_+^m)$ *, which steers the state of the system from* x_0 *to* x_f *, i.e.,* $x_f = x\ \left(t_f, x_0, u\right)$.

Consider the mapping

$$a_{x_0} : t \in [0,t_f] \longmapsto e^{At}x_0 \in \mathbb{R}_+^n.$$

Let's consider the nonlinear operator G defined by

$$G : x \in L^2([0,t_f],\mathbb{R}^n) \mapsto \int_0^{\cdot} e^{A(\cdot-\tau)} f(x(\tau))\,d\tau \in L^2([0,t_f],\mathbb{R}^n).$$

and D denote the linear operator defined by

$$D : u \in L^2([0,t_f],\mathbb{R}^m) \mapsto \int_0^{\cdot} e^{A(\cdot-\tau)} Bu(\tau)\,d\tau \in L^2([0,t_f],\mathbb{R}^n).$$

Then, the solution of system (1) has the form

$$x = a_{x_0} + G(x) + Du.$$

3.1. Characterization of controllability - First mapping.

The aim of this subsection is to establish a necessary and sufficient condition for the controllability of system based on fixed points of a mapping appropriately chosen. Also, we characterize the set U_+ of nonnegative controls which steer the state of system (1) from a initial state $x_0 \in \mathbb{R}^n_+$ at $t = 0$ to a desired final state $x_f \in \mathbb{R}^n_+$ at $t = t_f$, i.e.,

$$U_+ = \left\{ u \in L^2([0,t_f],\mathbb{R}^m_+) : x\ \left(t_f, x_0, u\right) = x_f \right\}.$$

Definition 9. *The positive image of the operator D is*

$$\mathrm{Im}_+ D = \left\{ Du \in L^2([0,t_f],\mathbb{R}^n_+) : u \in L^2([0,t_f],\mathbb{R}^m_+) \right\}.$$

Let $P : L^2([0,t_f],\mathbb{R}^n) \to \mathrm{Im}_+ D$ be any projection on $\mathrm{Im}_+ D$, $\widetilde{x}$ be any fixed element of $\mathrm{Im}_+ D$ different from zero.

We define

$$\begin{array}{cccc} g_{\widetilde{x}}: & L^2([0,t_f],\mathbb{R}^n) & \to & \mathrm{Im}_+ D \\ & x & \longmapsto & \begin{cases} 0, \text{ if } x_f = x\left(t_f\right) \\ \widetilde{x}, \text{ otherwise} \end{cases} \end{array}$$

and

$$\zeta : x \in L^2([0,t_f],\mathbb{R}^n) \longmapsto x - a_{x_0} - G(x) \in L^2([0,t_f],\mathbb{R}^n).$$

We consider the operator

$$F : x \in L^2([0,t_f],\mathbb{R}^n) \longmapsto a_{x_0} + G(x) + P\zeta(x) + g_{\widetilde{x}}(x) \in L^2([0,t_f],\mathbb{R}^n).$$

then we have the following proposition.

Proposition 10. *The nonlinear system (1) is controllable in time t_f if and only if, for all $x_0, x_f \in \mathbb{R}^n_+$, F has a fixed point.*

Proof. $(Sufficiency)$ Let $x_0, x_f \in \mathbb{R}^n_+$. $x \in S_F$ then

$$x = a_{x_0} + G(x) + P\zeta(x) + g_{\widetilde{x}}(x). \qquad (5)$$

Thus

$$\zeta(x) = P\zeta(x) + g_{\widetilde{x}}(x) \in \mathrm{Im}_+ D.$$

which implies that $\zeta(x) = P\zeta(x)$ and $g_{\widetilde{x}}(x) = 0$, which ensures that $x(t_f) = x_f$.

Consequently, the equation (5) becomes

$$x = a_{x_0} + G(x) + \zeta(x)$$

Since $\zeta(x) \in \mathrm{Im}_+ D$, then there exists an input $u \in L^2([0,t_f],\mathbb{R}^n_+)$ such that $\zeta(x) = Du$, that means

$$x = a_{x_0} + G(x) + Du,$$

and then

$$x(t_f) = x_f = a_{x_0}(t_f) + (G(x))(t_f) + (Du)(t_f) = x(t_f, x_0, u),$$

i.e., the system (1) is controllable in time t_f .

$(Necessity)$ Let $x_0, x_f \in \mathbb{R}^n_+$. Since the system (1) is controllable in time t_f , there exists an input $u \in L^2([0,t_f],\mathbb{R}^n_+)$ such that $x(t_f, x_0, u) = x_f$. Then we get

$$x(t) := x(t, x_0, u) = a_{x_0}(t) + (G(x))(t) + (Du)(t),\ t \in [0,t_f],$$

and hence

$$x(t_f) := x(t_f, x_0, u) = x_f.$$

Consequently

$$\zeta(x) = Du \in \mathrm{Im}_+ D \text{ and } g_{\widetilde{x}}(x) = 0,$$

then

$$P\zeta(x) = \zeta(x).$$

Hence, we obtain

$$F(x) = a_{x_0} + G(x) + P\zeta(x) + g_{\widetilde{x}}(x) = a_{x_0} + G(x) + Du = x$$

Then x is a fixed point of the operator F. The proposition is proved. □

Remark 11. *The fixed points of F are independent of the choice of the projection operator P and the element $\widetilde{x}$. Indeed, let P_1 and P_2 be two projections on Im_+D and $\widetilde{x}_1$ and $\widetilde{x}_2$ two any elements not equal to zero of Im_+D. Let's consider the operators*

$$F_1 : x \in L^2([0,t_f],\mathbb{R}^n) \longmapsto a_{x_0} + G(x) + P_1\zeta(x) + g_{\widetilde{x}_1}(x) \in L^2([0,t_f],\mathbb{R}^n).$$

and

$$F_2 : x \in L^2([0,t_f],\mathbb{R}^n) \longmapsto a_{x_0} + G(x) + P_2\zeta(x) + g_{\widetilde{x}_2}(x) \in L^2([0,t_f],\mathbb{R}^n).$$

Let x be a fixed point of F_1. By proof of Proposition 10, we have $x\left(t_f\right) = x_f$ and $\zeta(x) \in Im_+D$. Consequently $P_2\zeta(x) = \zeta(x)$ and $g_{\widetilde{x}_2}(x) = 0$, then

$$F_2(x) = a_{x_0} + G(x) + \zeta(x) = x$$

Hence, if x is a fixed point of F_1, then it is also a fixed point of F_2.

In the following, we shall need to inverse the operator D. But D is not inversible in a general case. Introduce then

$$\widetilde{D} : y \in (\ker D)^{\perp} \mapsto \widetilde{D}y = Dy \in \operatorname{Im} D,$$

this operator is inversible and its inverse, which is defined on $\operatorname{Im} D$ can be extended to $\operatorname{Im} D \oplus (\operatorname{Im} D)^{\perp}$ as follows

$$D^{\dagger} : y + z \in \operatorname{Im} D \oplus (\operatorname{Im} D)^{\perp} \mapsto \widetilde{D}^{-1}y \in L^2([0,t_f],\mathbb{R}^m).$$

The operator $D^{\dagger}$ is known as the pseudo inverse operator of D. In particular, the mapping $D^{\dagger}$ satisfies

$$\begin{cases} DD^{\dagger}y = y \quad \text{for all } y \in \operatorname{Im} D, \\ D^{\dagger}Dz = z \quad \text{for all } z \in (\ker D)^{\perp}. \end{cases}$$

Remark 12. *If $\operatorname{Im} D$ is closed, then $L^2([0,t_f],\mathbb{R}^n) = \operatorname{Im} D \oplus (\operatorname{Im} D)^{\perp}$ and $D^{\dagger}$ satisfies* [21]

$$DD^{\dagger}D = D,\ D^{\dagger}DD^{\dagger} = D^{\dagger},\ \left(DD^{\dagger}\right)^* = DD^{\dagger} \text{ and } \left(D^{\dagger}D\right)^* = D^{\dagger}D,$$

with D^* is the adjoint of D.

Now, we characterize the set U_+ by the following result.

Proposition 13. *We have*

$$U_+ = \left\{D^{\dagger}\zeta(x) + \Psi \in L^2([0,t_f],\mathbb{R}^m_+) : x \in S_F \text{ and } \Psi \in \ker D\right\}.$$

Proof. If $u \in U_+$, then by proof of Proposition 10, the trajectory x of system (1) corresponding to control u is a fixed point of F and $\zeta(x) = Du \in \operatorname{Im}_+ D$. Hence, we can write

$$u = D^\dagger \zeta(x) + \Psi,$$

with $\Psi = u - D^\dagger \zeta(x)$, and we have

$$D\Psi = \zeta(x) - DD^\dagger \zeta(x) = \zeta(x) - \zeta(x) = 0,$$

i.e., $\Psi \in \ker D$.

Conversely, let $u = D^\dagger \zeta(x) + \Psi \in L^2([0,t_f],\mathbb{R}^m_+)$, with $x \in S_F$ and $\Psi \in \ker D$, then $\zeta(x) \in \operatorname{Im}_+ D$, and hence $Du = DD^\dagger \zeta(x) = \zeta(x)$. Thus $x = a_{x_0} + G(x) + Du$, then $u \in U_+$. This finishes the proof. □

3.2. Characterization of controllability - Second mapping.

In this subsection, we shall characterize the controllability of system (1) and the set U_+ using another mapping. For this, let

$$\widehat{a}_{x_0} : t \in [0,t_f] \longmapsto \begin{pmatrix} a_{x_0}(t) \\ a_{x_0}(t_f) \end{pmatrix} \in \mathbb{R}^n_+ \times \mathbb{R}^n_+,$$

$$\widehat{G} : x \in L^2([0,t_f],\mathbb{R}^n) \longmapsto \begin{pmatrix} Gx \\ (Gx)(t_f) \end{pmatrix} \in L^2([0,t_f],\mathbb{R}^n) \times \mathbb{R}^n,$$

and we define

$$\widehat{D} : u \in L^2([0,t_f],\mathbb{R}^m) \longmapsto \begin{pmatrix} Du \\ (Du)(t_f) \end{pmatrix} \in L^2([0,t_f],\mathbb{R}^n) \times \mathbb{R}^n.$$

We consider the following operators

$$\begin{array}{rccc} \widehat{\zeta} : & L^2([0,t_f],\mathbb{R}^n) \times \mathbb{R}^n & \to & L^2([0,t_f],\mathbb{R}^n) \times \mathbb{R}^n \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \end{pmatrix} - \widehat{a}_{x_0} - \widehat{G}(x), \end{array}$$

and

$$\begin{array}{rccc} \widehat{F} : & L^2([0,t_f],\mathbb{R}^n) \times \mathbb{R}^n & \to & L^2([0,t_f],\mathbb{R}^n) \times \mathbb{R}^n \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \widehat{a}_{x_0} + \widehat{G}(x) + \widehat{P}\widehat{\zeta}\begin{pmatrix} x \\ x_f \end{pmatrix} + \begin{pmatrix} 0 \\ y - x_f \end{pmatrix} \end{array}$$

with $\widehat{P} : L^2([0,t_f],\mathbb{R}^n) \times L^2([0,t_f],\mathbb{R}^n) \to \mathrm{Im}_+ \widehat{D}$ any projection on $\mathrm{Im}_+ \widehat{D}$.

Proposition 14. *The nonlinear system* (1) *is controllable in time* t_f *if and only if, for all* $x_0, x_f \in \mathbb{R}^n_+$, $\widehat{F}$ *has a fixed point.*

Proof. Let $x_0, x_f \in \mathbb{R}^n_+$. If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_{\widehat{F}}$, then we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \widehat{a}_{x_0} + \widehat{G}(x) + \widehat{P}\widehat{\zeta}\begin{pmatrix} x \\ x_f \end{pmatrix} + \begin{pmatrix} 0 \\ y - x_f \end{pmatrix}$$

Hence

$$\widehat{P}\widehat{\zeta}\begin{pmatrix} x \\ x_f \end{pmatrix} = \begin{pmatrix} x \\ x_f \end{pmatrix} - \widehat{a}_{x_0} - \widehat{G}(x) = \widehat{\zeta}\begin{pmatrix} x \\ x_f \end{pmatrix},$$

which implies that $\widehat{\zeta}\begin{pmatrix} x \\ x_f \end{pmatrix} \in \mathrm{Im}_+ \widehat{D}$, so there exists an input $u \in L^2([0,t_f],\mathbb{R}^m_+)$ such that

$$\widehat{\zeta}\begin{pmatrix} x \\ x_f \end{pmatrix} = \widehat{D}u.$$

The rest of the proof is similar to that of Proposition 10. □

Proposition 15. *The set* U_+ *is given by*

$$U_+ = \left\{ \widehat{D}^\dagger \widehat{\zeta}\begin{pmatrix} x \\ x_f \end{pmatrix} + \Psi \in L^2([0,t_f],\mathbb{R}^m_+) : x \in \widetilde{S}_{\widehat{F}} \text{ and } \Psi \in \ker \widehat{D} \right\},$$

with

$$\widetilde{S}_{\widehat{F}} := \left\{ x \in L^2([0,t_f],\mathbb{R}^n) : \text{there exists } y \in L^2([0,t_f],\mathbb{R}^n) \text{ such that } \begin{pmatrix} x \\ y \end{pmatrix} \in S_{\widehat{F}} \right\}.$$

Proof. It is similar to that of Proposition 13. □

4. Observability

In this section we discuss the concept of observability problem for nonlinear positive continuous systems. Consider the nonlinear systems $(1)-(2)$ with $u(t)=0$ for $t\in\mathbb{R}_+$, and x_0 is assumed to be unknown.

Definition 16. *System $(1)-(2)$ is said to be observable in a finite time $t_f>0$ if it is possible to determine uniquely the nonnegative initial state $x_0\in\mathbb{R}^n_+$ from the knowledge of the output $y\in L^2([0,t_f],\mathbb{R}^r_+)$.*

Consider the operator

$$S: z\in\mathbb{R}^n \mapsto e^{A\cdot}z\in L^2([0,t_f],\mathbb{R}^n),$$

then, the solution of system (1) has the form

$$x(t)=(Sx_0)(t)+(G(x))(t)\,,\ t\in[0,t_f],$$

and the output (2) can be rewritten as

$$y(t)=C(Sx_0)+C(G(x))(t)\,,\ t\in[0,t_f],$$

The goal of this section is to give a characterization of the set Θ_+ of states of system (1) such that $y_g=Cx$ where $y_g\in L^2([0,t_f],\mathbb{R}^r_+)$ is the given output, i.e.,

$$\Theta_+=\left\{x\in L^2([0,t_f],\mathbb{R}^n_+): x=Sx_0+G(x) \text{ and } y_g=Cx\right\},$$

and consequently we shall establish a necessary and sufficient condition for the observability of system $(1)-(2)$.

4.1. Characterization of observability - First mapping.

Let $P: L^2([0,t_f],\mathbb{R}^n)\to \mathrm{Im}_+ S$ be any projection on $\mathrm{Im}_+ S$ and $\widetilde{x}$ be any fixed element of $\mathrm{Im}_+ S$ different from zero.

we define

$$\zeta: x\in L^2([0,t_f],\mathbb{R}^n)\longmapsto x-G(x)\in L^2([0,t_f],\mathbb{R}^n).$$

and we consider the operator

$$H: x\in L^2([0,t_f],\mathbb{R}^n)\longmapsto G(x)+P\zeta(x)+h_{\widetilde{x}}(x)\in L^2([0,t_f],\mathbb{R}^n).$$

with

$$h_{\widetilde{x}}: \begin{array}{ccc} L^2([0,t_f],\mathbb{R}^n) & \to & \mathrm{Im}_+ S \\ x & \longmapsto & \begin{cases} 0, \text{ if } y_g = Cx \\ \widetilde{x}, \text{ otherwise} \end{cases} \end{array}$$

The following proposition gives a characterization of the set Θ_+.

Proposition 17. *Let $x \in L^2([0,t_f],\mathbb{R}^n)$. Then x is an element of Θ_+ if and only if x is a fixed point of H.*

Proof. It is similar to that of Proposition 10. □

Thus a necessary and sufficient condition for the observability of our system is given by

Proposition 18. *The system $(1)-(2)$ is observable in time t_f if and only if, for every given output $y_g \in L^2([0,t_f],\mathbb{R}^r_+)$, S_H has at most one element.*

Proof. System $(1)-(2)$ is observable in time t_f if and only if for all $y_g \in L^2([0,t_f],\mathbb{R}^r_+)$, there exists at most one $x_0 \in \mathbb{R}^n_+$ such that

$$\begin{cases} x = Sx_0 + G(x), \\ y_g = Cx, \end{cases}$$

where x is the trajectory of system (1) coresponding to the initial state x_0. Consequently, the system $(1)-(2)$ is observable if and only if the set Θ_+, and hence S_H, contains at most one element. □

4.2. Characterization of observability - Second mapping.

The aim of this subsection is to give a second characterization of the set Θ_+ and of the observability of system (1)-(2) based on the fixed points of another function appropriately chosen.

$$\widehat{S} : z \in \mathbb{R}^n \mapsto \begin{pmatrix} Sz \\ CSz \end{pmatrix} \in L^2([0,t_f],\mathbb{R}^n) \times L^2([0,t_f],\mathbb{R}^n),$$

and

$$\widehat{G} : x \in L^2([0,t_f],\mathbb{R}^n) \longmapsto \begin{pmatrix} Gx \\ C(Gx) \end{pmatrix} \in L^2([0,t_f],\mathbb{R}^n) \times L^2([0,t_f],\mathbb{R}^n),$$

We consider the following operators

$$\begin{array}{rccc} \widehat{\zeta}: & L^2([0,t_f],\mathbb{R}^n)\times L^2([0,t_f],\mathbb{R}^n) & \to & L^2([0,t_f],\mathbb{R}^n)\times L^2([0,t_f],\mathbb{R}^n) \\ & \begin{pmatrix} x \\ z \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ z \end{pmatrix} - \widehat{G}(x), \end{array}$$

and

$$\begin{array}{rccc} \widehat{H}: & L^2([0,t_f],\mathbb{R}^n)\times L^2([0,t_f],\mathbb{R}^n) & \to & L^2([0,t_f],\mathbb{R}^n)\times L^2([0,t_f],\mathbb{R}^n) \\ & \begin{pmatrix} x \\ z \end{pmatrix} & \mapsto & \widehat{G}(x)+\widehat{P}\widehat{\zeta}\begin{pmatrix} x \\ y_g \end{pmatrix}+\begin{pmatrix} 0 \\ z-y_g \end{pmatrix}, \end{array}$$

with $\widehat{P}: L^2([0,t_f],\mathbb{R}^n)\times L^2([0,t_f],\mathbb{R}^n)\to \mathrm{Im}_+\widehat{S}$ any projection on $\mathrm{Im}_+\widehat{S}$.

Proposition 19. *The set Θ_+ is given by*

$$\Theta_+ = \widetilde{S}_{\widehat{H}} = \left\{ x\in L^2([0,t_f],\mathbb{R}^n) : \text{there exists } z\in L^2([0,t_f],\mathbb{R}^n) \text{ such that } \begin{pmatrix} x \\ z \end{pmatrix} \in S_{\widehat{H}} \right\}.$$

Proof. It is similar to that of Proposition 14. □

Proposition 20. *The system $(1)-(2)$ is observable in time t_f if and only if for every given output $y_g\in L^2([0,t_f],\mathbb{R}^r_+)$, $\widetilde{S}_{\widehat{H}}$ has at most one element.*

Proof. It is similar to that of Proposition 18. □

5. CONCLUSION

In this work we have employed a technique based on the fixed point theory for resolving the controllability and observability problem for nonlinear positive discrete systems. Sufficient conditions for the positivity of continuous nonlinear positive systems have been established (Proposition 6). Criteria for the controllability (Propositions 10, 14) and observability (Propositions 18, 20) have been proved. A characterization of nonnegative controls which drives the state of the system from its initial value to a given desired final state is given (Propositions 13, 15). The set of all nonnegative states which correspond to the given output is also characterized (Propositions 17, 19). In our future work, we investigate the controllability and observability of fractional positive nonlinear continuous systems.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] R.E. Kalman, On the General Theory of Control Systems, Proceedings of the First IFAC Congress, Vol. 1, Butterworths, London, 1961.

[2] R.E. Kalman, P.L. Falb, M.A. Arbib, Topics in mathematical system theory, McGraw-Hill, New York, 1969.

[3] R. Hermann, A. Krener, Nonlinear controllability and observability, IEEE Trans. Autom. Contr. 22 (1977), 728–740.

[4] E.D. Sontag, Mathematical Control Teory: Deterministic Finite Dimensional Systems, vol. 6, Springer, New York, USA, 1990.

[5] D.G. Luenberger, Observers for multivariable systems, IEEE Trans. Autom. Control, 11 (1966), 190-197.

[6] T. Kaczorek, Positive 1D and 2D systems, Springer-Verlag, London, 2002.

[7] D.G. Luenberger, Introduction to dynamic systems: Theory, Models and Applications, Academic Press, New York, 1979.

[8] A. Berman, R.J. Plemmons. Nonnegative matrices in the mathematical sciences, vol. 9, Society for Industrial and Applied Mathematics, Philadelphia, USA, 1994.

[9] L. Farina, S. Rinaldi, Positive linear systems: Theory and Applications, John Wiley & Sons, New York, 2000.

[10] T. Kaczorek, Decoupling zeros of positive continuous-time linear systems, Bull. Polish Acad. Sci., Techn. Sci. 61 (2013), 557-562.

[11] D.N.P. Murthy, Controllability of a linear positive dynamic system, Int. J. Syst. Sci. 17 (1986), 49-54.

[12] M. Fanti, B. Maione, B. Turchiano, Controllability of multi input positive discrete-time systems, Int. J. Control, 51 (1990), 1295-1308.

[13] M.E. Valcher, Controllability and reachability criteria for discrete-time positive systems, Int. J. Control, 65 (1996), 511-536.

[14] M.E. Valcher, Reachability properties for continuous-time positive systems, IEEE Trans. Autom. Control, 54 (2009), 1586-1590.

[15] L. Caccetta, V. Rumchev, A survey of reachability and controllability for positive linear systems, Ann. Oper. Res. 98 (2000), 101-122.

[16] Y. Ohta, H. Maeda, S. Kodama, Reachability, observability and realizability of continuous-time positive systems. SIAM J. Control Optim. 22 (1984), 171-180.

[17] Z. Bartosiewicz, Local Positive Reachability of Nonlinear Continuous-Time Systems, IEEE Trans. Autom. Control, 61 (2016), 4217–4221.

[18] Z. Bartosiewicz, Positive reachability of discrete-time nonlinear systems, in: 2016 IEEE Conference on Control Applications (CCA), IEEE, Buenos Aires, Argentina, 2016: pp. 1203–1208.

[19] L.A. Metzler, Stability of multiple markets: the hick conditions, Econometrica, 13 (1945), 277-292.

[20] V.I. Arnold, Ordinary Differential Equations, Springer-Verlag, Berlin, 1992.

[21] S.R. Caradus, Generalized Inverses and Operator Theory, Queen's Papers in Pure and Applied Mathematics no. 50, Queen's University, Kingston, Ontario, 1978.