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MATHEMATICAL MODELING AND OPTIMAL CONTROL STRATEGY FOR A DISCRETE TIME MODEL OF COVID-19 VARIANTS

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Abstract. In this study, we analyze the transmission dynamics of several variants of Covid-19 that have appeared around the world. Our aim is to propose a discrete mathematical model that describes the dynamics of different infectious compartments, namely, Susceptible (S), Exposed (E), Individuals infected with the Alpha variant (I_1), Individuals infected with the Beta variant (I_2), Individuals infected with the Gamma variant (I_3), Individuals infected with the Delta variant (I_4), Hospitalized (H), Quarantined (Q) and Recovered (R). We also focus on the importance of people infected with the Alpha, Beta, Gamma and Delta variants, with the aim of finding optimal strategies to minimize the number of people infected with the different variants of Covid-19. We used three controls which represent: 1) awareness programs through media and civil society to urge uninfected people to stay away from infected people, as well as to encourage individuals to be vaccinated, 2) encouraging people infected with Covid-19 variants to self-isolate at home or join quarantine centers and encouraging severe cases go to hospitals and in the last control we use medical and psychological treatment to increase the immunity of people infected with different variants and reduce the number of people in hospitals and in isolation centers. We use the principle of the Pontryagin's maximum principle in discrete time to characterize these optimal controls. The resulting optimality system is solved numerically using Matlab. Therefore, the results obtained confirm the performance of the optimization strategy.

Keywords: discrete mathematical model; Covid-19 model with four variant; optimal control.

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1. Introduction

Throughout history, the world has witnessed many deadly diseases and epidemics, some of which were epidemics confined to certain countries or geographic areas, and some of them were global epidemics.These epidemics claimed tens if not hundreds of millions of lives and caused demographic, social and economic changes around the world. The Black Plague was one of the most famous and deadly of these epidemics in ancient and medieval times as well as the Plague of Justinian and the Plague of Emmaus in the Levant region. In the modern era, we find Cholera, Smallpox and the Spanish flu. The methods of confronting these epidemics differed according to the different times and available possibilities. The effects of these epidemics on the societies they invaded also varied. The world is currently facing the worst pandemic in history. The Coronavirus or Covid-19 pandemic, also known as the Coronavirus pandemic, is an ongoing global pandemic caused by severe acute respiratory syndrome Coronavirus (SARS-CoV-2). The disease first broke out in the Chinese city of Wuhan in early December 2019. The virus has now affected the world in a way or another with over 185 million cases of Covid-19 reported in over 188 countries and territories as of July 9, 2021, with over 4.01 million deaths and over 1 million recoveries. See the figure (1). It represents the number of cases infected with Covid-19, and the weekly number of deaths in the world according to WHO [22].

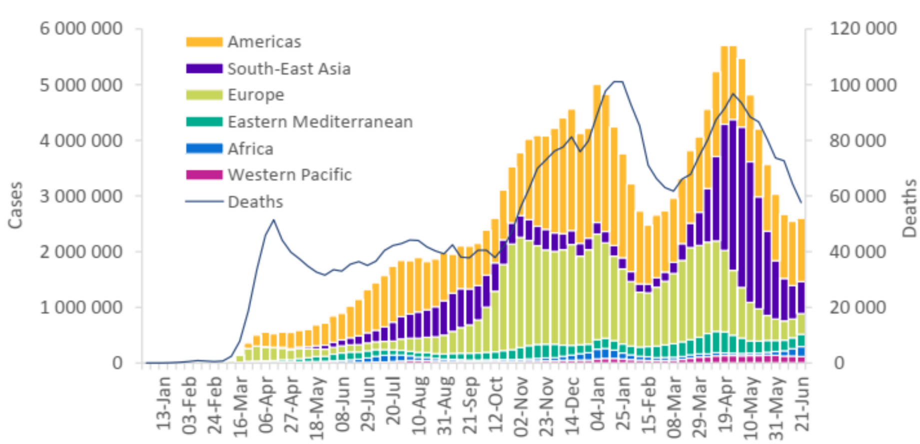

FIGURE 1. Covid-19 cases reported weekly by WHO Region, and global deaths, as of 27 June 2021

Several mathematical modeling studies have been conducted to understand the Coronavirus with its variants and to describe its dynamics. For example, [4] studied the transmission of (Covid-19) in the human population and used a compartmental model to describe the spread of this infectious disease. [5] formulated a mathematical model to study the impact of a new, more communicable disease. SARS-CoV-2 variant on the prevalence, hospitalizations and deaths associated with the SARS-CoV-2 virus, [6] constructed a dynamic model based on detailed data from the World Organization (WHO) mortality and the spread actual epidemic, [7] formulated a mathematical model that addresses the transmission of two variants of SARS-CoV-2 to hospitalizations observed Covid-19, hospital bed occupancy and intensive care and deaths ; SARS-CoV-2 PCR prevalence, [8].

Additionally, most of this previous research has focused on continuous-time modeling. In this research, we will adopt a discrete-time modeling because statistical data is collected at discrete times (day, week, month and year) as well as the treatment and vaccination of some patients is done in discrete time. It is therefore more direct, more practical and more precise to describe the phenomena using discrete-time modeling than continuous-time modeling and the use of discrete-time models make it possible to avoid certain mathematical complexities such as the choice of a space of functions and the regularity of the solutions. Therefore, equation differences appear to be a more natural way to describe epidemic patterns. Moreover, the numerical solutions of differential equations use discretization, which encourages the direct use of the difference equations. Numerical exploration of discrete-time models is quite simple and can therefore be easily implemented by non-mathematicians. In addition to this work, we will study the dynamics of a mathematical model of Covid-19 variants. SE$I_1I_2I_3I_4$QHR.

The population is divided into nine compartments: susceptible individuals (S), exposed individuals (E), individuals infected with the Alpha variant (I1), individuals infected with the Bêta variant (I2), individuals infected with the Gamma variant (I3) and individuals infected with the Delta variant (I4). The SARS-CoV-2 virus has a genome, which is a ribonucleic acid or RNA molecule. When it enters an organism, the virus replicates to infect new cells and its genetic makeup is reproduced. Sometimes there is an error in the replication of the virus resulting in mutations. This is called a variant virus. Variants can give it new properties. This is the case of the variants currently circulating around the world. The World Health Organization (WHO) has classified the variants into two categories: The worrying variants (VOC): they are characterized by an increase in the transmissibility or detrimental evolution of the epidemiology of Covid-19, an increase in virulence or modification of the clinical picture or a "decrease in the effectiveness of the measures health and social care tools or diagnostic tools, vaccines and treatments available, currently the Alpha (English), Beta (South African), Gamma (Brazilian) and Delta (Indian) variants: The Alpha variant, previously referred to as B.1.1.7, was first detected in the UK in September 2020, according to the World Health Organization, and by December 2020 it had made its appearance in the US. It has spread to at least 114 countries, according to the Global Virus Network. It is also responsible for around 95% of new corona cases in the UK. Between May 23 and June 5, 2021, Alpha caused about 60% of all cases. The Beta variant previously called B.1.351, was first discovered in South Africa in May 2020 and identified as a variant of concern in December 2020. It has spread to at least 48 countries and 23 US states, according to Global Virus Network. Additionally, it contains eight distinct mutations that can affect the way the virus invades human cells. The Gamma variant is also known as P.1, appeared in Brazil in November 2020, according to data from the World Health Organization. Scientists first discovered the Gamma variant in Japan in early January 2021, when four travelers tested positive for the virus after traveling to Brazil. The researchers then found evidence that the Gamma variant was indeed prevalent in Brazil. According to the United Nations website, the mutant has appeared in 74 countries around the world and was first discovered in the United States in January 2021, and it has spread to at least 30 American states. The Delta variant previously called B.1.617.2 first appeared in India in October 2020 and was classified as a variant of concern in May 2021, according to the World Health Organization. The Delta variant quickly spread to over 100 countries and quickly became the dominant variant around the world. Additionally, the Delta variant accounted for more than half of all cases in the United States. The variants of interest: they are characterized by genetic markers that can affect the transmission of the virus, its diagnosis, its treatment or even the immune response; evidence that these variants cause an increase in the number of cases or clusters; a distribution which is limited to a single country. These are the variants: Epsilon (American), Zeta (another Brazilian variant), Eta (several countries), Theta (Philippines), Lota (another American variant), Kappa (another Indian variant) and Lambda (Peruvian). Throughout this research, we seek to find optimal strategies to reduce the number of infected individuals, limit and prevent the movement of individuals infected with the different variants of Covid-19. We use three optimal controls that represent awarness programs through media and civil society to urge uninfected people to stay away from infected and likely people as well as encouraging the susceptible individuals to be vaccinated, government efforts to urge people infected with Covid-19 variants to self-isolate at home or join quarantine centers and also encourage severe cases to be hospitalized and in the final control, we use medical treatment and psychological support. The objective of this control is to increase the immunity of people infected with the different variants and to reduce the number people in hospitals and in isolation centers. This article is organized as follows. In section 2, we present our discrete mathematical model of Covid-19 variants which describes the dynamics of propagation and transmission of Corona virus variants. In section 3, we present the optimal control problem for the proposed model and we characterize these optimal controls using the Pontryagin's maximum principle in discrete time. Numerical simulations and cost-effectiveness analysis are presented in section 4. Finally, we conclude the article in section 5.

2. A Mathematical Model and Basic Properties

2.1. A Mathematical Model.

In this section, we present a discrete $\mathbf{S}_k, \mathbf{E}_k$ $, \mathbf{I}_{1,k}, \mathbf{I}_{2,k}, \mathbf{I}_{3,k}, \mathbf{I}_{4,k}, \mathbf{H}_k, Q_k, R_k$. We propose a discrete model to describe the interaction within a population where the disease Covid-19 variant exists. The population under study is divided into eight compartments: Susceptible individuals exposed to have new Corona virus $S(t)$, Asymptomatic infected cases or cases with mild symptoms E(t),$\mathbf{I}_1(\mathbf{t}), \mathbf{I}_2(\mathbf{t}), \mathbf{I}_3(\mathbf{t})$and $I_4(t)$ multiple variants ,$\mathbf{I}_1(\mathbf{t})$: represents individuals infected with the Alpha variant (detected in UK), $\mathbf{I}_2(\mathbf{t})$: represents individuals infected with the Béta variant (detected in South African), $\mathbf{I}_3(\mathbf{t})$: represents individuals infected with the Gamma variant (detected in Brazilian), $\mathbf{I}_4(\mathbf{t})$: represents individuals infected with the Delta variant (detected in Indian), Hospitalized cases H(t),Quarantaine cases Q(t), the recovered cases R(t). The total number of the population at time t is given by $N(t) = S(t) + E(t) + I_1(t) + I_2(t) + I_3(t) + I_4(t) + H(t) + Q(t) + R(t)$: In this section, we present a discrete S_k, E_k, $I_{1,k}$, $I_{2,k}$, $I_{3,k}$, $I_{4,k}$, H_k, Q_k, R_k. We propose a discrete model to describe the interaction within a population where the disease Covid-19 variant exists. The population under study is divided into eight compartments: Susceptible individuals exposed to have new Corona virus S(t), Asymptomatic infected cases or cases with mild symptoms $E(t)$, $I_1(t)$, $I_2(t)$, $I_3(t)$ and $I_4(t)$ multiple variants, $I_1(t)$: represents individuals infected with the Alpha variant (detected in UK), $I_2(t)$: represents individuals infected with the Béta variant (detected in South African), $I_3(t)$: represents individuals infected with the Gamma variant (detected in Brazilian), $I_4(t)$: represents individuals infected with the Delta variant (detected in Indian), Hospitalized cases H(t),Quarantaine cases Q(t), the recovered cases R(t). The total number of the population at time t is given by $N(t) = S(t)+E(t)+I_1(t)+I_2(t)+I_3(t)+I_4(t)+H(t)+Q(t)+R(t)$:

The following diagram will show the direction of movement of individuals between compartments in figure 2

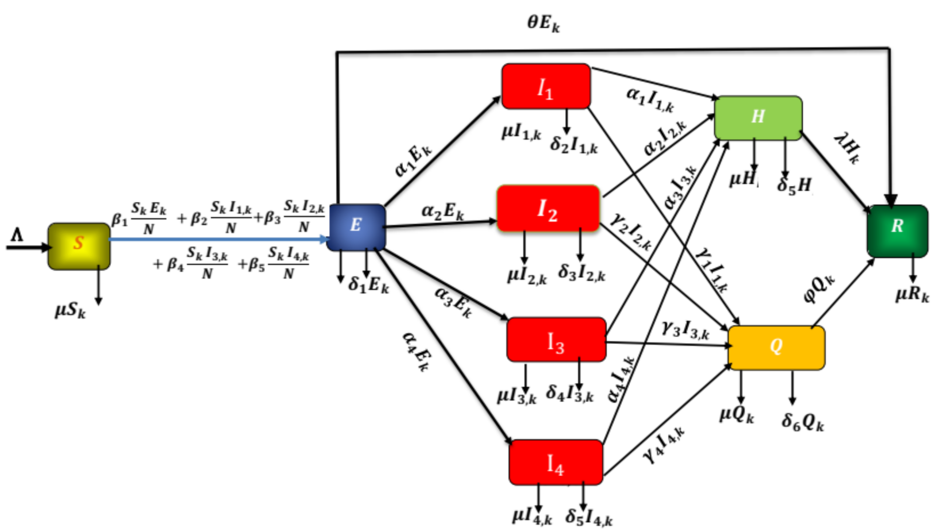

Figure 2: Schematic diagram of the nine compartments in the model.

2.2. **Model Equations.**

Adding the rates at which the stages of Covid-19 variants enter the compartment and also by subtracting the rates at which people leave a compartment, we obtain a difference equation for the rate at which patients change in each compartment during separate times. Therefore, we present the model of Covid-19 variants with the following system of difference equations:

$$
(1) \qquad \begin{cases}
\mathbf{S}_{k+1} = \Lambda + (1-\mu)S_k - \beta_1 \frac{S_k E_k}{N} - \beta_2 \frac{S_k I_{1,k}}{N} - \beta_3 \frac{S_k I_{2,k}}{N} - \beta_4 \frac{S_k I_{3,k}}{N} - \beta_5 \frac{S_k I_{4,k}}{N} \\
\mathbf{E}_{k+1} = (\mathbf{1} - \mu - \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)\mathbf{E}_k + \beta_1 \frac{S_k E_k}{N} \\
\mathbf{I}_{1,k+1} = (\mathbf{1} - \mu - \delta_1 - \lambda_1 - \gamma_1)\mathbf{I}_{1,k} + \alpha_1 \mathbf{E}_k + \beta_2 \frac{S_k I_{1,k}}{N} \\
\mathbf{I}_{2,k+1} = (\mathbf{1} - \mu - \delta_2 - \lambda_2 - \gamma_2)\mathbf{I}_{2,k} + \alpha_2 \mathbf{E}_k + \beta_3 \frac{S_k I_{2,k}}{N} \\
\mathbf{I}_{3,k+1} = (\mathbf{1} - \mu - \delta_3 - \lambda_3 - \gamma_3)\mathbf{I}_{3,k} + \alpha_3 \mathbf{E}_k + \beta_4 \frac{S_k I_{3,k}}{N} \\
\mathbf{I}_{4,k+1} = (\mathbf{1} - \mu - \delta_4 - \lambda_4 - \gamma_4)\mathbf{I}_{4,k} + \alpha_4 \mathbf{E}_k + \beta_5 \frac{S_k I_{4,k}}{N} \\
\mathbf{H}_{k+1} = (\mathbf{1} - \mu - \delta_5 - \lambda)\mathbf{H}_k + \gamma_1 \mathbf{I}_{1,k} + \gamma_2 \mathbf{I}_{2,k} + \gamma_3 \mathbf{I}_{3,k} + \gamma_4 \mathbf{I}_{4,k} \\
\mathbf{Q}_{k+1} = (\mathbf{1} - \mu - \delta_6 - \varphi)\mathbf{Q}_k + \lambda_1 \mathbf{I}_{1,k} + \lambda_2 \mathbf{I}_{2,k} + \lambda_3 \mathbf{I}_{3,k} + \lambda_4 \mathbf{I}_{4,k} \\
\mathbf{R}_{k+1} = (\mathbf{1} - \mu)\mathbf{R}_k + \lambda \mathbf{H}_k + \theta \mathbf{E}_k + \varphi \mathbf{Q}_k
\end{cases}
$$

where $S_0 \geq 0, E_0 \geq 0, I_{1,0} \geq 0, I_{1,0} \geq 0, I_{2,0} \geq 0, I_{3,0} \geq 0, I_{4,0} \geq 0, Q_0 \geq 0, H_0 \geq 0, and\ R_0 \geq 0$, are the given initial states.

Comartement "S": represents people likely to be infected with the disease Covid-19 or one of the variants of this epidemic.

This compartment is increased by the recruitment rate noted Λ. It is reduced by a μ natural mortality rate. It is also reduced by effective contact with exposed individuals "E" with a β_1 rate. It is also reduced by effective contact by patients with the following variants:

– I1by a β_2 rate (the rate of patients who become infected with the variant Alpha of Covid-19).

– I2 by a β_3 rate (the rate of patients who become infected with the variant Béta of Covid-19).

– I3 by a β_4 rate (the rate of patients who become infected with the variant Gamma of Covid-19)..

– I4 by a β_5 rate (the rate of patients who become infected with the variant Delta of Covid-19).

Comartement "E": represents individuals carrying the disease without symptoms; it is increased by the rate β_1 (rate of effective contact with susceptible individuals "S"). It is reduced by a natural mortality rateμ and by the rates α_1, α_2 ,α_3 and α_4 (which represent respectively the rates of attack by the variants I1, I2,I3 and I4 and also by theta (the rate of individuals recovered).

Comartement "I_1": represents patients infected with the Alpha variant. It is reduced by a natural mortality rate μ and by the mortality rate δ_1 due to the Alpha variant, and it is also decreased by the rate of hospitalization λ_1 and also by the rate of quarantine γ_1. It is increased by the rate $\alpha_1$due to exposed individuals.

Comartement "I_2": represents patients infected with the Béta variant. It is reduced by a natural mortality rate μ and by the mortality rate δ_2 due to the Béta variant, and it is also decreased by the hospitalization rate λ_2 and also by the quarantine rate γ_2. It is increased by the rate α_2 due to exposed individuals.

Comartement "I_3": represents patients infected with the Gamma variant. It is reduced by a natural mortality rate mu and by the mortality rate δ_3 due to the Gamma variant, and it is also decreased by the rate d hospitalization λ_3 and also by the quarantine rate γ_3. It is increased by the rate α_3 due to exposed individuals.

Comartement "I_4": represents patients infected with the Delta variant. It is reduced by a natural mortality rate μ and by the mortality rate δ_4 due to the Delta variant, and it is also decreased by the hospitalization rate λ_4 and also by the quarantine rate γ_4. It is increased by the rate α_4 due to exposed individuals.

Comartement "H": represents hospitalized patients; It is decreased by a natural mortality rate μ and by the mortality rate δ_5 due to the variant, and it is also decreased by the rate of recovered individuals λ. H is increased by the rates λ_1,λ_2,λ_3 and λ_4 due to those infected with the variants.

Comartement "Q": represents confined patients; It is decreased by a natural mortality rate μ and by the mortality rate δ_6 due to the variant, and it is also decreased by the rate of recovered individuals φ.Q is increased by the levels γ_1, γ_2, γ_3 and γ_4 due to those infected with the variants.

Comartement "R": represents the recovered patients. It is reduced by a natural mortality rate mu. It is increased by the θ, λ and φ rates due respectively to exposed individuals, hospitalized patients and confined individuals.

3. The Optimal Control Problem

The control strategies we are adopting consist of an awareness program through media and civil society to urge uninfected people to stay away from infected and likely infected people as well as encouraging people to be vaccinated, government efforts to urge people infected with Covid-19 variants to self-isolate at their homes or join quarantine centers and also encouraging severe cases to hospitalized and in th last control we use treatment with medication and psychological support. Our main objective in adopting these strategies is to minimize the number of infections caused by the different variants of Covid-19, during the time steps k = 0 to T-1 and also by minimizing the costs of implementing these strategies. In this model we include the three controls u_k, v_k and w_k which consecutively represent awareness programs through media and civil society, second control represents the encouragement of individuals infected with Covid-19 variants to join hospitals and quarantine centers or to undergo self-isolation in their their homes, the last control is devoted to treatment with drugs and psychological support, these checks also seem to be effective against Covid-19 variants followed as measures at time k. Thus, the controlled mathematical system is given by the following system of difference equations:

$$
(2)\quad \begin{cases}
\mathbf{S}_{k+1} = \Lambda + (\mathbf{1}-\mu)\mathbf{S}_k - (\beta_1 \frac{S_k E_k}{N} + \beta_2 \frac{S_k I_{1,k}}{N} + \beta_3 \frac{S_k I_{2,k}}{N} + \beta_4 \frac{S_k I_{3,k}}{N} + \beta_5 \frac{S_k I_{4,k}}{N})(\mathbf{1}-\mathbf{u}_k) \\
\mathbf{E}_{k+1} = ((\mathbf{1}-\mu-\theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\mathbf{E}_k + \beta_1 \frac{S_k E_k}{N}(\mathbf{1}-\mathbf{u}_k) \\
\mathbf{I}_{1,k+1} = (\mathbf{1}-\mu-\delta_1-\gamma_1-\lambda_1)\mathbf{I}_{1,k} + \alpha_1 \mathbf{E}_k + \beta_2 \frac{S_k I_{1,k}}{N}(\mathbf{1}-\mathbf{u}_k) - \mathbf{v}_k \mathbf{I}_{1,k} \\
\mathbf{I}_{2,k+1} = (\mathbf{1}-\mu-\delta_2-\gamma_2-\lambda_2)\mathbf{I}_{2,k} + \alpha_2 \mathbf{E}_k + \beta_3 \frac{S_k I_{2,k}}{N}(\mathbf{1}-\mathbf{u}_k) - \mathbf{v}_k \mathbf{I}_{2,k} \\
\mathbf{I}_{3,k+1} = (\mathbf{1}-\mu-\delta_3-\gamma_3-\lambda_3)\mathbf{I}_{3,k} + \alpha_3 \mathbf{E}_k + \beta_4 \frac{S_k I_{3,k}}{N}(\mathbf{1}-\mathbf{u}_k) - \mathbf{v}_k \mathbf{I}_{3,k} \\
\mathbf{I}_{4,k+1} = (\mathbf{1}-\mu-\delta_4-\gamma_4-\lambda_4)\mathbf{I}_{4,k} + \alpha_4 \mathbf{E}_k + \beta_5 \frac{S_k I_{4,k}}{N}(\mathbf{1}-\mathbf{u}_k) - \mathbf{v}_k \mathbf{I}_{4,k} \\
\mathbf{H}_{k+1} = (\mathbf{1}-\mu-\delta_5-\lambda)\mathbf{H}_k + \gamma_1 \mathbf{I}_{1,k} + \gamma_2 \mathbf{I}_{2,k} + \gamma_3 \mathbf{I}_{3,k} + \gamma_4 \mathbf{I}_{4,k} + \rho_1 \mathbf{v}_k \mathbf{I}_{1,k} \\
+\rho_2 \mathbf{v}_k \mathbf{I}_{2,k} + \rho_3 \mathbf{v}_k \mathbf{I}_{3,k} + \rho_4 \mathbf{v}_k \mathbf{I}_{4,k} - \mathbf{w}_k \mathbf{H}_k \\
\mathbf{Q}_{k+1} = (\mathbf{1}-\mu-\delta_6-\varphi)\mathbf{Q}_k + \lambda_1 \mathbf{I}_{1,k} + \lambda_2 \mathbf{I}_{2,k} + \lambda_3 \mathbf{I}_{3,k} + (\mathbf{1}-\rho_1)\mathbf{v}_k \mathbf{I}_{1,k} \\
+(\mathbf{1}-\rho_2)\mathbf{v}_k \mathbf{I}_{2,k} + (\mathbf{1}-\rho_3)\mathbf{v}_k \mathbf{I}_{3,k} + (\mathbf{1}-\rho_4)\mathbf{v}_k \mathbf{I}_{4,k} - \mathbf{w}_k \mathbf{Q}_k \\
\mathbf{R}_{k+1} = (\mathbf{1}-\mu)\mathbf{R}_k + \lambda \mathbf{H}_k + \theta \mathbf{E}_k + \varphi \mathbf{Q}_k + \mathbf{w}_k \mathbf{Q}_k + \mathbf{w}_k \mathbf{H}_k
\end{cases}
$$

Where $S_0 \geq 0, E_0 \geq 0, I_{1,0} \geq 0, I_{2,0} \geq 0, I_{3,0} \geq 0, I_{4,0} \geq 0, H_0 \geq 0, Q_0 \geq 0, \textbf{and } \mathbf{R}_0 \geq 0$, are the given initial states.

There are tree controls $u_k = (u_0, u_1, ..., u_{T-1})$, $v_k = (v_0, v_1, ..., v_{T-1})$, $w_k = (w_0, w_1, ..., w_{T-1})$, The first control can be interpreted as the proportion to adopt in awareness programs through media and civil society and the encouragement of individuals likely to be vaccinated. Thus, we note that (1-u_k) ((S_k E $_k$) / N) is the proportion of contacts between susceptible individuals and exposed individuals at time step k and also as a control between susceptible individuals and individuals infected with variants of covid-19 (1-u_k) ((S_k $I_{1,k}$) / N), (1-u_k) ((S_k $I_{2,k}$) / N), (1-u_k) ((S_k $I_{3,k}$) / N) and (1-u_k) ((S_k $I_{4,k}$) / N). The second control represents the encouragement of individuals infected with covid-19 variants to join hospitals and quarantine centers or to undergo self-isolation in their their homes to reduce the rate of spread in time step k. The third control entails the effort made by the government to provide effective treatment by drugs and psychological support to improve the morale and finally increase the immunities of the patients and recover the infected people as soon as possible. Thus, the occupancy rate of people infected with covid-19 variants in hospitals and quarantine centers will be reduced at time step k. The challenge that we face here is how to minimize the objective functional:

$$
\begin{aligned}
J(v_k, v_k, w_k) &= A_T E_T + B_T I_{1,T} + C_T I_{2,T} + + D_T I_{3,T} + F_T I_{4,T} + P_T H_T + G_T Q_T \\
&+ \sum_{k=0}^{T-1} \left(A_k E_k + B_k I_{1,k} + C_k I_{2,k} + D_k I_{3,k} + F_k I_{4,k} + P_k H_k + G_k Q_k \right) \\
&+ \sum_{k=0}^{T-1} \left(\frac{M_k}{2} u_k^2 + \frac{N_k}{2} v_k^2 + \frac{L}{2} w_k^2 \right) \qquad (3)
\end{aligned}
$$

Where the parameters $A_k > 0$, $B_k > 0$, $C_k > 0$, $D_k > 0$, $F_k > 0$, $P_k > 0, G_k >, M_k > 0, N_k > 0$, and $L_k > 0$ are the cost coefficients, they are selected to weigh the relative importance of S_k, $E_k, I_{1,k}$, $I_{1,k}, I_{2,k}, I_{3,k}, I_{4,k}$,u_k, v_k and w_k at time k. T is the final time.

In other words, we seek the optimal controls u_k, v_k and w_k such that:

$$J(u_k^*, v_k^*, w_k^*) = \min_{(u,v,w) \in U_{ad}} J(u_k, v_k, w_{,k}), \qquad (4)$$

Where U_{ad} is the set of admissible controls defined by:where Uad is the set of admissible control defined by

$$
\begin{aligned}
U_{ad} &= \{(u,v,w) : u = (u_0, u_1, ..., u_{T-1}), v = (v_0, v_1, ..., v_{T-1}), w = (w_0, w_1, ..., w_{T-1}) : \\
a_i &\leq u_{i,k} \leq b_i \; ; \; c_i \leq v_{i,k} \leq d_i ; e_i \leq w_{i,k} \leq f_i ; k = 0, 1, 2 ... T-1\} \qquad (5)
\end{aligned}
$$

The sufficient condition for the existence of optimal controls u, v and w for problems (2) and (3) come from the following theorem.

Theorem 1. *There exists the optimal controls* (u_k^*, v_k^*, w_k^*) *such that:*

$$J(u_k^*, v_{,k}^*, w_k^*) = \min_{(u,v,w)\in U_{ad}} J(u,v,w) \tag{6}$$

subject to the control system (2) with initial conditions.

Proof: Since the coefficients of the state equations are bounded and there is a finite number of time steps, $S = (S_0, S_1, ..., S_T), E = (E_0, E_1, ..., E_T), I_1 = (I_{1,0}, I_{1,1}, ..., I_{1,T}), I_2 = (I_{2,0}, I_{2,1}, ..., I_{2,T}), I_3 = (I_{3,0}, I_{3,1}, ..., I_{3,T}), I_3 = (I_{4,0}, I_{4,1}, ..., I_{4,T})$,

$H = (H_0, H_1, ..., H_T), Q = (Q_0, Q_1, ..., Q_T)$ and $R = (R_0, R_1, ..., R_T)$ are uniformly bounded for all (u,v,w) in the control set U_{ad}; thus $J(u,v,w)$ is bounded for all $(u,v,w)\,U_{ad}$.

Since $J(u,v,w)$ is bounded, $\inf J(u,v,w)$ is finite, and there exists a sequence

$(u^j; v^j; w^j) \in Uad$ such that $\lim_{j\to+\infty} J(u^j; v^j; w^j) = \inf J(u,v,w)$ and corresponding sequences of states $S^j, E^j, I_1^j, I_2^j, I_3^j, I_4^j, H^j, Q^j$ and R^j. Since there is a finite number of uniformly bounded sequences, there exist $(u^*, v^*, w^*) \in U_{ad}$ and $S^*, E^*, I_1^*, I_2^*, I_3^*, I_4^*, Q^*, H^*$and $R^* \in IR^{T+1}$ such that on a subsequence, $(u^j; v^j; w^j) \to (u^*, v^*, w^*)$, $S^j \to S^*$,$E^j \to E^*$, $I_1^j \to I_1^*$, $I_2^j \to I_2^*, I_3^j \to I_3^*, I_4^j \to I_4^*, H^j \to H^*$, $Q^j \to Q^*$and $R^j \to R^*$. Finally, due to the finite dimensional structure of system (2) and the objective function $J(u,v,w)$; (u^*, v^*, w^*) is an optimal control with corresponding states S^*, E^*, $I_1^*, I_2^*, I_3^*, I_4^*, H^*, Q^*$ and R^*. Therefore $\inf_{(u,v,w)\in U_{ad}} J(u,v,w)$ is achieved.

4. Characterisation of the Optimal Control

We apply the discrete version of Pontryagin's Maximum Principle [1], [3], [4], [5], [6], [9],[14], [15], [19]. The key idea is introducing the adjoint function to attach the system of difference equations to the objective functional resulting in the formation of a function called the Hamiltonian. This principle converts the problem of finding the control to optimize the objective functional subject to the state difference equation with initial condition to find the control to optimize the Hamiltonian pointwise (with respect to the control).

We have the Hamiltonian H_k at time step k, defined by:

$$\begin{aligned} H_k &= A_k E_k + B_k I_{1,k} + C_k I_{2,k} + D_k I_{3,k} + F_k I_{4,k} + P_k H_k + G_k Q_k \\ &\quad + \frac{M_k}{2} u_k^2 + \frac{N_k}{2} v_k^2 + \frac{L_k}{2} w_k^2 + \sum_{j=1}^{9} \lambda_{j,k+1} f_{j,k+1}, \end{aligned} \tag{7}$$

Where $f_{i,k+1}$ is the right side of the system of difference equations (2) of the i^{th} state variable at time step $k+1$.

Theorem 2. *Given the optimal controls* $(u_k^*, v_k^*, w_k^*) \in U_{ad}^3$ *and the solutions* S^*, $E^*, I_{1,j}^*, I_{2,j}^*, I_{3,j}^*$, $I_{4,j}^*, H^*, Q^*$ *and* R^* *of the corresponding state system* (2), *there exist adjoint functions* $\lambda_{1,k}$, $\lambda_{2,k}$, $\lambda_{3,k}$, $\lambda_{4,k}, \lambda_{5,k}, \lambda_{6,k}$, $\lambda_{7,k}$, $\lambda_{8,k}$, *and* , $\lambda_{9,k}$ *satisfying*

$$\begin{aligned}
\lambda_{1,k} &= \lambda_{1,k+1}((1-\mu)-(\beta_1\frac{E_k}{N}+\beta_2\frac{I_{1,k}}{N}+\beta_3\frac{I_{2,k}}{N}+\beta_4\frac{I_{3,k}}{N})+\beta_5\frac{I_{4,k}}{N})(1-u_k)) \\
&\quad +\lambda_{2,k+1}(\beta_1\frac{E_k}{N})(1-u_k)+\lambda_{3,k+1}(\beta_2\frac{I_{1,k}}{N})(1-u_k)+\lambda_{4,k+1}(\beta_3\frac{I_{2,k}}{N})(1-u_k) \\
&\quad +\lambda_{5,k+1}(\beta_4\frac{I_{3,k}}{N})(1-u_k)+\lambda_{6,k+1}(\beta_5\frac{I_{4,k}}{N})(1-u_k) \\
\lambda_{2,k} &= A_k-\lambda_{1,k+1}(\beta_1\frac{S_k}{N})(1-u_k)+\lambda_{2,k+1}((1-\mu-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\theta) \\
&\quad +\beta_1\frac{S_k}{N}(1-u_k))+\lambda_{3,k+1}\alpha_1+\lambda_{4,k+1}\alpha_2+\lambda_{5,k+1}\alpha_3+\lambda_{6,k+1}\alpha_4+\lambda_{9,k+1}\theta \\
\lambda_{3,k} &= B_k+\lambda_{1,k+1}(\beta_2\frac{S_k}{N})(1-u_k)+\lambda_{3,k+1}(1-\mu-\delta_1-\lambda_1-\gamma_1-v_k) \\
&\quad +\lambda_{7,1+k}((\gamma_1+\rho_1 v_k)+\lambda_{8,k+1}(\lambda_1+(1-\rho_1)v_{1,k}) \\
\lambda_{4,k} &= C_k+\lambda_{1,k+1}(\beta_3\frac{S_k}{N})(1-u_k)+\lambda_{4,k+1}(1-\mu-\delta_2-\lambda_2-\gamma_2-v_k) \\
&\quad +\lambda_{7,1+k}((\gamma_2+\rho_2 v_k)+\lambda_{8,k+1}(\lambda_2+(1-\rho_2)v_k) \\
\lambda_{5,k} &= D_k-\lambda_{1,k+1}(\beta_4\frac{S_k}{N})(1-u_k)+\lambda_{5,k+1}(1-\mu-\delta_3-\lambda_3-\gamma_3-v_k) \\
&\quad +\lambda_{7,1+k}((\gamma_3+\rho_3 v_k)+\lambda_{8,k+1}(\lambda_3+(1-\rho_3)v_k) \\
\lambda_{6,k} &= F_k-\lambda_{1,k+1}(\beta_5\frac{S_k}{N})(1-u_k)+\lambda_{6,k+1}(1-\mu-\delta_4-\lambda_4-\gamma_4-v_k) \\
&\quad +\lambda_{7,1+k}((\gamma_4+\rho_4 v_k)+\lambda_{8,k+1}(\lambda_4+(1-\rho_4)v_k) \\
\lambda_{7,k} &= P_k+\lambda_{7,k+1}(1-\mu-\delta_5-\lambda-w_k)-\lambda_{9,k+1}(\lambda+w_k) \\
\lambda_{8,k} &= G_k+\lambda_{8,k+1}(1-\mu-\delta_6-\varphi-w_k)+\lambda_{9,k+1}(\varphi+w_k) \\
\lambda_{9,k} &= \lambda_{9,k+1}(1-\mu) \qquad (8)
\end{aligned}$$

With the transversality conditions at time T.

$$\begin{aligned}\lambda_{1,T} &= 0, \lambda_{2,T} = A_T, \lambda_{3,T} = B_T, \lambda_{4,T} = C_T, \lambda_{5,T} = D_T, \lambda_{6,T} = F_T, \lambda_{7,T} = P_T,\\ \lambda_{8,T} &= G_T, \textit{and } \ \lambda_{9,T} = 0. \end{aligned} \tag{9}$$

Furthermore, for $k = 0, 1, 2..., T-1$ the optimal controls u_k^* ,v_k^* and w_k^* are given by:

$$\begin{aligned}
u_k^* &= \min\left[b;\max\begin{pmatrix} a, \beta_1 \frac{S_kE_k}{NM_k}\left(\lambda_{2,k+1}-\lambda_{1,k+1}\right)+\beta_2\frac{S_kI_{1,k}}{NM_k}\left(\lambda_{3,k+1}-\lambda_{1,k+1}\right)\\ +\beta_3\frac{S_kI_{2,k}}{NM_k}\left(\lambda_{4,k+1}-\lambda_{1,k+1}\right)+\beta_4\frac{S_kI_{3,k}}{NM_k}\left(\lambda_{5,k+1}-\lambda_{1,k+1}\right)\\ +\beta_5\frac{S_kI_{4,k}}{NM_k}\left(\lambda_{6,k+1}-\lambda_{1,k+1}\right)\end{pmatrix}\right]\\
v_k^* &= \min\left[d;\max\left(c, \frac{1}{N_k}\begin{bmatrix} ((\lambda_{3,k+1}-\lambda_{7,k+1}\rho_1-\lambda_{8,k+1}(1-\rho_1))I_{1,k}\\ +((\lambda_{4,k+1}-\lambda_{7,k+1}\rho_2-\lambda_{8,k+1}(1-\rho_2))I_{2,k}\\ +(\lambda_{5,k+1}-\lambda_{7,k+1}\rho_3-\lambda_{8,k+1}(1-\rho_3))I_{3,k}\\ +(\lambda_{6,k+1}-\lambda_{7,k+1}\rho_4-\lambda_{8,k+1}(1-\rho_4))I_{4,k}\end{bmatrix}\right)\right]\\
w_k^* &= \min\left[f;\max\left(e, \frac{1}{L_k}\left[\left(\lambda_{7,k+1}-\lambda_{9,k+1}\right)H_k+\left(\lambda_{8,k+1}-\lambda_{9,k+1}\right)Q_k\right]\right)\right]
\end{aligned} \tag{10}$$

Proof: The Hamiltonian at time step k is given by:

$$\begin{aligned}
H_k &= A_kE_k+B_kI_{1,k}+C_kI_{2,k}+D_kI_{3,k}+F_kI_{4,k}+P_kH_k+G_kQ_k\\
&\quad+\frac{M_k}{2}u_k^2+\frac{N_k}{2}v_k^2+\frac{L_k}{2}W_k^2+\lambda_{1,k+1}f_{1,k+1}+\lambda_{2,k+1}f_{2,k+1}+\lambda_{3,k+1}f_{3,k+1}\\
&\quad+\lambda_{4,k+1}f_{4,k+1}+\lambda_{5,k+1}f_{5,k+1}+\lambda_{6,k+1}f_{6,k+1}+\lambda_{7,k+1}f_{7,k+1}+\lambda_{8,k+1}f_{8,k+1}\\
&\quad+\lambda_{9,k+1}f_{9,k+1}\\
&= A_kE_k+B_kI_{1,k}+C_kI_{2,k}+D_kI_{3,k}+F_kI_{4,k}+P_kH_k+G_kQ_k\\
&\quad+\frac{M_k}{2}u_k^2+\frac{N_k}{2}v_k^2+\frac{L_k}{2}w_k^2\\
&\quad+\lambda_{1,k+1}\begin{bmatrix}\Lambda+(1-\mu)S_k-(\beta_1\frac{S_kE_k}{N}+\beta_2\frac{S_kI_{1,k}}{N}+\beta_3\frac{S_kI_{2,k}}{N}\\ +\beta_4\frac{S_kI_{3,k}}{N})(1-u_k)+\beta_5\frac{S_kI_{4,k}}{N})(1-u_k)\end{bmatrix}\\
&\quad+\lambda_{2,k+1}\left[(1-\mu-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\theta)E_k+\beta_1\frac{S_kE_k}{N}(1-u_k)\right]\\
&\quad+\lambda_{3,k+1}\left[(1-\mu-\delta_1-\gamma_1-\lambda_1)I_{1,k}+\alpha_1E_k+\beta_2\frac{S_kI_{1,k}}{N}(1-u_k)-v_kI_{1,k}\right]
\end{aligned} \tag{11}$$

$$+\lambda_{4,k+1}\left[(1-\mu-\delta_2-\gamma_2-\lambda_2)I_{2,k}+\alpha_2E_k+\beta_3\frac{S_kI_{2,k}}{N}(1-u_k)-v_kI_{2,k}\right]$$
$$+\lambda_{5,k+1}\left[(1-\mu-\delta_3-\gamma_3-\lambda_3)I_{3,k}+\alpha_3E_k+\beta_4\frac{S_kI_{3,k}}{N}(1-u_k)-v_kI_{3,k}\right].$$
$$+\lambda_{6,k+1}\left[(1-\mu-\delta_4-\gamma_4-\lambda_4)I_{4,k}+\alpha_4E_k+\beta_5\frac{S_kI_{4,k}}{N}(1-u_k)-v_kI_{4,k}\right]$$
$$+\lambda_{7,k+1}\left[\begin{array}{c}(1-\mu-\delta_5-\lambda)H_k+\gamma_1I_{1,k}+\gamma_2I_{2,k}+\gamma_3I_{3,k}+\gamma_4I_{4,k}\\ +\rho_1v_kI_{1,k}+\rho_2v_kI_{2,k}+\rho_3v_kI_{3,k}+\rho_4v_kI_{4,k}-w_kH_k\end{array}\right]$$
$$+\lambda_{8,k+1}\left[\begin{array}{c}(1-\mu-\delta_6-\varphi)Q_k+\lambda_1I_{1,k}+\lambda_2I_{2,k}+\lambda_3I_{3,k}\\ +\lambda_4I_{4,k}+(1-\rho_1)v_kI_{1,k}+(1-\rho_2)v_kI_{2,k}+(1-\rho_3)v_kI_{3,k}\\ +(1-\rho_4)v_kI_{4,k}-w_kQ_k\end{array}\right]$$
$$+\lambda_{9,k+1}\left[(1-\mu)R_k+\theta E_k+\lambda H_k+\varphi Q_k+w_kH_k+w_kQ_k\right]$$

For $k=0,1...T-1$ the optimal controls u_k, v_k and w_k can be solved from the optimality condition,

$$\frac{\partial H_k}{\partial u_k}=0,\frac{\partial H_k}{\partial v_k}=0,\ \text{and}\ \frac{\partial H_k}{\partial w_k}=0 \tag{12}$$

That are,

So, we have

$$\begin{aligned}u_k &= \beta_1\frac{1}{NM_k}\left(\lambda_{2,k+1}-\lambda_{1,k+1}\right)S_kE_k+\beta_2\frac{1}{NM_m}\left(\lambda_{3,k+1}-\lambda_{1,k+1}\right)S_kI_{1,k}\\ &\quad+\beta_3\frac{1}{NM_k}\left(\lambda_{4,k+1}-\lambda_{1,k+1}\right)S_kI_{2,k}+\beta_4\frac{1}{NM_k}\left(\lambda_{5,k+1}-\lambda_{1,k+1}\right)S_kI_{3,k}\\ &\quad+\beta_5\frac{1}{NM_k}\left(\lambda_{6,k+1}-\lambda_{1,k+1}\right)S_kI_{4,k}\\ v_k &= \frac{1}{N_k}\left[\begin{array}{c}((\lambda_{3,k+1}-\lambda_{7,k+1}\rho_1-\lambda_{8,k+1}(1-\rho_1))I_{1,k}\\ +((\lambda_{4,k+1}-\lambda_{7,k+1}\rho_2-\lambda_{8,k+1}(1-\rho_2))I_{2,k}\end{array}\right]+\\ &\quad\frac{1}{N_k}\left[\begin{array}{c}((\lambda_{5,k+1}-\lambda_{7,k+1}\rho_3-\lambda_{8,k+1}(1-\rho_3))I_{3,k}+\\ (\lambda_{6,k+1}-\lambda_{7,k+1}\rho_4-\lambda_{8,k+1}(1-\rho_4))I_{4,k}\end{array}\right]\\ w_k &= \frac{1}{L_k}\left[\left(\lambda_{7,k+1}-\lambda_{9,k+1}\right)H_k+\left(\lambda_{8,k+1}-\lambda_{9,k+1}\right)Q_k\right]\end{aligned} \tag{13}$$

$$\frac{\partial H_k}{\partial u_k} = Mu_k + \beta_1 \tfrac{S_k E_k}{N}\left(\lambda_{1,k+1} - \lambda_{2,k+1}\right) + \beta_2 \tfrac{S_k I_{1,k}}{N}\left(\lambda_{1k+1} - \lambda_{3,k+1}\right) +$$
$$\beta_3 \tfrac{S_k I_{2,k}}{N}\left(\lambda_{1,k+1} - \lambda_{4,k+1}\right) + \beta_4 \tfrac{S_k I_{3,k}}{N}\left(\lambda_{1,k+1} - \lambda_{5,k+1}\right) + \beta_5 \tfrac{S_k I_{4,k}}{N}\left(\lambda_{1,k+1} - \lambda_{5,k+1}\right) = 0$$
$$\frac{\partial H_k}{\partial v_k} = N_k v_k + I_{1,k}(\lambda_{7,k+1}\rho_1 + \lambda_{8,k+1}(1-\rho_1) - \lambda_{3,k+1})$$
$$+I_{2,k}(\lambda_{7,k+1}\rho_2 + \lambda_{8,k+1}(1-\rho_2) - \lambda_{4,k+1}) + I_{3,k}(\lambda_{7,k+1}\rho_3 + \lambda_{8,k+1}(1-\rho_3) - \lambda_{5,k+1})$$
$$+I_{4,k}\left(\lambda_{7,k+1}\rho_4 + \lambda_{8,k+1}(1-\rho_4) - \lambda_{6,k+1}\right) = 0$$
$$\frac{\partial H_k}{\partial w} = L_k w_k + H_k\left(\lambda_{9,k+1} - \lambda_{7,k+1}\right) + Q_k\left(\lambda_{9,k+1} - \lambda_{8,k+1}\right) = 0$$

By the bounds in U_{ad} of the controls, it is easy to obtain u_k^* , v_k^* and w_k^* in the form of (10). With the transversality conditions at time T.

$$\begin{aligned} \lambda_{1,T} &= 0, \lambda_{2,T} = A_T, \lambda_{3,T} = B_T, \lambda_{4,T} = C_T, \lambda_{5,T} = D_T, \lambda_{6,T} = F_T, \lambda_{7,T} = P_T, \\ \lambda_{8,T} &= G_T, and \;\; \lambda_{9,T} = 0. \end{aligned} \tag{14}$$

Furthermore, for $k = 0,1,2...,T-1$ the optimal controls u_k^* ,v_k^* and w_k^* are given by:

$$\begin{aligned} u_k &= \beta_1 \frac{1}{NM_k}\left(\lambda_{2,k+1} - \lambda_{1,k+1}\right) S_k E_k + \beta_2 \frac{1}{NM_m}\left(\lambda_{3,k+1} - \lambda_{1,k+1}\right) S_k I_{1,k} \\ &\quad + \beta_3 \frac{1}{NM_k}\left(\lambda_{4,k+1} - \lambda_{1,k+1}\right) S_k I_{2,k} + \beta_4 \frac{1}{NM_k}\left(\lambda_{5,k+1} - \lambda_{1,k+1}\right) S_k I_{3,k} \\ &\quad + \beta_5 \frac{1}{NM_k}\left(\lambda_{6,k+1} - \lambda_{1,k+1}\right) S_k I_{4,k} \\ v_k &= \frac{1}{N_k}\left[\begin{array}{c} ((\lambda_{3,k+1} - \lambda_{7,k+1}\rho_1 - \lambda_{8,k+1}(1-\rho_1))I_{1,k} \\ +((\lambda_{4,k+1} - \lambda_{7,k+1}\rho_2 - \lambda_{8,k+1}(1-\rho_2))I_{2,k} \end{array}\right] + \\ &\quad \frac{1}{N_k}\left[\begin{array}{c} ((\lambda_{5,k+1} - \lambda_{7,k+1}\rho_3 - \lambda_{8,k+1}(1-\rho_3))I_{3,k} + \\ (\lambda_{6,k+1} - \lambda_{7,k+1}\rho_4 - \lambda_{8,k+1}(1-\rho_4))I_{4,k} \end{array}\right] \\ w_k &= \frac{1}{L_k}\left[\left(\lambda_{7,k+1} - \lambda_{9,k+1}\right) H_k + \left(\lambda_{8,k+1} - \lambda_{9,k+1}\right) Q_k\right] \end{aligned} \tag{15}$$

$$\frac{\partial H_k}{\partial u_k} = Mu_k + \beta_1 \tfrac{S_k E_k}{N}\left(\lambda_{1,k+1} - \lambda_{2,k+1}\right) + \beta_2 \tfrac{S_k I_{1,k}}{N}\left(\lambda_{1k+1} - \lambda_{3,k+1}\right) +$$
$$\beta_3 \tfrac{S_k I_{2,k}}{N}\left(\lambda_{1,k+1} - \lambda_{4,k+1}\right) + \beta_4 \tfrac{S_k I_{3,k}}{N}\left(\lambda_{1,k+1} - \lambda_{5,k+1}\right) + \beta_5 \tfrac{S_k I_{4,k}}{N}\left(\lambda_{1,k+1} - \lambda_{5,k+1}\right) = 0$$
$$\frac{\partial H_k}{\partial v_k} = N_k v_k + I_{1,k}(\lambda_{7,k+1}\rho_1 + \lambda_{8,k+1}(1-\rho_1) - \lambda_{3,k+1})$$
$$+I_{2,k}(\lambda_{7,k+1}\rho_2 + \lambda_{8,k+1}(1-\rho_2) - \lambda_{4,k+1}) + I_{3,k}(\lambda_{7,k+1}\rho_3 + \lambda_{8,k+1}(1-\rho_3) - \lambda_{5,k+1})$$
$$+I_{4,k}\left(\lambda_{7,k+1}\rho_4 + \lambda_{8,k+1}(1-\rho_4) - \lambda_{6,k+1}\right) = 0$$

$$\frac{\partial H_k}{\partial w} = L_k w_k + H_k \left(\lambda_{9,k+1} - \lambda_{7,k+1}\right) + Q_k \left(\lambda_{9,k+1} - \lambda_{8,k+1}\right) = 0$$

By the bounds in U_{ad} of the controls, it is easy to obtain u_k^* , v_k^* and w_k^* in the form of (10).

5. Simulation

In this section, we present the results obtained by solving numerically the optimality system. This system consists of the state system, adjoint system, initial and final time conditions and the controls characterization.

In this formulation, there were initial conditions for the state variables and terminal conditions for the adjoints. That is, the optimality system is a two-point boundary value problem with separated boundary conditions at time steps $k = 0$ and $k = T$. We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the controls at the first iteration and then before the next iteration we update the controls by using the characterization. We continue until convergence of successive iterates is achieved.

6. Discussion

In this section, we study and analyze numerically the effects of the optimal control strategies such as media awareness programs and education, limiting contact between infected and susceptible individuals, encouraging infected individuals with variants of Covid-19 to go to quarantine centers or at home and to take severe cases to hospitals,and providing medical treatment and psychological support to increase the immune system of those infected. The numerical solution of the model (2) is executed using Matlab with the following parameter values and the initial values of the state variable in table (1).

Table (1): The parameters used for the model (1).

S_0	E_0	$I_{1,0}$	$I_{2,0}$	$I_{3,0}$	$I_{4,0}$	Q_0	H_0	R_0	Λ	α_1	α_2
5.10^3	3.10^3	$1,5.10^3$	1.10^3	2.10^3	2.10^3	1.10^3	1.10^3	1.10^3	5.10^2	0.45	0.5
α_3	α_4	β_1	β_2	β_3	β_4	β_5	λ_1	λ_2	λ_3	λ_4	λ
0.04	0.025	0.75	0.45	0.35	0.45	0.85	0.035	0.035	0.045	0.65	0.015
γ_1	γ_2	γ_3	γ_4	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	φ	θ
0.035	0.035	0.02	0.05	0.03	0.025	0.03	0.03	0.03	0.07	0.25	0.25

The proposed control strategies in this work help to achieve several objectives:

6.0.1. *Objective A: Protecting individuals against the spread of Covid-19 variants, we must limit and reduce contact between exposed individuals and encourage individuals who are likely to be vaccinated.*

Due to the importance of awareness programs through media and other means for those at risk not to come into contact to limit the spread of Covid-19 variants, we propose an optimal strategy in wich we activate the optimal control variable u that represents the awareness programs of exposed individuals so as not to propagate the Covid-19 variants and also to limit the probable infected cases. Figure (3) represents the evolution of exposed individuals without optimal control u and the evolution of exposed individuals with optimal control u, in which the effect of awareness programs offered by media was positve in reducting the number of individuals exposed to Covid-19 variants and prevent exposed individuals from coming into contact with susceptible individuals.

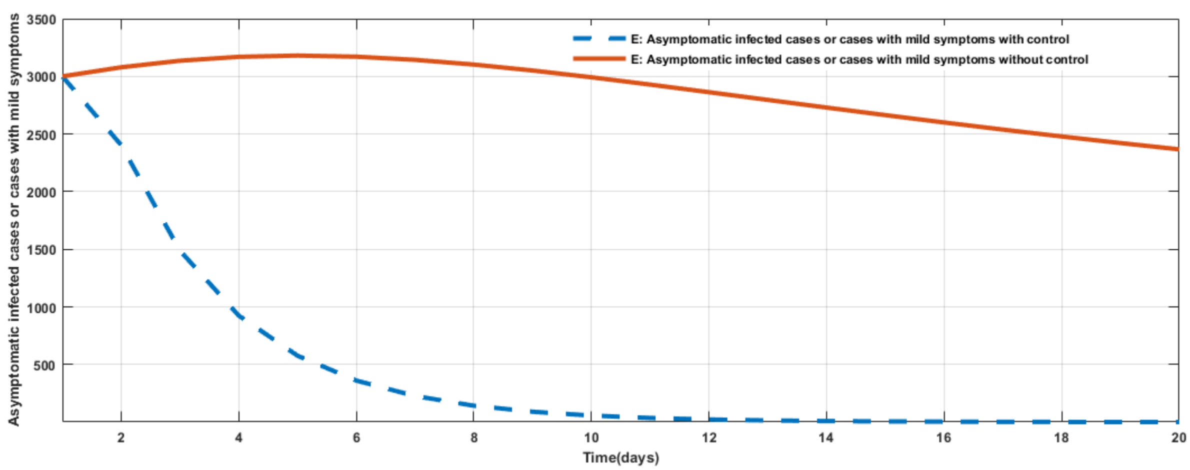

Figure 3: represents the evolution of exposed individuals with and without optimal control.

From the pace, we observe that the number of exposed individuals decreases in a significant way when using the control u on the other hand the number of the exposed individuals increase in the case where there is no control; this explains the importance of control use.

6.0.2. *Objective B : Isolation and hospitalization of individuals infected with variants of covid-19.*

When the number of individuals infected with the Covid-19 variants is high, it is mandatory to use certain strategies such as encouragement and awareness of individuals infected with the covid-19 variants to isolate themselves at their homes or in quarantine centers as well as encourage severe cases to be hospitalized to limit the spread of Covid-19 variants and reduce the number of infected individuals. For this, we propose an optimal strategy using the optimal control v, whose objective is to encourage infected individuals to isolate themselves at home or in specialized quarantine centers and to sensitize severe cases to go hospitals once the optimal control is executed, has been established when there is a significant decrease in Covid-19 variants - compared to a situation when there is no control yet as shown in figure 4(a); 4(b);4(c) and 4(d).

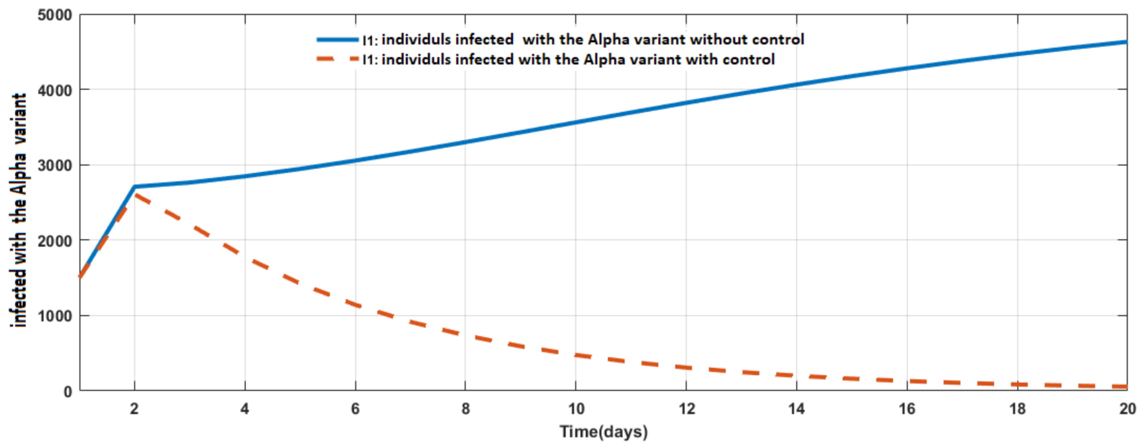

Figure 4 (a): Represents the evolution of individuals infected with the Alpha variant with and without optimal control.

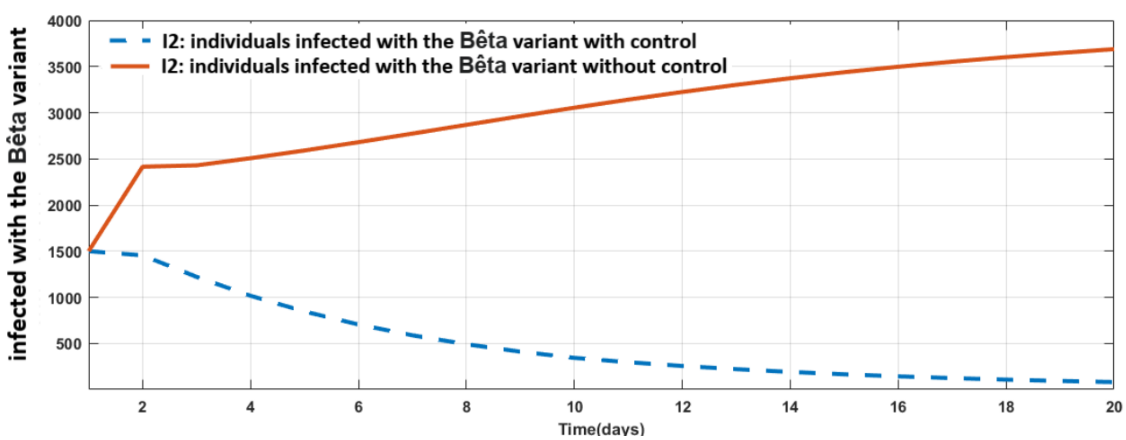

Figure 4 (b): Represents the evolution of individuals infected with the Bêta variant with and without optimal control.

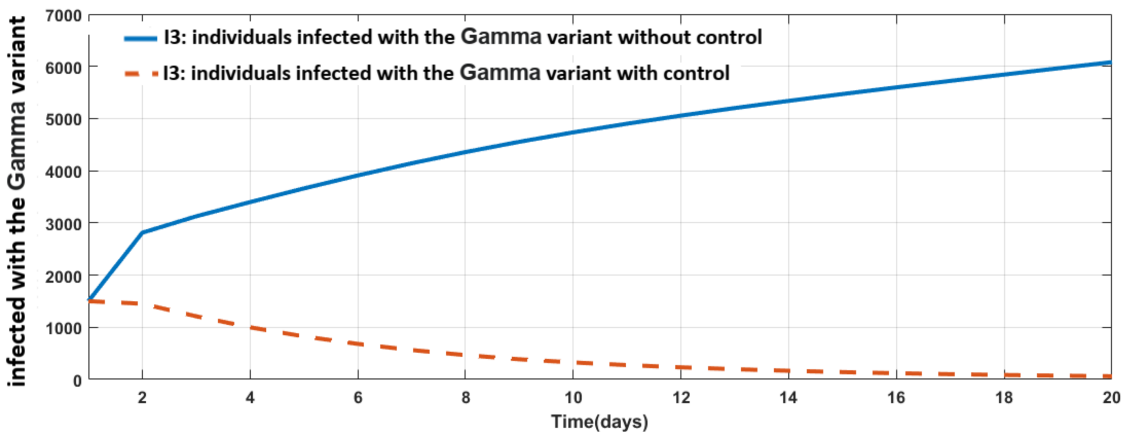

Figure 4 (c): Represents the evolution of individuals infected with the Gamma variant with and without optimal control.

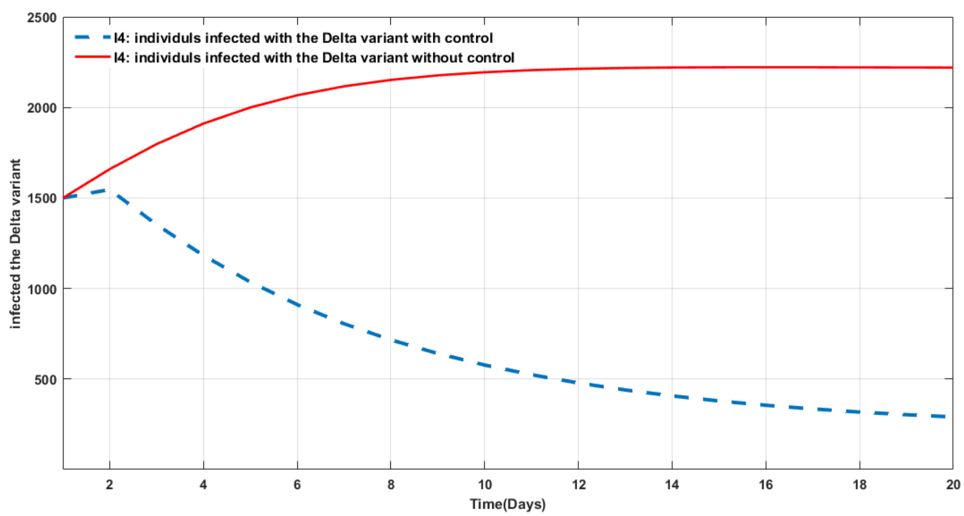

Figure 4 (d): Represents the evolution of individuals infected with the Delta variant with and without optimal control.

6.0.3. *Objective C: Providing follow-up with psychological support and medical treatment to increase the immunity of individuals infected with variants of covid-19.*

To increase the immunity of individuals infected with the different variants of covid-19 and reduce their occupancy rate in quarantine centers and hospitals, we use the control strategy w which consists of treating the number of individuals infected with the different variants of covid-19 found in hospitals and quarantine centers either with drug treatment or psychological support. In figure 5 (a) and figure 5 (b), there is a very significant decrease in the number of hospitalized infected individuals who are in isolation centers. So, this control has a positive effect on reducing of the number of infected individuals compared to the case where there is no control.

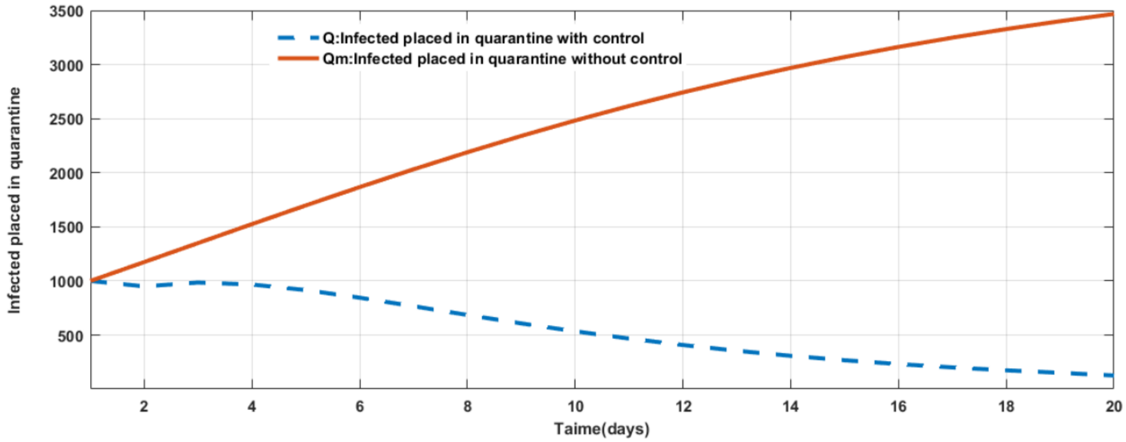

Figure 5 (a): Represents the evolution of individuals infected with the different variants who are isolated with and without optimal control.

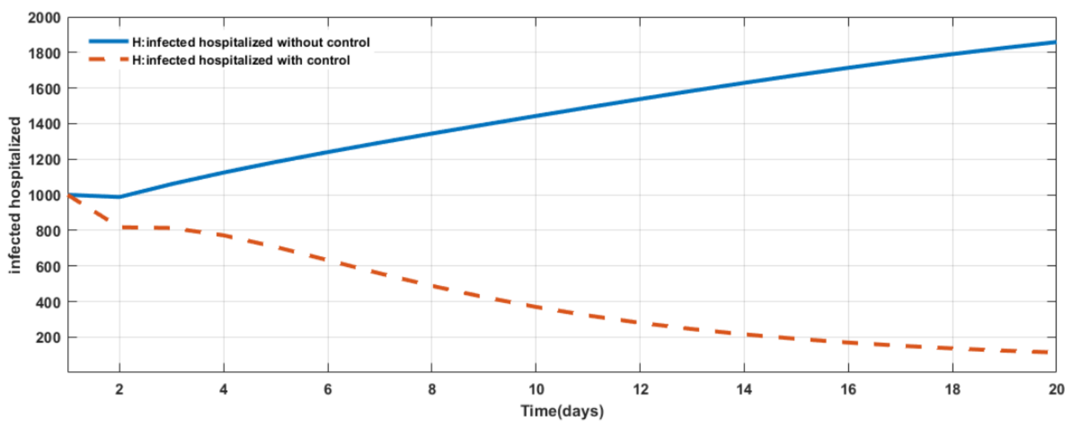

Figure 5 (b): Represents the evolution of individuals infected with the different variants who are hospitalized with and without optimal control.

6.0.4. *The optimal strategy used has a positive effect on increasing the number of the recovered individuals.*

The optimal strategy used for the optimal control had a positive effect which makes it possible to increase the number of recovered individuals. Initially, the number of the recovered individuals

without control is lower compared to the individuals provided with follow-up, psychological treatment and medication. The proposed strategy has an effective impact on increasing the immunity of the infected individuals and it also makes it possible to considerably increase the number of recovered individuals. Figure 6 (a) shows the evolution of the recovered individuals with and without optimal control.

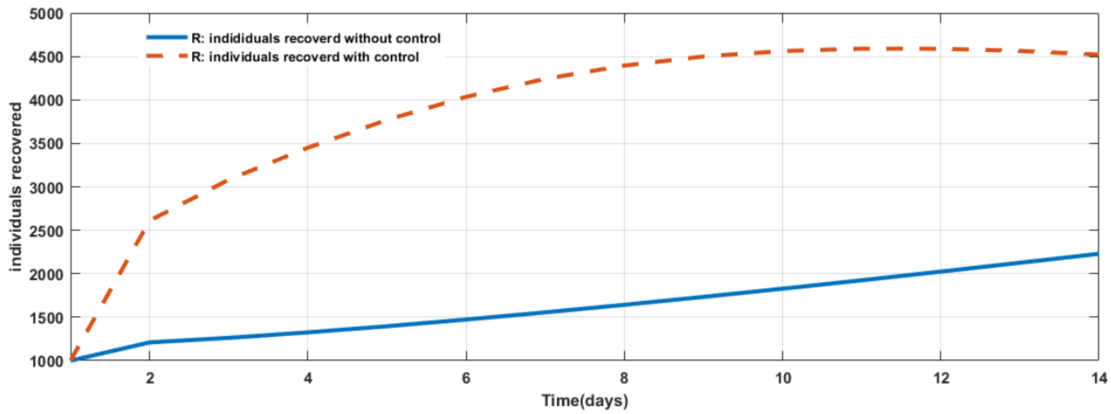

Figure 6: represents the evolution of the recovered individuals with and without optimal control.

Note:

Several optimal controls can be combined to achieve other objectives and implement other strategies depending on the phenomenon and the particularity of each society.

The optimal strategy used has a positive effect on increasing the number of recovered individuals.

7. Conclusion

In this article, we have introduced a discrete modeling of individuals infected with different covi-19 variants, in order to minimize the number of exposed individuals, individuals infected with different covid-19 variants. We have also introduced three controls which represent respectively awareness programs by the media and civil society to limit contact with susceptible individuals and individuals infected with the variants of covid-19, the encouragement of individuals infected with the variants of covid -19 to do isolation in quarantines and hospital hospitals, treatment with drugs and follow-up psychological support We applied the results of the control theory and we succeeded in obtaining the characterizations of the optimal controls. Numerical simulation of the results obtained showed the effectiveness of the proposed control strategies.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] O. Balatif, A. Labzai, M. Rachik, A discrete mathematical modeling and optimal control of the electoral behavior with regard to a political party, Discrete Dyn. Nat. Soc. 2018 (2018), 9649014.

[2] A.M. Ramos, M. Vela-Pérez, M.R. Ferrández, A.B. Kubik, B. Ivorra, Modeling the impact of SARS-CoV-2 variants and vaccines on the spread of COVID-19, Commun. Nonlinear Sci. Numer. Simul. 102 (2021), 105937.

[3] D. Kada, A. Labzai, O. Balatif, M. Rachik, E.H. Labriji, Spread of COVID-19 in morocco discrete mathematical modeling: optimal control strategies and cost-effectiveness analysis, J. Math. Comput. Sci. 10 (2020), 2070-2093.

[4] Z. Rachik, A. Labzai, O. Balatif, M. Rachik, A multi-region discrete-time mathematical modeling and optimal control of an electoral behavior, J. Math. Comput. Sci. 10 (2020), 2579-2598.

[5] A. Labzai, A. Kouidere, O. Balatif, M. Rachik, Stability analysis of mathematical model new corona virus (COVID-19) disease spread in population, Commun. Math. Biol. Neurosci. 2020 (2020), Article ID 41.

[6] A. Labzai, B. Khajji, O. Balatif, M. Rachik, Discrete mathematical modeling of the propagation of religious ideas: Optimal control approach, Commun. Math. Biol. Neurosci. 2020 (2020), Article ID 21.

[7] V. Guibout, A. Bloch, A discrete maximum principle for solving optimal control problems, in: 2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No.04CH37601), IEEE, Nassau, Bahamas, 2004: pp. 1806-1811.

[8] World Health Organization, Global health sector strategy on HIV 2016–2021.
http://apps.who.int/iris/bitstream/10665/246178/1/WHO-HIV-2016.05-eng.pdf

[9] A. Labzai, A. Kouidere, B. Khajji, O. Balatif, M. Rachik, Mathematical modeling and optimal control strategy for a discrete time drug consumption model, Discrete Dyn. Nat. Soc. 2020 (2020), 5671493.

[10] C.L. Hwang, L.T. Fan, A discrete version of Pontryagin's maximum principle, Oper. Res. 15 (1967), 139–146.

[11] A. Labzai, O. Balatif, M. Rachik, Optimal control strategy for a discrete time smoking model with specific saturated incidence rate, Discrete Dyn. Nat. Soc. 2018 (2018), 5949303.

[12] A. Lahrouz, L. Omari, D. Kiouach, A. Belmaâti, Deterministic and stochastic stability of a mathematical model of smoking, Stat. Probab. Lett. 81 (2011), 1276–1284.

[13] S. Mushayabasa, G. Tapedzesa, Modeling illicit drug use dynamics and its optimal control analysis, Comput. Math. Methods Med. 2015 (2015), 383154.

[14] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The mathematical theory of optimal processes, Wiley, New York, 1962.

[15] M.D. Rafal, W.F. Stevens, Discrete dynamic optimization applied to on-line optimal control, AIChE J. 14 (1968), 85–91.

[16] N.G. Davies, S. Abbott, R.C. Barnard, et al. Estimated transmissibility and impact of SARS-CoV-2 lineage B.1.1.7 in England, Science. 372 (2021), eabg3055.

[17] W. Ding, R. Hendon, B. Cathey, E. Lancaster, R. Germick, Discrete time optimal control applied to pest control problems, Involve. 7 (2014), 479–489.

[18] G. Gonzalez-Parra, D. Martínez-Rodríguez, R. Villanueva-Micó, Impact of a new SARS-CoV-2 variant on the population: A mathematical modeling approach, Math. Comput. Appl. 26 (2021), 25.

[19] D. Zhang, B. Shi, Oscillation and global asymptotic stability in a discrete epidemic model, J. Math. Anal. Appl. 278 (2003), 194-202.

[20] G. Giordano, M. Colaneri, A. Di Filippo, et al. Modeling vaccination rollouts, SARS-CoV-2 variants and the requirement for non-pharmaceutical interventions in Italy, Nat Med. 27 (2021), 993–998.

[21] F. Ndaïrou, I. Area, J.J. Nieto, D.F.M. Torres, Mathematical modeling of COVID-19 transmission dynamics with a case study of Wuhan, Chaos Solitons Fractals. 135 (2020), 109846.

[22] World Health Organization, COVID-19 weekly epidemiological update, edition 45, 22 June 2021, (2021).