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A CAREFUL STUDY OF THE EFFECT OF THE INFECTIOUS DISEASES AND REFUGE ON THE DYNAMICAL BEHAVIOR OF PREY-SCAVENGER MODELING

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Abstract: In this paper, the dynamics of scavenger species predation of both susceptible and infected prey at different rates with prey refuge is mathematically proposed and studied. It is supposed that the disease was spread by direct contact between susceptible prey with infected prey described by Holling type-II infection function. The existence, uniqueness, and boundedness of the solution are investigated. The stability constraints of all equilibrium points are determined. In addition to establishing some sufficient conditions for global stability of them by using suitable Lyapunov functions. Finally, these theoretical results are shown and verified with numerical simulations.

Keywords: prey-scavenger model; SIS epidemics disease; prey refuge; nonlinear incidence rate.

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1. INTRODUCTION

The terms mathematical model is one of the most important subjects for study and has a wider scope. The first mathematical model in the field of ecology that involves the interactions between

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biological species was modeled and studied, by Lotka and Volterra in the middle of 1920. On the other hand, the mathematical model in the field of an epidemic which describes the spread of disease from susceptible to infected and then to removal individuals has been formulated by Kermack and McKendric in 1927. Recently, many articles dealing with scavenge population in ecology model, scavenger represented an animal that consumed carcasses of other animals those which are dead naturally or killed by other animals. Different types of prey-predator and/or scavenge models including different biological factors were proposed and studied. Researchers successfully introduce how much exists of scavenger population effected on prey-predator and/or harvest model and studied the behavior of these system with different functional response, see [1-6] and the reference cited therein. On the other hand, refuge which is defined as the place that provides shelter or protection, as well as stage structure, of one or more species have been a wider subject to study see [7-16]. It's well-known that in nature no species can survive alone; and the species not only spreads the disease but also competes with other species for space or food or is predated by other species. In most previous studies, the prey interacts with predator and/or scavenger with effects of infectious disease on this model have become problems of major subject for study by many researchers. Recently, Abdul Satar and Naji [17] suggested and studied ecological model consisting of prey, predator, and scavenger involving toxicant and harvesting. While, Marwah and Hassan [18] proposed and analyzed prey-predator-scavenger model contented migration and spreading infectious disease.

Based on the above discussion, we formulated a three-dimensional system for the prey-scavenger model (where infectious disease SIS spread among prey population). Positivity and boundedness of all solutions of the proposed model are discussed along with both local and global stability as well as, the persistence conditions at each equilibrium point are investigated. Finally, to verify the analytic results we solve the model by numerical simulation for different values of parameters and represented them graphically.

2. THE MATHEMATICAL MODEL FORMULATION

In this section an eco-epidemiological model consisting of a prey-scavenger model incorporating prey refuge with infectious disease in the prey is proposed for study. In order to construct our model the following hypotheses considered:

1. In the absence of disease, the prey population grows logistically with carrying capacity K and intrinsic birth rate r.
2. In the existence of SIS infectious disease, the prey population is divided into two groups, namely susceptible prey denoted by $S(t)$ and infected prey denoted by $I(t)$. Therefore at time t, the total population is $\mathcal{N}(t) = S(t) + I(t)$.
3. Disease spreads among the prey population and it transmitted between the prey individuals (but not the scavenger) by contact, according to Holling type-II infection function with maximum incidence rate β and half saturation constant α. Further the disease disappears and the infected prey becomes susceptible prey again at a recover rate b_1.
4. The susceptible prey is capable of reproducing only and the infected prey is removed by death at a natural rate d_1.
5. The susceptible prey species are assumed to take a refuge. That is $(1 - \varepsilon)S$, ε is a prey refuge constant, of the susceptible prey is available for feeding by scavenger.
6. According to nature of scavenger, we assume scavenger feeds upon susceptible prey killed by other animals or dead naturally according to ratio-dependent functional response with maximum attack rate b and half saturation constant m or linearly with maximum attack rate γ_1, respectively. The consumed susceptible prey, which killed by other animals, is converted into scavenger with efficiency e_1. Also, we assume scavenger feeds upon infected prey killed by other animals or dead naturally by linear functional response with maximum attack rates c or γ_2, respectively. The consumed infected prey, which killed by other animals, is converted into scavenger with efficiency e_2.
7. Finally in the absence of the prey the scavenger decay exponentially with natural death rate d_2 and intra-specific competition rate γ_3.

According to the above assumptions the prey-scavenger model (1) can be modified to the following set of differential equations.

$$\begin{aligned}\frac{dS}{dt} &= rS\left(1-\frac{S+I}{K}\right)-\frac{\beta SI}{(1+\alpha I)}-\frac{b(1-\varepsilon)SP}{(mP+S)}+b_1 I\\ \frac{dI}{dt} &= \frac{\beta SI}{(1+\alpha I)}-b_1 I-d_1 I-cIP\\ \frac{dP}{dt} &= \frac{e_1 b(1-\varepsilon)SP}{(mP+S)}+e_2 cIP+\gamma_1(1-\varepsilon)SP+\gamma_2 IP-\gamma_3 P^2-d_2 P\end{aligned} \quad (1)$$

3. BOUNDEDNESS OF THE MODEL

Theorem (1): All the solutions of system (1), which initiate in $\mathfrak{R}^3_+$ are uniformly bounded provided that the following condition holds

$$d_1 d_2 > (r+d_1)K\mathcal{M} \quad \text{where} \quad \mathcal{M} = \max\{\gamma_1(1-\mathcal{E});\ \gamma_2\}.$$

Proof: note that the prey population is $\mathcal{N}(t) = S(t)+I(t)$, so when $I=0$ the first equation of system (1) can be rewritten as: $\frac{dS}{dt} \le rS(1-\frac{S}{K})$

The right handside must be positive that implies $rS(1-\frac{S}{K}) > 0$.

Since $S>0$ then $S(t) \le K$

when $I \ne 0$, consider the function $\mathcal{N}(t) = S(t)+I(t)$ and the derivative with respect time is:

$$\frac{d\mathcal{N}}{dt} = (r+d_1)S-\frac{rS^2}{K}-\frac{rSI}{K}-d_1(S+I) \le (r+d_1)K-d_1\mathcal{N}(t)$$

Hence, by using Gronwall lemma we get

$$\mathcal{N}(t) \le \mathcal{N}(0)e^{-d_1 t}+\frac{(r+d_1)K}{d_1}[1-e^{-d_1 t}]$$

So, as $t\to\infty$ then $\mathcal{N}(t) \le \delta$ where $\delta = \frac{(r+d_1)K}{d_1}$.

Now, define the function $\mathcal{W}(t) = S(t)+I(t)+P(t)$ then

$$\frac{d\mathcal{W}}{dt} = \frac{dS}{dt}+\frac{dI}{dt}+\frac{dP}{dt}$$

$$= rS-\frac{rS^2}{K}-\frac{rSI}{K}-\frac{b(1-\varepsilon)(1-e_1)SP}{(mp+S)}-d_1 I-cIP(1-e_2)$$
$$+\gamma_1(1-\varepsilon)SP+\gamma_2 IP-\gamma_3 P^2-d_2 P$$

$$\le (r+1)S-S-d_1 I-d_2 P+[\gamma_1(1-\varepsilon)S+\gamma_2 I]P$$

$\frac{d\mathcal{W}}{dt} \leq (r+1)S - S - d_1 I - d_2 P + \mathcal{M}P(S+I)$

$\leq (r+1)S - S - d_1 I - (d_2 - \mathcal{M}\delta)P$

then $\frac{d\mathcal{W}}{dt} \leq (r+1)K - Q\mathcal{W}$ where $Q = \min\{1; d_1; (d_2 - \mathcal{M}\delta)\}$

So, by using Gronwall, it's obtained that: $\mathcal{W}(t) \leq \mathcal{W}(0)e^{-Qt} + \frac{(r+1)K}{Q}[1 - e^{-Qt}]$

Hence, $\lim_{n\to\infty} sup\, \mathcal{W}(t) \leq \frac{(r+1)K}{Q}$ that is independent of the initial conditions.

4. EXISTENCE OF EQUILIBRIUM POINTS

The system (1) has at most five non negative equilibrium points, namely $E_i = (S_i, I_i, P_i)$ where $i = 0, \cdots, 4$. The existence conditions for each of these equilibrium points are established in the following:

1. The vanishing equilibrium point $E_0 = (0,0,0)$ always exists.
2. The axial equilibrium point $E_1 = (S_1, 0,0)$ where $S_1 = K$ always exists.
3. The scavenger free equilibrium point $E_2 = (S_2, I_2, 0)$ when I_2 is the positive root of the following quadratic equation and $\beta S_2 = (b_1 + d_1)(1 + \alpha I_2)$

$$\mathcal{A}_1 I_2^2 + \mathcal{A}_2 I_2 + \mathcal{A}_3 = 0 \tag{2}$$

Where:

$\mathcal{A}_1 = -\alpha r \varphi_1(\alpha(b_1 + d_1) + \beta)$

$\mathcal{A}_2 = r\alpha\varphi_1(\beta K - 2\varphi_1) - \beta\big(d_1\beta K - r(b_1 + d_1)\big)$

$\mathcal{A}_3 = r\varphi_1\big(\beta K - (b_1 + d_1)\big)$

Obviously, E_2 exists uniquely in the int.$\Re_+^3$ if and only if $\beta K > (b_1 + d_1)$. (3)

4. The disease free equilibrium point $E_3 = (S_3, 0, P_3)$ where $P_3 = \frac{rS_3(K - S_3)}{K[b(1-\varepsilon) - rm] + rmS_3}$, while S_3 represents a positive root of the following quadratic equation:

$$h_1 S_3^2 + h_2 S_3 + h_3 = 0 \tag{4}$$

Here

$h_1 = r[re_1 m^2 + \gamma_1 mK + \gamma_3 K]$

$$h_2 = (b(1-\varepsilon) - rm)[2e_1 rmK - \gamma_1 K^2] - rK(md_2 + \gamma_3 K)$$

$$h_3 = K^2(b(1-\varepsilon) - rm)[e_1(b(1-\varepsilon) - rm) - d_2]$$

Clearly, E_3 exists uniquely in the int.$\Re^3_+$, provided that the following condition holds

$$rm < b(1-\varepsilon) < \left(rm + \frac{d_2}{e_1}\right) \tag{5}$$

5. The positive equilibrium point $E_4 = (S_4, I_4, P_4)$ where $P_4 = \frac{\beta S_4 - (b_1+d_1)(1+\alpha I_4)}{c(1+\alpha I_4)}$ and (S_4, I_4) represents a positive intersection point of the following two isoclines:

$$\begin{aligned} f(S,I) = {} & R_1 SI^3 + R_2 I^3 + R_3 S^2 I^2 + R_4 SI^2 + R_5 I^2 \\ & + R_6 S^2 I + R_7 SI + R_8 I + R_9 S^2 + R_{10} S + R_{11} \end{aligned} \tag{6.a}$$

$$\begin{aligned} g(S,I) = {} & J_1 S^2 I^3 + J_2 SI^3 + J_3 I^3 + J_4 S^3 I^2 + J_5 S^2 I^2 + J_6 SI^2 + J_7 I^2 \\ & + J_8 S^3 I + J_9 S^2 I + J_{10} SI + J_{11} I + J_{12} S^3 + J_{13} S^2 + J_{14} S \end{aligned} \tag{6.b}$$

Where:

$R_1 = \alpha^2 c^2 (e_2 c + \gamma_2) > 0;$

$R_2 = -\alpha^2 mc(e_2 c + \gamma_2)(b_1 + d_1) < 0;$

$R_3 = \gamma_1 \alpha^2 c^2 (1-\varepsilon) > 0;$

$$R_4 = \begin{matrix} \alpha c[(e_2 c + \gamma_2)(2c + m\beta) + e_1 \alpha cb(1-\varepsilon) \\ -\alpha cd_2 + \alpha(b_1 + d_1)(\gamma_3 - m\gamma_1(1-\varepsilon))] \end{matrix};$$

$R_5 = \alpha m(b_1 + d_1)\left[\alpha\left(cd_2 - \gamma_3(b_1 + d_1)\right) - 2c(e_2 c + \gamma_2)\right];$

$R_6 = \gamma_1 \alpha c(1-\varepsilon)[m\beta + 2c] - \alpha c\beta\gamma_3;$

$$R_7 = \begin{matrix} 2\alpha(b_1 + d_1)[m\gamma_3\beta + c\gamma_3 - mc\gamma_1(1-\varepsilon)] - \alpha mc\beta d_2 \\ +c(c + m\beta)(e_2 c + \gamma_2) + 2\alpha c^2(e_1 b(1-\varepsilon) - d_2) \end{matrix};$$

$R_8 = m(b_1 + d_1)\left[2\alpha\left(cd_2 - \gamma_3(b_1 + d_1)\right) - c(e_2 c + \gamma_2)\right];$

$R_9 = [c\gamma_1(1-\varepsilon) - \gamma_3\beta](c + m\beta);$

$$R_{10} = \begin{matrix} (b_1 + d_1)[\gamma_3(c + 2m\beta) - mc\gamma_1(1-\varepsilon)] \\ +c^2(e_1 b(1-\varepsilon) - d_2) - mc\beta d_2 \end{matrix};$$

$R_{11} = m(b_1 + d_1)\left(cd_2 - \gamma_3(b_1 + d_1)\right);$

$J_1 = -rc\alpha^2 < 0;$

$J_2 = \alpha^2[cKb_1 + mr(b_1 + d_1)] > 0;$

$J_3 = \alpha^2 mKb_1(b_1 + d_1) > 0;$

$J_4 = -\alpha^2 rc < 0;$

$J_5 = \alpha^2 r[cK + m(b_1 + d_1)] - \alpha[r(m\beta + 2c) + c\beta K];$

$J_6 = \alpha(b_1 + d_1)[rm(2 - \alpha K) + K(\alpha b + \beta m)] + \alpha K b_1(2c + m\beta);$

$J_7 = -2\alpha m K b_1(b_1 + d_1) < 0;$

$J_8 = -r\alpha(2c + m\beta) < 0;$

$$J_9 = \begin{array}{l} r(c + m\beta)(\alpha K - 1) + 2\alpha rm(b_1 + d_1) \\ +K\big[r\alpha c - \beta\big(m\beta + c + \alpha b(1 - \varepsilon)\big)\big] \end{array};$$

$J_{10} = \varphi_1[rm + K\alpha b + m\beta K + K\alpha b(1 - \varepsilon) - 2r\alpha m K] + K b_1(c + m\beta);$

$J_{11} = -mKb_1(b_1 + d_1) < 0;$

$J_{12} = -r(c + m\beta) < 0;$

$J_{13} = rK(c + m\beta) + rm(b_1 + d_1) - Kb\beta(1 - \varepsilon);$

$J_{14} = K(b_1 + d_1)[b(1 - \varepsilon) - rm]$

clearly as $I \to 0$ and due to descarte rule the isocline (6.a) has a unique positive root, say S_1^*, if the following conditions hold

$$\left.\begin{array}{l} R_9 > 0 \text{ and } R_{11} < 0 \\ \text{or} \\ R_9 < 0 \text{ and } R_{11} > 0 \end{array}\right\} \quad (7.a)$$

Moreover as $I \to 0$ the isocline (6.b) has a unique positive root, say S_2^*, if the following condition holds

$$J_{14} > 0 \quad (7.b)$$

Consequently, these two isoclines (6.1) and (6.2) have an intersection point in the int.$\Re_+^2$, namely $(S_4.I_4)$, provided that the following conditions are satisfied:

$$S_1^* < S_2^*; \quad (7.c)$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial S} < 0 \text{ and } \frac{\partial f}{\partial I} > 0 \\ \text{or} \\ \frac{\partial f}{\partial S} > 0 \text{ and } \frac{\partial f}{\partial I} < 0 \end{array}\right\} \quad (7.d)$$

$$\left.\begin{array}{l} \frac{\partial g}{\partial S} > 0 \text{ and } \frac{\partial g}{\partial I} > 0 \\ \text{or} \\ \frac{\partial g}{\partial S} < 0 \text{ and } \frac{\partial g}{\partial I} < 0 \end{array}\right\} \quad (7.e)$$

Therefore, the positive equilibrium point E_4 exists uniquely in the int.$\Re_+^3$ if in addition to above conditions (7.a)-(7.e) the following conditions are satisfied too:

$$\beta S_4 > (b_1 + d_1)(1 + \alpha I_4) \quad (7.f)$$

5. Stability of the Model

At equilibrium points E_i ; $i = 1, \cdots, 4$ the Jacobian matrix of the system (1) is:

$$J_i = \begin{pmatrix} \sigma_{11}^{[i]} & \sigma_{12}^{[i]} & \sigma_{13}^{[i]} \\ \sigma_{21}^{[i]} & \sigma_{22}^{[i]} & \sigma_{23}^{[i]} \\ \sigma_{31}^{[i]} & \sigma_{32}^{[i]} & \sigma_{33}^{[i]} \end{pmatrix}$$

Here:

$$\sigma_{11}^{[i]} = r - \frac{r(2S_i + I_i)}{K} - \frac{\beta I_i}{(1 + \alpha I_i)} - \frac{mb(1-\varepsilon)P_i^2}{(mP_i + S_i)^2} ;$$

$$\sigma_{12}^{[i]} = \frac{-rS_i}{K} - \frac{\beta S_i}{(1 + \alpha I_i)^2} + b_1 ;$$

$$\sigma_{13}^{[i]} = \frac{-b(1-\varepsilon)S_i^2}{(mP_i + S_i)^2} ;$$

$$\sigma_{21}^{[i]} = \frac{\beta I_i}{(1 + \alpha I_i)} ;$$

$$\sigma_{22}^{[i]} = \frac{\beta S_i}{(1 + \alpha I_i)^2} - b_1 - d_1 - cP_i ;$$

$$\sigma_{23}^{[i]} = -cI_i ;$$

$$\sigma_{31}^{[i]} = \frac{e_1 mb(1-\varepsilon)P_i^2}{(mP_i + S_i)^2} + \gamma_1(1-\varepsilon)P_i ;$$

$$\sigma_{32}^{[i]} = e_2 cP_i + \gamma_2 P_i ;$$

$$\sigma_{33}^{[i]} = \frac{e_1 b(1-\varepsilon)S_i^2}{(mP_i + S_i)^2} + e_2 cI_i + \gamma_1(1-\varepsilon)S_i + \gamma_2 I_i - 2\gamma_3 P_i - d_2$$

6. Dynamics of the System Around Equilibrium Point E_0

Theorem (2): The trivial equilibrium point $E_0 = (0,0,0)$ is always unstable.

Proof: the first and third equations of system (1) have a finite value at E_0, that means:

$$\lim_{(S,I,P)\to(0,0,0)} F_i(S,I,P) = 0 \; ; i = 1,2,3$$

Hence, these functions are continuous on the extended domain

$$\Re_+^3 = \{(S,I,P): S(t) \geq 0, I(t) \geq 0; P(t) \geq 0\}.$$

In fact, they are Lipschizian on $\Re_+^3$. Accordingly, the solution of the system (1) with nonnegative initial condition exists and is unique. Thus, the int.$\Re_+^3$ is invariant for system (1). Clearly, the system (1) can't be linearized about E_0. So, local stability of E_0 can't be studied directly. However, by using the transformation of variables $x(t) = S(t);\; y(t) = \frac{I(t)}{S(t)}$ and $z(t) = \frac{P(t)}{I(t)}$ the transformed system is obtained as:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[r - \frac{rx(1+y)}{K} - \frac{\beta xy}{(1+\alpha xy)} - \frac{b(1-\varepsilon)yz}{(myz+1)} + b_1 y\right] = H_1(x,y,z) \\
\frac{dy}{dt} &= y\left[\frac{\beta x(1+y)}{(1+\alpha xy)} - b_1(1+y) - d_1 - cxyz - r \right. \\
&\quad \left. + \frac{rx(1+y)}{K} + \frac{b(1-\varepsilon)yz}{(myz+1)}\right] = H_2(x,y,z) \\
\frac{dz}{dt} &= z\left[\frac{e_1 b(1-\varepsilon)}{(myz+1)} + \gamma_1(1-\varepsilon)x + (c-\gamma_3)xyz + (e_2 c + \gamma_2)xy \right. \\
&\quad \left. + (d_1 + b_1 - d_2) - \frac{\beta x}{(1+\alpha xy)}\right] = H_3(x,y,z)
\end{aligned} \tag{8}$$

Functions $H_i(x,y,z); i = 1,2,3$ are continuous and have second order derivatives on $\Re_+^3$. Accordingly, the solution of the system (8) with nonnegative initial condition exist and is unique.

The Jacobian matrix $J \equiv \frac{d}{dt}\mathcal{H}_i(x,y,z)$ for system (8) is:

$$J = \begin{bmatrix} \partial_{11} & \partial_{12} & \partial_{13} \\ \partial_{21} & \partial_{22} & \partial_{23} \\ \partial_{31} & \partial_{32} & \partial_{33} \end{bmatrix}$$

where:

$$\partial_{11} = r - \frac{2rx(1+y)}{K} - \frac{(2+\alpha xy)\beta xy}{(1+\alpha xy)^2} - \frac{b(1-\varepsilon)yz}{(myz+1)} + b_1 y \; ;$$

$$\partial_{12} = \frac{-rx^2}{K} - \frac{\beta x^2}{(1+\alpha xy)^2} - \frac{b(1-\varepsilon)xz}{(myz+1)^2} + b_1 y \; ;$$

$$\partial_{13} = \frac{-b(1-\varepsilon)xy}{(myz+1)^2} \; ;$$

$$\partial_{21} = \frac{ry(1+y)}{K} + \frac{\beta y(1+y)}{(1+\alpha xy)^2} - cy^2 z \; ;$$

$$\partial_{22}=\frac{\beta x(1+2y+\alpha xy^2)}{(1+\alpha xy)^2}-b_1(1+2y)-d_1-2cxyz$$
$$-r+\frac{rx(1+2y)}{K}+\frac{b(1-\varepsilon)(myz+2)yz}{(myz+1)^2}$$

$$\partial_{23}=\frac{b(1-\varepsilon)y^2}{(myz+1)^2}-cxy^2\ ;$$

$$\partial_{31}=(e_2c+\gamma_2)yz+\gamma_1(1-\varepsilon)z-\frac{\beta z}{(1+\alpha xy)^2}+(c-\gamma_3)yz^2\ ;$$

$$\partial_{32}=\frac{-e_1mb(1-\varepsilon)z^2}{(myz+1)^2}+(e_2c+\gamma_2)xz+\frac{\alpha\beta x^2z}{(1+\alpha xy)^2}+(c-\gamma_3)xz^2\ ;$$

$$\partial_{33}=\frac{e_1b(1-\varepsilon)}{(myz+1)^2}+(e_2c+\gamma_2)xy+\gamma_1(1-\varepsilon)x-\frac{\beta x}{(1+\alpha xy)^2}$$
$$+2(c-\gamma_3)xyz+(b_1+d_1-d_2)$$

Then, the Jacobian matrix of system (8) at the equilibrium point E_0 is:

$$J=\begin{bmatrix} r & 0 & 0 \\ 0 & -(r+b_1+d_1) & 0 \\ 0 & 0 & e_1b(1-\varepsilon)+(b_1+d_1-d_2) \end{bmatrix}$$

And the characteristic equation is

$$(r-\lambda)(-(r+b_1+d_1)-\lambda)(e_1b(1-\varepsilon)+(b_1+d_1-d_2)-\lambda)=0$$

since we have positive and negative eigenvalues then E_0 is saddle point.

6.1. DYNAMICS OF THE SYSTEM AROUND EQUILIBRIUM POINT E_1

The Jacobian matrix at equilibrium point E_1 is:

$$\mathrm{J}_1=\begin{bmatrix} -\mathrm{r} & \mathrm{b}_1-(\mathrm{r}+\beta\mathrm{K}) & -\mathrm{b}(1-\varepsilon) \\ 0 & \beta\mathrm{K}-(\mathrm{b}_1+\mathrm{d}_1) & 0 \\ 0 & 0 & (\mathrm{e}_1\mathrm{b}+\gamma_1\mathrm{K})(1-\varepsilon)-\mathrm{d}_2 \end{bmatrix}$$

And the characteristic equation is:

$$\left(-r-\lambda^{[1]}\right)\left(\beta K-(b_1+d_1)-\lambda^{[1]}\right)\left((e_1b+\gamma_1K)(1-\varepsilon)-d_2-\lambda^{[1]}\right)=0 \quad (9)$$

Then, the equilibrium point E_2 is asymptotically stable if the conditions hold

$$\beta K<(b_1+d_1) \qquad (10.a)$$

$$(e_1b+\gamma_1K)(1-\varepsilon)<d_2 \qquad (10.b)$$

Otherwise the equilibrium point E_2 is saddle point.

6.2. DYNAMICS OF THE SYSTEM AROUND EQUILIBRIUM POINT E_2

The Jacobian matrix at equilibrium point E_2 is:

$$J_2 = \begin{bmatrix} \sigma_{11}^{[2]} & \sigma_{12}^{[2]} & \sigma_{13}^{[2]} \\ \sigma_{21}^{[2]} & \sigma_{22}^{[2]} & \sigma_{23}^{[2]} \\ 0 & 0 & \sigma_{33}^{[2]} \end{bmatrix}$$

Where:

$$\sigma_{11}^{[2]} = r - \frac{r(2S_2+I_2)}{K} - \frac{\beta I_2}{(1+\alpha I_2)} \ ; \quad \sigma_{12}^{[2]} = \frac{-rS_2}{K} - \frac{(b_1+d_1)}{(1+\alpha I_2)} + b_1$$

$$\sigma_{13}^{[2]} = -b(1-\varepsilon) \ ; \quad \sigma_{21}^{[2]} = \frac{\beta I_2}{(1+\alpha I_2)} \ ; \quad \sigma_{22}^{[2]} = \frac{-\alpha I_2(b_1+d_1)}{(1+\alpha I_2)}$$

$$\sigma_{23}^{[2]} = -cI_2 \ ; \ \sigma_{33}^{[2]} = e_1 b(1-\varepsilon) + (e_2 c + \gamma_2)I_2 + \gamma_1(1-\varepsilon)S_2 - d_2$$

And the characteristic equation is:

$$\left(\sigma_{33}^{[2]} - \lambda^{[2]}\right)\left(\left(\lambda^{[2]}\right)^2 + \mathcal{A}_1^{[2]}\left(\lambda^{[2]}\right) + \mathcal{A}_2^{[2]}\right) = 0 \qquad (11)$$

Where:

$$\mathcal{A}_1^{[2]} = -\left(\sigma_{11}^{[2]} + \sigma_{22}^{[2]}\right);$$

$$\mathcal{A}_2^{[2]} = \sigma_{11}^{[2]}\sigma_{22}^{[2]} - \sigma_{12}^{[2]}\sigma_{21}^{[2]};$$

So, the necessary and sufficient conditions to ensure all the eigenvalues of the Jacobian matrix J_2 lie in left complex plane when we have

$$\sigma_{11}^{[2]} < 0 \ ; \ \sigma_{33}^{[2]} < 0 \text{ and } \sigma_{12}^{[2]} < 0. \qquad (12)$$

Implies equilibrium point E_2 is asymptotically stable and it's Saddle point otherwise.

6.3. DYNAMICS OF THE SYSTEM AROUND EQUILIBRIUM POINT E_3

The Jacobian matrix at equilibrium point E_3 is:

$$J_3 = \begin{bmatrix} \sigma_{11}^{[3]} & \sigma_{12}^{[3]} & \sigma_{13}^{[3]} \\ 0 & \sigma_{22}^{[3]} & 0 \\ \sigma_{31}^{[3]} & \sigma_{32}^{[3]} & \sigma_{33}^{[3]} \end{bmatrix}$$

Where:

$\sigma_{11}^{[3]} = r - \frac{2rS_3}{K} - \frac{mb(1-\varepsilon)P_3^2}{(mP_3+S_3)^2}$; $\sigma_{12}^{[3]} = \frac{-rS_3}{K} - \beta S_3 + b_1$

$\sigma_{13}^{[3]} = \frac{-b(1-\varepsilon)S_3^2}{(mP_3+S_3)^2}$; $\sigma_{22}^{[3]} = \beta S_3 - (b_1 + d_1 + cP_3)$

$\sigma_{31}^{[3]} = \frac{e_1 mb(1-\varepsilon)P_3^2}{(mP_3+S_3)^2} + \gamma_1(1-\varepsilon)P_3$; $\sigma_{32}^{[3]} = e_2 cP_3 + \gamma_2 P_3$

$\sigma_{33}^{[3]} = \frac{e_1 b(1-\varepsilon)S_3^2}{(mP_3+S_3)^2} + \gamma_1(1-\varepsilon)S_3 - 2\gamma_3 P_3 - d_2.$

And the characteristic equation is:

$$\left(\sigma_{22}^{[3]} - \lambda^{[3]}\right)\left(\left(\lambda^{[3]}\right)^2 + \mathcal{A}_1^{[3]}\left(\lambda^{[3]}\right) + \mathcal{A}_2^{[3]}\right) = 0 \qquad (13)$$

Where:

$\mathcal{A}_1^{[3]} = -\left(\sigma_{11}^{[3]} + \sigma_{33}^{[3]}\right)$;

$\mathcal{A}_2^{[3]} = \sigma_{11}^{[3]}\sigma_{33}^{[3]} + -\sigma_{13}^{[3]}\sigma_{31}^{[3]}$;

Obviously, the equilibrium point E_3 is asymptotically stable if the following conditions hold, and E_3 Saddle point otherwise.

$$\sigma_{11}^{[3]} < 0 \text{ ; } \sigma_{22}^{[3]} < 0 \text{ and } \sigma_{33}^{[3]} < 0. \qquad (14)$$

6.4. DYNAMICS OF THE SYSTEM AROUND EQUILIBRIUM POINT E_4

The Jacobian matrix at equilibrium point E_4 is:

$$J_4 = \begin{bmatrix} \sigma_{11}^{[4]} & \sigma_{12}^{[4]} & \sigma_{13}^{[4]} \\ \sigma_{21}^{[4]} & \sigma_{22}^{[4]} & \sigma_{23}^{[4]} \\ \sigma_{31}^{[4]} & \sigma_{32}^{[4]} & \sigma_{33}^{[4]} \end{bmatrix}$$

Where:

$\sigma_{11}^{[4]} = r - \frac{r(2S_4+I_4)}{K} - \frac{\beta I_4}{(1+\alpha I_4)} - \frac{mb(1-\varepsilon)P_4^2}{(mP_4+S_4)^2}$;

$\sigma_{12}^{[4]} = \frac{-rS_4}{K} - \frac{\beta S_4}{(1+\alpha I_4)^2} + b_1$; $\sigma_{13}^{[4]} = \frac{-b(1-\varepsilon)S_4^2}{(mP_4+S_4)^2}$;

$\sigma_{21}^{[4]} = \frac{\beta I_4}{(1+\alpha I_4)}$; $\sigma_{22}^{[4]} = \frac{-\alpha\beta S_4 I_4}{(1+\alpha I_4)^2}$; $\sigma_{23}^{[4]} = -cI_4$;

$\sigma_{31}^{[4]} = \frac{e_1 mb(1-\varepsilon)P_4^2}{(mP_4+S_4)^2} + \gamma_1(1-\varepsilon)P_4$; $\sigma_{32}^{[4]} = e_2 cP_4 + \gamma_2 P_4$;

$$\sigma_{33}^{[4]} = \frac{e_1 b(1-\varepsilon)S_4^2}{(mP_4+S_4)^2} + e_2 c I_4 + \gamma_1(1-\varepsilon)S_4 + \gamma_2 I_4 - 2\gamma_3 P_4 - d_2.$$

And the characteristic equation is:

$$\left(\lambda^{[4]}\right)^3 + \mathcal{A}_1^{[4]}\left(\lambda^{[4]}\right)^2 + \mathcal{A}_2^{[4]}\left(\lambda^{[4]}\right) + \mathcal{A}_3^{[4]} = 0 \tag{15}$$

Where:

$$\mathcal{A}_1^{[4]} = -\left(\sigma_{11}^{[4]} + \sigma_{22}^{[4]} + \sigma_{33}^{[4]}\right);$$

$$\mathcal{A}_2^{[4]} = \sigma_{11}^{[4]}\sigma_{33}^{[4]} + \sigma_{22}^{[4]}\sigma_{33}^{[4]} + \sigma_{11}^{[4]}\sigma_{22}^{[4]} - \sigma_{12}^{[4]}\sigma_{21}^{[4]} - \sigma_{13}^{[4]}\sigma_{31}^{[4]} - \sigma_{23}^{[4]}\sigma_{32}^{[4]};$$

$$\mathcal{A}_3^{[4]} = \sigma_{11}^{[4]}\sigma_{23}^{[4]}\sigma_{32}^{[4]} + \sigma_{22}^{[4]}\sigma_{13}^{[4]}\sigma_{31}^{[4]} + \sigma_{33}^{[4]}\sigma_{12}^{[4]}\sigma_{21}^{[4]} - \sigma_{11}^{[4]}\sigma_{22}^{[4]}\sigma_{33}^{[4]} - \sigma_{12}^{[4]}\sigma_{23}^{[4]}\sigma_{31}^{[4]}\sigma_{13}^{[4]}\sigma_{32}^{[4]}\sigma_{21}^{[4]} \quad ;$$

$$\Delta = \mathcal{A}_1^{[4]}\mathcal{A}_2^{[4]} - \mathcal{A}_3^{[4]} = -\left(\sigma_{11}^{[4]}\right)^2\sigma_{22}^{[4]} - \left(\sigma_{11}^{[4]}\right)^2\sigma_{33}^{[4]} - \sigma_{11}^{[4]}\left(\sigma_{22}^{[4]}\right)^2 - \left(\sigma_{22}^{[4]}\right)^2\sigma_{33}^{[4]} - \sigma_{11}^{[4]}\left(\sigma_{33}^{[4]}\right)^2 - \sigma_{22}^{[4]}\left(\sigma_{33}^{[4]}\right)^2 - 2\sigma_{11}^{[4]}\sigma_{22}^{[4]}\sigma_{33}^{[4]} + \sigma_{12}^{[4]}\sigma_{23}^{[4]}\sigma_{31}^{[4]} + \sigma_{33}^{[4]}\sigma_{23}^{[4]}\sigma_{32}^{[4]} + \sigma_{13}^{[4]}\sigma_{31}^{[4]}\left(\sigma_{11}^{[4]}+\sigma_{33}^{[4]}\right) + \sigma_{12}^{[4]}\sigma_{21}^{[4]}\left(\sigma_{11}^{[4]}+\sigma_{22}^{[4]}\right) + \sigma_{32}^{[4]}\left(\sigma_{22}^{[4]}\sigma_{23}^{[4]}+\sigma_{21}^{[4]}\sigma_{13}^{[4]}\right)$$

So, by using Routh-Hurwitz criterion, the equilibrium point E_4 is asymptotically stable if the following conditions hold

$$\sigma_{11}^{[4]} < 0\ ;\ \sigma_{33}^{[4]} < 0\ ;\ \sigma_{12}^{[4]} < 0 \tag{16.a}$$

$$\sigma_{22}^{[4]} < \min\left\{\frac{-\sigma_{13}^{[4]}\sigma_{21}^{[4]}}{\sigma_{23}^{[4]}}\ ;\ \frac{\sigma_{12}^{[4]}\sigma_{23}^{[4]}}{\sigma_{13}^{[4]}}\right\} \tag{16.b}$$

Otherwise the equilibrium point E_4 is Saddle point.

Theorem (3): The equilibrium point E_1 is a globally asymptotically stable provided that the following conditions hold

$$(b_1(S_1+1)+d_1) > \left(\frac{rS_1}{K} + \beta(S_1+1) + b_1\right)K \tag{17.a}$$

$$c(1-e_2) > \gamma_2 \tag{17.b}$$

$$d_2 > (b(S_1-e_1)+\gamma_1)(1-\varepsilon)K \tag{17.c}$$

Proof: consider the following positive definite real valued function:

$$\mathfrak{w}_1(t) = \frac{(S-S_1)^2}{2} + I(t) + P(t)$$

And the derivative of $\mathfrak{w}_1(t)$ with respect to time can be written as

$$\frac{d\mathfrak{w}_1}{dt} = (S - S_1)\frac{dS}{dt} + \frac{dI}{dt} + \frac{dP}{dt}$$

So, by using system (1) with some algebraic manipulations we get

$$\frac{d\mathfrak{w}_1}{dt} = \frac{-rS}{K}(S - S_1)^2 - \left[(b_1(S_1 + 1) + d_1) - \left(\frac{rS_1}{K} + \beta(S_1 + 1) + b_1\right)K\right]I$$
$$-[c(1 - e_2) - \gamma_2]IP - [d_2 - (b(S_1 - e_1) + \gamma_1)(1 - \varepsilon)K]P$$

Clearly, $\frac{d\mathfrak{w}_1}{dt}$ is negative definite function under the conditions (17.a-17.c). Moreover it's clear that the function $\mathfrak{w}_1(t)$ is radially unbounded; then according to the Lyapunov first theorem the equilibrium point E_1 is a globally asymptotically stable point.

Theorem (4): The equilibrium point E_2 is globally asymptotically stable provided that the following sufficient conditions hold

$$d_2 > cI_2 + K(1 - \varepsilon)(bS_2 + e_1 b + \gamma_1) \qquad (18.a)$$

$$c(1 - e_2) > \gamma_2 \qquad (18.b)$$

$$4Z_{11}Z_{22} > Z_{12}^2 \qquad (18.c)$$

Where $Z_{11},\ Z_{12},\ Z_{22}$ given in the prove.

Proof: consider the following positive definite real valued function:

$$\mathfrak{w}_2(t) = \frac{(S - S_1)^2}{2} + \left(I - I_2 - I_2\, ln\frac{I}{I_2}\right) + P(t)$$

And the derivative of $\mathfrak{w}_2(t)$ with respect to time can be written as

$$\frac{d\mathfrak{w}_2}{dt} = (S - S_2)\frac{dS}{dt} + \frac{(I - I_2)}{I}\frac{dI}{dt} + \frac{dP}{dt}$$
$$\leq -[Z_{11}(S - S_2)^2 + Z_{12}(S - S_2)(I - I_2) + Z_{22}(I - I_2)^2] - [c(1 - e_2) - \gamma_2]IP$$
$$-[d_2 - cI_2 - K(1 - \varepsilon)(bS_2 + e_1 b + \gamma_1)]P$$

Hence by doing some algebraic manipulations and the conditions (18.a-18.c), we get that

$$\frac{d\mathfrak{w}_2}{dt} \leq -\left[\sqrt{Z_{11}}(S - S_2) + \sqrt{Z_{22}}(I - I_2)\right]^2 - [c(1 - e_2) + \gamma_2]IP$$
$$-[d_2 - cI_2 - K(1 - \varepsilon)(bS_2 + e_1 b + \gamma_1)]P$$

Where:

$$Z_{11} = r\left[\frac{(S+S_2)}{K} - 1\right] + \frac{rI_2}{K} + \beta I_2(1+\alpha I)$$
$$Z_{12} = b_1 + \beta(1+\alpha I) - \frac{S}{K}(r+K\beta)$$
$$Z_{22} = \alpha\beta S_2$$

Now, by using conditions (18.a-18.c) guarantees that $\frac{d\mathfrak{w}_2}{dt} < 0$. It's clear that the equilibrium point E_2 is a globally asymptotically stable point.

Theorem (5): The equilibrium point E_3 is globally asymptotically stable that satisfied the following conditions

$$\gamma_3 > e_1 mbS_3(1-\varepsilon) \quad (19.a)$$

$$c(1-e_2) > \gamma_2 \quad (19.b)$$

$$(b_1+d_1) > \beta K(1+S_3) + rS_3 \quad (19.c)$$

$$4L_{11}L_{22} > L_{12}^2 \quad (19.d)$$

Where $L_{11},\ L_{12},\ L_{22}$ given in the prove.

Proof: consider the following positive definite real valued function:

$$\mathfrak{w}_3(t) = \frac{(S-S_3)^2}{2} + I(t) + \left(P - P_3 - P_3\, ln\frac{P}{P_3}\right)$$

Then the derivative of $\mathfrak{w}_3(t)$ with respect to time can be written as

$$\frac{d\mathfrak{w}_3}{dt} = (S-S_3)\frac{dS}{dt} + \frac{dI}{dt} + \frac{(P-P_3)}{P}\frac{dP}{dt}$$
$$< -[L_{11}(S-S_3)^2 + L_{12}(S-S_3)(P-P_3) + L_{22}(P-P_3)^2] - [c(1-e_2)-\gamma_2]IP$$
$$-[b_1+d_1-rS_3-\beta K(1+S_3)]I - [e_2c+\gamma_2]IP_3$$

So, by doing some algebraic manipulations we get that

$$\frac{d\mathfrak{w}_3}{dt} < -\left[\sqrt{L_{11}}(S-S_3) + \sqrt{L_{22}}(P-P_3)\right]^2 - [c(1-e_2)-\gamma_2]IP$$
$$-[b_1+d_1-rS_3-\beta K(1+S_3)]I - [e_2c+\gamma_2]IP_3$$

Where

$L_{11} = r\left[\frac{(S+S_3)}{K} - 1\right] + mbPP_3(1-\varepsilon)$
$L_{12} = (bSS_3 - \gamma_1 + e_1 mbP_3)(1-\varepsilon)$
$L_{22} = \gamma_3 - e_1 mbS_3(1-\varepsilon)$

Obviously, $\frac{d\mathfrak{w}_3}{dt}$ is negative definite function with the conditions (19.a-19.d). Moreover it's clear that the function $\mathfrak{w}_3(t)$ is radially unbounded, then according to the Lyapunov first theorem E_3 is a globally asymptotically stable point.

Theorem (6): The equilibrium point E_4 is globally asymptotically stable that satisfied the following conditions

$$G_{12}^2 < G_{11}G_{22} \tag{20.a}$$

$$G_{13}^2 < G_{11}G_{33} \tag{20.b}$$

$$G_{23}^2 < G_{22}G_{33} \tag{20.c}$$

Where $G_{11}, G_{12}, G_{13}, G_{22}, G_{23}, G_{33}$ given in the prove.

Proof: consider the following positive definite real valued function:

$$\mathfrak{w}_4(t) = \frac{(S-S_4)^2}{2} + \left(I - I_4 - I_4 \ln\frac{I}{I_4}\right) + \left(P - P_4 - P_4 \ln\frac{P}{P_4}\right)$$

And the derivative of $\mathfrak{w}_4(t)$ with respect to time can be written as

$$\frac{d\mathfrak{w}_4}{dt} = (S-S_4)\frac{dS}{dt} + \frac{(I-I_4)}{I}\frac{dI}{dt} + \frac{(P-P_4)}{P}\frac{dP}{dt}$$

$$\begin{aligned}\frac{d\mathfrak{w}_4}{dt} = {} & (S-S_4)\left[rS - \frac{rS^2}{K} - \frac{rSI}{K} - \frac{\beta SI}{(1+\alpha I)} - \frac{b(1-\varepsilon)SP}{(mP+S)} + b_1 I\right] \\ & +(I-I_4)\left[\frac{\beta S}{(1+\alpha I)} - b_1 - d_1 - cP\right] \\ & +(P-P_4)\left[\frac{e_1 b(1-\varepsilon)S}{(mP+S)} + e_2 cI + \gamma_1(1-\varepsilon)S + \gamma_2 I - \gamma_3 P - d_2\right]\end{aligned}$$

$$\begin{aligned}\frac{d\mathfrak{w}_4}{dt} = {} & -\left[\frac{G_{11}}{2}(S-S_4)^2 + G_{12}(S-S_4)(I-I_4) + \frac{G_{22}}{2}(I-I_4)^2\right] \\ & -\left[\frac{G_{11}}{2}(S-S_4)^2 + G_{13}(S-S_4)(P-P_4) + \frac{G_{33}}{2}(P-P_4)^2\right] \\ & -\left[\frac{G_{22}}{2}(S-S_4)^2 + G_{23}(I-I_4)(P-P_4) + \frac{G_{33}}{2}(P-P_4)^2\right]\end{aligned}$$

Consequently, by using conditions (20.a-20.c) we get that

$$\frac{d\mathfrak{w}_4}{dt} \leq -\left[\sqrt{\frac{G_{11}}{2}}(S-S_4)+\sqrt{\frac{G_{22}}{2}}(I-I_4)\right]^2 - \left[\sqrt{\frac{G_{11}}{2}}(S-S_4)+\sqrt{\frac{G_{33}}{2}}(P-P_4)\right]^2 - \left[\sqrt{\frac{G_{22}}{2}}(I-I_4)+\sqrt{\frac{G_{33}}{2}}(P-P_4)\right]^2$$

where

$$G_{11} = \left(\frac{(S+S_4)}{K}-1\right)+\frac{rI_4}{K}+\frac{\beta I_4}{(1+\alpha I_4)}+\frac{mbPP_4(1-\varepsilon)}{(mP+S)(mP_4+S_4)}$$

$$G_{12} = b_1+\frac{\beta}{(1+\alpha I)}-\frac{rS}{K}-\frac{\beta S}{(1+\alpha I)(1+\alpha I_4)}$$

$$G_{22} = \frac{\alpha\beta S_4}{(1+\alpha I)(1+\alpha I_4)};\ G_{13} = (1-\varepsilon)\left[\frac{b(e_1 mP_4 - SS_4)}{(mP+S)(mP_4+S_4)}+\gamma_1\right]$$

$$G_{23} = \gamma_2 - c(1-e_2);\ G_{33} = \gamma_3+\frac{e_1 mbS_4(1-\varepsilon)}{(mP+S)(mP_4+S_4)}$$

Clearly, $\frac{d\mathfrak{w}_4}{dt}$ is negative definite under conditions (20.a-20.c). Moreover it's clear that the function $\mathfrak{w}_4(t)$ is radially unbounded, then according to the Lyapunov first theorem E_4 is a globally asymptotically stable point.

7. NUMERICAL SIMULATION

In order to verify theoretical analytical results in our proposed model we have solved model (1) numerically by using Matlab program. Numerical simulations are solved by choosing the parametric values from the following set

$$\begin{array}{l} r=1;\ K=20;\ \beta=0.2;\ \alpha=0.2;\ b=0.6;\ \varepsilon=0.5; \\ m=0.5;\ b_1=0.1;\ d_1=0.1;\ c=0.2;\ e_1=0.3; \\ e_2=0.5;\ \gamma_1=0.05;\ \gamma_2=0.02;\ \gamma_3=0.9;\ d_2=0.7 \end{array} \quad (21)$$

It's clear that starting from three different sets of initial values, the solutions of system (1) approaches asymptotically to positive equilibrium point $E_4 = (3.674,7.705,0.446)$ as shown in phase portrait and their series given in figure (1). This matched with the analytical result obtained in theorem (6), which determined the sufficient condition (20) for globally stable positive equilibrium point E_4.

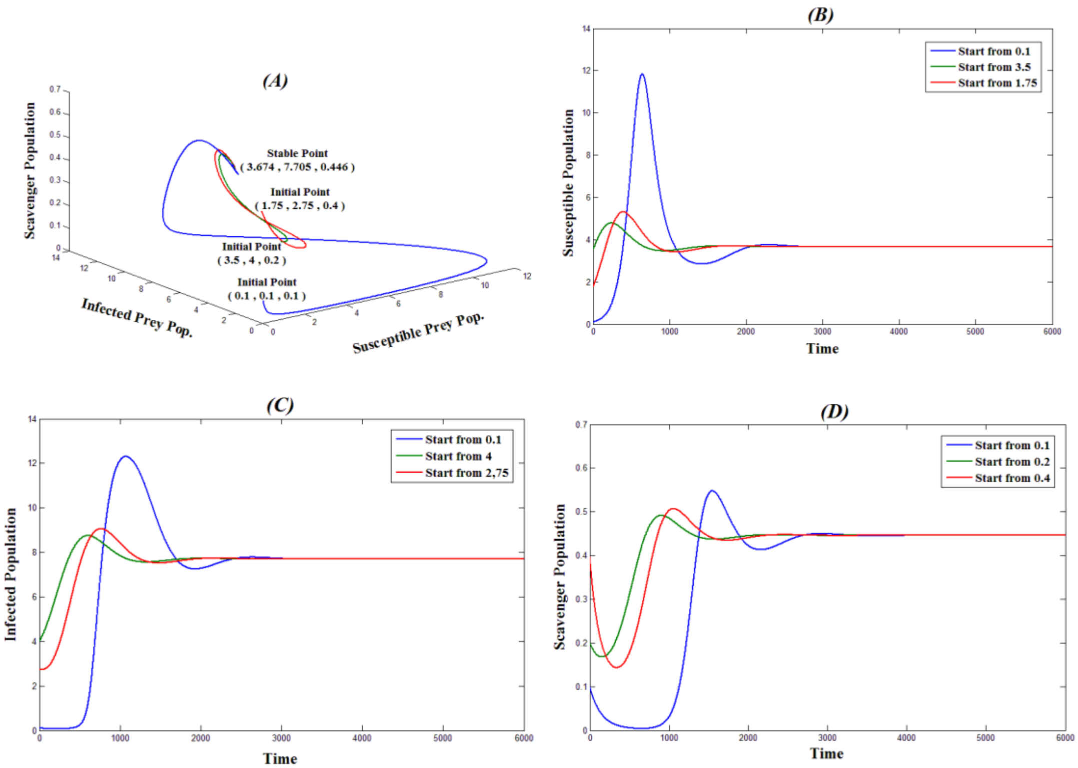

Fig. (1): Trajectories of system (1) started from different initial points approaches asymptotically to globally stable positive equilibrium point E_4. (A) Phase portrait; (B) Time series of susceptible prey; (C) Time series of infected prey; (D) Time series of scavenger.

Further numerical simulations have been verified for data given by eq. (21) with varying parameters $\beta = 0.002$ and $\varepsilon = 0.05$ then the trajectory of system (1), starting from different sets of initial points, is approaching asymptotically to globally stable disease free equilibrium point $E_3 = (19.742,0,0.452)$ as shown in phase portrait and their series given in figure (2).

While the solutions of system (1) approach asymptotically to the globally stable scavenger free equilibrium point $E_2 = (3.047,10.235,0)$ as shown in figure (3) drawing from different set of initial points, and data given in eq. (21) with vary the parameter $c = 0.02$.

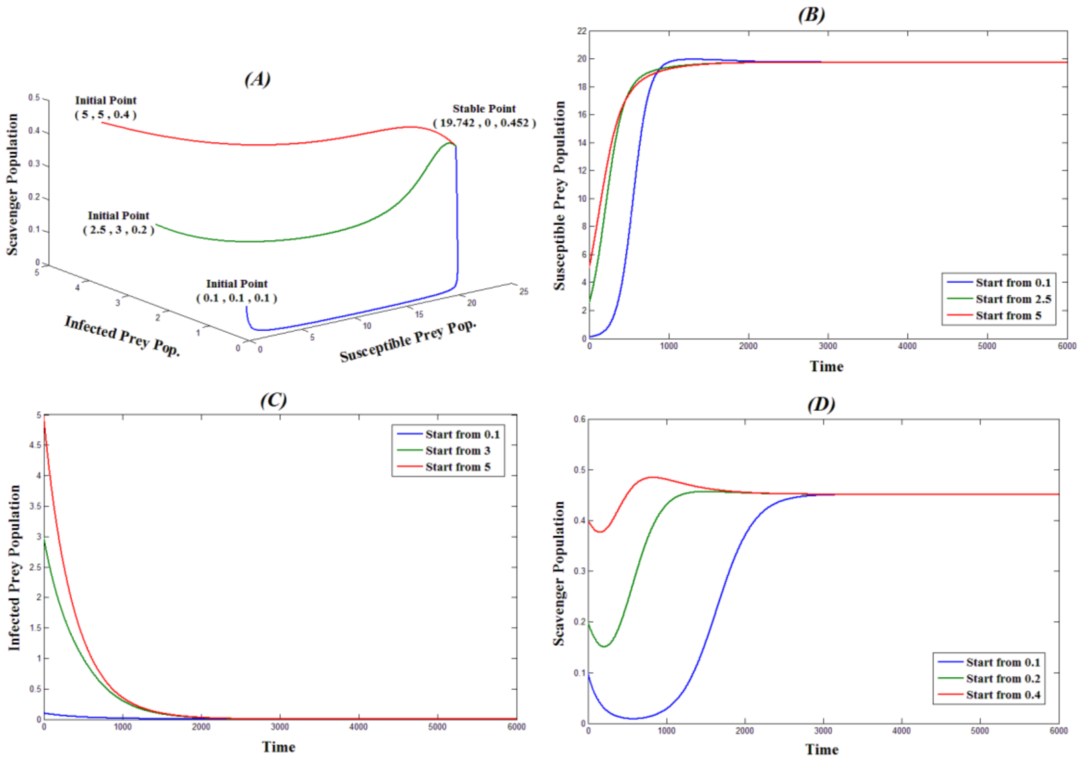

Fig. (2): Trajectories of system (1) started from different initial points approaches asymptotically to globally stable disease free equilibrium point E_3. (A) Phase portrait; (B) Time series for susceptible prey; (C) Time series for infected prey; (D) Time series for scavenger.

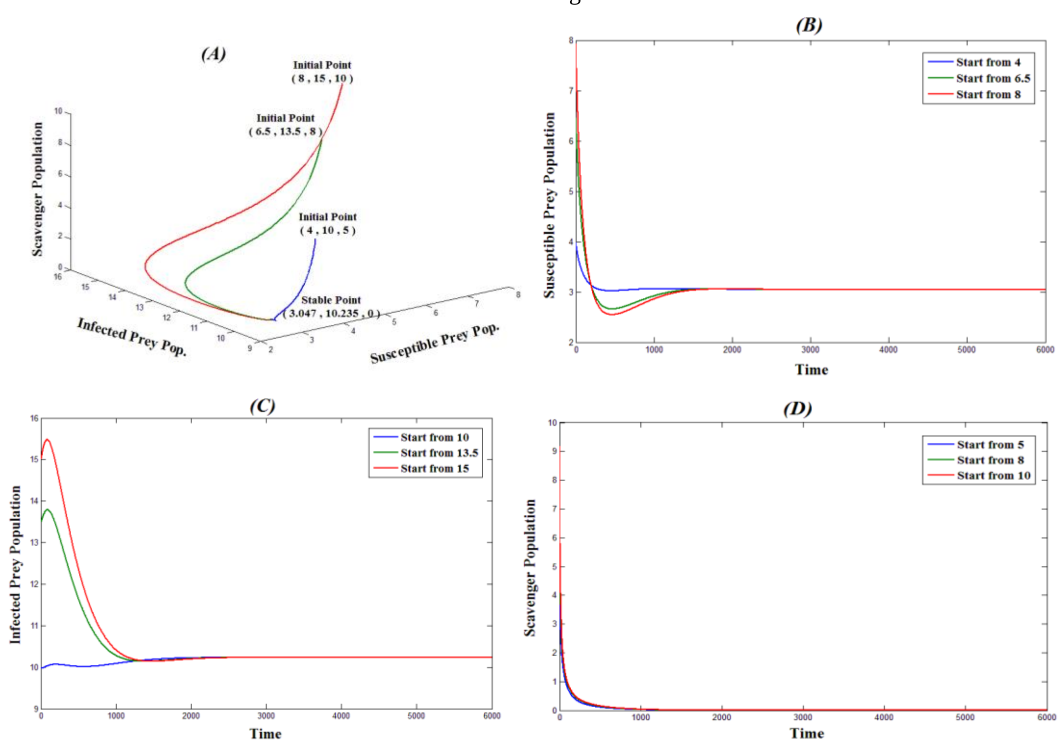

Fig. (3): Trajectories of system (1) started from different initial points approaches asymptotically to globally stable scavenger free equilibrium point E_2. (A) Phase portrait; (B) Time series for susceptible prey; (C) Time series for infected prey; (D) Time series for scavenger.

Finally, for the parameters values given in eq. (21) with $\beta = 0.002$ and from different sets of initial points, it's easy to verify the trajectories of system (1) approaches asymptotically to the globally stable axial equilibrium point $E_1 = (20,0,0)$ as shown in the figure (4).

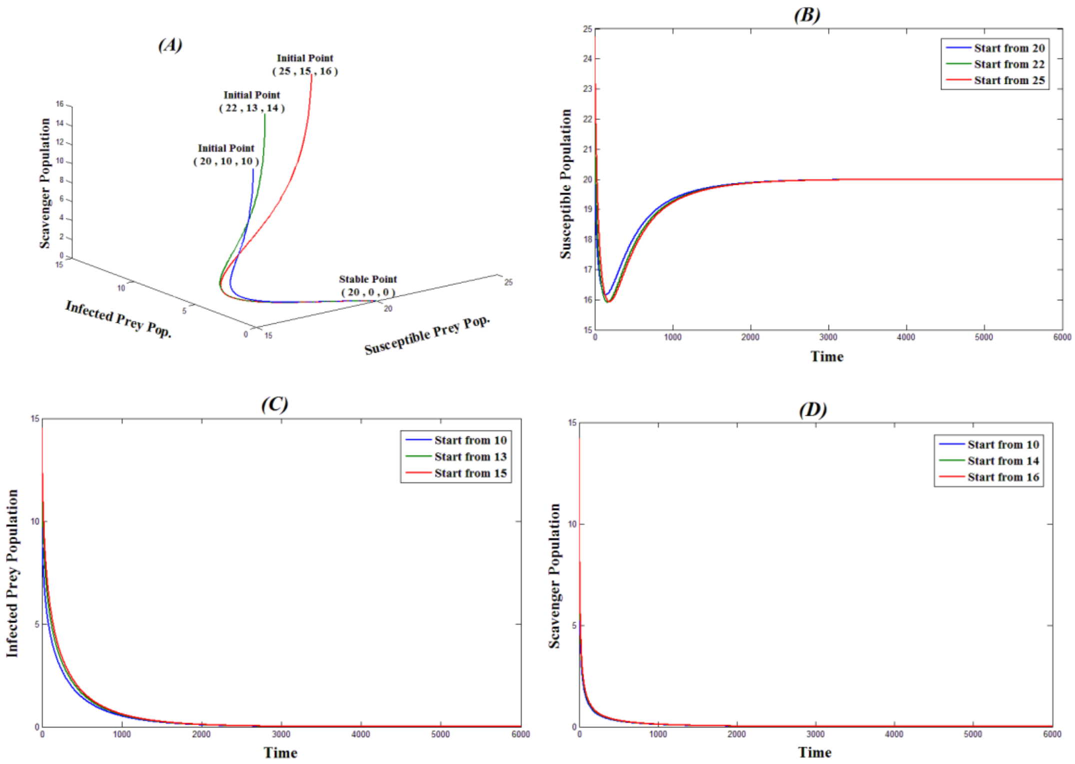

Fig. (4): Trajectories of system (1) started from different initial points approaches asymptotically to globally stable axial equilibrium point E_1. (A) Phase portrait; (B) Time series for susceptible prey; (C) Time series for infected prey; (D) Time series for scavenger.

7. Discussion and Results

In this paper, the interaction dynamics of prey and scavenger proposed and analyzed. Spread infection disease represented by Holling type-II infection function in prey population and prey refuge are considering. The model included both ratio-dependent and linear functional responses with different rates. The existences and boundedness of solutions of suggested model have been discussed. local stability has been investigated around each of the equilibrium point. Also, investigate the global dynamics at each equilibrium point by using suitable Lyapunov functions. The qualitative dynamical behavior as a function of varying the sets of parameters values is studied analytically as well as numerically. Finally, for the biologically feasible set of hypothetical data as given in Eq. (21), the system (1) is solved numerically and the obtained results are explained in some typical figures and we will summarize as follows:

1. System (1) has no periodic solution, instead of that the solution approaching asymptotically to one of their Four possible equilibrium points depending on their set of parameters values.
2. If we take $\beta = 0.002$ and $\varepsilon = 0.05$ and keeping all parameters value in eq.(21), the positive equilibrium point E_4 becomes unstable and the trajectory of system (1) approaches asymptotically to the disease free equilibrium point E_3.
3. Moreover, the positive equilibrium point E_4 becomes unstable and the trajectory of system (1) approaches asymptotically to the axial equilibrium point E_1 as keeping data given in eq.(21) with $\beta = 0.002$.
4. It's observed that, in case of maximum attack parameter varying choose $c = 0.02$ with keeping the rest of parameters as in eq.(21) the positive equilibrium point E_4 becomes unstable and the trajectory of system (1) approaches asymptotically to the scavenger free equilibrium pointE_2.
5. According to the above discussion, it’s observed that system (1) is sensitive to varying in many of its parameters and hence there is higher possibility to control.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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