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SENSITIVITY AND MATHEMATICAL MODEL ANALYSIS ON SMOKING WITH HEALTH EDUCATION EFFECT

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Abstract. In this article we are given a mathematical model of smoking with health education effect. The solution of the model is positive (if the initial population is positive) and bounded. The reproductive number R_0 is calculated from the model, and we show if $R_0 > 1$ the endemic equilibrium point exists and is unique, and if $R_0 \leq 1$ the endemic equilibrium point does not exist. Smoking free equilibrium point is stable when $R_0 < 1$, and endemic equilibrium point is stable when $R_0 > 1$. Numerical simulation has been included to show stability and instability of equilibrium points. Furthermore, we study the relative change in R_0 if the value of the parameter changes, this is called sensitivity analysis of parameters involving in the reproductive number.

Keywords: health education; smoking; reproduction number; endemic equilibrium point.

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1. Introduction

Tobacco smoking is the major issue that cause death and disease (for example, lung cancer) in many countries. We can mention studies in US and Australia. In US, smoking causes more

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than 400,000 deaths per year [1] and in Australia, 15 % of all deaths were as a result of tobacco smoking [2]. Eventhough tobacco smoking is a killer disease, in some countries the prevalence rate of smoking is increased [12].

Thus forming policies minimizing the proportion of people that start smoking or decrease the duration of smoking is a big problem for public health. Usually people agree that health education may be an attractive policy to overcome this difficulty because many studies showed that better educated individuals have a better health and a lower risk of mortality [6].

V. Maralani in [9] studied the link between education and smoking and conclude that educational inequalities in smoking are better understood as a bundling of advantageous statuses that develops in childhood. D. Walque in [15] collected data from smoking population, and concluded that education does affect smoking decisions: educated individuals are less likely to smoke, and among those who initiated smoking, they are more likely to have stopped.

Studies in [7] find that education decreases the probability of ever having smoked substantially, but the evidence on quitting smoking is mixed. P. Koning and his colleges conducted a research in Australian twins and conclude that a higher educational attainment increases the probability of smoking cessation, rather than decreasing the probability of starting smoking [8].

All studies mentioned above showed education affect smoking and thus decreases the number of smoker and also prevent people to join the smoker state. There are a number of studies on mathematical model of smoking by taking different assumption. We direct refer the reader to the papers [3, 4, 13, 14].

In this paper we extend the work of P. Xiao, Z. Zhang, and X. Sun in [17]. In their work smoker group does not have relation directly to permanent quit group. We propose a model in which smoker group directly related to the permanent quit group and we drop the assumption that smoker with associated disease. Furthermore, sensitivity analysis of the reproduction number and stability of the equilibrium solution will be included. The latter is done by Routh Hurith Criteria.

We organized the paper as follows. In section two we propose the mathematical model and we show positivity and boundedness of the solution. We discuss about equilibrium point, free equilibrium point, reproduction number, sensitivity analysis and endemic equilibrium point in section three, four, five, six and seven respectively. In section eight stability of free and endemic equilibrium point are seen. In section nine we will see the numerical simulation by taking initial values for the variables, and appropriate values for the parameters. Finally, conclusion and declaration have been included.

2. METHODS

2.1. Formulation of the model.

Model assumption. To prepare our model we assume the following.

(1) Every state have the same death rate, represented by μ.
(2) New recruitment rate of the system to be same as the death rate.
(3) The new population recruited into the population is divided into two proportions: educated and uneducated, represented by P_E and P_N, respectively.
(4) Educated people have lower chance to be come smoker than uneducated people.
(5) Smoker (represented by S) can turn into temporary quitters (represented by Q_T) by getting treatment or self abstaining.
(6) Smoker group has a death rate ε as a result of smoking tobacco in addition to natural death rate.
(7) Temporary quitters can relapse.
(8) Temporary quitters can turn into permanent quitters (represented by Q_P).
(9) Smoker can turn into permanent quitters by getting treatment or self abstaining.

Description of variables and parameters. In table (1), variables and parameters are described to form mathematical model that shows the dynamics of smoking.

TABLE 1. Variables and its descriptions

No	Variables/ Parameters	Description	Value
1	P_N	The number people who do not smoke and do not get health education, may become smoker in the future	$P_N \geq 0$
2	P_E	The number people who get health education and do not smoke, may become smoker in the future	$P_E \geq 0$
3	S	The number of people who smoke tobacco	$S \geq 0$
4	Q_T	The number of people who are currently abstaining smoking, but may not succeed	$Q_T \geq 0$
5	Q_P	The number of people who are permanently quitting smoking, never smoke again	$Q_P \geq 0$
6	μ	Natural death rate or new recruitment rate of the system	$0 < \mu < 1$
7	ξ	The proportion ξ of new recruitment is uneducated and $1-\xi$ is educated	$0 < \xi < 1$
8	δ	Reflects educated people have lower chance to become smoker	$0 < \delta < 1$
9	β	Infection rate from educated and uneducated state to smoker state	$0 \leq \beta \leq 1$
10	κ	Infection rate from temporary quitters to smoker	$0 \leq \kappa \leq 1$
11	α	The rate at which smoker to temporary quitters and permanent quitters P to S	$0 \leq \alpha \leq 1$
12	ζ	The proportion of ζ of abstaining smoking enters to temporary quitters and $1-\zeta$ to permanent quitters	$0 \leq \zeta \leq 1$
13	ν	The number people transferred from temporary quitters to permanent quitters	$0 \leq \nu \leq 1$
14	ε	Death rate as a result of smoking	$0 < \varepsilon < 1$

The dynamic system. As we observe in figure(1),

(1) P_N and P_E are increases by $\mu\xi$ and $\mu(1-\xi)$ respectively.

(2) P_N decreases due to natural death(μP_N and the influence of factors that cause a person to move to smoker state ($\beta P_N S$).

(3) P_E decreases due to natural death(μP_N and the influence of factors that cause a person to move to smoker state ($\beta\delta P_E S$).

(4) S increases by the impact of $\beta P_N S$, $\beta\delta P_E S$, and κQ_T.

(5) S decreases due to natural death (μS), death rate as a result of smoking εS, the influence of factors that cause a person to leave the population active smokers and join the population of people who have stopped smoking both temporary($\alpha\zeta S$) and permanently($\alpha(1-\zeta)S$).

(6) Q_T increases due to the influence of factors that cause a person to leave the population active smokers and join the population of people who have stopped smoking temporary($\alpha\zeta S$).

(7) Q_T decreases due to κQ_T, natural death (μQ_T) and stopping smoking permanently (νQ_T).

(8) Q_P increases by $\alpha(1-\xi)S$ and νQ_T.

(9) Q_P decreases by natural death rate (μQ_P).

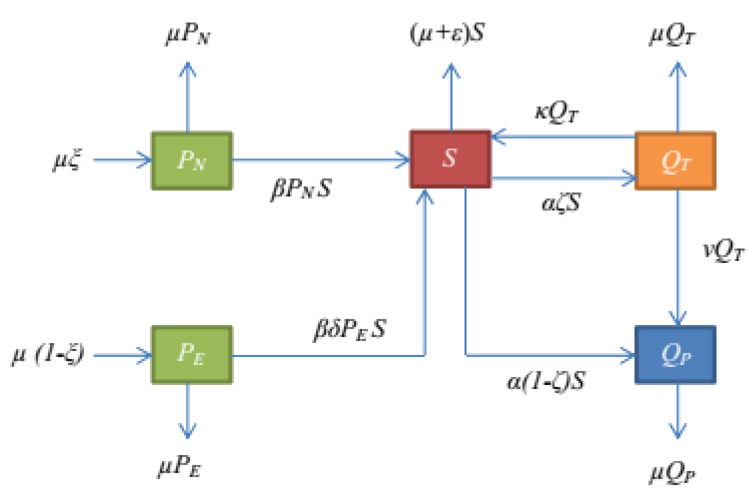

FIGURE 1. Smoking Model

$$\frac{dP_N}{dt} = \mu\xi - \mu P_N - \beta P_N S \tag{1}$$

$$\frac{dP_E}{dt} = \mu(1-\xi) - \mu P_E - \beta\delta P_E S \tag{2}$$

$$\frac{dS}{dt} = \beta P_N S + \beta\delta P_E S + \kappa Q_T - \mu S - \varepsilon S - \alpha S \tag{3}$$

$$\frac{dQ_T}{dt} = \alpha\zeta S - \kappa Q_T - \mu Q_T - \nu Q_T \tag{4}$$

$$\frac{dQ_P}{dt} = \alpha(1-\zeta)S + \nu Q_T - \mu Q_P \tag{5}$$

Positivity and boundedness of the Solution.

Theorem 2.1. *If the initial population sizes of the model are positive, then the population sizes at any time are non negative. In other words, if $P_E(0) > 0, P_N(0) > 0, S(0) > 0, Q_T(0) > 0$ and $Q_P(0) > 0$ then $P_E(t) > 0, P_N(t) > 0, S(t) > 0, Q_T(t) > 0$ and $Q_P(t) > 0$ for all t.*

Proof. Equation (1) can be expressed as an in equality

$$\frac{dP_N}{P_N} \geq -(\mu + \beta S)dt.$$

Integrated both sides from 0 to t the solution is obtained as

$$P_N(t) \geq P_N(0)e^{-\int(\mu+\beta S)dt}.$$

Since $P_N(0) > 0, P_N(t) > 0$. In the same manner from equation (2),(3),(4) and (5) we obtain respectively

$$P_E(t) \geq P_E(0)e^{-\int(\mu+\beta\delta S)dt} > 0,$$

$$S(t) \geq S(0)e^{-\int(\beta P_N+\beta\delta P_E-\mu-\varepsilon-\alpha)dt} > 0,$$

$$Q_T(t) \geq Q_T(0)e^{-\int(\kappa+\mu+\nu)dt} > 0, \text{ and}$$

$$Q_P(t) \geq Q_P(0)e^{-\int(\mu)dt} > 0.$$

□

Theorem 2.2. *All the solutions $P_E(t), P_N(t), S(t), Q_T(t)$ and $Q_P(t)$ of the system* (1),(2),(3),(4) *and* (5) *are bounded.*

Proof. We denote the total population size at time t by $N(t.)$ Then $N(t) = P_E(t) + P_N(t) + S(t) + Q_T(t) + Q_P(t)$. We assume $N(t)$ is constant, and $P_E(t), P_N(t), S(t)$, $Q_T(t)$ and $Q_P(t)$ are proportions of $N(t)$ where $P_E(t) + P_N(t) + S(t) + Q_T(t) + Q_P(t) = 1$. Since the variable Q_P does not appear in the first four equations of the dynamic system, we will only consider the subsystem:

$$\frac{dP_N}{dt} = \mu\xi - \mu P_N - \beta P_N S \tag{6}$$

$$\frac{dP_E}{dt} = \mu(1-\xi) - \mu P_E - \beta\delta P_E S \tag{7}$$

$$\frac{dS}{dt} = \beta P_N S + \beta\delta P_E S + \kappa Q_T - \mu S - \varepsilon S - \alpha S \tag{8}$$

$$\frac{dQ_T}{dt} = \alpha\zeta S - \kappa Q_T - \mu Q_T - \nu Q_T \tag{9}$$

In this subsystem

$$\begin{aligned} \frac{dP_N}{dt} + \frac{dP_E}{dt} + \frac{dS}{dt} + \frac{dQ_T}{dt} &= \mu - \mu(P_N + P_E + S + Q_T) - \\ &\quad (\varepsilon S + \alpha S(1-\zeta) + \nu Q_T) \\ &\leq \mu - \mu(P_N + P_E + S + Q_T) \end{aligned} \tag{10}$$

If we let $N_1(t) = P_N(t) + P_E(t) + S(t) + Q_T(t)$, then (10) becomes

$$\frac{dN_1(t)}{dt} \leq \mu - \mu N_1(t).$$

The solution of the initial value problem

$$\phi' = \mu - \mu\phi$$
$$\phi(0) = N_1(0)$$

is $\phi(t) = (N_1(0) - 1)e^{-\mu t} + 1$, and $\lim_{t\to\infty}\phi(t) = 1$. Hence $N_1(t) \leq \phi(t)$, it follows that

$$\lim_{t\to\infty} \sup N_1(t) \leq 1.$$

Thus, the solution of the system (1),(2),(3),(4) and (5) is bounded. □

Therefore, the region we consider here is

$$\Omega = \{(P_N, P_E, S, Q_T) \in \mathbb{R}^4 : P_N + P_E + S + Q_T \le 1, P_E > 0, P_N > 0, S \ge 0, Q_T > 0\}.$$

2.2. Equilibrium point. Taking the right hand side of equations (1)-(5) zero we get the following.

$$P_N = \frac{\mu\xi}{\mu + \beta S} \tag{11}$$

$$P_E = \frac{\mu(1-\xi)}{\mu + \beta\delta S} \tag{12}$$

$$S = \frac{\kappa Q_T}{\mu + \alpha + \varepsilon - \beta P_N - \beta\delta P_E} \tag{13}$$

$$Q_T = \frac{\alpha\zeta S}{\kappa + \mu + \nu} \tag{14}$$

$$Q_P = \frac{\alpha(1-\zeta)S + \nu Q_T}{\mu} \tag{15}$$

The point (P_N, P_E, S, Q_T, Q_P) is called equilibrium point of the dynamic system (1)-(5) where P_N, P_E, S, Q_T and Q_P as described in (11)-(15).

Free equilibrium point. It is obtained by taking $S = 0$ in the equilibrium point. So the point $(P_{N_0}, P_{E_0}, 0, 0, 0)$, where $P_{N_0} = \xi, P_{E_0} = 1 - \xi$ is free equilibrium point of the dynamic system (1)-(5).

Reproduction number. In this section we will calculate reproduction number using the next generation method. We have applied the method as follows. Let $X = (S, Q_T, Q_P, P_N, P_E)$. Then

$$\frac{dX}{dt} = F(X) - V(X),$$

where terms that describe appearances of new infections belong in

$$F(X) = \begin{pmatrix} \beta P_N S + \beta\delta P_E S \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and terms that describe a transfer of existing infections belong in

$$V(X)=\begin{pmatrix}(\mu+\alpha+\varepsilon)S-\kappa Q_T\\ (\nu+\mu+\kappa)Q_T-\alpha\zeta S\\ \mu Q_P-\nu Q_T-\alpha(1-\zeta)S\\ \beta P_N S+\mu P_N-\mu\xi\\ \beta\delta P_E S+\mu P_E-\mu(1-\xi)\end{pmatrix}$$

Let $M_0=(0,0,0,P_{N_0},P_{E_0})$. Then the Jacobian matrix of $F(X)$ and $V(X)$ at M_0 are respectively

$$DF(M_0)=\begin{pmatrix}F_{3\times3} & O_{3\times2}\\ O_{2\times3} & O_{2\times2}\end{pmatrix}\text{ and }DV(M_0)=\begin{pmatrix}V_{3\times3} & O_{3\times2}\\ J_{2\times3} & T_{2\times2}\end{pmatrix},$$

where

$$F=\begin{pmatrix}\beta P_{N_0}+\beta\delta P_{E_0} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ V=\begin{pmatrix}\mu+\alpha+\varepsilon & -\kappa & 0\\ -\alpha\zeta & \nu+\mu+\kappa & 0\\ -\alpha(1-\zeta) & -\nu & \mu\end{pmatrix},$$

$$J=\begin{pmatrix}\beta P_{N_0} & 0 & 0\\ \beta\delta P_{E_0} & 0 & 0\end{pmatrix}\text{ and }T=\begin{pmatrix}\mu & 0\\ 0 & \mu\end{pmatrix}.$$

Here

$$det(DV)=\mu[(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta].$$

Hence

$$DV^{-1}=\frac{1}{det(DV)}\begin{pmatrix}\mu(\nu+\mu+\kappa) & \kappa\mu & 0\\ \alpha\mu\zeta & \mu(\mu+\alpha+\varepsilon) & 0\\ A & B & C\end{pmatrix},$$

where

$$A=\alpha\nu\zeta+\alpha(1-\zeta)(\nu+\mu+\kappa),$$
$$B=\nu(\mu+\alpha+\varepsilon)-\kappa\alpha(1-\zeta)\text{ and}$$
$$C=(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta.$$

And

$$DFDV^{-1}=\frac{1}{det(DV)}\begin{pmatrix}Z\mu(\nu+\mu+\kappa) & Z\alpha\mu\zeta & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

where $Z=\beta P_{N_0}+\beta\delta P_{E_0}$.

Here

$$R_0=\rho(DFDV^{-1})=\frac{(\beta P_{N_0}+\beta\delta P_{E_0})(\nu+\mu+\kappa)}{(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta}$$

Endemic equilibrium point. To determine the endemic equilibrium point of the dynamic system (1)-(5) where at least one of the infected components is non-zero, we need to take the following steps:

From (13) we have

$$S(\mu+\alpha+\varepsilon-\beta P_N-\beta\delta P_E)-\kappa Q_T=0. \tag{16}$$

Substituting (14) in (16) we get

$$S(\mu+\alpha+\varepsilon-\beta P_N-\beta\delta P_E-\kappa\frac{\alpha\zeta}{\kappa+\mu+\nu})=0. \tag{17}$$

If $S\neq 0$, equation (17) becomes

$$\beta P_N+\beta\delta P_E=\mu+\alpha+\varepsilon-\kappa\frac{\alpha\zeta}{\kappa+\mu+\nu}. \tag{18}$$

Substituting (11) and (12) in (18) we have

$$\beta\frac{\mu\xi}{\mu+\beta S}+\beta\delta\frac{\mu(1-\xi)}{\mu+\beta\delta S}=\mu+\alpha+\varepsilon-\kappa\frac{\alpha\zeta}{\kappa+\mu+\nu}.$$

Let

$$G(S)=\frac{\xi(\mu+\beta\delta S)+\delta(1-\xi)(\mu+\beta S)}{(\mu+\beta S)(\mu+\delta\beta S)}-\frac{1}{\mu\beta}\left[\frac{(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta}{\kappa+\mu+\nu}\right]=0.$$

We now have the following lemma.

Lemma 2.3. *(1) $G(S)$ is decreasing for $S>0$,*

(2) $\lim_{S\to 0^+}G(S)=\dfrac{[(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta]}{\mu\beta(\kappa+\mu+\nu)}(R_0-1)$ and

(3) $G(1)<0$.

Proof. (1) Differentiating G with respect to S, we have

$$\begin{aligned} G^{'}(S) &= \frac{\beta}{(\mu+\beta S)^2(\mu+\delta\beta S)^2}\begin{bmatrix} \delta[(\mu+\beta S)(\mu+\delta\beta S)]- \\ [\xi(\mu+\beta\delta S)+\delta(1-\xi)(\mu+\beta S)]\times \\ [(\mu+\beta\delta S)+\delta(\mu+\beta S)] \end{bmatrix} \\ &= \frac{\beta[\delta^2(\xi-1)(\mu+\beta S)^2-\xi(\mu+\beta\delta S)^2]}{(\mu+\beta S)^2(\mu+\delta\beta S)^2} \\ &< 0 \text{ since } 0\leq\xi\leq 1 \end{aligned}$$

It follows that G is decreasing for $S>0$.

(2) Now,

$$\begin{aligned} \lim_{S\to 0^+} G(S) &= \frac{\xi+\delta(1-\xi)}{\mu}-\frac{1}{\mu\beta}\left[\frac{(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta}{\kappa+\mu+\nu}\right] \\ &= \frac{\beta(\xi+\delta(1-\xi))(\kappa+\mu+\nu)-[(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta]}{\mu\beta(\kappa+\mu+\nu)} \\ &= \frac{[(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta]}{\mu\beta(\kappa+\mu+\nu)}\left[\frac{\beta(\xi+\delta(1-\xi))(\kappa+\mu+\nu)}{(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta}- \right. \\ &\left. 1\right] \\ &= \frac{[(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta]}{\mu\beta(\kappa+\mu+\nu)}(R_0-1) \end{aligned}$$

(3) We note that

$$(\mu+\beta S)(\mu+\delta\beta S)>\beta S(\mu+\delta\beta S) \tag{19}$$

$$\xi(\mu+\beta\delta S)+\delta(1-\xi)(\mu+\beta S)\leq\mu+\beta\delta S \tag{20}$$

For if $\xi(\mu+\beta\delta S)+\delta(1-\xi)(\mu+\beta S)>\mu+\beta\delta S$, then $(\xi-1)(\mu+\beta\delta S)>(\xi-1)\delta(\mu+\beta S)$. This gives $\mu+\beta\delta S<\delta(\mu+\beta S)$, which is impossible. Substituting inequalities (19) and (20) in $G(S)$, we have

$$G(S)<\frac{1}{\beta S}-\frac{1}{\mu\beta}\left[\frac{(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta}{\kappa+\mu+\nu}\right]$$

So,

$$\begin{aligned} G(1) &< \frac{1}{\beta}-\frac{1}{\mu\beta}\left[\frac{(\mu+\alpha+\varepsilon)(\kappa+\mu+\nu)-\kappa\alpha\zeta}{\kappa+\mu+\nu}\right] \\ &= \frac{-1}{\beta}\left[\frac{\alpha(\mu+\nu)+\alpha\kappa(1-\xi)+\varepsilon(\mu+\nu+\kappa)}{\mu(\kappa+\mu+\nu)}\right] \\ &< 0 \end{aligned}$$

□

Lemma 2.4. *If $R_0 > 1$, $G(S)$ has a unique root in $(0,1)$. However, if $R_0 \leq 1$, $G(S)$ has no a root in $(0,1)$.*

Proof. If $R_0 > 1$, then $\lim_{S\to 0^+} G(S) > 0$. Since G is decreasing for $S > 0$, there is a unique root in $(0,1)$. However, if $R_0 \leq 1$, then $\lim_{S\to 0^+} G(S) < 0$ and hence G has no a root in $(0,1)$. □

In (2.4) we have seen the existence of endemic equilibrium point for $R_0 > 1$. Thus the endemic equilibrium point will be

$$(P_N^*, P_E^*, S^*, Q_T^*, Q_P^*) = (P_N, P_E, Q_T, Q_P),$$

where P_N, P_E, S, Q_T and Q_P are given as in (11),(12),(14)and (15), respectively; and S is the unique zero of $G(S)$. We immediately have the following theorem.

Theorem 2.5. *If $R_0 > 1$, the endemic equilibrium point exist and unique. If $R_0 \leq 1$, then the endemic equilibrium point does not exist*

Stability of Free Equilibrium Point. The Jacobian matrix of the given system is

$$\begin{pmatrix} -\mu-\beta S & 0 & -\beta P_N & 0 \\ 0 & -\mu-\beta\delta S & -\beta\delta P_E & 0 \\ \beta S & \beta\delta S & \beta P_N+\beta\delta P_E-\mu-\alpha-\varepsilon & \kappa \\ 0 & 0 & \alpha\zeta & -\kappa-\mu-\nu \end{pmatrix}$$

At the free equilibrium point the Jacobian matrix becomes

$$\begin{pmatrix} -\mu & 0 & -\beta P_{N_0} & 0 \\ 0 & -\mu & -\beta\delta P_{E_0} & 0 \\ 0 & 0 & \beta P_{N_0}+\beta\delta P_{E_0}-\mu-\alpha-\varepsilon & \kappa \\ 0 & 0 & \alpha\zeta & -\kappa-\mu-\nu \end{pmatrix}$$

The characteristic equation is

$$(\gamma+\mu)^2[\gamma^2+(N+M-L)\gamma+(D-MN)(R_0-1)]=0,$$

where

$$L=\beta P_{N_0}+\beta\delta P_{E_0},\ M=\mu+\alpha+\varepsilon,\ N=\kappa+\mu+\nu,\ D=\alpha\kappa\zeta.$$

Let $\mathscr{P}(\gamma)=\gamma^2+(N+M-L)\gamma+(D-MN)(R_0-1)=0$. If $R_0>1$, all roots of γ have not negative real part. Thus the free equilibrium point is unstable. If $R_0<1$ and $N+M>L$, then the two roots of γ have negative real part. In this case the free equilibrium point is stable. Furthermore, if $R_0<1$, then $N+M>L$; and when $\kappa+2\mu+\nu+\alpha\leq\beta(\xi+\delta-\delta\xi), R_0\geq 1$. This discussion gives the following result.

Theorem 2.6. *If $R_0<1$, then the free equilibrium point is stable. If $R_0>1$, then the free equilibrium point is unstable.*

Stability of Endemic Equilibrium Point. Evaluating the Jacobian matrix at the endemic equilibrium point we obtain

$$\begin{pmatrix} -\mu-\beta S^* & 0 & -\beta P_N^* & 0 \\ 0 & -\mu-\beta\delta S^* & -\beta\delta P_E^* & 0 \\ \beta S^* & \beta\delta S^* & \beta P_N^*+\beta\delta P_E^*-\mu-\alpha-\varepsilon & \kappa \\ 0 & 0 & \alpha\zeta & -\kappa-\mu-\nu \end{pmatrix}$$

The characteristic equation of this Jacobian matrix is

$$\mathscr{P}(\gamma)=\gamma^4+U\gamma^3+V\gamma^2+W\gamma+X=0,$$

where

$$\begin{aligned}
U &= -[H+I+J+K] \\
V &= HI+JK+HJ+HK+IJ+IK+\beta^2 P_N^* S^* - \kappa\alpha\zeta \\
W &= -[HIJ+HIK+JKH+JKI-\kappa\alpha\zeta H-\kappa\alpha\zeta I+\beta^2\delta^2 S^* P_E^* + \\
&\quad \beta^2 S^* P_N^* I+\beta^2 S^* P_N^* K] \\
X &= HIJK+\beta^2\delta^2 S^* P_E^* K+\beta^2 S^* P_N^* IK-\kappa\alpha\zeta HI \\
H &= -\mu-\beta S^* \\
I &= -\mu-\beta\delta S^* \\
J &= \beta P_N^* + \beta\delta P_E^* - \mu-\alpha-\varepsilon \\
K &= -\kappa-\mu-\nu
\end{aligned}$$

All the roots of $\mathscr{P}(\gamma)$ have negative real part if

$$U>0, UV-W>0, \begin{vmatrix} U & W & 0 \\ 1 & V & X \\ 0 & U & W \end{vmatrix} > 0, \text{ and } \begin{vmatrix} U & W & 0 & 0 \\ 1 & V & X & 0 \\ 0 & U & W & 0 \\ 0 & 1 & V & X \end{vmatrix} > 0. \tag{21}$$

From this discussion we now have the following theorem.

Theorem 2.7. *If $R_0 > 1$ and (21) is satisfied, the endemic equilibrium point is stable.*

3. Results and Discussion

3.1. Sensitivity analysis. Sensitivity analysis permits to investigate how uncertainty in the input parameters affects the model outputs. C.J Silva and D.F.M Torres in [11] and H.S Rodrigues and his colleges in [10] did sensitivity of the basic reproduction number for a tuberculosis model and dengue epidemiological model, respectively. B.Fekede and B.Mebrate discussed sensitivity analysis in recent paper[16] on secondhand smoker. The sensitivity analysis of R_0 with respect to the parameter p is defined by $\frac{p}{R_0}\left[\frac{\partial R_0}{\partial p}\right]$ [5]. We denote it by $\mathscr{S}_p^{R_0}$. Thus,

$$\mathscr{S}_p^{R_0} = \frac{p}{R_0}\left[\frac{\partial R_0}{\partial p}\right].$$

We now calculate the sensitivity analysis of R_0 with respect to the parameters involved in R_0 as follows.

$$\mathscr{S}_{\beta}^{R_0} = \frac{\beta}{R_0}\frac{(P_{N_0}+\delta P_{E_0})(\nu+\mu+\kappa)}{(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta} > 0 \tag{22}$$

$$\mathscr{S}_{\nu}^{R_0} = \frac{\nu}{R_0}\frac{(\beta P_{N_0}+\beta\delta P_{E_0})(-\alpha\kappa\zeta)}{[(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta]^2} < 0 \tag{23}$$

$$\mathscr{S}_{\delta}^{R_0} = \frac{\delta}{R_0}\frac{\beta P_{E_0}(\nu+\mu+\kappa)}{(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta} > 0 \tag{24}$$

$$\mathscr{S}_{\alpha}^{R_0} = \frac{\alpha}{R_0}\frac{-(\nu+\mu+\kappa-\kappa\zeta)(\beta P_{N_0}+\beta\delta P_{E_0})(\nu+\mu+\kappa)}{(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta} < 0 \tag{25}$$

$$\mathscr{S}_{\kappa}^{R_0} = \frac{\kappa}{R_0}\frac{(\beta P_{N_0}+\beta\delta P_{E_0})\alpha\zeta(\nu+\mu)]}{[(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta]^2} > 0 \tag{26}$$

$$\mathscr{S}_{\mu}^{R_0} = \frac{\mu}{R_0}\frac{(\beta P_{N_0}+\beta\delta P_{E_0})[-\alpha\kappa\zeta-(\nu+\mu+\kappa)^2]}{[(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta]^2} < 0 \tag{27}$$

$$\mathscr{S}_{\zeta}^{R_0} = \frac{\zeta}{R_0}\frac{\alpha\kappa(\beta P_{N_0}+\beta\delta P_{E_0})(\nu+\mu+\kappa)}{[(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta]^2} > 0 \tag{28}$$

$$\mathscr{S}_{\xi}^{R_0} = \frac{\xi}{R_0}\frac{(\beta-\delta\beta)(\nu+\mu+\kappa)}{(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta} > 0 \tag{29}$$

$$\mathscr{S}_{\varepsilon}^{R_0} = \frac{\varepsilon}{R_0}\frac{-(\beta P_{N_0}+\beta\delta P_{E_0})(\nu+\mu+\kappa)^2}{(\mu+\alpha+\varepsilon)(\nu+\mu+\kappa)-\alpha\kappa\zeta} < 0 \tag{30}$$

As we see in (22), (24), (26), (28) and (29), the partial derivative of R_0 with respect to $\beta, \delta, \kappa, \zeta$ and ξ greater than zero. It follows that R_0 is sensitive to those parameters. Thus if R_0 increases or decreases, theses parameters increases or decreases, respectively. This means R_0 proportional to these parameters. However, in (23), (25) and (27),the partial derivative of R_0 with respect to ν, α, μ and ε less than zero. Hence, an increases or decreases in R_0 yields decreases or increases in the parameter involved in R_0, respectively. The relationship between R_0 and the parameters within it can be described graphically as showen in figure (2).

For the purpose of numerical simulation we provide the following initial values(IV) for

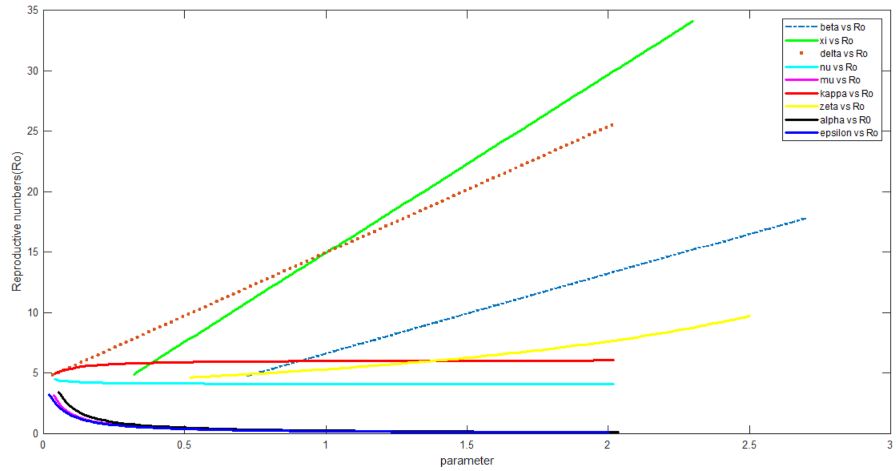

FIGURE 2. Parameters versus reproductive number

P_N, P_E, S, Q_T, and Q_P as follows.

	IV1	IV2	IV3	IV4	IV5
P_N	0.20	0.40	0.15	0.30	0.10
P_E	0.30	0.20	0.15	0.10	0.05
S	0.25	0.10	0.40	0.20	0.30
Q_T	0.10	0.05	0.25	0.20	0.15
Q_P	0.15	0.25	0.05	0.20	0.40

We will take the parameters $\mu = 0.017, \xi = 0.3, \delta = 0.0135, \kappa = 0.02, \alpha = 0.035, \zeta = 0.5, \varepsilon = 0.001$ and $\nu = 0.02$. The parameter β can be taken as shown bellow.

	β	R_0
Case 1	0.02	< 1
Case 2	0.7	> 1

Since the free equilibrium does not depend on β, in both cases we have

$$(P_N, P_E, S, Q_T, Q_P) = (0.3, 0.7, 0, 0, 0).$$

For numerical simulation we use Rung-kutta 4-5 methods and MatLab 2018 software. It will be seen separately for free and endemic equilibrium point.

3.2. Free Equilibrium Point. We will construct the graphs in each case as follows.

i)$\beta = 0.3$

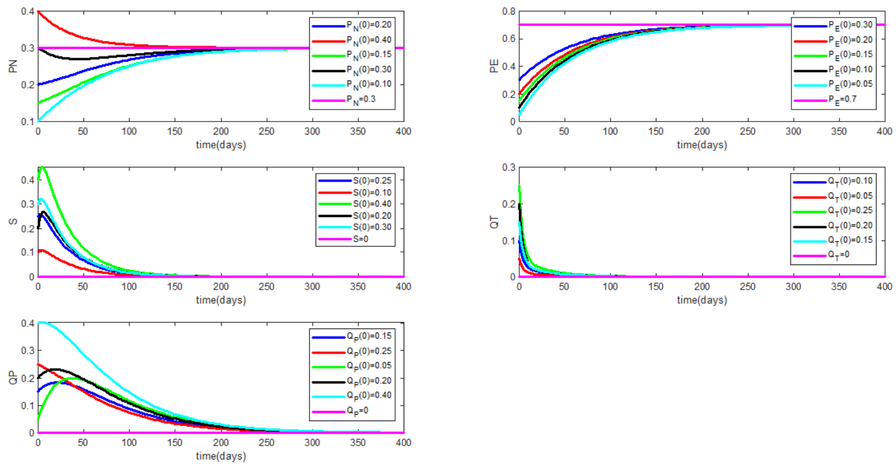

FIGURE 3. $R_0 < 1$

In the figure (3) it can be seen that (P_N, P_{E,S,Q_T,Q_P}) approaches to free equilibrium point

$$(0.3, 0.7, 0, 0, 0)$$

for the given initial values as $t \to \infty$. Hence, the free equilibrium point is stable.

ii)$\beta = 0.7$

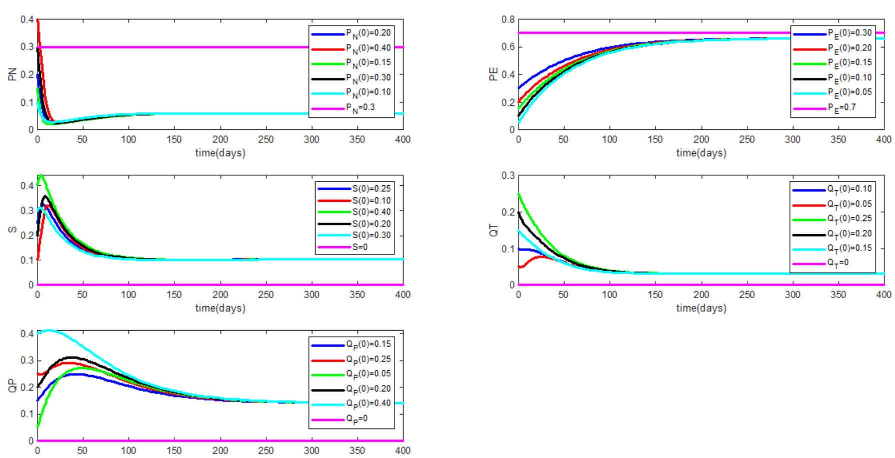

FIGURE 4. $R_0 > 1$

In the figure (4) we see that (P_N, P_{E,S,Q_T,Q_P}) does not approaches to free equilibrium point $(0.3, 0.7, 0, 0, 0)$ for the given initial values as $t \to \infty$. Hence, the free equilibrium point is unstable.

3.3. Endemic Equilibrium Point.

We take $\beta = 0.7$. We recall that $G(S)$ has a root between 0 and 1. Since $G(S)$ is continuous on $[0,1]$, by bisection method the root of $G(S)$ can be approximated as $S = 0.1045$. Thus, we can calculate P_N, P_E, Q_T and Q_P respectively $0.0566, 0.6616$, 0.0321 and 0.1453.

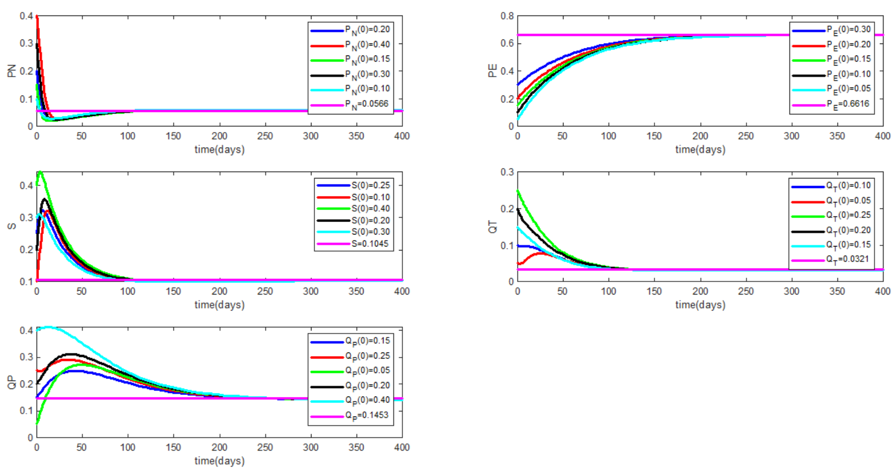

FIGURE 5. $R_0 > 1$

In the figure (5) (P_N, P_{E,S,Q_T,Q_P}) approaches to the endemic equilibrium point

$$(0.0566, 0.6616, 0.1045, 0.0321, 0.1453)$$

for the given initial values as $t \to \infty$. Hence, the endemic equilibrium point is stable.

4. Conclusion

We have obtained the reproduction number, that is, the average number of people who catch a smoking habit from a single addicted smoker, is given by

$$R_0 = \frac{(\beta P_{N_0} + \beta \delta P_{E_0})(\nu + \mu + \kappa)}{(\mu + \alpha + \varepsilon)(\nu + \mu + \kappa) - \alpha \kappa \zeta}.$$

If $S = 0$ and $R_0 < 1$, then the population of smokers disappears over time. And if $S = 0$ and $R_0 > 1$, then the smoking population persists. An increase (or decrease) in $\beta, \xi, \delta, \kappa$ and ζ leads an increase(or decrease) the average number of secondary cases of infection in the community. But, an increase(or decrease) in the parameters ν, μ, α and ε leads to minimizing(or maximizing) the endemic nature of smoking in the community.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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