Available online at http://scik.org

Commun. Math. Biol. Neurosci. 2021, 2021:72

https://doi.org/10.28919/cmbn/6405

ISSN: 2052-2541

THE EFFECT OF EPIDEMIC DISEASE OUTBREAKS ON THE DYNAMIC BEHAVIOR OF A PREY-PREDATOR MODEL WITH HOLLING TYPE II FUNCTIONAL RESPONSE

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Abstract: We formulate and analyze the dynamism of the epidemic problem consisting of the prey-predator interaction with a susceptible–infected–susceptible (*SIS*) epidemic disease in prey species has been suggested for study in the present work. It is supposed that the disease was spread in two separate ways by direct contact between susceptible prey with infected prey and external sources of infective such as (water, food, contamination environment). The uniqueness and boundedness of the trajectory of this model has been discussed and the existence of all the fixed points (FPs) are determined. The local and global stability (LS and GS) conditions for all of the feasible (EPs) are established. Finally, to confirm the analytic results we solve the model by numerical simulation for different values of parameters and represent them graphically.

Keywords: prey-predator; epidemic diseases; immigration; stability analysis; external sources of infection.

2010 AMS Subject Classification: 92D30, 92D40, 92C42, 92D25, 34D20, 34D23.

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Received June 26, 2021

1. INTRODUCTION

Since there are the presence of several species in the world, which are in constant contact with each other in various ways. This helps the disease transfer, show the organism's rapid interest in the study of disease prey-predator interaction has increased. On the other hand, from both an ecological and an economic viewpoint, the effect of vaccination on the population is very significant. Also, in an ecological environment, the presence of disease in the prey-predator or both is common. Recently, many diseases have been eradicated due to the progressing in modern medicine and the advent in antibiotics and vaccination [1].

On the other hand, the health safety of the society is subjected to a serious risk, when resistant strains of bacteria or infectious viral species appear. A significant threat to human life or animals can occur due to the rapid spread of infectious disease between people groups such as, *Spanish influenza*, *AIDS*, *MERS*, the *Black death* and the recent pandemic of *Avian and Swine influenza*. The effect can be greatly amplified if an infectious disease spreads to densely populated urban areas or has long existed in humans [2]. Of this purpose, it is important to consider the spatial-temporal dynamics of infection transmission to control and mitigate the risk of disease outbreaks. Researchers have sought to analyze disease transmission using mathematical models [3,4].

The influence of disease on the ecological system is an important issue in mathematical and ecological problems. Thus, ecologists and researchers have been paying increasing attention in recent times to the development of important tools along with experimental ecology and describing how ecological species are being infected. A nonlinear differential equations which are known as prey-predator model to describe the population dynamics of two interacting species have been proposed by the Italian mathematician *Vito Volterra* and the chemist *Alfred Lotka*. The *Lotka-Volterra* model is based on sound mathematical logic and it consists of 4 factors that are (i) the growth rate of prey, (ii) mortality rate of predator, (iii) the predation rate and (iv) the conversion rate.

On the other hand, almost all of the models for infectious disease transmission originated from

Kermack's and *Mc Kendrick's* classic work [5]. Both mathematical ecology and mathematical epidemiology in the study of biology and applied mathematics are two separate fields. The study of a combination of these two fields is called eco-epidemiology. Many researchers have been studied and analyzed eco-epidemiological models and suggested disease in prey species only. Hu and Li [6] have proposed and studied prey–predator model with delayed this model consist of three dimensional model with disease in the prey. Johri et al. [7], have proposed a prey–predator model with disease in the prey that considered a *Lotka-Volterra* type and has been studied in local and global stability.

However, Bezabih et al. [8,9] have proposed and studied eco-epidemiological model of prey –predator model. Kang et al. [10], formulating and studied a prey-predator system with allee and disease effects in the prey. Also, there are a number of authors have recently proposed and debated eco-epidemiological models with some assumptions for more information see the following references

Peter [11] studied and developmental prey-predator model with disease effect in prey on the dynamical behavior of this model. Naji and Mustafa [12] described an eco-epidemiological model's dynamics with a nonlinear incidence rate. Silva [13] proved the presence of periodic solutions for prey disease in eco-epidemic models. Xi et al. [14] has been studied the impulsive prey-predator interaction with communicable disease. Sinha et al. [15], has been studied the prey-predator interaction in a toxic setting. Furthermore, vaccination is vital in preventing infectious disease. A vaccine is a biological preparation that provides the active immunity gained for a particular disease. This has become an important way of raising the burden of disease and is a vital instrument for sustaining health and welfare. All the animals including the human population are given vaccination. Animal vaccines are part of a group of veterinary biologics known as animal medicine. The vaccines continue to play an increasingly important role in animal management systems for preventive health and disease. The vaccination aims to reduce the number of individuals in the community who become sick. Migration is also a major demographic phenomenon that is present in all animals. Migration is called the physical transfer from one location to another. One explanation for animal migration is due to the seasonal transition. For example, bird migration is the normal seasonal movement, often along a flyway to the north and south.

The reasons for migration depend on the species and hence, we have taken the effect of migration into consideration while formulating the mathematical equations of the prey-predator model. The word migration for various species has been clarified by Dingle and Drake [16]. They regarded migration as an adaptation to resources that either fluctuate seasonally or less predictably on a spatiotemporal basis. Several authors studied the predator-prey model by taking migration in prey species. For example, Kant and Kumar [17] have been analyzed the eco-epidemiological model with disease in prey and predator with migrating effect. In this study, the epidemic problem that consists of prey–predator interaction with *SIS*–type spread disease in prey species was analyzed and studied. The epidemic of disease is contagious by contact and external sources. The prey has migration and the prey are given a vaccine to protect them have been suggested in this work . The boundedness of the trajectory are studied. The existence for all (FPs) as well as, the stability analysis of the our model is studied.

2. PROBLEM FORMULATION

The epidemic problem consists of the prey–predator interaction with an SIS epidemic disease in prey species is suggested for study of this paper. This type of disease divides the prey species has density $M(T)$ at time T into two class population: The susceptible individuals $S(T)$ at time T and the infected individuals $I(T)$ at time T. And therefore at time T the total prey population will be $M(T) = S(T) + I(T)$. The predator species has density $P(T)$ at time.

In addition, the following principal hypotheses are followed when formulating the model's dynamic equations:

1. The prey species ($S(T)$ and $I(T)$) are grows ,according to logistic fashion with intrinsic growth rate $\alpha_1 > 0$ and $\alpha_2 > 0$ respectively and carrying capacity δ. Furthermore, the disease prevents the $I(T)$ from competing with the $S(T)$, but the $S(T)$ have the ability to compete.

2. The disease in prey species is transmitted through direct contact between $S(T)$ *and* $I(T)$ according to mass action law with force of infection $\beta_1 > 0$ or indirect way from Outside sources for examples food, water, air,….with indirect infection $\beta_2 > 0$. The disease epidemic disappears and infected individuals are again susceptible at recovery rate $c > 0$. Furthermore, the disease can cause death in $I(T)$ with the death rate of the disease $d_1 > 0$.
3. The predator's functional response to the prey is presumed to be of modified Holling type II with maximum attack rate $\alpha_1 > 0$ *and* $\alpha_2 > 0$ from ($S(T)$ *and* $I(T)$) respectively and the constant $\gamma > 0$ represented the half-saturation constant. Moreover, $\theta > 0$ describes the preference rate between the prey $S(T)$ *and* $I(T)$) of the predator. In addition that it is converted from prey ($S(T)$ *and* $I(T)$ to predator with conversion rate $c_1 > 0$ and $c_2 > 0$ respectively.
4. In the absence of prey species, the predator species decay exponentially at a normal death rate of $d_2 > 0$.
5. The species of prey has migration rates of μ_1 *and* μ_2 corresponding to the prey ($S(T)$ *and* $I(T)$), respectively.
6. A vaccine is provided to the prey species to immunize them from the disease incidence with a vaccination rate of $0 < m < 1$ and $(1 - m)$ represented the liability to the disease.
7. Prey (susceptible and infection) may have out migration ,they can migrate to other geographical zone. Let m_1 and m_2 are the rate of migration of susceptible and infective populations, respectively. Also ecology suggested that $m_1 > m_2$. It is natural factor that susceptible (healthy/ sound). Prey are more strong as compared to infected one therefore the probability of migration of healthy prey is more than that of infected prey and hence based on these assumptions for epidemic prey–predator interaction, the model of differential equations that represent this model can be written as:

$$\begin{aligned}
\frac{dS}{dT} &= \alpha_1 S\left(1-\frac{S}{\delta}\right)-\frac{\alpha_1 SP}{\gamma+\theta I+S}-(1-m)(\beta_1 I+\beta_2)S-m_1 S+cI,\\
\frac{dI}{dT} &= \alpha_2 I\left(1-\frac{S+I}{\delta}\right)-\frac{\alpha_2 IP}{\gamma+\theta I+S}+(1-m)(\beta_1 I+\beta_2)S-m_2 I-cI-d_1 I,\\
\frac{dP}{dT} &= \frac{c_1\alpha_1 SP}{\gamma+\theta I+S}+\frac{c_2\alpha_1 IP}{\gamma+\theta I+S}-d_2 P.
\end{aligned} \tag{1}$$

Clearly, the model (1) contains 14 parameters which make the model (1) very difficult for mathematical analysis. The dimensions-less variables are being used to simplify the model proposed

$$t=Ta_1,\ x=\frac{S}{\delta},\ y=\frac{I}{\delta},\ z=\frac{P}{\delta},\ g_1=\frac{\alpha_1}{a_1},\ g_2=\frac{\gamma}{\delta},\ g_3=\frac{\beta_1\delta}{a_1},\ g_4=\frac{\beta_2}{a_1},$$
$$g_5=\frac{m_1}{a_1},\ g_6=\frac{c}{a_1},\ g_7=\frac{a_2}{a_1},\ g_8=\frac{\alpha_2}{a_1},\ g_9=\frac{m_2}{a_1}, g_{10}=\frac{d_1}{a_1},\ g_{11}=\frac{d_2}{a_1}.$$

Where $g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11}$ are the dimensionless parameters of model 2.

The model (1) can be expressed in the form of the following dimensionless:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x)-\frac{g_1 xz}{g_2+\theta y+x}-(1-m)(g_3 y+g_4)x-g_5 x+g_6 y,\\
\frac{dy}{dt} &= g_7 y(1-(x+y))-\frac{g_8 yz}{g_2+\theta y+x}+(1-m)(g_3 y+g_4)x-g_9 y-g_6 y-g_{10} y,\\
\frac{dz}{dt} &= \frac{c_1 g_1 xz}{g_2+\theta y+x}+\frac{c_2 g_8 yz}{g_2+\theta y+x}-g_{11} z.
\end{aligned} \tag{2}$$

In the right side of the model (2), the interaction functions are continuous and also continuous partial derivatives then are Lipschitizian functions and the model (2) has a unique trajectory. The necessary condition for the uniformly bounded trajectory of the model (2) is provide in the theorem (2.1).

Theorem 2.1: All the trajectories of the model (2) are bounded uniformly.

Proof: Let $M=x+y+z$, where x, y, z be any trajectory of the model (2), then take derivative of M at the time along the trajectory of the model (2) we will get :

$$\frac{dM}{dt} \le 1+g_5-\mathcal{L}q=\mathcal{H}-\mathcal{L}q.$$

Here

$$\mathcal{L}=1+g_7,$$

$$\mathcal{H}=1+g_5,$$

$$q=min\{\mathcal{H}\quad, g_7+g_9+g_{10}, g_{11}\}.$$

Using *Gronwell lemma* [18], you get the following:

$$0 < M(t) \le M(0)e^{-\mathcal{L}t} + \frac{\mathcal{H}}{\mathcal{L}}.$$

Thus $\lim_{t\to\infty} M(t) \le \frac{\mathcal{H}}{\mathcal{L}}$.

3. The Existence and the Local Stability (LS) Analysis of Fixed Points (FPs)

The existence of (FPs) and (LS) of the model (2) are discussed. The model (2) has at most three of (FPs)

- The vanishing fixed point is denoted by $(VFP) = (0,0,0)$ always exists.
- The predator free fixed point is denoted by $(PFFP) = (\tilde{x}, \tilde{y}, 0)$, where

$$\tilde{y} = \frac{\tilde{x}[1-(\tilde{x}+(1-m)g_4+g_5\ \ )]}{(1-m)g_3\tilde{x}-g_6}, \tag{3}$$

$\tilde{x}$ has been represented a non-negative root in the 3rd order of the following equation:

$$\tilde{A}_1 \mathrm{x}^3 + \tilde{A}_2 \mathrm{x}^2 + \tilde{A}_3 \mathrm{x} + \tilde{A}_4 = 0. \tag{4}$$

Here

$$\begin{aligned}
\tilde{A}_1 &= (1-m)g_3[g_7 - (1-m)g_3(1-(1-m)g_4)] - g_7,\\
\tilde{A}_2 &= (1-m)\big[(-2+(1-m)g_4+g_5)g_3\, g_7 - 2g_4 + g_3\big((1-m)g_3 + g_6 - (1-m)g_3g_5 + \sigma\big)\big]\\
&\quad + g_7(2-g_6-2g_5),\\
\tilde{A}_3 &= (1-m)(g_7(g_3[1-(1-m)g_4-g_5] - g_4[g_6+2-(1-m)g_4-2g_5])\\
&\quad + g_3g_6\big((1-m)g_4+g_5-2(1-m)g_4-1\big) + \sigma g_3((1-m)g_4\\
&\quad + g_5 - 1)) + g_7[(2-g_5)(g_5+g_6)-1] - g_6\sigma,\\
\tilde{A}_4 &= g_6[g_4(1-m)(g_7+g_6-\sigma) + g_7(g_5-1) + g_6\sigma[1+g_5],
\end{aligned}$$

with

$$\sigma = g_4 + g_6 + g_{10}. \tag{5}$$

The (PFFP) exists uniquely in $\mathrm{Int}\mathfrak{R}^3_+$ If one of the following conditions are satisfied.

$$\left.\begin{aligned}
&\tilde{x} + (1-m)g_4 + g_5 < 1 \quad \text{and} \quad g_6 < (1-m)g_1\tilde{x},\\
&\qquad\qquad\qquad\qquad OR\\
&\tilde{x} + (1-m)g_4 + g_5 > 1 \quad \text{and} \quad g_6 > (1-m)g_1\tilde{x}.
\end{aligned}\right\} \tag{6a}$$

As well as, one of the following cases

$$\left.\begin{aligned}
&\tilde{A}_1 > 0, \tilde{A}_2 > 0 \quad and \quad \tilde{A}_4 < 0,\\
&\tilde{A}_1 > 0, \tilde{A}_2 < 0 \quad and \quad \tilde{A}_4 < 0,\\
&\tilde{A}_1 < 0, \tilde{A}_2 > 0 \quad and \quad \tilde{A}_4 > 0,\\
&\tilde{A}_1 < 0, \tilde{A}_2 < 0 \quad and \quad \tilde{A}_4 > 0.
\end{aligned}\right\} \tag{6b}$$

- The positive fixed point is denoted by $(PFP) = (x^*, y^*, z^*)$, where

$$x^* = \frac{a_1 y^* + g_2 g_{11}}{a_2}, \tag{7}$$

$$z^* = \frac{[a_2(g_2+\theta y^*)+a_1 y^*+g_2 g_{11}][g_7 y^*\big((1-y^*)-(a_1 y^*+g_2 g_{11})\big)+(1-m)(g_3 y^*+g_4)(a_1 y^*+g_2 g_{11})-g a_2 y^*]}{a_2^2 \quad g_8 y^*}. \tag{8}$$

Here

$$a_1 = g_{11}\theta - c_2 g_8 \quad and \quad a_2 = c_1 g_1 - g_{11},$$

while, y^* represents a non-negative root of the following fourth order polynomial equation

$$k_1 y^{*4} + k_2\, y^{*3} + k_3\, y^{*2} + k_4 y^* + k_5 = 0. \tag{9}$$

where

$$\begin{aligned}
k_1 &= a_1 g_1(a_1 + a_2\theta)[g_7(a_1 + a_2) - a_1 g_3(1-m)] - a_1 g_8(a_1 + a_2\theta)[a_1 + a_2 g_3(1-m)],\\
k_2 &= -a_1 g_2 g_8\,(a_2 + g_{11})[a_1 + a_2 g_3(1-m)] + (a_1 + a_2\theta)[a_1 g_8(a_2 - 2g_2 g_{11})\\
&\quad -a_1 g_1 g_7(a_2 - g_2 g_{11}) - a_2 g_8(a_1 g_5 - a_2 g_6) + a_1 a_2 \sigma g_1 + (g_2 g_3\, g_{11} + a_1 g_4)(1-m)\\
&\quad (a_2 g_8 - a_1 g_1)] + [g_1 g_2(a_2(a_1 + g_{11}\theta) + 2a_1 g_{11}][g_7(a_1 + a_2) + a_1(1-m)g_3],\\
k_3 &= (a_{2+} g_{11})\big[g_2 g_8\big(a_1(a_2 - 2g_2 g_{11}) - a_2(a_1 g_5 - a_2 g_6)\big)\big(g_7(a_1 + a_2)\\
&\quad -a_1 g_3(1-m)] - g_2(1-m)(g_2 g_3 g_{11} + a_1 g_4)[a_2 g_8(a_2 + g_{11})\\
&\quad +g_{11}(a_1(a_2 + 2g_{11}\ \ ) + g_{11} a_2\theta] - g_2 g_{11}(a_1 + a_2\theta)\big[a_2 g_8\big((1-m)g_4 + g_5\big)\\
&\quad +a_1 g_4 g_1(1-m)] + a_2 \sigma g_1 g_2[a_1(a_2 + 2g_{11}) + g_{11} a_2 \theta],\\
k_4 &= g_2^2 g_{11}(a_{2+} g_{11})\big[(a_2 - g_2 g_{11})(g_8 - g_1 g_7) - a_2 g_8\big((1-m)g_4 + g_5\big) + g_1(-a_1 g_4(1-m)\\
&\quad +a_2\sigma) - g_1(1-m)(g_2\ \ g_3\ \ g_{11} + a_1 g_4)] - g_1 g_4 g_2^2 g_{11}^2(a_1 + a_2\theta)(1-m)],\\
k_5 &= -g_1 g_2^3 g_{11}^2 g_4(1-m)(a_2 + g_{11}).
\end{aligned}$$

The positive fixed point $(\text{PFP}) = (x^*, y^*, z^*)$, exists if and only if the one of the following cases is holds

$$\left.\begin{array}{l}
k_1 > 0, k_2 > 0 \ \ and \ \ k_3 > 0,\\
k_1 > 0, k_2 > 0 \ \ and \ \ k_4 < 0,\\
k_1 > 0, k_3 < 0 \ \ and \ \ k_4 < 0\,.
\end{array}\right\} \tag{10}$$

The next, the (LS) analysis of the above (FPs) of the model (2) is discussed by using a linearization method. Note that the Varational Matrix (*V*) of the model (2) at

$(VFP) = (0,0,0)$ can be written as

$$V_0 = V(VFP) = \left(e_{ij}\right)_{3\times 3} \ \ .i,j = 1,2,3. \tag{11}$$

where

$$e_{11} = 1 - (1-m)g_4 - g_5 \quad ; \quad e_{22} = g_7 - \sigma;$$
$$e_{33} = -g_{11} < 0 \quad , \quad e_{12} = e_{13} = e_{21} = e_{23} = e_{31} = e_{32} = 0.$$

Then the eigenvalues of V_0 is:

$$\gamma_{0x} = 1 - (1-m)g_4 - g_5 \quad , \quad \gamma_{0y} = g_7 - \sigma \quad , \quad \gamma_{0z} = -g_{11} < 0$$

If the following two conditions are satisfied then (VFP) is (LS) in the $\mathrm{Int}\mathfrak{R}_+^3$:

$$1 < (1-m)g_4 + g_5, \tag{12}$$

$$\sigma < g_7. \tag{13}$$

Now, the (V) of model (2) at $(PFFP) = (\tilde{x}, \tilde{y}, 0)$ can be written in the form:

$$V_1 = V(\mathrm{PFFP}) = \left(r_{ij}\right)_{3\times 3} \quad , \quad i,j = 1,2,3. \tag{14}$$

where

$$r_{11} = -\tilde{x}\left(1 + \frac{g_6\tilde{y}}{\tilde{x}^2}\right) < 0 \quad ; \quad r_{12} = \tilde{x}\left(-g_3(1-m) + \frac{g_6}{\tilde{y}}\right)$$
$$r_{13} = \frac{-g_1\tilde{x}}{\tilde{b}} < 0 \quad ; \quad r_{21} = \tilde{y}\left(-g_7 + \frac{(g_3\tilde{y}+g_4)(1-m)}{\tilde{y}}\right)$$
$$r_{22} = -\left(g_7 + \frac{g_{4(1-m)\tilde{x}}}{\tilde{y}^2}\right)\tilde{y} \quad ; \quad r_{23} = \frac{-g_8\tilde{y}}{\tilde{b}}$$
$$r_{31} = r_{32} = 0 \quad ; \quad r_{33} = \frac{(c_1 g_1\tilde{x})(c_2 g_8\tilde{y})}{\tilde{b}^2} - g_4$$

where $\tilde{b} = g_2 + \theta\tilde{y} + \tilde{x}$

The characteristic equation of V_1 is:

$$(\hat{\gamma}_1 - r_{33})[\hat{\gamma}^2 - T\hat{\gamma}_1 + D] = 0.$$

Clearly, the eigenvalues of V_1 satisfy the following relations:

$$T = r_{11} + r_{12}, \tag{15}$$

$$D = r_{11}r_{22} - r_{12}r_{21}, \tag{16}$$

$$\hat{\gamma}_{1z} = r_{33.} \tag{17}$$

The conditions (18) and (19) give each of the eigenvalues real negative parts,

$$r_{12}r_{21} < r_{11}r_{22} \tag{18}$$

$$r_{33} < 0 \tag{19}$$

Thus, (PFFP) is (LS) in $\mathrm{Int}\mathfrak{R}_+^3$.

Theorem (3.1): The positive fixed point $(\mathrm{PFP}) = (x^*, y^*, z^*)$ is (LS) if satisfy the following conditions:

$$z^* < min\left\{\begin{matrix}\frac{b^{*^2}\left(x^{*2}+g_6y^*\right)}{g_1x^{*2}}, \frac{b^{*^2}(g_3(1-m)x^*-g_6)}{g_1\theta x^*}, & \frac{b^{*^2}(g_3y^*-(g_3y^*+g_4)(1-m))}{g_8y^*}, \\ \frac{b^{*^2}\left(g_8y^{*^2}+g_4(1-m)x^*\right)}{g_8\theta y^{*^2}} & \end{matrix}\right\}, \tag{20}$$

$$c_1g_1\theta x^* < c_2g_8(g_2+x^*), \tag{21}$$

$$\sqrt{\frac{g_8 \ E_1y^*}{g_1E_3}} < x^* < min\left\{\frac{g_8 \ E_2y^{*^2}}{g_1E_4}, \quad \frac{E_1E_4}{E_2E_3y^*}\right\}. \tag{22}$$

where $\quad b^* = g_2 + \theta y^* + x^*.$

Proof: The (V) for (PFP) of the model (2) may be written as follows :

$$V_1 = V(\text{PFP}) = \left(a_{ij}\right)_{3\times 3} \quad , \quad i,j = 1,2,3. \tag{23}$$

Here

$$a_{11} = \frac{E_1}{b^{*^2}x^*} \; ; \; a_{12} = \frac{E_2}{b^{*^2}} \quad ; \; a_{13} = \frac{-g_1x^*}{b^*}$$
$$a_{21} = \frac{E_3}{b^{*^2}} \; ; \; a_{22} = \frac{E_4}{b^{*^2}y^*} \; ; \; a_{23} = \frac{-g_8y^*}{b^*}$$
$$a_{31} = \frac{E_5z^*}{b^{*^2}} \; ; \; a_{32} = \frac{E_6z^*}{b^{*^2}} \; ; \; a_{33} = 0,$$

with

$$\begin{aligned} E_1 &= -b^{*^2}\left(x^{*^2} + g_6y^*\right) + g_1z^*x^*, \\ E_2 &= g_1\theta z^*x^* - b^{*^2}(g_3(1-m)x^* - g_6), \\ E_3 &= -b^{*^2}\left(g_4y^* - (g_3y^* + g_4)(1-m)\right) + g_8z^*y^*, \\ E_4 &= -b^{*^2}\left(g_7y^{*2} + g_4(1-m)x^*\right) + g_8\theta z^*y^{*^2} , \\ E_5 &= (g_2 + \theta y^*)c_1g_1, \\ E_6 &= c_2g_8(g_2 + x^*) - c_1g_1\theta x^*. \end{aligned} \tag{24}$$

Hence we can write the characteristic equations of V_1 as follows:

$$\gamma_2^3 + A_1^*\gamma_2^2 + A_2^*\gamma_2 + A_3^* = 0.$$

Here

$$\begin{aligned} A_1^* &= -(a_{11} + a_{22}), \\ A_2^* &= a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}, \\ A_3^* &= -a_{31}[a_{12}a_{23} - a_{13}a_{22}] + a_{32}[a_{11}a_{23} - a_{13}a_{21}]. \end{aligned}$$

And

$$\begin{aligned} \Delta &= A_1^*A_2^* - A_3^* \\ &= -(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21}) + a_{31}(a_{11}a_{13} + a_{12}a_{23}) + a_{32}[a_{22}a_{23} + a_{13}a_{21}]. \end{aligned}$$

So, by substituting the values of a_{ij}, and the simplifying the resulting terms we obtain:

$$A_1^* = \frac{-1}{b^{*2}}\left[\frac{E_1}{x^*} + \frac{E_4}{y^*}\right]$$
$$A_3^* = \frac{z^*}{b^{*2}} \quad \left[-\frac{E_5}{y^*}\left(-g_8E_2y^{*2} + g_1E_4x^*\right) + \frac{E_6}{x^*}\left(-g_8E_1y^* + g_1E_3x^{*2}\right)\right]$$

And

$$\Delta = \frac{-1}{b^*x^*y^*}\left[\frac{E_1}{x^*} + \frac{E_4}{y^*}\right][E_1E_2 - E_1E_2x^*y^*] - \frac{z^*}{b^*}[E_5(g_1E_1 + g_6E_2y^*)$$
$$-E_6(g_8E_4 + g_1E_3x^*)]$$

If the conditions (20)-(22) are satisfies then $A_i^* > 0, i = 1,2,3 \; and \; \Delta > 0$. Thus the (PFP) is (LS) according to the results of *Routh – Hawartiz*.

4. GLOBAL STABILITY ANALYSIS

In this section, the region of global stability (basin of attraction) of all (FPs) of the model (2) is presented as shown in the following theorems.

Theorem (4.1): The (VFP) is a globally asymptotically stable provided that the following conditions hold

$$1 < g_5, \tag{25a}$$

$$g_7 < g_9 + g_{10}. \tag{25b}$$

Proof: Consider the following positive function

$$L_0 = x + y + z.$$

Clearly, the above function is a continuously differentiable function such that

$L_0(0,0,0) = 0$ and $L_0(x, y, z) > 0 \;\forall\, (x, y, z) \neq (0,0,0)$. Further,

$$\frac{dL_0}{dt} = \left[x(1 - \mathrm{x}) - \frac{g_1xz}{g_2+\theta y+x} - (1 - m)(g_3y + g_4)x - g_5x + g_6y\right]$$
$$+\left[g_7y(1 - (\mathrm{x} + \mathrm{y})) - \frac{g_8yz}{g_2+\theta y+x} + (1 - m)(g_3y + g_4)x - g_9y\right.$$
$$\left.-g_6y \quad - g_{10}y\right] + \left[\frac{c_1g_1xz}{g_2+\theta y+x} + \frac{c_2g_8yz}{g_2+\theta y+x} - g_{11}z\right].$$

Now, by doing some algebraic manipulation and using the conditions (25a) and (25b), we get

$$\frac{dL_0}{dt} \leq -[x + g_5 - 1]x - [g_9 + g_{10} - g_7]y - g_{11}z.$$

Consequently, due to condition above is $\frac{dL_0}{dt} < 0$ negative. Thus, the (VFP) is a globally asymptotically stable and the proof is complete.

Theorem (4.2): The $(PFFP) = (\tilde{x}, \tilde{y}, 0)$, is a globally asymptotically stable provided that the following sufficient conditions hold

$$1 < (1-m)g_4 + g_5 + x + \tilde{x} + (1-m)g_3\tilde{y}, \tag{26a}$$

$$g_7 < g_6 + g_9 + g_{10} + g_7(y + \tilde{y}) + g_7 x, \tag{26b}$$

$$d_{12}^2 < 4d_{11}d_{22}. \tag{26c}$$

Proof: Consider the following positive definite real valued function

$$L_1 = \frac{(x-\tilde{x})^2}{2} + \frac{(y-\tilde{y})^2}{2} + z.$$

Then the derivative of this function with respect to time can be written as

$$\begin{aligned}\frac{dL_1}{dt} &= (x-\tilde{x})\left[x(1 - x - \tfrac{g_1 z}{g_2+\theta y+x}) - (1-m)(g_3 y + g_4)x - g_5 x + g_6 y\right] \\ &\quad (y-\tilde{y})\left[(g_7 - g_7\mathrm{x} - g_7 y - \tfrac{g_8 z}{g_2+\theta y+x})y + (1-m)(g_3 y + g_4)x \right. \\ &\quad \left. -(g_9 + g_6 + g_{10})y\right] + \left(\tfrac{c_1 g_1 x}{g_2+\theta y+x} + \tfrac{c_2 g_8 y}{g_2+\theta y+x} - g_{11}\right)z, \\ &= -[d_{11}(x-\tilde{x})^2 + d_{12}(x-\tilde{x})(y-\tilde{y}) + d_{22}(y-\tilde{y})^2] - g_{11}z \\ &\quad -(1-c_1)\tfrac{g_1 xz}{g_2+\theta y+x}(x-\tilde{x}) - (1-c_2)\tfrac{g_8 yz}{g_2+\theta y+x}(y-\tilde{y}).\end{aligned}$$

Now, by doing some algebraic manipulation and using the conditions (26a–26c), we get that

$$\frac{dL_1}{dt} \le -\left[\sqrt{d_{11}}(x-\tilde{x}) + \sqrt{d_{22}}(y-\tilde{y})\right]^2 - g_{11}z.$$

Where

$$\begin{aligned} d_{11} &= (1-m)g_4 + g_5 - 1 + x + \tilde{x} + (1-m)g_3\tilde{y} \\ d_{12} &= (1-m)g_3 x + g_7\tilde{y} - g_6 - (1-m)g_3\tilde{y} - (1-m)g_4 \\ d_{22} &= g_6 + g_9 + g_{10} - g_7 + g_7(y+\tilde{y}) + g_7 x \end{aligned}$$

Obviously, $\frac{dL_1}{dt} \le 0$ when $\tilde{x} \le x$ *and* $\tilde{y} \le y$, with the conditions (26a-26c) are hold. We get the $(PFFP)$ is globally asymptotically stable in the sub region.

Theorem (4.3): The $(\mathrm{PFP}) = (x^*, y^*, z^*)$ is a globally asymptotically stable in the sub region of $\mathfrak{R}_+^3$ that satisfied the following conditions

$$q_{12}^2 < q_{11}q_{22} \tag{27a}$$

$$q_{13}^2 < q_{11}q_{33} \tag{27b}$$

$$q_{23}^2 < q_{22}q_{33} \tag{27c}$$

Proof: Consider the following positive function

$$L_2 = \frac{(x-x^*)^2}{2} + \frac{(y-y^*)^2}{2} + \frac{(z-z^*)^2}{2}.$$

Then the derivative of this function with respect to time can be written as

$$\begin{aligned}
\frac{dL_2}{dt} &= (x-x^*)\left[x - x^2 - \frac{g_1 xz}{g_2+\theta y+x} - (1-m)(g_3 y + g_4)x - g_5 x + g_6 y\right] \\
&\quad +(y-y^*)\left[g_7 y - g_7 \mathrm{xy} - g_7 y^2 - \frac{g_8 yz}{g_2+\theta y+x} + (1-m)(g_3 y+g_4)x \right.\\
&\quad \left. -(g_9+g_6+g_{10})y\right] + (z-z^*)\left[\frac{c_1 g_1 xz}{g_2+\theta y+x} + \frac{c_2 g_8 yz}{g_2+\theta y+x} - g_{11} z\right], \\
&= -\left[\frac{q_{11}}{2}(x-x^*)^2 + q_{12}(x-x^*)(y-y^*) + \frac{q_{22}}{2}(y-y^*)^2\right] \\
&\quad -\left[\frac{q_{11}}{2}(x-x^*)^2 + q_{13}(x-x^*)(z-z^*) + \frac{q_{33}}{2}(z-z^*)^2\right] \\
&\quad -\left[\frac{q_{22}}{2}(y-y^*)^2 + q_{23}(y-y^*)(z-z^*) + \frac{q_{33}}{2}(z-z^*)^2\right] \\
&\quad -\left[\frac{g_1(\theta y^* xz - \theta x^* z^* y)}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)}\right](x-x^*) - \left[\frac{g_8(x^* yz - y^* z^* x)}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)}\right](y-y^*) \\
&\quad -\left[\frac{c_1 g_1(\theta x^* z^* y - \theta y^* xz) + c_2 g_8(y^* z^* x - x^* yz)}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)}\right](z-z^*).
\end{aligned}$$

Consequently by using (27a)-(27c) conditions we get that

$$\begin{aligned}
\frac{dL_2}{dt} &\le -\left[\sqrt{\frac{q_{11}}{2}}(x-x^*) + \sqrt{\frac{q_{22}}{2}}(y-y^*)\right]^2 - \left[\sqrt{\frac{q_{11}}{2}}(x-x^*) + \sqrt{\frac{q_{33}}{2}}(z-z^*)\right]^2 \\
&\quad -\left[\sqrt{\frac{q_{22}}{2}}(y-y^*) + \sqrt{\frac{q_{33}}{2}}(z-z^*)\right]^2 - \left[\frac{g_1(\theta y^* xz - \theta x^* z^* y)}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)}\right](x-x^*) \\
&\quad -\left[\frac{g_8(x^* yz - y^* z^* x)}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)}\right](y-y^*) - \left[\frac{c_1 g_1(\theta x^* z^* y - \theta y^* xz) + c_2 g_8(y^* z^* x - x^* yz)}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)}\right](z-z^*).
\end{aligned}$$

where

$$q_{11} = x + x^* + (1-m)g_4 + g_5 + (1-m)g_3 y^* + \frac{g_1 g_2 z^*}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)},$$

$$q_{12} = (g_7 - (1-m)g_3)y^* - (1-m)g_4 + (1-m)g_3 x,$$

$$g_{13} = \frac{g_1(g_2+x^*)x - c_1 g_1 g_2 z^*}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)},$$

$$g_{22} = g_6 + g_9 + g_{10} - g_7 + g_7(y+y^*) + (g_7 - (1-m)g_3)x + \frac{g_2 g_8 z^*}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)},$$

$$q_{23} = \frac{g_8(g_2+\theta y^*)y - c_2 g_2 g_8 z^*}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)},$$

$$q_{33} = g_{11} - \frac{c_1 g_1(g_2+x^*)x + c_2 g_8(g_2+\theta y^*)y}{(g_2+\theta y+x)(g_2+\theta y^*+x^*)}.$$

Obviously, $\frac{dL_2}{dt} \le 0$ when $x \ge x^*, y \ge y^*, z \ge z^*$ and $q_{ii} > 0,\ i = 1,2,3$ with the conditions (27a)-(27c) are hold. Thus, the (PFP) is a globally asymptotically stable in the sub region.

5. NUMERICAL SIMULATION

To confirm the above analytical findings and understand the effect of varying the parameters on the global dynamics of model (2), numerical simulation is done in this section. The objectives of this study are confirming our obtained analytical results and detecting the set of control parameters that affect the dynamics of the system. Consequently, model (2) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that for the following set of hypothetical parameters the model (2) has a globally asymptotically stable to (PFP) as shown in the below figure 1:

$$\begin{gathered} g_1 = 0.2,\ \ g_2 = 0.3,\ \ g_3 = 0.2, g_4 = 0.1,\ \ g_5 = 0.01 \\ g_6 = 0.2, g_7 = 1\ , g_8 = 1, g_9 = 0.001,\ \ g_{10} = 0.005 \\ g_{11} = 0.1,\ \ c_1 = 0.5,\ \ c_2 = 0.7,\ \ \theta = 0.5,\ \ m = 0.1 \end{gathered} \tag{28}$$

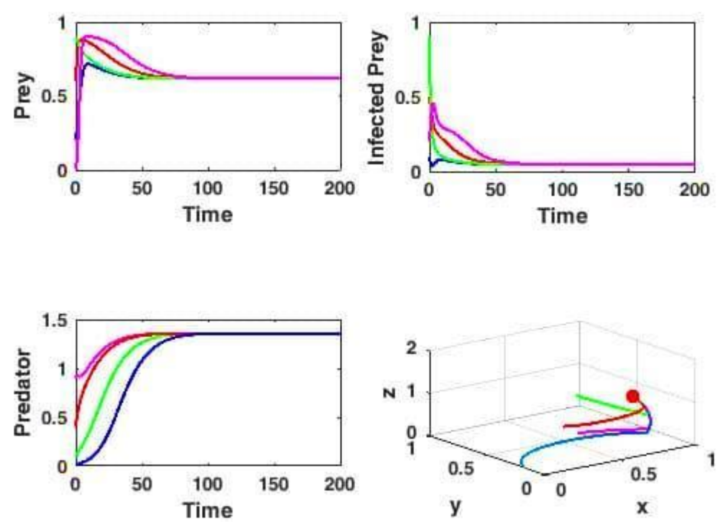

Figure 1: Globally asymptotically stable to (PFP) of model (2).

Clearly, Figure 1, confirm our obtained analytical results in theorem (4.3), regarding to existence that (PFP) is a globally asymptotically stable.

However, for the data by equation (28) with $g_1 = 0.1$ and $g_8 = 0.1$, the solution of model (2) approaches asymptotically to the (PFFP) with reducing the ratio of maximum attack to the of logistic growth rate as shown in the following typical, Figure 2.

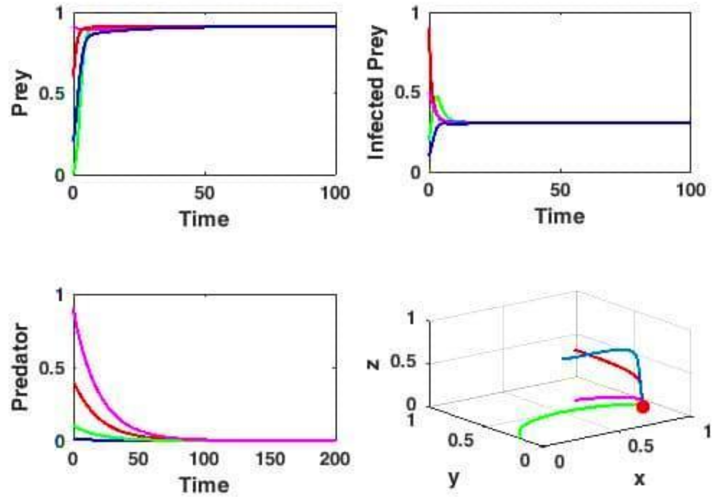

Figure 2: Globally asymptotically stable of (PFFP) of the model (2).

Now, in order to investigate the effect of varying parameters value at a time on the dynamical behavior of the model (2), the following results are observed. According to the Figure 3, it is clear that the solution of the model (2) approaches to the periodic of (PFP) for the parameters values given in Eq. (28) with varying $g_5 \geq 1.1$, see Figure 3.

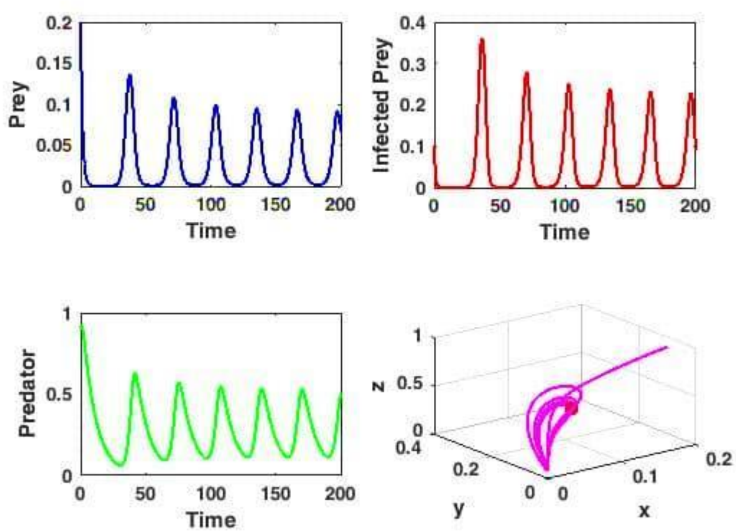

Figure 3: Periodic attractor of (PFP) of the model (2).

Again, we choose the ratio of vaccination and logistic growth rate of infected prey to the logistic growth rate of susceptible prey values $g_5 = 1.1 \; and \; g_7 = 0.15$, respectively. Keeping other parameters fixed as given in equation (28), we get the trajectories of the model (2) approaches to the (VFP), see Figure 4.

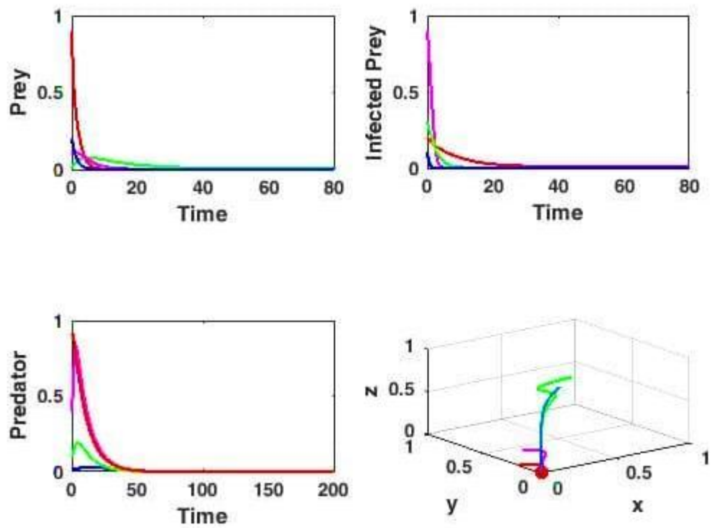

Figure 4: Globally asymptotically stable of (VFP) of the model (2).

In addition, varying the parameter $g_{11} = 3$ with other data as in Eq. (28) the solution of the model (2) approaches asymptotically to (PFFP) , as shown in the Figure (6).

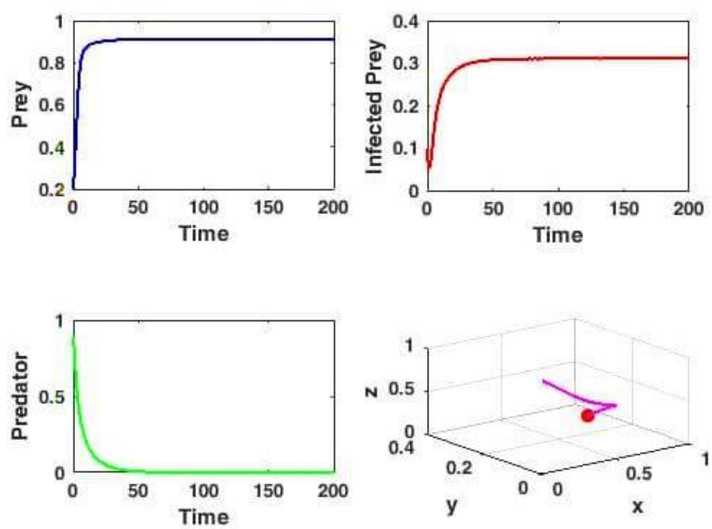

Figure 5: Time series of the solution of the model (2), for the data given by Eq. (28) with, $g_{11} = 3$, that approaches asymptotically to (PFFP).

6. CONCLUSION AND DISCUSSION

In conclusion, the prey–predator interaction with (SIS) kind of epidemic disease is suggested and studied in this work. This model consists of prey–predator interactions, the first population is called prey species that are infected with the kind SIS epidemic disease while the second population is called predator species which is healthy. The suggested model is expressed by three differential equations which describes the dynamic of prey–predator interactions. The susceptible prey denoted by S, the infected prey denoted by I and the susceptible predator denoted by p. It is supposed that the disease is spreading between the prey population by contact and external sources of infective with linear incidence rate. The effect of the disease, the migration and the vaccine to protect them in prey on the dynamic behavior of the epidemic problem have been analyzed and discussed. The proposed model has three FPs that are the vanishing (FP) is denoted (VFP) always exists, the predator free (FP) is denoted (PFFP) and the positive (FP) is denoted (PFP).The existence of the proposed model's trajectory is investigated, and the bounds for the trajectory are proved. The existing conditions for all of FPs are studied. Finally the (LS) analysis of each possible FP for this model is studied by using the linearization method.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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