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GLOBAL STABILITY OF A FRACTIONAL-ORDER GAUSE-TYPE PREDATOR-PREY MODEL WITH THRESHOLD HARVESTING POLICY IN PREDATOR

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Abstract. Lyapunov function gives a major contribution in studying the dynamics of biological models. In this paper, we study the global stability of a fractional-order Gause-type predator-prey model with threshold harvesting policy in predator by using Lyapunov function. We initiate our work by investigating the existence and uniqueness of solution, and then prove the non-negativity and boundedness of solution. Furthermore, we show that the model has four equilibrium points, where the non-trivial equilibrium points are conditionally globally asymptotically stable. At the end, we demonstrate some numerical simulations by using the generalized Adam–Basforth–Moulton method to support theoretical results. We show numerically that the conversion efficiency rate of predation and the order of the derivative influence the dynamics of the model. We also present the existence of forward and Hopf bifurcation numerically driven by conversion efficiency rate of predation and the order of the derivative respectively.

Keywords: fractional-order; predator-prey; global stability; threshold harvesting policy.

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1. INTRODUCTION

Mathematical modeling is considered as an efficacious scientific tool in learning the most of biological process. One of the exceedingly famous among scientists is modeling the interaction between two populations where a population acts as a prey while the other carries out its role as a predator. This mathematical model is usually known as the predator-prey model and has been used by many researchers to better understand the complex biological phenomena in a meaningful way [1, 2, 3, 4]. There are at least two inventions that have become the foundation in predator-prey modeling nowadays, namely Lotka-Volterra [5] and Leslie-Gower [6, 7] models, see the fundamental evolution of predator-prey model in Berryman [8] and references therein. The predator-prey model is tremendously developed and successfully applied in various fields such as plantations, agriculture, forestry, and fisheries [9, 10, 11, 12, 13]. In this fields, the modeling is focused in developing the biocontrol strategies to preserve the sustainable biological resources such as the pest management [14, 15, 16, 17, 18, 19, 20, 21, 22] as well as the optimal harvesting policy [23, 24, 25, 26, 27, 28].

If we review further, harvesting management holds a vital role in predator-prey interaction since it is directly related to the demands of human needs. Therefore, optimal harvesting must also consider the sustainability of biological resources. There are several types of research studies about harvesting in predator-prey interaction as in [29, 30, 31, 32, 33], which investigate the dynamics of the models and examine the biological conditions so that the population exists or extinct. Nevertheless, their model assumes that the harvesting still exists for all conditions of the population density. Therefore, for relatively small population densities, harvesting would logically threaten the existence of the population. To solve this problem, several researchers have developed a harvesting management called the threshold harvesting policy (THP) [34, 35, 36, 37, 9, 38, 2, 39, 40, 41, 13]. THP is considered capable of maintaining a balance between large yields and protect the exploited biological resources [38]. Such policy is generally applied to prevent overexploitation on the renewable resources while still maximizing the harvesting profits [35]. The policy works as follows: when the population density higher than a determined positive threshold level, the harvesting is permissible; when the population density lower than a determined positive threshold level, the harvesting is not allowed.

In [37], the authors study the impact of THP on the dynamics of a Rosenzweig-MacArthur predator-prey model. They use the integer-order derivative as the operator of the model. Various phenomena are presented such as the stability of equilibrium points and the existence of several bifurcations. In [13], the authors investigate a similar model as in [37] but using fractional-order derivative as the operator to take into account the effects of time memory and transmissible properties which exists in most of the biological process [42, 43, 44, 45, 46]. The model with fractional-order derivative is considered capable of expressing the current condition as a result of capturing the whole time state of the biological process which cannot be provided by the integer-order derivative's model [47]. In [41], the authors study the local stability of a fractional-order predator-prey model with THP in predator as in [37, 13], but assume that the predation process proportional to the density of prey. This model is defined as follows.

$$\begin{aligned} {}^{C}\mathscr{D}_t^{\alpha} x &= rx\left(1-\frac{x}{K}\right) - mxy \equiv F_1, \\ {}^{C}\mathscr{D}_t^{\alpha} y &= nxy - dy - H(y) \equiv F_2, \end{aligned} \tag{1}$$

where

$$H(y) = \begin{cases} 0 & , \text{ if } y < T, \\ \dfrac{h(y-T)}{c+(y-T)} & , \text{ if } y \geq T. \end{cases}$$

The biological interpretations of variables and parameters are given in Table 1. The Caputo fractional-order derivative ${}^{C}\mathscr{D}_t^{\alpha}$ is used as the operator defined by

$${}^{C}\mathscr{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \tag{2}$$

where $0 < \alpha \leq 1$, $t \geq 0$, $f \in C^n([0,+\infty),\mathbb{R})$, and Γ is the Gamma function [48].

In [41], the equilibrium points of the model (1) and their local dynamics have been studied. Since the asymptotic stability is defined locally, then there exists a region so that for any initial condition in this region, the solution evolves to a specific equilibrium point. This explains that although the equilibrium point is locally asymptotically stable, it is not guaranteed that all of the solutions convergent to it. In fact, we need to describe the dynamics for any initial condition so that all the orbit of the solution can be described globally. So far, the global dynamics of model (1) have not been investigated. Therefore, we interest to investigate the global stability condition of equilibrium points of the model (1) by employing a Lyapunov function.

TABLE 1. Description of variables and parameters.

Variable/Parameter	Description
$x(t)$	Density of prey at time t
$y(t)$	Density of predator at time t
r	Intrinsic growth rate of prey
K	Environmental carrying capacity of prey
m	Attack rate of predator on prey
n	Conversion efficiency rate of predation
d	Natural death rate of predator
h	Harvesting rate on predator
c	Half saturation constant of harvesting
T	Harvesting threshold on predator

This paper is organized as follows. To support our analytical study, in Section 2, we present some previous theoretical results in fractional calculus. For the biological purpose, we investigate the existence and uniqueness of solutions, and the non-negativity and boundedness of solutions of the model (1) in Sections 3 and 4. In Section 5, we give the global stability condition of equilibrium points. In Section 6, some numerical simulations including the phase-portraits, time series, and bifurcation diagram of the model (1) are presented to illustrate the theoretical results. We end our works by giving the conclusion in Section 7.

2. PRELIMINARIES

In this section, we present some important results in fractional calculus which are associated with our analytical study.

Lemma 1. [49] *Suppose that* ${}^{C}\mathscr{D}_t^{\alpha}u(t) \in C[a,b]$ *where* $0 < \alpha \leq 1$ *and* $u(t) \in C[a,b]$.

(i) If ${}^{C}\mathscr{D}_t^{\alpha}u(t) \geq 0$, $\forall t \in (a,b)$, *then* $u(t)$ *is a non-decreasing function for each* $t \in [a,b]$.

(ii) If ${}^{C}\mathscr{D}_t^{\alpha}u(t) \leq 0$, $\forall t \in (a,b)$, *then* $u(t)$ *is a non-increasing function for each* $t \in [a,b]$.

Lemma 2. [50] *Consider a Caputo fractional-order equation*

$$ {}^{C}\mathscr{D}_t^{\alpha}u(t) = f(t,u(t)),\ t > 0,\ u(0) \geq 0,\ \alpha \in (0,1], \tag{3}$$

where $f:(0,\infty)\times\Omega\to\mathbb{R}^n, \Omega\subseteq\mathbb{R}^n$. *A unique solution of eq.* (3) *on* $(0,\infty)\times\Omega$ *exists if* $f(t,u(t))$ *satisfies the locally Lipschitz condition with respect to u.*

Lemma 3. *(Standard comparison theorem for Caputo fractional-order derivative* [51]*). Let* $u(t)\in C([0,+\infty))$. *If* $u(t)$ *satisfies* ${}^{C}\mathscr{D}_t^{\alpha}u(t)\le a-bu(t)$, $u(0)=u_0$, *where* $\alpha\in(0,1]$, $(a,b)\in\mathbb{R}^2$ *and* $b\neq 0$, *then*

$$u(t)\le\left(u_0-\frac{a}{b}\right)E_{\alpha}[-bt^{\alpha}]+\frac{a}{b}.$$

Lemma 4. [52] *Let* $u(t)\in C(\mathbb{R}_+)$, $u^*\in\mathbb{R}_+$, *and its Caputo fractional derivatives of order-*α *exist for any* $\alpha\in(0,1]$. *Then, for any* $t>0$, *we have*

$${}^{C}\mathscr{D}_t^{\alpha}\left[u(t)-u^*-u^*\ln\frac{u(t)}{u^*}\right]\le\left(1-\frac{u^*}{u(t)}\right){}^{C}\mathscr{D}_t^{\alpha}u(t).$$

Lemma 5. *(Generalized LaSalle Invariance Principle* [53]*). Suppose* Ω *is a bounded closed set and every solution of system*

$$ {}^{C}\mathscr{D}_t^{\alpha}u(t)=f(u(t)), \tag{4}$$

which starts from a point in Ω *remains in* Ω *for all time. If* $\exists V(u):\Omega\to\mathbb{R}$ *with continuous first order partial derivatives satisfies* ${}^{C}\mathscr{D}_t^{\alpha}V|_{Eq.(4)}\le 0$, *then every solution* $u(t)$ *originating in* Ω *tends to* M *as* $t\to\infty$, *where* M *is the largest invariant set of* $E:=\left\{u|{}^{C}\mathscr{D}_t^{\alpha}V|_{Eq.(4)}=0\right\}$.

3. EXISTENCE AND UNIQUENESS

Let $\Theta:=\left\{(x,y)\in\mathbb{R}^2:\max(|x|,|y|)\le\sigma\right\}$, $T_+\le+\infty$, and $\mathbb{R}^2_+:=\{(x,y):x\ge0,\ y\ge0\}$. The existence and uniqueness of solution of the model (1) are stated in the following theorem.

Theorem 6. *Model* (1) *with initial value in* $\mathbb{R}^2_+\subseteq\Theta$ *has a unique solution in* $\Theta\times[0,T_+]$.

Proof. By considering that the system is switched when the solution passes through the threshold level, we divide the proof into two cases, i.e when $y<T$ and $y\ge T$. We adopt the similar approach that has been used in [13]. We denote $X=(x,y)$ and $\bar{X}=(\bar{x},\bar{y})$. We start for $y\ge T$. Consider a mapping $F(X)=(F_1(X),F_2(X))$. For any $X,\bar{X}\in\Theta$, it follows from eq. (1) that

$$\begin{aligned}\|F(X)-F(\bar{X})\| &= |F_1(X)-F_1(\bar{X})|+|F_2(X)-F_2(\bar{X})|\\ &=\left|\left(rx\left(1-\frac{x}{K}\right)-mxy\right)-\left(r\bar{x}\left(1-\frac{\bar{x}}{K}\right)-m\bar{x}\bar{y}\right)\right|+\end{aligned}$$

$$
\begin{aligned}
&\left|\left(nxy-dy-\frac{h(y-T)}{c+(y-T)}\right)-\left(n\bar{x}\bar{y}-d\bar{y}-\frac{h(\bar{y}-T)}{c+(\bar{y}-T)}\right)\right|\\
&=\left|r(x-\bar{x})-\frac{r}{K}(x+\bar{x})(x-\bar{x})-my(x-\bar{x})-m\bar{x}(y-\bar{y})\right|+\\
&\quad\left|ny(x-\bar{x})+n\bar{x}(y-\bar{y})-d(y-\bar{y})-\frac{ch(y-\bar{y})}{(c+y-T)(c+\bar{y}-T)}\right|\\
&\leq r|x-\bar{x}|+\frac{2\sigma r}{K}|x-\bar{x}|+\sigma m|x-\bar{x}|+\sigma m|y-\bar{y}|+\sigma n|x-\bar{x}|+\\
&\quad\sigma n|y-\bar{y}|+d|y-\bar{y}|+\frac{h}{c}|y-\bar{y}|\\
&=L_1|x-\bar{x}|+L_2|y-\bar{y}|\\
&\leq L|_{y\geq T}\,\|X-\bar{X}\|
\end{aligned}
$$

where $L_1 = r+\left(\frac{2r}{K}+m+n\right)\sigma$, $L_2 = (m+n)\sigma+d+\frac{h}{c}$, $L|_{y\geq T} = \max\{L_1,L_2\}$. Therefore, $F(X)$ satisfies the Lipschitz condition when $y\geq T$. Hence, according to Lemma 2, for any non-negative initial condition, there exists a unique solution $X(t)$ of model (1) when $y\geq T$. We can also easily reveal that $\|F(X)-F(\bar{X})\|\leq L|_{y<T}\,|\,\|X-\bar{X}\|$ for $y<T$, where $L|_{y<T} = \max\{L_1,L_3\}$ with $L_3=(m+n)\sigma+d$. Thus for $y\geq T$, $F(X)$ is also satisfies the Lipschitz condition. Consequently, Lemma 2 states that when $y<T$, the solution of model (1) also exists and unique. □

4. NON-NEGATIVITY AND BOUNDEDNESS

For the biological reason, the solutions of model (1) for any initial conditions in $\mathbb{R}^2_+$ must be in $\mathbb{R}^2_+$ and bounded. Such properties of solutions are presented in the following theorem.

Theorem 7. *The solutions of model* (1) *are non-negative and uniformly bounded if their initial conditions are in* $\mathbb{R}^2_+$.

Proof. We begin by showing the non-negativity of solutions using *"reductio ad absurdum"*. Suppose that there exists a constant $t_1>t_0$ such that

$$
\left\{\begin{array}{rcl}
x(t) & > & 0,\ t_0\leq t<t_1\\
x(t_1) & = & 0\\
x(t_1^+) & < & 0
\end{array}\right. \tag{5}
$$

According to eq. (5) and (1), we obtain

$$ {}^{C}\mathscr{D}_t^{\alpha} x(t_1)\Big|_{x(t_1)=0} = 0. \tag{6}$$

Based on Lemma 1, we acquire $x(t_1^+) = 0$, which contradicts with eq. (5). Therefore, for all $t \geq t_0$, we have $x(t) \geq 0$. Using similar approach, we can prove that $y(t) \geq 0$ for all $t \geq t_0$. Thus for initial conditions in $\mathbb{R}_+^2$, the solutions of model (1) are always in $\mathbb{R}_+^2$ as $t \to \infty$.

Now, to prove the boundedness of model (1), we first define a function $\mathscr{H}(t) = x + \frac{my}{n}$. Therefore, we acquire

$$\begin{aligned}
{}^{C}\mathscr{D}_t^{\alpha}\mathscr{H}(t) + d\mathscr{H}(t) &= {}^{C}\mathscr{D}_t^{\alpha}x + \frac{m}{n}{}^{C}\mathscr{D}_t^{\alpha}y + dx + \frac{dmy}{n}\\
&= rx\left(1-\frac{x}{K}\right) - mxy + \frac{m}{n}\left(nxy - dy - \frac{h(y-T)}{c+(y-T)}\right) + dx + \frac{dmy}{n}\\
&= (d+r)x - \frac{rx^2}{K} - \frac{hm(y-T)}{n(c+y-T)}\\
&= -\frac{r}{K}\left(x - \frac{(d+r)K}{2r}\right)^2 + \frac{(d+r)^2K}{4r} - \frac{hm(y-T)}{n(c+y-T)}\\
&\leq \frac{(d+r)^2K}{4r}
\end{aligned}$$

Based on Lemma 3, we obtain $\mathscr{H}(t) \leq \left(\mathscr{H}(0) - \frac{(d+r)^2K}{4dr}\right)E_{\alpha}\left[-d(t)^{\alpha}\right] + \frac{(d+r)^2K}{4dr}$. Clearly that $\mathscr{H}(t) \to \frac{(d+r)^2K}{4dr}$ for $t \to \infty$. Therefore, all solutions of model (1) are confined to the region Υ where

$$\Upsilon := \left\{(x,y) \in \mathbb{R}_+^2 : x + \frac{my}{n} \leq \xi\right\}, \tag{7}$$

where $\xi = \frac{(d+r)^2K}{4dr} + \varepsilon$ and $\varepsilon > 0$. We have completed the prove of Theorem 7. ☐

5. EQUILIBRIUM POINT AND LYAPUNOV STABILITY

To compare the local stability with the global stability conditions of equilibrium points, we rewrite the existence and local stability condition of the equilibrium points of model (1) given by Theorem 2 and 3 in [41] as the following commensurate lemma.

Lemma 8. [41] *The existence and local asymptotic stability of equilibrium points of model* (1) *are given as follows.*

(i) Both predator and prey extinction point $\Phi_0 = (0,0)$ *always exists and is a saddle point.*

(ii) The predator extinction point $\Phi_1 = (K,0)$ *always exists and is locally asymptotically stable if* $n < \frac{d}{K}$.

(iii) The unharvested co-existence point $\Phi_2 = \left(\frac{d}{n}, \frac{(nK-d)r}{mnK}\right)$ *exists if* $n > \frac{d}{K}$ *and* $(r-mT)n < \frac{dr}{K}$*, which is always locally asymptotically stable.*

(iv) The harvested co-existence point $\Phi_3 = (x^*, y^*)$ *exists if* $y^* < \min\left\{\frac{r}{m}, T\right\}$ *and is locally asymptotically stable if (i)* $\xi_1 < 0$ *and* $\xi_2 > 0$*, or; (2)* $\xi_1 > 0$*,* $\xi_1^2 < 4\xi_2$*, and* $\alpha < \alpha^*$*; where* $x^* = \frac{(r-my^*)K}{r}$*,* y^* *is positive solution of polynomial equation*

$$mnKy^3 + ((c-T)mnK + (d-nK)r)y^2 + ((c-T)(d-nK)+h)ry - hrT = 0,$$

and

$$\xi_1 = \frac{h(y^*-T)^2 - chT}{(c+y^*-T)^2 y^*},$$

$$\xi_2 = \left(mny^* - \frac{h(y^*-T)^2 r - chrT}{(c+y^*-T)^2 K y^*}\right)x^*,$$

$$\alpha^* = \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{4\xi_2 - \xi_1^2}}{\xi_1}\right).$$

In this section, we investigate the global stability of Φ_1, Φ_2, and Φ_3. Since the harvesting is disabled when the density of predator below the threshold level, we divide the global stability into the following two regions

$$\Omega_1 := \{(x,y) : x \geq 0,\ y < T\},\ \text{and}\ \Omega_2 := \{(x,y) : x \geq 0,\ y \geq T\}.$$

We give the global stability condition of equilibrium points Φ_1, Φ_2, and Φ_3 in the following theorem.

Theorem 9. *If* $n \leq \frac{d}{K}$ *then the predator extinction point* $\Phi_1 = (K,0)$ *is globally asymptotically stable in the region* Ω_1.

Proof. Let $\mathscr{V}_1(x,y) = \left[x - K - K\ln\frac{x}{K}\right] + \frac{my}{n}$ be the Lyapunov function. By utilizing Lemma 4, we acquire

$$^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_1(x,y) \leq \left(\frac{x-K}{x}\right){}^{C}\mathscr{D}_t^{\alpha}x + \frac{m}{n}{}^{C}\mathscr{D}_t^{\alpha}y$$

$$
\begin{aligned}
&= (x-K)\left(r-\frac{rx}{K}-my\right)+\frac{m}{n}(nxy-dy)\\
&= (x-K)\left(-\frac{r}{K}(x-K)-my\right)+\frac{m}{n}(nxy-dy)\\
&= -\frac{r}{K}(x-K)^2-\left(\frac{d}{n}-K\right)my
\end{aligned}
$$

Thus, ${}^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_1(x,y)\le 0$ when $n\le\dfrac{d}{K}$ for all $(x,y)\in\mathbb{R}^2_+$. To achieve ${}^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_1(x,y)=0$, we have two cases:

(i) When $n<\dfrac{d}{K}$, ${}^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_1(x,y)=0$ implies that $(x,y)=(K,0)$.

(ii) When $n=\dfrac{d}{K}$, ${}^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_1(x,y)=0$ implies that $x=K$. By subtituting $x=K$ to the first equation of model (1), we have $0={}^{C}\mathscr{D}_t^{\alpha}x=-mKy$ and thus $y=0$. Therefore, ${}^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_1(x,y)=0$ implies that $(x,y)=(K,0)$.

According to those two cases, the only invariant set on which $\mathscr{V}_1(x,y)=0$ is the singleton $\{\Phi_1\}$ and hence Lemma 5 says that Φ_1 is a globally asymptotically stable in the region Ω_1. □

Theorem 10. *The unharvested co-existence point* $\Phi_2=\left(\dfrac{d}{n},\dfrac{(nK-d)r}{mnK}\right)$ *is always globally asymptotically stable in the region* Ω_1.

Proof. Consider a Lyapunov function defined by

$$
\mathscr{V}_2(x,y)=\left[x-\frac{d}{n}-\frac{d}{n}\ln\frac{nx}{d}\right]+\frac{m}{n}\left[y-\left(1-\frac{\delta}{\eta}\right)-\frac{(nK-d)r}{mnK}\ln\frac{mnKy}{(nK-d)r}\right].
$$

We apply Lemma 4 to obtain

$$
\begin{aligned}
{}^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_1(x,y) &\le \left(\frac{x-\dfrac{d}{n}}{x}\right){}^{C}\mathscr{D}_t^{\alpha}x+\frac{m}{n}\left(\frac{y-\dfrac{(nK-d)r}{mnK}}{y}\right){}^{C}\mathscr{D}_t^{\alpha}y\\
&= \left(x-\frac{d}{n}\right)\left(r-\frac{rx}{K}-my\right)+\frac{m}{n}\left(y-\frac{(nK-d)r}{mnK}\right)(nx-d)\\
&= \left(x-\frac{d}{n}\right)\left(-\frac{r}{K}\left(x-\frac{d}{n}\right)-\left(my-\frac{(nK-d)r}{nK}\right)\right)\\
&\quad+\left(my-\frac{(nK-d)r}{nK}\right)\left(x-\frac{d}{n}\right)\\
&= -\frac{r}{K}\left(x-\frac{d}{n}\right)^2-\left(my-\frac{(nK-d)r}{nK}\right)\left(x-\frac{d}{n}\right)\\
&\quad+\left(my-\frac{(nK-d)r}{nK}\right)\left(x-\frac{d}{n}\right)
\end{aligned}
$$

$$= -\frac{r}{K}\left(x-\frac{d}{n}\right)^2$$

Therefore, ${}^C\mathscr{D}_t^\alpha \mathscr{V}_2(x,y) \leq 0$ for all $(x,y) \in \mathbb{R}_+^2$. Denote that ${}^C\mathscr{D}_t^\alpha \mathscr{V}_2(x,y) = 0$ implies that $x = \frac{d}{n}$. By solving the first equation when $x = \frac{d}{n}$, we achieve $0 = {}^C\mathscr{D}_t^\alpha x = \frac{dr}{n}\left(1-\frac{d}{nK}\right) - \frac{dmy}{n}$ and thus $y = \frac{(nK-d)r}{mnK}$. Therefore ${}^C\mathscr{D}_t^\alpha \mathscr{V}_2(x,y) = 0$ implies that $(x,y) = \left(\frac{d}{n}, \frac{(nK-d)r}{mnK}\right)$. This means the only invariant set on which ${}^C\mathscr{D}_t^\alpha \mathscr{V}_2(x,y) = 0$ is the singleton $\{\Phi_2\}$. Following Lemma 5, the unharvested co-existence point $\Phi_2 = \left(\frac{d}{n}, \frac{(nK-d)r}{mnK}\right)$ is always globally asymptotically stable in the region Ω_1. □

Theorem 11. *Let* $n < \frac{(y^*-T)mT}{(c+y^*-T)\xi} - \frac{my^*}{\xi}$. *The harvested co-existence point* $\Phi_3 = (x^*, y^*)$ *is globally asymptotically stable in the region* Ω_2.

Proof. It is clear that Φ_3 is the equilibrium point of (1) if

$$mnK(y^*)^3 + ((c-T)mnK + (d-nK)r)(y^*)^2 + ((c-T)(d-nK)+h)r(y^*) - hrT = 0. \tag{8}$$

Now, by taking a positive Lyapunov function

$$\mathscr{V}_3(x,y) = \left[x - x^* - x^* \ln\frac{x}{x^*}\right] + \frac{m}{n}\left[y - y^* - y^* \ln\frac{y}{y^*}\right],$$

and obeying Lemma 4, we achieve

$$\begin{aligned}
{}^C\mathscr{D}_t^\alpha \mathscr{V}_3(x,y) &\leq \left(\frac{x-x^*}{x}\right){}^C\mathscr{D}_t^\alpha x + \frac{m}{n}\left(\frac{y-y^*}{y}\right){}^C\mathscr{D}_t^\alpha y \\
&= (x-x^*)\left(r - \frac{rx}{K} - my\right) + \frac{m}{n}(y-y^*)\left(nx - d - \frac{h(y-T)}{y(c+y-T)}\right) \\
&= (x-x^*)\left(\frac{rx^*}{K} + my^* - \frac{rx}{K} - my\right) + \\
&\quad \frac{m}{n}(y-y^*)\left(nx - \left(nx^* - \frac{h(y^*-T)}{y^*(c+y^*-T)}\right) - \frac{h(y-T)}{y(c+y-T)}\right) \\
&= (x-x^*)\left(-\frac{r}{K}(x-x^*) - m(y-y^*)\right) + \\
&\quad \frac{m}{n}(y-y^*)\left(n(x-x^*) + \frac{h(y^*-T)}{y^*(c+y^*-T)} - \frac{h(y-T)}{y(c+y-T)}\right) \\
&= -\frac{r}{K}(x-x^*)^2 + \frac{mh(y^*-T)(y-y^*)}{ny^*(c+y^*-T)} - \frac{mh(y-T)(y-y^*)}{ny(c+y-T)} \\
&= -\frac{r}{K}(x-x^*)^2 + \frac{mh(y^*-T)y}{ny^*(c+y^*-T)} - \frac{mh(y^*-T)}{n(c+y^*-T)}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{mh(y-T)}{n(c+y-T)} + \frac{mh(y-T)y^*}{ny(c+y-T)} \\
&\leq -\frac{r}{K}(x-x^*)^2 + \frac{mhy}{ny^*} - \frac{mh(y^*-T)}{n(c+y^*-T)} + \frac{mhy^*}{ny} \\
&= -\frac{r}{K}(x-x^*)^2 - \left(\frac{y^*-T}{c+y^*-T} - \frac{y}{y^*} - \frac{y^*}{y}\right)\frac{mh}{n}
\end{aligned}$$

Since $y < \dfrac{\xi n}{m}$ (see eq. (7)) and $\{y, y^*\} \geq T$, we achieve

$$^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_3(x,y) \leq -\frac{r}{K}(x-x^*)^2 - \left(\frac{y^*-T}{c+y^*-T} - \frac{\xi n}{mT} - \frac{y^*}{T}\right)\frac{mh}{n}.$$

Therefore, because $n < \dfrac{(y^*-T)mT}{(c+y^*-T)\xi} - \dfrac{my^*}{\xi}$, we conclude

$$^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_3(x,y) \leq -\frac{r}{K}(x-x^*)^2. \tag{9}$$

Clearly, $^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_3(x,y) \leq 0$ for all $(x,y) \in \mathbb{R}^2_+$. From eq. (9), $^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_3(x,y) = 0$ when $x = x^*$. By applying $x = x^*$ to model (1), we acquire

$$\begin{aligned}
0 = {}^{C}\mathscr{D}_t^{\alpha}x &= rx^*\left(1-\frac{x^*}{K}\right) - mx^*y, \\
0 = {}^{C}\mathscr{D}_t^{\alpha}y &= nx^*y - dy - \frac{h(y-T)}{c+(y-T)},
\end{aligned}$$

which gives cubic polynomial equation

$$mnKy^3 + ((c-T)mnK + (d-nK)r)y^2 + ((c-T)(d-nK)+h)ry - hrT = 0. \tag{10}$$

Comparing eq. (10) and (8), we obtain $y = y^*$. Hence, $^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_3(x,y) = 0$ implies that $(x,y) = (x^*, y^*)$ and the only invariant set on which $^{C}\mathscr{D}_t^{\alpha}\mathscr{V}_3(x,y) = 0$ is the singleton $\{\Phi_3\}$. Applying Lemma 5, the harvested co-existence point $\Phi_3 = (x^*, 1-x^*)$ is globally asymptotically stable in the region Ω_2. □

Based on Lemma 8 and Theorem 9 to 11, we have the following remarks.

Remark 1. The global stability properties of Φ_1 given by Theorem 9 same as its local stability properties given by Lemma 8. This means for every non-negative initial condition in Ω_1, the solutions are convergent to Φ_1 and hence the predator will extinct whenever $n \leq \dfrac{d}{K}$.

Remark 2. Since Φ_2 is always globally asymptotically stable in Ω_2, we confirm that the locally asymptotically stable condition is always satisfied, which supports the validity of Lemma 8. This means if the conversion efficiency rate of predation is greater than the ratio of natural death rate and environmental carrying capacity of prey, then both prey and predator populations will survive.

Remark 3. If the globally asymptotically stable requirement of Φ_3 is satisfied, then the existence of both prey and predator are guaranteed. This means that THP is a suitable policy for the renewable resources management.

6. NUMERICAL SIMULATIONS

We explore the dynamics of model (1) numerically by performing some numerical simulations using a Caputo fractional-order predictor-corrector scheme developed by Diethelm et al. [54]. Due to field data limitations, we use hypothetical parameter values for the numerical simulations. We divide the numerical simulations into two cases to describe the dynamics of the model (1) as follows.

(i) The influence of the conversion efficiency rate of predation (n), and

(ii) The influence of the order of the derivative (α).

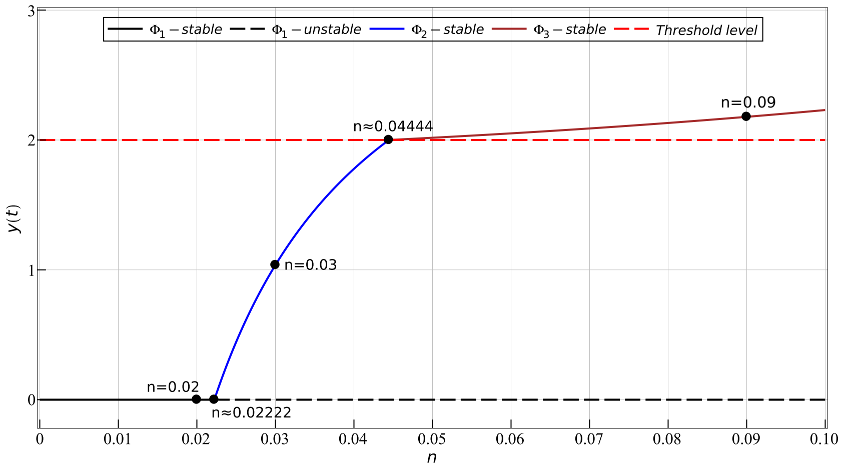
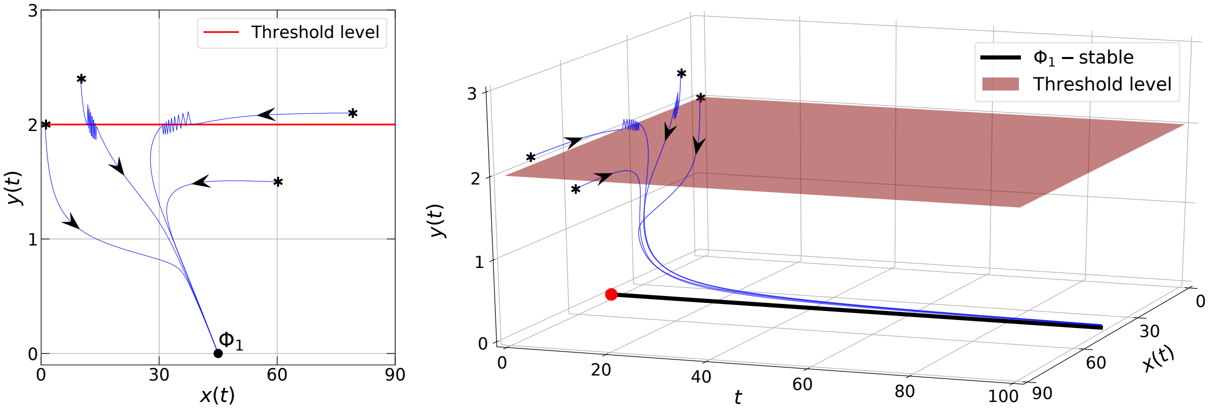
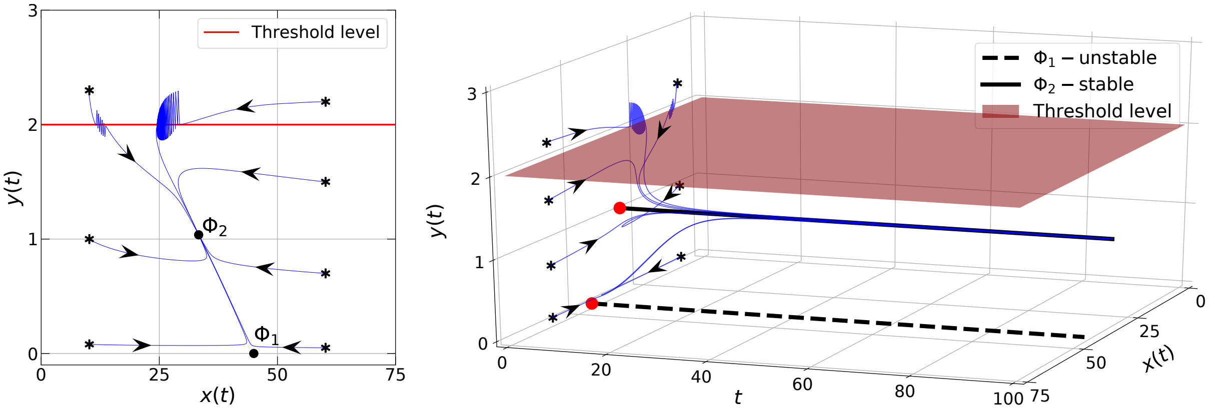


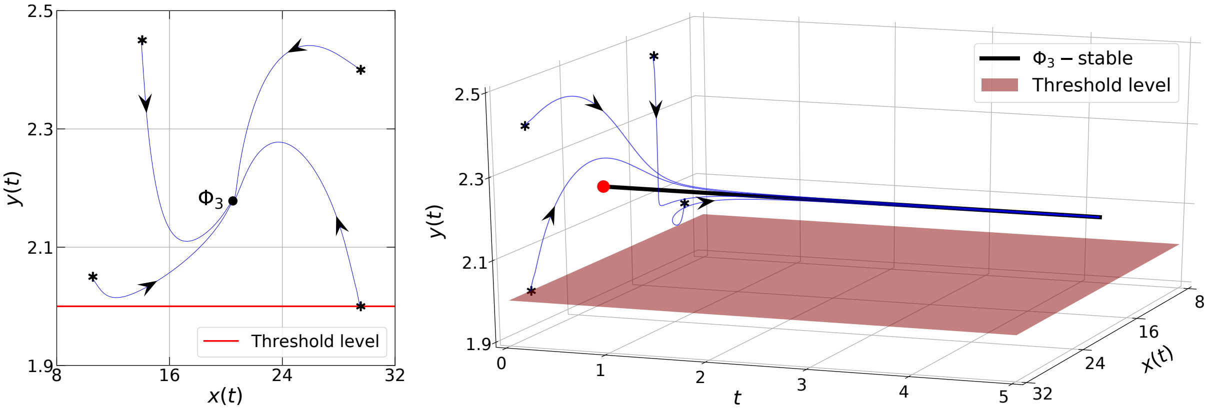
FIGURE 1. Bifurcation diagram of model (1) driven by n in interval $[0, 0.1]$ with parameter values: $r = 6$, $K = 45$, $m = 1.5$, $d = 1$, $h = 7$, $c = 0.5$, $T = 2$, and $\alpha = 0.92$.



(A) Phase portrait when $n = 0.02$ (B) 3D- time series when $n = 0.02$

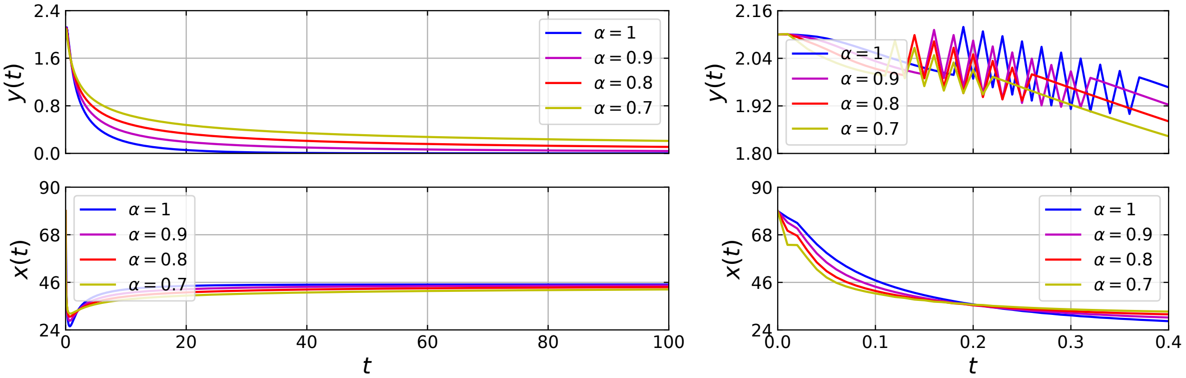


(C) Phase portrait when $n = 0.03$ (D) 3-D time series when $n = 0.03$

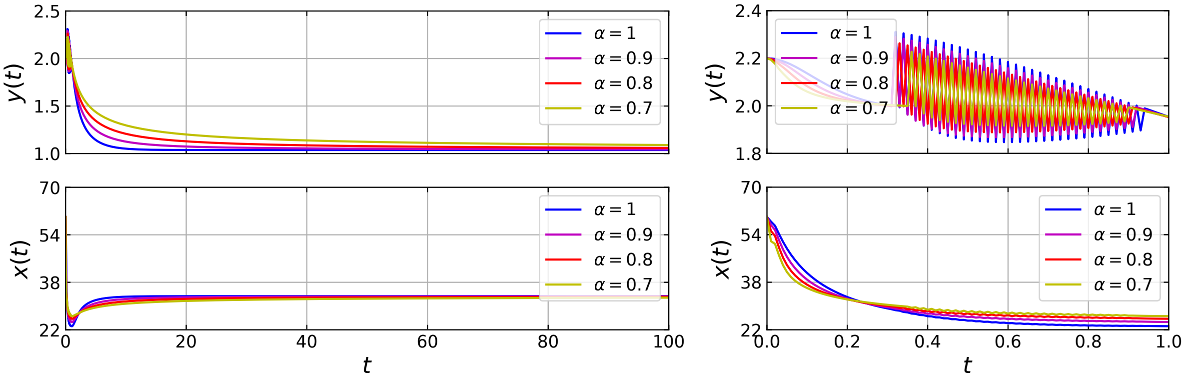


(E) Phase portrait when $n = 0.09$ (F) 3-D time series when $n = 0.09$

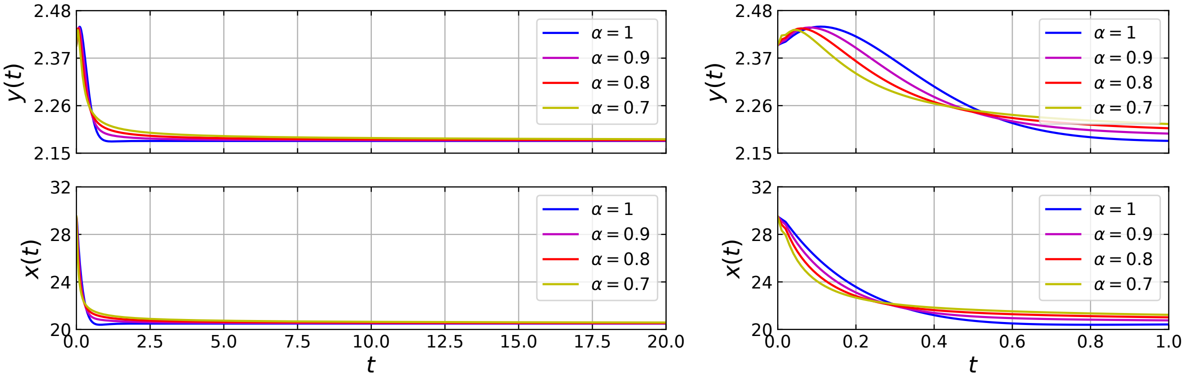
FIGURE 2. Numerical simulations of model (1) with parameter values: $r = 6$, $K = 45$, $m = 1.5$, $d = 1$, $h = 7$, $c = 0.5$, $T = 2$, and $\alpha = 0.92$.



(A) $n = 0.02$ (B) Local amplification to (a)

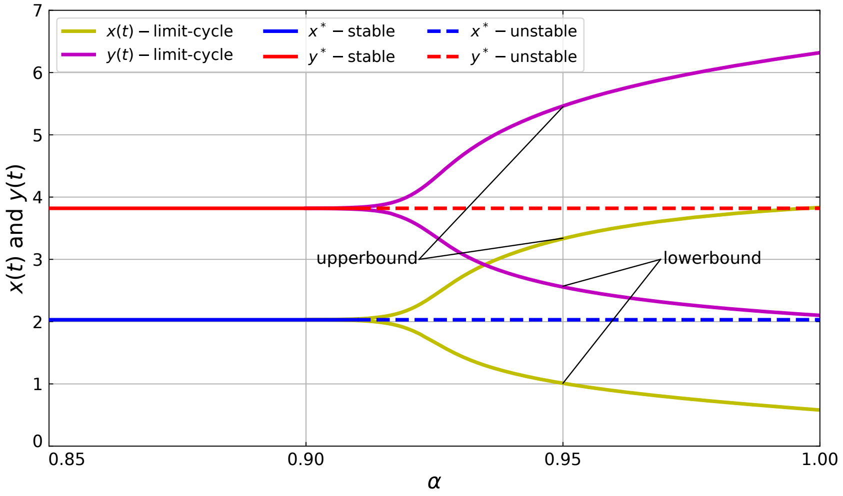


(C) $n = 0.03$ (D) Local amplification to (c)

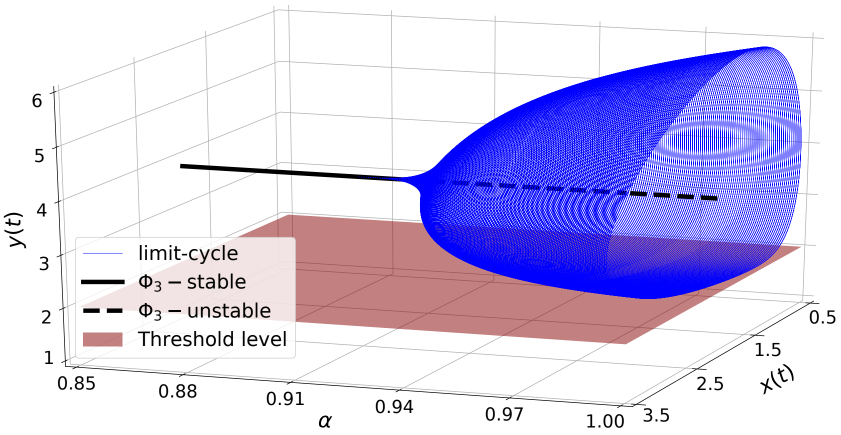


(E) $n = 0.09$ (F) Local amplification to (e)

FIGURE 3. Time series of model (1) with parameter values: $r = 6$, $K = 45$, $m = 1.5$, $d = 1$, $h = 7$, $c = 0.5$, and $T = 2$.

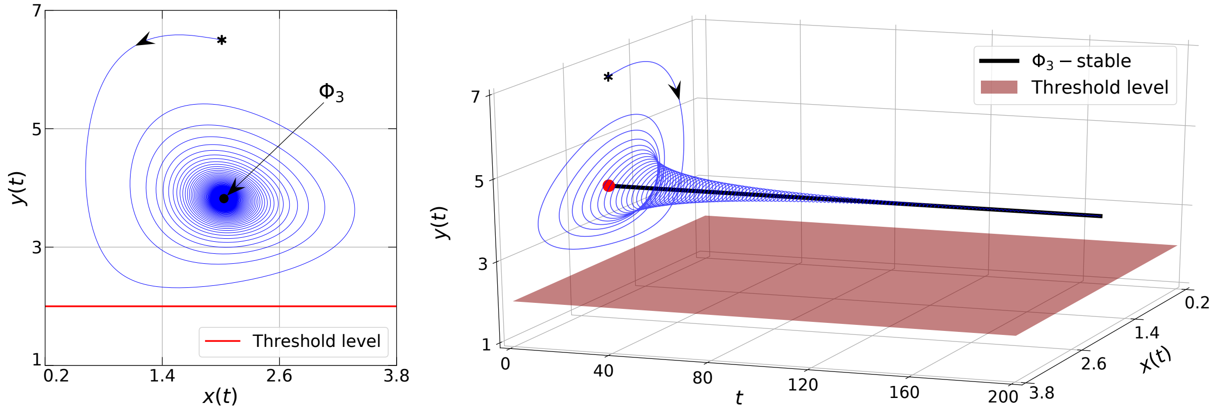


(A) Bifurcation diagram

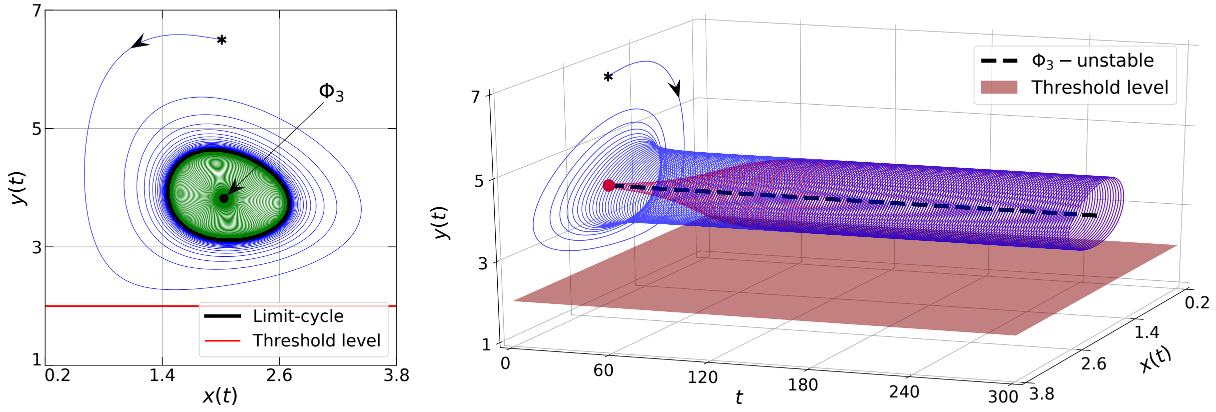


(B) Evolution of limit-cycle

FIGURE 4. The diagram of Hopf bifurcation and evolution of limit-cycle of model (1) around $\Phi_2 = (2.03136, 3.81943)$ with parameter values: $r = 6$, $K = 45$, $m = 1.5$, $n = 1.2$, $d = 1$, $h = 7$, $c = 0.5$, and $T = 2$.



(A) Phase portrait when $\alpha = 0.92$ (B) 3-D time series when $\alpha = 0.92$



(C) Phase portrait when $\alpha = 0.93$ (D) 3-D time series when $\alpha = 0.93$

FIGURE 5. Numerical simulations of model (1) around $\Phi_3 = (2.03136, 3.81943)$ with parameter values: $r = 6$, $K = 45$, $m = 1.5$, $n = 1.2$, $d = 1$, $h = 7$, $c = 0.5$, and $T = 2$.

6.1. The influence of the conversion efficiency rate of predation. From previous analytical results, it can be noticed that the global stability of equilibrium points depends on the value of the conversion efficiency rate of predation (n). To show the influence of n on the dynamics of model (1), we first plot in Figure 1 the bifurcation diagram driven by n in the interval $[0, 0.1]$, while other parameter values are arranged as follows.

$$r = 6,\ K = 45,\ m = 1.5,\ d = 1,\ h = 7,\ c = 0.5,\ T = 2,\ \text{and } \alpha = 0.92. \tag{11}$$

From Figure 1, it is seen that when $0 < n \lessapprox 0.02222$, Φ_1 is globally asymptotically stable while Φ_2 does not exists. When $0.02222 \lessapprox n \lessapprox 0.04444$, Φ_1 losses its stability and a globally asymptotically stable equilibrium point Φ_2 appears. This phenomenon is called forward bifurcation, where a stable equilibrium point splits into two equilibrium points where the original equilibrium point losses its stability and the new equilibrium point is stable. Furthermore, when $0.04444 < n \leq 0.1$, Φ_2 vanishes and there appears a globally asymptotically stable equilibrium point Φ_3. To show the orbit of solutions for each case, we portray the numerical simulations including the phase portraits and the time series. When $n = 0.02$, the solutions with the initial values below the threshold level are directly convergent to Φ_1, while if the initial values above the threshold level, the solutions oscillate and moving toward Φ_1; see Figures 2a and 2b. If we replace n with 0.03, Φ_1 becomes a saddle-point and Φ_2 is globally asymptotically stable. The similar behavior occurs in Φ_2. When the initial values above the threshold level, the solution oscillates around the threshold value and convergent to Φ_2 while others are directly convergent; see Figures 2c and 2d. Finally, when we take $n = 0.09$, Φ_1 is still unstable, and Φ_2 does not exists and there exists an asymptotically stable Φ_3. As in Figures 2e and 2f, all solutions are convergent to Φ_3. From the biological point of view, if the predator has adequate food for its growth, the existence of both prey and predator can be maintained although the harvesting is permitted. Even if the conversion efficiency rate of predation decreases down to a certain value limit, by obeying the THP, the existence of both populations are still preserved.

6.2. The influence of the order of the derivative. To show numerically the influence of the order of the derivative (α), we first set the similar parameter values as in Figure 2 and some different values of α. It is shown that in this case, α does not change the stability of

$\Phi_i,\ i = 1,2,3$, except the convergence rate of the solution. The solution with a higher value of α is convergent more rapidly to the equilibrium point than that with a smaller value of α, see Figure 3. Especially, from Figures 3b and 3d, we examine that for higher value of α, the amplitude of oscillations are wider compared to the smaller value of α. The time to oscillate is also reduced when α goes down.

The interesting phenomenon which occurs in the model (1) is the occurrence of Hopf bifurcation. See Theorem 4 in [41]. In our works, we present the existence of Hopf bifurcation around Φ_3 by portraying the bifurcation diagram driven by α. We set the parameter values as follows: $r = 6$, $K = 45$, $m = 1.5$, $n = 1.2$, $d = 1$, $h = 7$, $c = 0.5$, and $T = 2$. If we vary α in interval $\alpha \in [0.85, 1]$, we obtain the bifurcation diagram as in Figure 4a. The asymptotically stable Φ_3 losses its stability via Hopf bifurcation when α passes through a bifurcation point. Although Φ_3 is unstable, the solutions are still bounded since they are convergent to a limit-cycle. We perform the evolution of the limit-cycle in Figure 4b. It is confirmed that the diameter of the limit-cycle is enlarged when α is increased. Furthermore, we demonstrate the stability of Φ_3 by taking two different values of α as in Figure 5. When $\alpha = 0.92$, ϕ_3 is asymptotically stable, where for the initial value above the threshold level, the solution oscillates towards Φ_3. If we take $\alpha = 0.93$, Φ_3 becomes unstable and all solutions are convergent to a limit-cycle. Although the interior equilibrium point above the threshold level losses its stability, both populations' densities are still maintained and change periodically.

7. CONCLUSIONS

The global stability of a fractional-order Gause-type predator-prey model with THP in predator has been studied. We have shown that when the predator density is below the threshold level, both local and global stability conditions have similar properties. If the interior equilibrium point exists, it is always globally asymptotically stable, which means the existence of both populations is maintained. When the population is above the threshold level, we have conditions that both population's densities are preserved. Although the equilibrium point losses its stability, both populations will not extinct, but change periodically via Hopf bifurcation. Hence, we confirm that THP is a suitable management policy to protect overexploitation on the renewable resources.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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