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MODELING AND ANALYSIS OF THE DYNAMICS OF COVID-19 TRANSMISSION IN PRESENCE OF IMMIGRATION AND VACCINATION

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Abstract. This paper aims to propose and study the dynamics of COVID-19 transmission model with immigration, vaccination and general incidence function. The global existence, positivity and boundedness of solutions are proved. Also, the sensitivity analysis is used to discover parameters that have impact on the threshold value of the basic reproduction number R_0. Furthermore, we construct appropriate Lyapunov functions to prove the global stability of equilibria.

Keywords: COVID-19; immigration; vaccination; general incidence rate; stability; sensitivity analysis.

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1. INTRODUCTION

In spite of the scientific progress and societies evolution, diseases still remain a great danger nowadays. In fact, these diseases can lead to huge epidemics that threaten hundreds of lives and subsequently result in a big population loss. COVID-19 is considered to be one of the deadliest epidemics that have noticeably disrupted our daily life and thus, should be a matter of

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high concern. COVID-19 is the name given by WHO on February 11, 2020 to a new respiratory infectious disease caused by coronavirus SARS-COV-2. The first confirmed cases of COVID-19 were detected on December 8, 2019 in Wuhan in China. Studies have shown that coronavirus and its variants are likely to cause serious problems for patients weakened by age or other chronic illnesses. The disease is transmitted by close contact with infected persons and also by asymptomatic patients.

There is currently no treatment able to eradicate the virus. The remedies provided to patients are only intended to treat symptoms. Because of COVID-19, more than 10 billion doses of vaccines have been injected in the world and more than 394 million people have tested positive. Despite all efforts, the disease has so far caused more than 5.7 million deaths [1].

Now, researchers around the world are exploring many avenues to find an antiviral drug or vaccine. For these reasons, the modeling of infectious diseases is a very important tool that will enable scientists not only to analyze their spread but also to control them and understand their transmission mechanism. Several epidemic models on theoretical developments are discussed by many researchers. In this context, it should be mentioned that the basic reproduction number R_0 and the incidence function (several expressions in the literature) are crucial keys in epidemiological models [2, 3, 4, 5, 6].

The SIR model in epidemiology gives us a simple dynamic description. Yet, despite its simplicity, the SIR model presents the basic structure generally associated to the spread of a disease in a population. Many variants of SIR model have been studied in recent years to more effectively model more complex infectious diseases [7, 8, 9, 10, 11]. In this paper, we develop a new SIR epidemic model for COVID-19 transmission in presence of immigration and vaccination.

The paper is organized as follows. In section 2, we present our model and prove the positive invariance of the feasible region. In section 3, we analyze the stability and sensitivity of the model without infected immigrants. The analysis of the model with infected immigrants is established in section 4. Finally, in the last section, a discussion is raised and some conclusions are provided.

2. MODEL FORMULATION

In this section, we first develop a mathematical model that takes into account the effects of immigration and vaccination. So, we divide the total population into three classes $S(t)$, $I(t)$ and $R(t)$ that represent susceptible, infected and recovered individuals at time t, respectively. The dynamics of the three classes is governed by the following nonlinear system of ODEs:

$$\left\{\begin{array}{ll} \frac{dS}{dt} & = \mathscr{A}+b-\mu S-\mathscr{F}(S,I)I-\nu S, \\ \frac{dI}{dt} & = \mathscr{F}(S,I)I+c-(\mu+d+r)I, \\ \frac{dR}{dt} & = rI+\nu S-\mu R, \end{array}\right. \tag{1}$$

where the susceptible individuals are recruited at a rate $\mathscr{A}$ and become infected by effective contact with infected individuals at rate $\mathscr{F}(S,I)I$. The natural death rate in all classes is denoted by μ, while d is the death rate due to COVID-19. The parameter ν is the rate of vaccination, and r denotes recovery rate of the infected individuals. Finally, b and c represent the immigrant to susceptible and the immigrant to infected, respectively.

Obviously, the first two equations of (1) do not depend on the variable R, model (1) can be rewrite by following system

$$\left\{\begin{array}{ll} \frac{dS}{dt} & = \mathscr{A}+b-\mu S-\mathscr{F}(S,I)I-\nu S, \\ \frac{dI}{dt} & = \mathscr{F}(S,I)I+c-(\mu+d+r)I. \end{array}\right. \tag{2}$$

Moreover and according to [3], we assume that the general incidence function $\mathscr{F}$ is continuously differentiable in the interior of $\mathbb{R}^2_+$ and satisfies the following conditions:

(H_1) $\mathscr{F}(0,I)=0$, for all $I\geq 0$.

(H_2) $\frac{\partial \mathscr{F}}{\partial S}(S,I)>0$, for all $S>0$ and $I\geq 0$.

(H_3) $\frac{\partial \mathscr{F}}{\partial I}(S,I)\leq 0$, for all $S\geq 0$ and $I\geq 0$.

Consider the biologically feasible region for the system (2):

$$\Gamma=\big\{(S,I)\in\mathbb{R}^2_+ : S+I\leq \frac{\mathscr{A}+b+c}{\mu}\big\}.$$

Theorem 2.1. *The feasible region Γ is positively invariant with respect to system (2).*

Proof. We have $\frac{dS}{dt}|_{S=0} = \mathscr{A} + b > 0$, $\frac{dI}{dt}|_{I=0} = c \geq 0$, and

$$\begin{aligned}\frac{d(S+I)}{dt} &= \mathscr{A} + b + c - \mu(S+I) - (d+r)I - \nu S \\ &\leq \mathscr{A} + b + c - \mu(S+I).\end{aligned}$$

Then $\limsup_{t\to\infty}(S(t)+I(t)) \leq \dfrac{\mathscr{A}+b+c}{\mu}$. Hence, Γ is positively invariant and the solutions are bounded. This completes the proof. □

3. Analysis of the Model Without Immigration of Infected Individuals

In this case, we have $c = 0$, and system (2) becomes

$$\begin{cases} \frac{dS}{dt} &= \mathscr{A} + b - \mu S - \mathscr{F}(S,I)I - \nu S, \\ \frac{dI}{dt} &= \mathscr{F}(S,I)I - (\mu + d + r)I. \end{cases} \tag{3}$$

3.1. Equilibria. Clearly, system (3) has always one disease-free equilibrium $\mathscr{E}_f(\frac{\mathscr{A}+b}{\mu+\nu}, 0)$. Then the basic reproduction number of the system (3) is given by

$$R_0 = \frac{\mathscr{F}(\frac{\mathscr{A}+b}{\mu+\nu}, 0)}{\mu + d + r}. \tag{4}$$

Theorem 3.1. *The disease-free equilibrium point of the system (3) is given by $\mathscr{E}_f(\frac{\mathscr{A}+b}{\mu+\nu}, 0)$ and it exists for all parameter values. If $R_0 > 1$, then (3) has a unique endemic equilibrium $\mathscr{E}^*(S^*, I^*)$ with $S^* \in (0, \frac{\mathscr{A}+b}{\mu+\nu})$ and $I^* > 0$.*

Proof. The steady state of system (3) satisfies the following equations

$$\mathscr{A} + b - \mu S - \mathscr{F}(S,I)I - \nu S = 0, \tag{5}$$

$$\mathscr{F}(S,I)I - (\mu + d + r)I = 0. \tag{6}$$

From (6), we have $I = 0$ or $\mathscr{F}(S,I) = \mu + d + r$.

If $I = 0$, then $S = \frac{\mathscr{A}+b}{\mu+\nu}$.

If $I \neq 0$, then $I = \frac{\mathscr{A}+b-(\mu+\nu)S}{\mu+d+r}$ and $\mathscr{F}(S, \frac{\mathscr{A}+b-(\mu+\nu)S}{\mu+d+r}) = \mu + d + r$.

We have $I = \frac{\mathscr{A}+b-(\mu+\nu)S}{\mu+d+r} \geq 0$ implies that $S \leq \frac{\mathscr{A}+b}{\mu+\nu}$. Hence, there is no positive equilibrium point if $S > \frac{\mathscr{A}+b}{\mu+\nu}$. So, we consider the following function $\mathscr{H}$ defined on $[0, \frac{\mathscr{A}+b}{\mu+\nu}]$ by

$$\mathscr{H}(S) = \mathscr{F}\big(S, \frac{\mathscr{A}+b-(\mu+\nu)S}{\mu+d+r}\big) - (\mu + d + r).$$

We have $\mathscr{H}(0) = -(\mu+d+r) < 0$ and $\mathscr{H}(\frac{\mathscr{A}+b}{\mu+\nu}) = (\mu+d+r)(R_0-1) > 0$ when $R_0 > 1$. According to hypotheses (H_2) and (H_3), we have

$$\mathscr{H}'(S) = \frac{\partial \mathscr{F}}{\partial S}(S,I) - \frac{\mu+\nu}{\mu+d+r} \times \frac{\partial \mathscr{F}}{\partial I}(S,I) > 0.$$

Hence, there exists a unique endemic equilibrium $\mathscr{E}^*(S^*,I^*)$ with $S^* \in (0, \frac{\mathscr{A}+b}{\mu+\nu})$ and $I^* > 0$. This completes the proof. □

3.2. Global stability of the disease-free equilibrium.

Theorem 3.2. *The disease-free equilibrium $\mathscr{E}_f$ is globally asymptotically stable if $R_0 \leq 1$ and unstable if $R_0 > 1$.*

Proof. Consider the following Lyapunov functional

$$\mathscr{V}(t) = S(t) - S^0 - \int_{S^0}^{S} \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(X,0)} dX + I(t),$$

where $S^0 = \frac{\mathscr{A}+b}{\mu+\nu}$. Let $a = \mu+d+r$. So, the time derivative of $\mathscr{V}$ computed along solutions of system (3) is given by

$$\begin{aligned}
\frac{d\mathscr{V}}{dt} &= \left(1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)}\right)\frac{dS}{dt} + \frac{dI}{dt} \\
&= \left(1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)}\right)\Big(\mathscr{A} + b - (\mu+\nu)S - \mathscr{F}(S,I)I\Big) + \mathscr{F}(S,I)I - aI \\
&= \Big(\mathscr{A} + b - (\mu+\nu)S\Big)\left(1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)}\right) + \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)}\mathscr{F}(S,I)I - aI \\
&= (\mu+\nu)(S^0 - S)\left(1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)}\right) + aI\left(\frac{\mathscr{F}(S,I)}{\mathscr{F}(S,0)}R_0 - 1\right) \\
&\leq (\mu+\nu)S^0\left(1 - \frac{S}{S^0}\right)\left(1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)}\right) + aI(R_0 - 1).
\end{aligned}$$

From (H_2), we have $1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)} \geq 0$ for $S \geq S^0$ and $1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)} \leq 0$ for $S \leq S^0$. Then

$$\left(1 - \frac{S}{S^0}\right)\left(1 - \frac{\mathscr{F}(S^0,0)}{\mathscr{F}(S,0)}\right) \leq 0.$$

Since $R_0 \leq 1$, we have $\frac{d\mathscr{V}}{dt} \leq 0$. Also, $\frac{d\mathscr{V}}{dt} = 0$ if and only if $S = S^0$ and $I = 0$. Thus, the disease-free equilibrium $\mathscr{E}_f$ is globally asymptotically stable. This completes the proof. □

3.3. Global stability of the endemic equilibrium. In the following, we assume that $R_0 > 1$ and the incidence function $\mathscr{F}$ satisfies, for all $S, I > 0$, the following assumption

$$(H_4) \qquad \left(1 - \frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I^*)}\right)\left(\frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)} - \frac{I}{I^*}\right) \leq 0.$$

Theorem 3.3. *Assume that $R_0 > 1$ and (H_4) holds. Then the endemic equilibrium $\mathscr{E}^*$ is globally asymptotically stable.*

Proof. We define a Lyapunov functional as follows

$$\mathscr{W}(t) = S(t) - S^* - \int_{S^*}^{S} \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(X,I^*)} dX + I^* \psi(\frac{I}{I^*}),$$

where $\psi(x) = x - 1 - \ln x$, for $x > 0$. It is evident that ψ attains its global minimum at $x = 1$ and $\psi(1) = 0$. Further, the function $\phi : x \mapsto x - S^* - \int_{S^*}^{x} \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(X,I^*)} dX$ has the global minimum on $\mathbb{R}_+^*$ at $x = S^*$ and $\phi(S^*) = 0$. Then $\phi(x) \geq 0$ for any $x > 0$. Thus, $\mathscr{W}(t) \geq 0$ with equality if and only if $S(t) = S^*$ and $I(t) = I^*$, for all t ≥ 0.

Since $\mathscr{A} + b = (\mu + \nu)S^* + \mathscr{F}(S^*,I^*)I^*$ and $\mathscr{F}(S^*,I^*) = a$, we have

$$\begin{aligned}
\frac{d\mathscr{W}}{dt} &= \big(\mathscr{A} + b - (\mu+\nu)S - \mathscr{F}(S,I)I\big)\big(1 - \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}\big) + \big(\mathscr{F}(S,I)I - aI\big)\big(1 - \frac{I^*}{I}\big) \\
&= (\mu+\nu)S^*(1 - \frac{S}{S^*})(1 - \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}) + aI^*[1 - \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)} - \frac{\mathscr{F}(S,I)}{\mathscr{F}(S^*,I^*)}\frac{I}{I^*} + \frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I^*)}\frac{I}{I^*}] \\
&\quad + aI^*(\frac{\mathscr{F}(S,I)}{\mathscr{F}(S^*,I^*)}\frac{I}{I^*} - \frac{\mathscr{F}(S,I)}{\mathscr{F}(S^*,I^*)} - \frac{I}{I^*} + 1) \\
&= (\mu+\nu)S^*(1 - \frac{S}{S^*})(1 - \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}) - aI^*[-3 + \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)} + \frac{\mathscr{F}(S,I)}{\mathscr{F}(S^*,I^*)} + \frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)}] \\
&\quad + aI^*[-1 - \frac{I}{I^*} + \frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)} + \frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I^*)}\frac{I}{I^*}] \\
&= (\mu+\nu)S^*(1 - \frac{S}{S^*})(1 - \frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}) - aI^*[(\frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)} - 1 + \ln(\frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)})) \\
&\quad + (\frac{\mathscr{F}(S,I)}{\mathscr{F}(S^*,I^*)} - 1 + \ln(\frac{\mathscr{F}(S,I)}{\mathscr{F}(S^*,I^*)})) + (\frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)} - 1 + \ln(\frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)}))] \\
&\quad + aI^*[-1 - \frac{I}{I^*} + \frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)} + \frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I^*)}\frac{I}{I^*}].
\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d\mathscr{W}}{dt} &= (\mu+\nu)S^*\left(1-\frac{S}{S^*}\right)\left(1-\frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}\right)\\ &\quad -aI^*\left[\psi(\frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)})+\psi(\frac{\mathscr{F}(S,I)}{\mathscr{F}(S^*,I^*)})+\psi(\frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)})\right]\\ &\quad +aI^*\left[-1-\frac{I}{I^*}+\frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)}+\frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I^*)}\frac{I}{I^*}\right].\end{aligned}$$

Since $1-\frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}\geq 0$ for $S\geq S^*$ and $1-\frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}\leq 0$ for $S\leq S^*$, we get

$$\left(1-\frac{S}{S^*}\right)\left(1-\frac{\mathscr{F}(S^*,I^*)}{\mathscr{F}(S,I^*)}\right)\leq 0.$$

Using (H_4), we obtain

$$-1-\frac{I}{I^*}+\frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)}+\frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I^*)}\frac{I}{I^*}=(1-\frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I^*)})(\frac{\mathscr{F}(S,I^*)}{\mathscr{F}(S,I)}-\frac{I}{I^*})\leq 0.$$

Since $\psi(x)\geq 0$ for $x>0$, then $\frac{d\mathscr{W}}{dt}\leq 0$. Thus, $\mathscr{E}^*$ is stable. Moreover, we have $\frac{d\mathscr{W}}{dt}=0$ if and only if $S=S^*$ and $I=I^*$. Finally, from LaSalle invariance principle [12], we conclude that $\mathscr{E}^*$ is globally asymptotically stable. This ends the proof. □

3.4. Sensitivity analysis. The sensitivity analysis is important to determine how best we can reduce the effect of disease. It indicates how crucial every parameter is to disease transmission and discovered parameters that have a high impact on the basic reproduction number R_0 which will require intervention strategies. The sensitivity of a variable with respect to model parameters is usually measured by sensitivity index. When the variable is a differentiable function of the parameter, the sensitivity index can be defined using partial derivatives.

Definition 3.4. *The normalized forward sensitivity index of R_0 that depends differentiably on a parameter p is defined by*

$$\rho_p^{R_0}=\frac{\partial R_0}{\partial p}\frac{p}{R_0}. \tag{7}$$

Note that the sensitivity index can depend on several parameters of the model, but also can be constant. We perform the sensitivity analysis using (7) with parameters $\mathscr{A}$, b, r, d, μ and ν.

We get the following sensitivity indices:

$$\rho_{\mathscr{A}}^{R_0} = \frac{\mathscr{A}}{R_0}\frac{\partial R_0}{\partial \mathscr{A}} = \frac{\mathscr{A}\frac{\partial \mathscr{F}}{\partial S}(\frac{\mathscr{A}+b}{\mu+\nu},0)}{(\mu+\nu)\mathscr{F}(\frac{\mathscr{A}+b}{\mu+\nu},0)} > 0,$$

$$\rho_{b}^{R_0} = \frac{b}{R_0}\frac{\partial R_0}{\partial b} = \frac{b\frac{\partial \mathscr{F}}{\partial S}(\frac{\mathscr{A}+b}{\mu+\nu},0)}{(\mu+\nu)\mathscr{F}(\frac{\mathscr{A}+b}{\mu+\nu},0)} > 0,$$

$$\rho_{r}^{R_0} = \frac{r}{R_0}\frac{\partial R_0}{\partial r} = \frac{-r}{\mu+d+r} < 0,$$

$$\rho_{d}^{R_0} = \frac{d}{R_0}\frac{\partial R_0}{\partial d} = \frac{-d}{\mu+d+r} < 0,$$

$$\rho_{\mu}^{R_0} = \frac{\mu}{R_0}\frac{\partial R_0}{\partial \mu} = \frac{-\mu}{\mu+d+r}[1+\frac{\mathscr{A}+b}{R_0(\mu+\nu)^2}\frac{\partial \mathscr{F}}{\partial S}(\frac{\mathscr{A}+b}{\mu+\nu},0)] < 0,$$

$$\rho_{\nu}^{R_0} = \frac{\nu}{R_0}\frac{\partial R_0}{\partial \nu} = \frac{-\nu(\mathscr{A}+b)}{\mathscr{F}(\frac{\mathscr{A}+b}{\mu+\nu},0)(\mu+\nu)^2}\frac{\partial \mathscr{F}}{\partial S}(\frac{\mathscr{A}+b}{\mu+\nu},0) < 0.$$

Equations above show that the parameters $\mathscr{A}$ and b are proportional to the threshold parameter R_0. Consequently, an increase or decrease in these parameters will increase or decrease the basic reproduction number R_0. On the other hand, the parameters r, d, μ and ν are inversely proportional to the threshold parameter R_0. So, an increase in these parameters will decrease R_0, while a decrease in these parameters will increase R_0.

4. Analysis of the Model With Immigration of Infected Individuals

Now, consider the case of $c > 0$. There is no disease-free equilibrium. In this section, we prove the existence and analyze the local and the global stability of the endemic equilibrium $\mathscr{E}_*$.

4.1. Equilibria.

Theorem 4.1. *Assume that $c > 0$. There exists a unique equilibrium $\mathscr{E}_*(S_*, I_*)$ of the system (2), where $I_* \in (\frac{c}{\mu+d+r}, \frac{\mathscr{A}+b+c}{\mu+d+r})$ and $S_* = \frac{\mathscr{A}+b+c-(\mu+d+r)I_*}{\mu+\nu}$.*

Proof. Solving $\frac{dS}{dt} = 0$ and $\frac{dI}{dt} = 0$, we get

$$\mathscr{A} + b - \mu S - \mathscr{F}(S,I)I - \nu S = 0, \tag{8}$$

$$\mathscr{F}(S,I)I + c - (\mu+d+r)I = 0. \tag{9}$$

By adding (8) and (9), we obtain $I = \frac{\mathscr{A}+b+c-(\mu+\nu)S}{\mu+d+r}$ which implies that $I \leq \frac{\mathscr{A}+b+c}{\mu+d+r}$.
From (9), we get $\mathscr{F}(S,I)I = (\mu+d+r)I - c$. Then $I \geq \frac{c}{\mu+d+r}$. Therefore,

$$\frac{c}{\mu+d+r} \leq I \leq \frac{\mathscr{A}+b+c}{\mu+d+r}.$$

If $I > \frac{\mathscr{A}+b+c}{\mu+d+r}$ or $I < \frac{c}{\mu+d+r}$, then there is no positive equilibrium point.

Now, we consider the function K defined on $[\frac{c}{\mu+d+r}, \frac{\mathscr{A}+b+c}{\mu+d+r}]$ by

$$K(I) = \mathscr{F}(h(I),I)I + c - (\mu+d+r)I,$$

where $h(I) = \frac{\mathscr{A}+b+c-(\mu+d+r)I}{\mu+\nu}$.
We have $K(\frac{c}{\mu+d+r}) = \frac{c}{\mu+d+r}\mathscr{F}(\frac{\mathscr{A}+b}{\mu+\nu}, \frac{c}{\mu+d+r}) > 0$ and $K(\frac{\mathscr{A}+b+c}{\mu+d+r}) = -(\mathscr{A}+b) < 0$.
Hence, there exists at least $I_* \in (\frac{c}{\mu+d+r}, \frac{\mathscr{A}+b+c}{\mu+d+r})$ such that $K(I_*) = 0$.
Moreover, we have $K'(I_*) = [h'(I_*)\frac{\partial \mathscr{F}}{\partial S} + \frac{\partial \mathscr{F}}{\partial I}]I_* + \mathscr{F}(h(I_*),I_*) - (\mu+d+r)$.
According to (H_2), (H_3) and (9), we get $K'(I_*) < 0$.
Hence, I_* is the unique solution of the equation $K(I) = 0$. Therefore, system (2) has a unique endemic equilibrium $\mathscr{E}_*(S_*,I_*)$ where $I_* \in (\frac{c}{\mu+d+r}, \frac{\mathscr{A}+b+c}{\mu+d+r})$ and $S_* = \frac{\mathscr{A}+b+c-(\mu+d+r)I_*}{\mu+\nu}$. □

4.2. Local stability of the endemic equilibrium.

Theorem 4.2. *The equilibrium $\mathscr{E}_*(S_*,I_*)$ of system (2) is locally asymptotically stable.*

Proof. The Jacobian matrix of system (2) at $\mathscr{E}_*$ is obtained as follows

$$J_* = \begin{pmatrix} -(\mu+\nu) - \frac{\partial \mathscr{F}}{\partial S}(S_*,I_*)I_* & -\mathscr{F}(S_*,I_*) - \frac{\partial \mathscr{F}}{\partial I}(S_*,I_*)I_* \\ \frac{\partial \mathscr{F}}{\partial S}(S_*,I_*)I_* & \mathscr{F}(S_*,I_*) + \frac{\partial \mathscr{F}}{\partial I}(S_*,I_*)I_* - a \end{pmatrix}.$$

Let $\det(J_*)$ and $tr(J_*)$ be respectively the determinant and the trace of the matrix J_*, which are given as

$$\det(J_*) = (\mu+\nu)[a - \mathscr{F}(S_*,I_*)] - (\mu+\nu)\frac{\partial \mathscr{F}}{\partial I}(S_*,I_*)I_* + a\frac{\partial \mathscr{F}}{\partial S}(S_*,I_*)I_*,$$

$$tr(J_*) = -(\mu+\nu) - \frac{\partial \mathscr{F}}{\partial S}(S_*,I_*)I_* + \frac{\partial \mathscr{F}}{\partial I}(S_*,I_*)I_* + [\mathscr{F}(S_*,I_*) - a].$$

According to (H_2),(H_3) and (9), we have $\det(J_*) > 0$ and $tr(J_*) < 0$. Hence, it follows that the endemic equilibrium point $\mathscr{E}_*$ is locally asymptotically stable. □

4.3. Global stability of the endemic equilibrium.

Theorem 4.3. *The equilibrium $\mathscr{E}_*(S_*,I_*)$ of system (2) is globally asymptotically stable.*

Proof. We define a Lyapunov function as follows

$$\mathscr{L}(t) = S(t) - S_* - \int_{S_*}^{S} \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(X,I_*)} dX + I_* \psi(\frac{I}{I_*}).$$

Calculating the time derivative of $\mathscr{L}$ along the positive solution of system (2), we get

$$\begin{aligned}
\frac{d\mathscr{L}}{dt} &= [\mathscr{A} + b - (\mu+\nu)S - \mathscr{F}(S,I)I](1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}) + (\mathscr{F}(S,I)I + c - aI)(1 - \frac{I_*}{I}) \\
&= [(\mu+\nu)S_*(1 - \frac{S}{S_*}) + \mathscr{F}(S_*,I_*)I_* - \mathscr{F}(S,I)I](1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}) \\
&\quad + [c + \mathscr{F}(S,I)I - \frac{c + \mathscr{F}(S_*,I_*)I_*}{I_*} I](1 - \frac{I_*}{I}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d\mathscr{L}}{dt} &= (\mu+\nu)S_*(1 - \frac{S}{S_*})(1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}) + (1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)})[\mathscr{F}(S_*,I_*)I_* - \mathscr{F}(S,I)I] \\
&\quad + [c(1 - \frac{I}{I_*}) + \mathscr{F}(S,I)I - \mathscr{F}(S_*,I_*)I](1 - \frac{I_*}{I}) \\
&= (\mu+\nu)S_*(1 - \frac{S}{S_*})(1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}) - c\frac{(I-I_*)^2}{II_*} \\
&\quad + (1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)})[\mathscr{F}(S_*,I_*)I_* - \mathscr{F}(S,I)I] + (1 - \frac{I_*}{I})[\mathscr{F}(S,I)I - \mathscr{F}(S_*,I_*)I] \\
&= (\mu+\nu)S_*(1 - \frac{S}{S_*})(1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}) - c\frac{(I-I_*)^2}{II_*} \\
&\quad + \mathscr{F}(S_*,I_*)I_*(1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)})(1 - \frac{\mathscr{F}(S,I)I}{\mathscr{F}(S_*,I_*)I_*}) \\
&\quad + \mathscr{F}(S_*,I_*)I_*(1 - \frac{I_*}{I})(\frac{\mathscr{F}(S,I)I}{\mathscr{F}(S_*,I_*)I_*} - \frac{I}{I_*}) \\
&= (\mu+\nu)S_*(1 - \frac{S}{S_*})(1 - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}) - c\frac{(I-I_*)^2}{II_*} \\
&\quad + \mathscr{F}(S_*,I_*)I_*[2 - \frac{I}{I_*} - \frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)} - \frac{\mathscr{F}(S,I)}{\mathscr{F}(S_*,I_*)} + \frac{\mathscr{F}(S,I)I}{\mathscr{F}(S,I_*)I_*}]
\end{aligned}$$

$$
\begin{aligned}
= \quad & (\mu+\nu)S_*(1-\frac{S}{S_*})(1-\frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)})-c\frac{(I-I_*)^2}{II_*} \\
& +\mathscr{F}(S_*,I_*)I_*[3-\frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}-\frac{\mathscr{F}(S,I)}{\mathscr{F}(S_*,I_*)}-\frac{\mathscr{F}(S,I_*)}{\mathscr{F}(S,I)}] \\
& +\mathscr{F}(S_*,I_*)I_*[-1-\frac{I}{I_*}+\frac{\mathscr{F}(S,I_*)}{\mathscr{F}(S,I)}+\frac{\mathscr{F}(S,I)I}{\mathscr{F}(S,I_*)I_*}].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{d\mathscr{L}}{dt} \quad = \quad & (\mu+\nu)S_*\left(1-\frac{S}{S_*}\right)\left(1-\frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}\right)-c\frac{(I-I_*)^2}{II_*}-\mathscr{F}(S_*,I_*)I_*\big[\psi(\frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}) \\
& +\psi(\frac{\mathscr{F}(S,I)}{\mathscr{F}(S_*,I_*)})+\psi(\frac{\mathscr{F}(S,I_*)}{\mathscr{F}(S,I)})\big]+\mathscr{F}(S_*,I_*)I_*\big[-1-\frac{I}{I_*}+\frac{\mathscr{F}(S,I_*)}{\mathscr{F}(S,I)}+\frac{\mathscr{F}(S,I)I}{\mathscr{F}(S,I_*)I_*}\big].
\end{aligned}
$$

According to (H_4), we have

$$
-1-\frac{I}{I_*}+\frac{\mathscr{F}(S,I_*)}{\mathscr{F}(S,I)}+\frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I_*)}\frac{I}{I_*}=\left(1-\frac{\mathscr{F}(S,I)}{\mathscr{F}(S,I_*)}\right)\left(\frac{\mathscr{F}(S,I_*)}{\mathscr{F}(S,I)}-\frac{I}{I_*}\right)\leq 0.
$$

Also, we have

$$
\left(1-\frac{S}{S_*}\right)\left(1-\frac{\mathscr{F}(S_*,I_*)}{\mathscr{F}(S,I_*)}\right)\leq 0.
$$

Therefore, $\frac{d\mathscr{L}}{dt}\leq 0$ with equality if and only if $S=S_*$ and $I=I_*$. It follows from LaSalle invariance principle that $\mathscr{E}_*$ is globally asymptotically stable. $\square$

5. Discussion and Conclusion

In this research, we have formulated a new and an adequate mathematical model to better describe the transmission of the COVID-19 disease epidemic. The model takes into account the immigration, vaccination and the general incidence function. First, we have proved the existence, positivity and boundedness of solutions. In absence of infected immigrants ($c=0$), the model has the disease-free equilibrium which is globally asymptotically stable when $R_0\leq 1$, it means that the disease is finally extinguished. When $R_0>1$, the disease persists and the model has a unique endemic equilibrium which is globally asymptotically stable. In addition, the sensitivity analysis is used to discover parameters that have impact on the threshold value R_0. When $c>0$, we have shown that the model has no disease-free equilibrium and proved that there exists a unique endemic equilibrium which is locally and globally asymptotically stable.

On the other hand, the study of the memory effect by using the new generalized Hattaf fractional derivative [13, 14] and also the impact of stochastic perturbations [15] on the dynamics of our developed model will be the main goal of our future works.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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