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PREDATOR-PREY INTERACTIONS WITH HARVESTING OF PREDATOR WITH PREY IN REFUGE

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Abstract: In this paper, we study predator prey interactions where the predator is exposed to the risk of disease and harvesting while the prey has the ability to use a refuge. We consider two models: the ability of prey to use a constant refuge and the ability to use random refuge. We found bounded, non-periodic solutions and the equilibrium points for both models. We then show the role of the refuges in the stability of the systems. The equilibrium was stable locally, but not globally, and we found some basin to these equilibrium points. Numerical simulations show several types of oscillations that occur due to the kinds of refuges and prey's ability to use these refuges. For both models, there exist an invariant region; the invariant region in the constant refuge is better than the invariant region in random refuge because it ensures the continuity of all populations and sustainability of the harvested species and controlling the disease without it becoming endemic. Finally the low density prey in refuges makes a limit cycle around the equilibrium in refuges whiles the small density is stable.

Keywords: Constant Refuge; Random Refuge; Basin of attraction; Limit cycle.

2000 AMS Subject Classification: 92B05

1. Introduction

Prey-predator models are of great interest to researchers in mathematics and ecology because they deal with environmental problems such as community's morbidity and how to control it, optimal harvest policy to sustain a community, and others. In the physical sciences, generic models can be constructed to explain a variety of

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phenomena. However, in the life sciences a model only describes a particular situation. Simple models such as the Lotka-Volterra are not able to tell us what is going on in the majority of cases. One of the reasons is due to the complexity of the biological ecosystem. Hence the needs for a variety of models to describe nature .

Theoretical and numerical studies of these models are able to give us an understanding of the interactions that is taking place. A particular class of models considers the existence of a disease in the predator or prey. Several models were constructed to study particular cases. To ensure the existence of the species involved, one of the steps taken is to harvest the infected species. Due to the need to survive, the prey has developed strategies to avoid the predator, one of them the use of refuge.

In this paper, we consider the case where the infected predator is harvested, while the prey has a refuge. Several related theoretical studies have been conducted.

Amongst them are studies on the disease spread among the prey and the epidemic among predators with action incidence [18], the role of transmissible disease in the Holling Tanner predator prey model [9], the analysis of prey predator model with disease in the prey [19], anothers study the disease in Lotka Volterra, [7]study the dynamics of a fisher resource system in an aquatic environment in two zones harvest in reserve area,[16] study the harvesting of infected prey,[2]show the stability analysis of harvesting,[6] Study the stability of harvested when the disease affects the predator by using the reproduction number, While some researchers took their studies using the refuge by the prey and the types of those refuges where used ,[4] idea refuge disk with response function type II functional response incorporation a constant prey refuge[3] studied the idea of using prey refuge random,[1,4]Some studies took the form of a special like the refuges effect on the stability of the models studied in [14,11, 10],[17, 8] the refuge protect a constant number of prey lead to a stable and stronger stabile ,[13 , 6 ,8] investigated that the a destabilizing effect through the occurrence of a stable limit cycle. Analysis [15] refers to the low density of prey at the refuge gives a limit cycle,[15] investigated the rate at which prey moves to refuge is proportional to predator density and show the stability,[12] per-formed an analytical study two kinds of refuge (constant and proportional) and investigated the role of these in different classes of functional responses.

The model is introduced after this section, followed by analysis on nature and properties of the solutions. Numerical simulations were performed to verify the theoretical discussions and to investigate further properties.

2. Mathematical model

The model is written as:

$$\begin{cases} \dot{x} = rx(1-x) - ax(1-m)\left(y+z\right) \\ \dot{y} = bx(1-m)y + \rho yz - q_1 y \\ \dot{z} = bx(1-m)z - \rho yz - q_2 z \end{cases} \tag{1}$$

where *x, y, z* are the prey, infected predator and susceptible predator respectively; r is the growth rate of prey; a ,b the capture rate $(a > b)$, ρ is the contact rate between the susceptible and infected predator; q_1, q_2 are the harvest rates of the infected and susceptible predator respectively, m is a constant (which describes the ability of the prey to use constant refuge) . We assume that the less effective predator shall be easier to harvest, so $q_1 > q_2$; we also assume infected predator not become susceptible again and finally the disease does not affect the ability of the infected predator attacking prey .

2.1 Nature of solutions

Theorem 1.The solution of system (1) is bounded.

Proof .

Let the function $w(x,y,z) = x(t) + y(t) + z(t)$ and take the positive number $0 < \mu < q_2$

Then $w'\left(t\right) + uw = rx(1-x) + \mu x - \left(a-b\right)x(1-m)\left(y+z\right) - \left(q_1 - \mu\right)y - \left(q_2 - \mu\right)z$

$$w'\left(t\right) + uw < -r\left(x^2 - \left(\frac{r+\mu}{r}\right)x + \left(\frac{r+\mu}{2r}\right)^2\right) + \frac{1}{r}\left(\frac{r+\mu}{2}\right)^2$$

Let $\frac{1}{r}\left(\frac{r+\mu}{2}\right)^2 = v$

$w'(t)+uw(t)\le v$

$0 < w(x,y,z) \le \frac{v}{u}\left(1-e^{-ut}\right)+e^{-ut}\,(x,y,z)|_{t=0}$ □

Theorem 2.The system (1) has no periodic solution

Proof.

To show there is no periodic orbit to this system, we use Dulac's criterion and first consider the xy-plane,

Let $h_1(x,y) = rx - rx^2 - ax(1-m)y$, $h_2(x,y) = bx(1-m)y - q_1 y$

$$\Delta(x,y) = \frac{\vartheta(h_1 H)}{\vartheta x} + \frac{\vartheta(h_2 H)}{\vartheta y} = -\frac{r}{y}$$

It's clear that is no change in sign, therefore this system cannot have any periodic solution in the xy - plane.

Let $H(x,z) = \frac{1}{xz}$, where $H(x,z)$ in the positive quadrant of the xz - plane

Let $h_3(x,z) = rx - rx^2 - ax(1-m)z$, $h_4(x,z) = bx(1-m)z - q_2 z$

$$\Delta(x,z) = \frac{\vartheta(h_3 H)}{\vartheta x} + \frac{\vartheta(h_4 H)}{\vartheta z} = -\frac{r}{z}$$

There is no change in sign; therefore there is no periodic solution in xz - plane.

Hence the system has no periodic solution. □

2.2 Equilibrium

Letting $(\dot{x} = \dot{y} = \dot{z} = 0)$ we get the equilibriums (non trivial):

(i) A predator free-equilibrium $P_{c1}(1,0,0)$ in the absent of the predator, the prey grows and tends to its carrying capacity.

(ii) A disease free equilibrium $P_{c2}(x_2,0,z_2)$, in this case the disease disappears from the system, where $x_2 = \dfrac{q_2}{b(1-m)}$, $z_2 = \dfrac{r(1-x_2)}{a(1-m)}$

(iii) The disease becomes epidemic i.e. all predators become infected, the equilibrium in this case $P_{c3}(x_3, y_3, 0)$, where $x_3 = \dfrac{q_1}{b(1-m)}$, $y_3 = \dfrac{r(1-x_3)}{a(1-m)}$

(iv) The positive equilibrium $P_c^*(x^*, y^*, z^*)$ where all population coexists and survives, from the second and third equations of system (1) we get

$$bx(1-m) = -\rho z + q_1$$
$$bx(1-m) = \rho y + q_2$$
$$y + z = \frac{q_1 - q_2}{\rho}$$

Then $x^* = 1 - \dfrac{a}{r\rho}(1-m)\left(q_1 - q_2\right)$, $y^* = \dfrac{bx^*(1-m) - q_2}{\rho}$, $z^* = \dfrac{q_1 - bx^*(1-m)}{\rho}$

With conditions $1 - \dfrac{r\rho}{a\left(q_1 - q_2\right)} < m$ and $1 - \dfrac{q_1}{bx^*} < m < 1 - \dfrac{q_2}{bx^*}$

2.2. Equilibrium

Letting $\left(\dot{x} = \dot{y} = \dot{z} = 0\right)$ we get the equilibriums (non trivial):

(i) A predator free-equilibrium $P_{c1}(1, 0, 0)$ in the absent of the predator, the prey grows and tends to its carrying capacity.

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With conditions $1 - \frac{r\rho}{a\left(q_1 - q_2\right)} < m$ and $1 - \frac{q_1}{bx^*} < m < 1 - \frac{q_2}{bx^*}$

2.3. Stability

The Jacobian matrix of system (1) is given by:

$$J_c = \begin{bmatrix} r - 2rx - a(1-m)(y+z) & -ax(1-m) & -ax(1-m) \\ b(1-m)y & bx(1-m) + \rho z - q_1 & \rho y \\ b(1-m)z & -\rho z & bx(1-m) - \rho y - q_2 \end{bmatrix}$$

First we study system (1) as a sub system (without disease); the system become

$$\begin{cases} \dot{x} = rx(1-x) - ax(1-m)z \\ \dot{z} = bx(1-m)z - q_2 z \end{cases} \tag{1-a}$$

The equilibrium (non trivial) are $E_c(1,0), \bar{E}_c(x_2, z_2))$, where

$$x_2 = \frac{q_2}{b(1-m)} \text{ and } z_2 = \frac{r}{a}\left(\frac{1-x_2}{1-m}\right)$$

The eigenvalues near the first equilibrium are $-r$ and $b(1-m) - q_2$.

This is stable when $m > 1 - \frac{q_2}{b}$ and unstable otherwise.

Let R_0 is denote the basic reproduction number of the susceptible predator, where $R_0 = \frac{b(1-m)}{q_2}$ and if $R_0 > 1$ the susceptible predator survive, but in this case $R_0 < 1$; therefore the second eigenvalues is negative then this equilibrium is stable. This stability can become unstable when we change one or all the parameters $\left(q_2, m, b\right)$.

The trace of Jacobian matrix near the equilibrium $\bar{E}_c(x_2, z_2)$ is $-rx_2$ (negative);

therefore it is stable without any condition. We cannot find the Lyapunov function at this point to proof a global asymptotically stable in R_+^2 , In the following theorem we show the basin of attraction of $\bar{E}_c(x_2, z_2)$.

Theorem 3. Assume that the equilibrium $\bar{E}_c(x_2, z_2)$ is locally stable, the basin of attraction of this equilibrium is denoted by $B(\bar{E}_c(x_2, z_2)$ where

$$B(\bar{E}_c(x_2, z_2) = \{(x, z) \in R_+^2 : x > x_2, z > z_2, \text{with} \left(x_2 z < z_2 x\right)\}$$

Proof.

Let $V_1(x, z)$ be a function where

$$V_1(x, z) = \left(x - x_2 - x_2 \log \frac{x}{x_2} \right) + \left(z - z_2 - z_2 \log \frac{z}{z_2} \right) \text{ ,then}$$

$$\frac{dV_1}{dt} = -r\left(x - x_2\right)^2 - \left(a - b\right)(1 - m)\left(x - x_2\right)\left(z - z_2\right) < 0 \ \square$$

The eigenvalues near $P_2(x_2, 0, z_2)$ are $\frac{-rx_2}{2} \pm \frac{\sqrt{r^2 x_2^2 - 4q_2 r\left(1 - x_2\right)}}{2}$ and $q_2 + \rho z_2 - q_1$

It is stability when $1 - \frac{r\rho(1 - x_2)}{a\left(q_1 - q_2\right)} > m$

When all predators become infected the subsystem become as

$$\begin{cases} \dot{x} = rx(1 - x) - ax(1 - m)y \\ \dot{y} = bx(1 - m)y - q_1 y \end{cases} \tag{1-b}$$

The equilibriums (non trivial) are $E_c(1, 0), \hat{E}_c(x_3, y_3)$ where,

$$x_3 = \frac{q_1}{b(1 - m)} \ , y_3 = \frac{r(1 - x_3)}{a(1 - m)}$$

And the eigenvalues near $E_c(1, 0)$ are $-r$ and $b(1 - m) - q_1$, this is stable when $m > 1 - \frac{q_1}{b}$ and unstable otherwise.

Let R_1 is denotes the basic reproduction number of the infected predator, where $R_1 = \frac{b(1 - m)}{q_1}$ and if $R_1 > 1$ the infected predator survive, and because in this case $R_1 < 1$

therefore the second eigenvalues is negative then this equilibrium is stable. This stability can transform to unstable when we decrease one or all parameters $\left(q_1, m, b\right)$.

The trace of Jacobian matrix near the equilibrium $\hat{E}_c(x_3, y_3)$ is $-rx_3$ (negative) therefore it is stable without any condition. We can find the basin of attraction to this point as in the following theorem.

Theorem 4. Assume that the equilibrium $\hat{E}_c(x_3, y_3)$ is locally stable, the basin of attraction of this equilibrium is denoted by $B\left(\hat{E}_c(x_3, y_3)\right)$ where

$$B(\bar{E}_c(x_3, y_3) = \{(x,z) \in R_+^2 : x > x_3, y > y_3\}$$

Proof.

The proof is same in theorem (3).

The equilibrium $P_{c3}(x_3, y_3, 0)$ is stable with condition $m > 1 - \dfrac{r\rho(1-x_3)}{a\left(q_1 - q_2\right)}$.

The stability near the equilibrium $P_c^*(x^*, y^*, z^*)$ is given by the equation

$\lambda^3 + A\lambda^2 + B\lambda + C = 0$ Where

$A = rx^* > 0$, $B = \left(ax^* b(1-m)^2 \left(y^* + z^*\right) + \rho^2 y^* z^*\right)$, $C = r\rho^2 x^* y^* z^* > 0$

$AB - C = rx^* \left(ax^* b(1-m)^2 \left(y^* + z^*\right)\right) > 0$

From Routh- Hurwitz stability criterion it is stable.

Theorem 5. Assume that the equilibrium $P_c^*(x^*, y^*, z^*)$ is locally stable, the basin of attraction of this equilibrium is denoted by $B\left(P_c^*(x^*, y^*, z^*)\right)$ where

$$B(P_c^*(x^*, y^*, z^*)) = \{(x,y,z) \in R_+^3 : x > x^*, y > y^*, z > z^*\}$$

Proof.

The proof is similar to proof of theorem (3).

3. The random refuge model

We write the model as follows, where the parameters are explained in section 2.

$$\begin{cases} \dot{x} = rx(1-x) - a(x-m)(y+z) \\ \dot{y} = b(x-m)y + \rho yz - q_1 y \\ \dot{z} = b(x-m)z - \rho yz - q_2 z \end{cases} \quad (2)$$

3.1 Nature of solution.

Theorem 6 . The solution of system (2) is bounded.

Proof.

Let the function $w(x,y,z) = x(t) + y(t) + z(t)$ and take the positive number $0 < \mu < q_2$

Then $w'(t) + uw = rx(1-x) + \mu x - (a-b)(x-m)(y+z) - (q_1 - \mu)y - (q_2 - \mu)z$

$$w'(t) + uw < -r\left(x^2 - \left(\frac{r+\mu}{r}\right)x + \left(\frac{r+\mu}{2r}\right)^2\right) + \frac{1}{r}\left(\frac{r+\mu}{2}\right)^2$$

Let $\frac{1}{r}\left(\frac{r+\mu}{2}\right)^2 = v$

$$w'(t) + uw(t) \le v$$

$$0 < w(x,y,z) \le \frac{v}{u}\left(1 - e^{-ut}\right) + e^{-ut}\left(x,y,z\right)|_{t=0} \ \square$$

Theorem 7.The system (2) has no periodic solution

Proof.

To show there is no periodic orbit to this system, we use Dulac's criterion and first consider the xy-plane,

Let $h_1(x,y) = rx - rx^2 - a(x-m)y$, $h_2(x,y) = b(x-m)y - q_1 y$, *and*

$H(x,y) = \frac{1}{xy}$, where $H(x,z)$ in the positive quadrant of the xy- plane, then

$$\Delta(x,y) = \frac{\vartheta(h_1 H)}{\vartheta x} + \frac{\vartheta(h_2 H)}{\vartheta y} = -\frac{r}{y} - \frac{am}{x^2}$$

It's clear that is no change in sign, therefore this system cannot have any periodic solution in the xy- plane.

Let $H(x,z)=\frac{1}{xz}$, where $H(x,z)$ in the positive quadrant of the xz - plane, and

$h_3(x,z)=rx-rx^2-a(x-m)z$, $h_4(x,z)=b(x-m)z-q_2 z$

$$\Delta(x,z)=\frac{\vartheta(h_3 H)}{\vartheta x}+\frac{\vartheta(h_4 H)}{\vartheta z}=-\frac{r}{z}-\frac{am}{x^2}$$

There is no change in sign; therefore there is no periodic solution in xz - plane.

Hence the system has no periodic solution. □

3.2 Equilibrium

Again by letting $(\dot{x}=\dot{y}=\dot{z}=0)$ we get the equilibriums (non trivial):

(i) A predator free-equilibrium $P_{r1}(1,0,0)$, in the absent of the predator, the prey grows and tends to its carrying capacity.

(ii) A disease free equilibrium $P_{r2}(x_2,0,z_2)$, in this case the disease disappears from the system and $x_2=\frac{q_2}{b}+m$, $z_2=\frac{rx_2(1-x_2)}{a(x_2-m)}$

(iii) The disease become an epidemic i.e. all predators become infected, the equilibrium in this case $P_{r3}(x_3,y_3,0)$, where $x_3=\frac{q_1}{b}+m$, $y_3=\frac{rx_3(1-x_3)}{a(x_3-m)}$

(iv) The positive equilibrium $P_r^*(x^*,y^*,z^*)$ all population coexists and survives, where $y+z=\frac{q_1-q_2}{\rho}$ .Then $x^*=\frac{R+\sqrt{R^2+4H}}{2}$

Where $R=\left(1-\frac{a}{r\rho}(q_1-q_2)\right)$ and $H=\frac{am}{r\rho}(q_1-q_2)$

$y^*=\frac{b(x^*-m)-q_2}{\rho}$, $z^*=\frac{q_1-b(x^*-m)}{\rho}$

It exists with conditions

$(i)\frac{r\rho}{a}>(q_1-q_2)$ (ii) $x^*-\frac{q_1}{b}<m<x^*-\frac{q_2}{b}$

3.3. Stability

The Jacobian matrix of system (2) is given by

$$J_r = \begin{bmatrix} r-2rx-a(y+z) & -a(x-m) & -a(x-m) \\ by & b(x-m)+\rho z-q_1 & \rho y \\ bz & -\rho z & b(x-m)-\rho y-q_2 \end{bmatrix}$$

The first subsystem (without disease) is

$$\begin{cases} \dot{x} = rx(1-x)-a(x-m)z \\ \dot{z} = b(x-m)z-q_2 z \end{cases} \tag{2-a}$$

The equilibrium (non trivial) are $E_r(1,0), \overline{E}_r(x_2, z_2)$ and the eigenvalues near the first equilibrium are $-r$ and $b(1-m)-q_2$ .This is stable when $m > 1-\frac{q_2}{b}$ and unstable otherwise. Let R_0' denote the basic reproduction number of the susceptible predator where $R_0' = \frac{b(x-m)}{q_2}$ and if $R_0' > 1$ the susceptible predator survive, and

because in this equilibrium there is no susceptible predator, so $R_0' < 1$;therefore it is necessary that $m > 1-\frac{q_2}{b}$,then the second eigenvalues is negative; this equilibrium is stable with condition $m > 1-\frac{q_2}{b}$. This stability can be transformed to unstable when we change one or all of the parameters (b, q_2, m). The Jacobian matrix near the equilibrium $\overline{E}_r(x_2, z_2)$ is

$$J_{r1} = \begin{bmatrix} r\left(1-2x_2-\frac{(q_2+bm)(R_0'-1)}{bq_2}\right) & -a\frac{q_2}{b} \\ bz_2 & 0 \end{bmatrix}$$

$$= \lambda^2 - r\left(1-2x_2-\frac{(q_2+bm)(R_0'-1)}{bq_2}\right)\lambda + aq_2 z_2$$

Let $B = r\left(1-2x_2-\frac{(q_2+bm)(R_0'-1)}{bq_2}\right)$

$$\lambda = \frac{B \pm \sqrt{B^2 - 4aq_2z_2}}{2}$$

It is locally asymptotically stable if $Tr(J_{r1}) < 0$ or $q_2\left(\frac{1-2x_2}{(R_0'-1)} - \frac{1}{b}\right) < m$ and unstable otherwise. We cannot find the Lyapunov function at this point to proof global asymptotically stable in R_+^2, so in the following theorem we show the basin of attraction of $\bar{E}_r(x_2, z_2)$.

Theorem 8. Assume that the equilibrium $\bar{E}_r(x_2, z_2)$ is locally stable; the basin of attraction of this equilibrium is

$$B(\bar{E}_r(x_2, z_2) = \{(x,z) \in R_+^2 : x > x_2, z > z_2, with\left(x_2 z < z_2 x\right)\}$$

Proof .

Let $V_4(x,z) = \left(x - x_2 - x_2 \log\frac{x}{x_2}\right) + \left(z - z_2 - z_2 \log\frac{z}{z_2}\right)$

Then $\frac{dV_4}{dt} = -r\left(x - x_2\right)^2 - \left(a-b\right)\left(z - z_2\right)\left(x - x_2\right) + am\left(\frac{x_2 z - z_2 x}{x x_2}\right)\left(x - x_2\right)$ □

The $P_2(x_2, 0, z_2)$ is stability with condition and $m < x_2 - \frac{r\rho x_2(1-x_2)}{a\left(q_1 - q_2\right)}$

The second subsystem when all population infected become as

$$\begin{cases} \dot{x} = rx(1-x) - a(x-m)y \\ \dot{y} = b(x-m)y - q_1 y \end{cases} \tag{2-b}$$

The equilibrium (non trivial) are $E(1,0)$ and $\hat{E}_r(x_3, y_3)$, the first is stable when $m > 1 - \frac{q_1}{b}$ and unstable otherwise. Let R_1' denotes the basic reproduction number of the infected predator where

$R_1' = \frac{b(1-m)}{q_1}$ and if $R_1' > 1$ the infected predator survive; but near this equilibrium no infected predator that means $R_1' < 1$ therefore this implies $m > 1 - \frac{q_1}{b}$ then the second eigenvalues is negative so this equilibrium is stable with condition $m > 1 - \frac{q_1}{b}$. This stability can be transformed to unstable when we change one or all of the parameters (q_1, b, m) .

The equilibrium $\hat{E}_r(x_3, y_3)$ is stable when $\frac{1}{b}\left(q_1 - \frac{1-2x_3}{(R_0'-1)}\right) > m$

Theorem 9. Assume that the equilibrium $\hat{E}_r(x_3, y_3)$ is locally stable; the basin of attraction of this equilibrium is

$$B(\hat{E}_r(x_3, y_3)) = \{(x, y) \in R_+^2 : x > x_3, y > y_3, with\left(xy_3 > x_3 y\right)\}$$

Proof. The proof as theorem (8) .

The equilibrium $P_{r3}(x_3, y_3, 0)$ is stable with conditions

$$\frac{1}{b}\left(q_1 - \frac{1-2x_3}{(R_0'-1)}\right) > m \text{ and } m > x_3 - \frac{r\rho x_3(1-x_3)}{a\left(q_1 - q_2\right)}$$

The stability near the equilibrium $P_r^*(x^*, y^*, z^*)$ is given by the equation

$$\lambda^3 - A\lambda^2 + B\lambda - C = 0 \text{Where } A = -\left(r - 2rx^* - \frac{a}{\rho}(q_1 - q_2)\right),$$

$$B = \frac{ab}{\rho}(x^* - m)\left(q_1 - q_2\right) + \rho^2 y^* z^* \text{ and} C = -\left(r - 2rx^* - a(\frac{q_1 - q_2}{\rho})\right)\rho^2 y^* z^*$$

It is stable if the conditions of Routh- Hurwitz stability criterion are satisfied i.e. $A > 0$, $C > 0$ *and* $AB - C > 0$

Theorem 10. Assume that the equilibrium $P_r^*(x^*, y^*, z^*)$ is locally stable; the basin of attraction of this equilibrium is

$$B(P_r^*(x, y, z)) = \{(x, y, z) \in R_+^3 : x > x^*, x^* y < y^* x, x^* z < z^* x\}$$

Proof .

Let C_1, C_2 and C_3 positive number and let the function $V_3(x, y, z)$ as

$$V_3(x, y, z) = \left(x - x^* - x^* \log\frac{x}{x^*}\right) + \left(y - y^* - y^* \log\frac{y}{y^*}\right) + \left(z - z^* - z^* \log\frac{z}{z^*}\right)$$

$$\frac{dV_3}{dt} = C_1\left(x - x^*\right)\left(r(1-x) - a\frac{(x-m)}{x}\left(y + z\right)\right) + C_2\left(y - y^*\right)\left(b(x-m) + \rho z\right) + C_3\left(z - z^*\right)\left(b(x-m) - \rho y\right)$$

Choose $C_2 = C_3$ and $C_1 = C_2 \dfrac{b}{a}$

$$\frac{dV_3}{dt} = -rC_2 \frac{b}{a}\left(x - x^*\right)^2 + C_2 bm\left(x - x^*\right)\left(\frac{y}{x} - \frac{y^*}{x^*} + \frac{z}{x} - \frac{z^*}{x^*}\right) < 0 \square$$

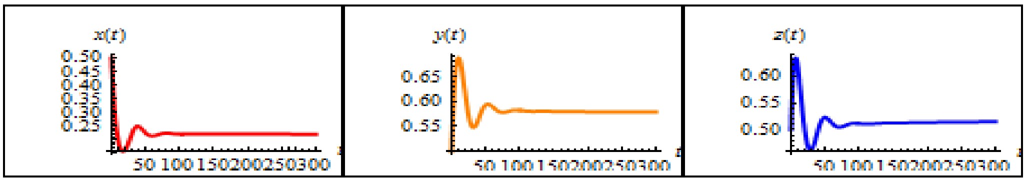
4. Numerical simulation

Here we study several cases of the constant and random refuges to show the effect these have on the behavior of the two systems. First, we study the constant refuge. To enable all population survive, we fixed the parameters as

$$(r = 0.5\ , a = 0.4\ \ , b = 0.3\ , \rho = 0.08\ ,\ \ q_1 = 0.1\ , q_2 = 0.0125\ , x(0) = y(0) = z(0) = 0.5)$$

and we show the following

Case 1: $m = 0.108$. Initially there are large oscillations, which then decrease in size very quickly to point equilibrium. Initially the value of m is very small which means that large number of prey is outside the refuge, so the predator can attack them quickly and this leads to sharp decrease in prey while the predator grows rapidly. As a result, the predator population decreases because lacks of food while the prey increase, but slowly. This cycle is repeated and is faster than previously but of smaller size. This shows it going to point equilibrium as in figure 1(a, b, c, d)



(a) (b) (c)

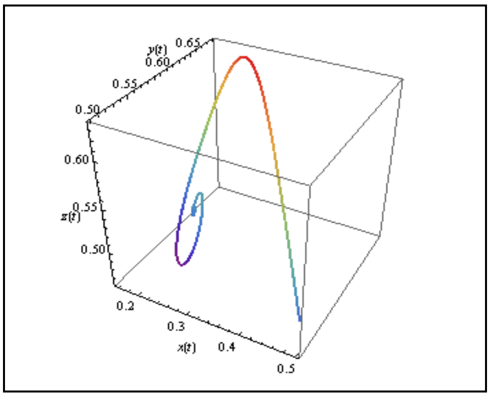
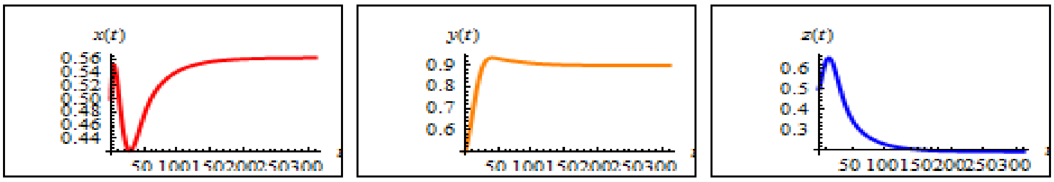


Figure 1(the behavior reached the equilibrium)

Case 2: when $m = 0.5$ we show there is little oscillations because $(1-0.5=0.5)$ mean the amount of prey is less than the first case and also tends to equilibrium as Figure 2 (a, b, c, d)



(a) (b) (c)

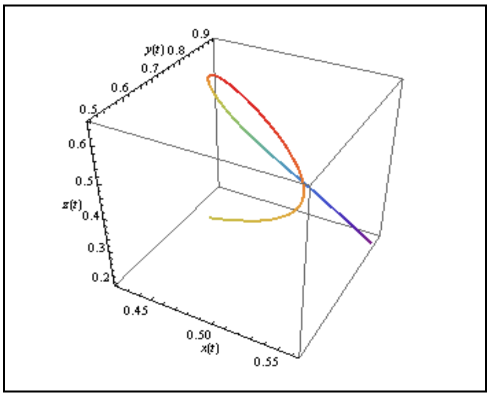
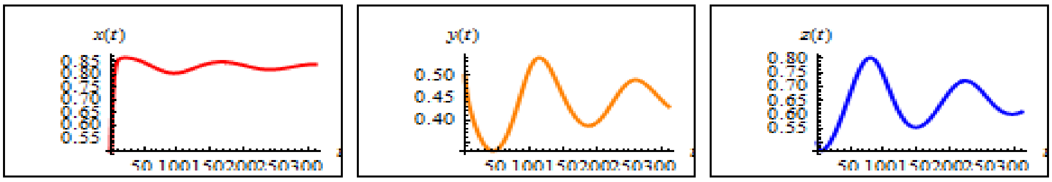


Figure 2 (the behavior reached the equilibrium)

Case 3: Increasing m to become $m =$ 0.85 shows sizeable oscillations that go to the equilibrium. The number of prey outside the refuge is small, the predator in the initial attack but after that the prey need more time to increasing and when prey increasing the predator attack them and continuous but weak vibration and become a limit cycle around equilibrium point figure 3(a, b, c, d)



(a) (b) (c)

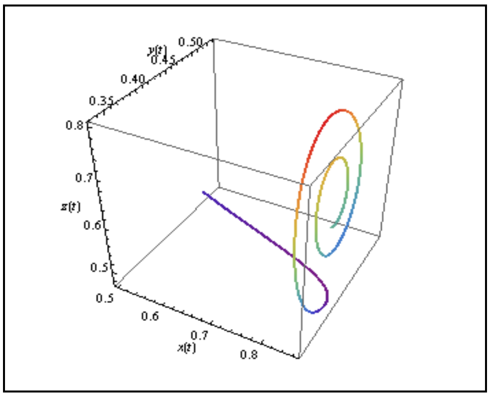
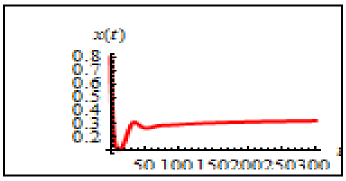


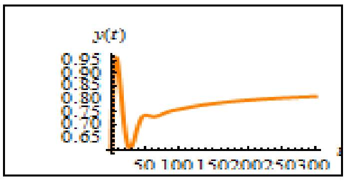
Figure 3 (low density prey indeed to the limit cyclic)

Second we study the random refuge. To keep all population survive we fixed the parameters as $(r=0.5, a=0.4, b=0.3, \rho=0.08,\ q_1=0.1, q_2=0.0125)$ and show the following

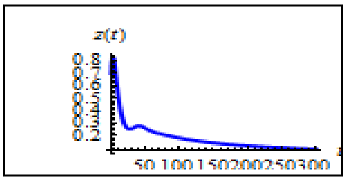
Case 1: m=0.02 in this case the $(x-m)$ is good quality food to predator so the predator attack it therefore the prey decreasing and predator increasing and same as case 1 of system (1) see figures 3 (a, b, c, d).



(a)



(b)



(c)

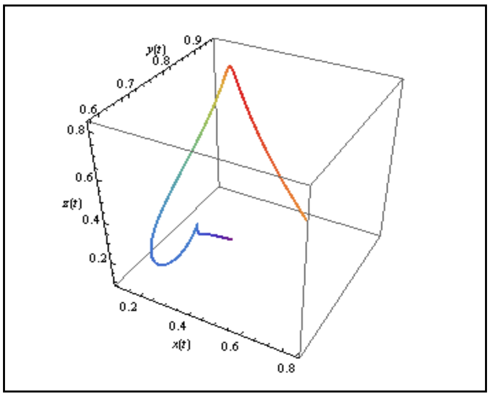
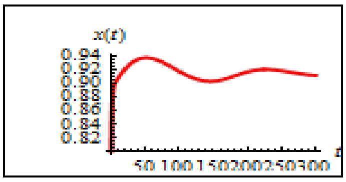
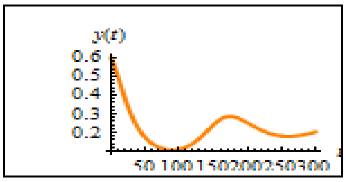


Figure 4 (high density of prey in refuge, the behavior reached the equilibrium)

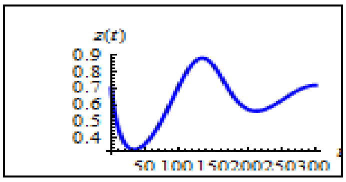
Case 2: increase m to $m=0.8$, in this case the prey outside the refuge is very little so there is no enough food to predator so the oscillations is little and weak than the first and recur more quickly and in a volume less than go in order to form the orbits around the equilibrium .see figure 4(a, b, c, d)



(a)



(b)



(c)

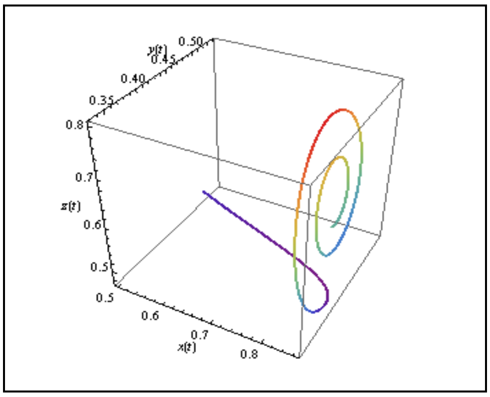


Figure 5 (low density of prey, the behavior as the limit cycle)

From above we can say that when high density of prey outside the refuge, the behavior of the systems in the initial start from a big oscillations and quickly tend to small oscillations and the equilibrium. When low density of prey outside the refuge, the oscillations tend to make circles around the equilibrium. Also we show that a small m in constant refuge is useful because this value give an oscillations area satisfying continuous harvesting, all population survive with increasing susceptible predator, decreasing infected predator and a control on disease see Fig(1.c). But the area in random refuge does not control disease because in small m, the infected predator increase and susceptible predator decrease see Fig (4.b).

Conclusion

In this paper we discussed and analysis model prey predator interaction with harvesting of predator and prey in refuge. We studied bounded solution, and discussed equilibriums points with its conditions. Show the role affected constant and random refuges on stability then we calculate the basin attraction of some of these points. In the numerical simulation we noticed behavior of models in the high size of prey in refuges tend to limit cycle around equilibrium. Finally constant refuge give us an area where continued harvest, all population survive and also to control the disease, but the random refuge does not guarantee us control the disease where infected predator increases while the susceptible predator decreasing.

Acknowledgements

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