Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2022, 2022:43
https://doi.org/10.28919/cmbn/7297
ISSN: 2052-2541

STABILITY ANALYSIS OF A FRACTIONAL-ORDER TWO-STRAIN EPIDEMIC MODEL WITH GENERAL INCIDENCE RATES

AMINA ALLALI*, SAIDA AMINE

Laboratory of Mathematics, Computer Science and Applications (LMCSA), Faculty of Sciences and Technology, Hassan II University of Casablanca, PO Box 146, 20650 Mohammedia, Morocco

Abstract. This paper is devoted to the study of the dynamics of a fractional order two-strain SEIR epidemic model with two general incidence rates. The basic results of the fractional-order calculus are recalled. Four equilibrium points for the model are given, namely the disease free equilibrium, the endemic equilibrium with respect to strain 1, the endemic equilibrium with respect to strain 2, and the total endemic equilibrium with respect to both strains. Local and global stability analysis is given using the basic reproduction rate. First, the local stability of the equilibrium point is proved by the Routh Hurwitz criterion for the fractional-order system (FR-H), and then the global stability is shown by using the Barbalat's lemma to the fractional-order system (FB). The Barbalat's lemma is a reliable method for the asymptotic analysis of the fractional dynamic systems. Finally, numerical simulations illustrated our analytical results.

Keywords: Caputo fractional derivative; SEIR two-strain epidemic model; general incidence function; stability analysis; local stability; FR-H; global stability; Lyapunov function; FB.

2010 AMS Subject Classification: 93A30, 65M12.

1. INTRODUCTION

Infectious diseases are responsible for one third of global mortality. Mathematical modelling in epidemiology is a source of knowledge for understanding the spread of an epidemic and an

*Corresponding author

E-mail address: aminaallali86@gmail.com

Received February 23, 2022

effective tool for controlling and predicting the dynamics of an outbreaks. Several diseases mutate and develop strains. Then many researchers study the dynamic of multi-strain of infection diseases. The global stability analysis of two-strain epidemic model with bilinear and non-monotone incidence rates is studied in [1]. The Lyapunov function and global stability of a two-strain SEIR model with bilinear and non monotone incidence is analysed in [2]. The global dynamics of two-strain model with a single vaccination and general incidence rate is remarked in [3]. The global dynamics of a multi-strain SEIR epidemic model with general incidence rates: application to covid-19 is discussed in [4]. These models have some limitations as they are local and do not possess the memory effects that appear in most of the biological systems.

Fractional calculus is a very efficient and suitable tool for modeling real world problems in different areas of mathematics, engineering, biology, finance, economics and social sciences. In recent years, many authors interested by the fractional calculus of mathematical epidemic model and the lot of researches can provide useful information's about the memory effects. The analysis of a Caputo fractional-order model for covid-19 is investigated in [5]. Mathematical model of SIR epidemic system (covid-19) with fractional derivative is presented in [6]. A fractional order SEIR model with general incidence is discussed in [7, 8]. The Modeling and analysis of covid-19 epidemics with treatment in fractional derivative using real data from Pakistan are shown in [9]. Fractional-order SEIQRDP model for simulating the dynamics of covid-19 epidemic is analyzed in [10]. A fractional order SIR epidemic model with nonlinear incidence rate can be found in [11]. The global stability analysis of a fractional differential system in hepatitis B can be observed in [12].

Some authors are working on multiple strains in fractional calculus. For example, in [13] the authors study multi-strain tuberculosis (TB) model of variable-order fractional derivatives. The two-strain epidemic model involving fractional derivative with Mittag-Leffler Kermel is demonstrated in [14]. A fractional-order two-strain epidemic model with two vaccinations is presented in [15]. The Analysis of two avian influenza epidemic models involving fractal-fractional derivatives with power and Mittag-Leffler memories is investigated in [16].

Several authors are using the different incidence functions in the fractional models. The bilinear incidence function βS is investigated in [17]. The saturated incidence function $\frac{\beta S}{1+\alpha_1 S}$

or $\frac{\beta S}{1+\alpha_2 I}$ is used in [18, 19, 20]. The Beddington-DeAngelis incidence function $\frac{\beta S}{1+\alpha_1 S+\alpha_2 I}$ is studied in [21]. The specific nonlinear incidence function $\frac{\beta S}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI}$ is applied in [22, 23, 24].

Nowadays, the authors use the general incidence function [25, 26] which represent a large set of infection incidence rate and give more information about many diseases transmission. Motivated by the above discussion, and inspired by [4], this paper aims to propose a model based on the memorability nature of Caputo fractional-order derivative with general incidence rates.

Our model will be described by the system of equations

$$\begin{cases} D^{\alpha}S(t) = A - f_1(S,I_1)I_1(t) - f_2(S,I_2)I_2(t) - \mu S(t), \\ D^{\alpha}E_1(t) = f_1(S,I_1)I_1(t) - (\beta_1+\mu)E_1(t), \\ D^{\alpha}E_2(t) = f_2(S,I_2)I_2(t) - (\beta_2+\mu)E_2(t), \\ D^{\alpha}I_1(t) = \beta_1 E_1(t) - (\lambda_1+\mu)I_1(t), \\ D^{\alpha}I_2(t) = \beta_2 E_2(t) - (\lambda_2+\mu)I_2(t), \\ D^{\alpha}R(t) = \lambda_1 I_1(t) + \lambda_2 I_2(t) - \mu R(t), \end{cases} \tag{1.1}$$

with the following non-negative initial conditions

$$\begin{aligned} &S(0)\geq 0, \quad E_1(0)\geq 0, \quad E_2(0)\geq 0, \\ &I_1(0)\geq 0, \quad I_2(0)\geq 0, \quad R(0)\geq 0. \end{aligned} \tag{1.2}$$

The total population is divided into classes $S(t)$ the susceptible class, $E_1(t)$ and $E_2(t)$ are respectively the strain exposed class for $i=1,2$, $I_1(t)$ and $I_2(t)$ are respectively the strain infected class for $i=1,2$ and $R(t)$ is the recovered class for all $t\geq 0$. The model parameters and the conditions of the function incidence rate $f_1(S,I_1)$ and $f_2(S,I_2)$ are detailed in Section 3. The local and global stability of the system with two strains is analyzed. To investigate these results, we use FR-H criteria and the extended Barbalat's lemma with Lyapunov function. This paper is organized as follows. In Section 2, the preliminary results are presented. In Section 3, the mathematical model is presented in term of fractional differential equations. In Section 4, the positivity and boundedness of solution are studied. The basic reproduction number and the equilibrium points are given in Section 5. The local and the global stability analysis are proved in Section 6. Numerical simulations are given to performed our theoretical results in Section 7. Finally, Section 8 brings the concluding remarks.

2. Preliminary Results

Fractional calculus plays an important role in modern science. In this part, we present some fractional calculus definitions and we introduce several important theorems.

Definition 2.1. [27] *Let $f:\mathbb{R}\to\mathbb{R}$ be a differential function and $\alpha\in\mathbb{C}$ such that* $\mathrm{Re}(\alpha)>0$. *The Riemann-Liouville fractional integral of order α of $f(t)$, $t\in\mathbb{R}$, denoted by $J^{\alpha}f(t)$, is defined as*

$$J^{\alpha}f(t)=\phi_{\alpha}(t)*f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau, \tag{2.1}$$

where the symbol $*$ denotes the convolution product, $\phi_{\alpha}(t)$ is the Gel'fand-Shilov function, defined for $\alpha\notin\mathbb{Z}_{-}$, as $\phi_{\alpha}(t)=\begin{cases}\frac{t^{\alpha-1}}{\Gamma(\alpha)} & , \text{ if } t\geq 0\\ 0 & , \text{ if } t<0.\end{cases}$ and $\Gamma(\alpha)$ is the Gamma function.

Definition 2.2. [27] *Let $f:\mathbb{R}\to\mathbb{R}$ be an integrable function, $\alpha\in\mathbb{C}$ with* $\mathrm{Re}(\alpha)>0$, *$\alpha\notin\mathbb{N}$ and $n-1<\mathrm{Re}(\alpha)\leq n$, $t>0$. The Riemann- Liouville fractional derivative of order α of f is*

$$D^{\alpha}_{RL}f(t)=D^{n}\left[J^{n-\alpha}f(t)\right]=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau. \tag{2.2}$$

Definition 2.3. [27] *Let $f:\mathbb{R}\to\mathbb{R}$ be an differential function, $\alpha\in\mathbb{C}$ with* $\mathrm{Re}(\alpha)>0$ *and m the natural number, such that $m-1<\mathrm{Re}(\alpha)\leq m$, $t>0$. The Caputo fractional derivative of order α of f is defined as*

$$D^{\alpha}f(t)=J^{m-\alpha}D^{m}f(t)=\phi_{m-\alpha}*D^{m}f(t), \tag{2.3}$$

In particularly, if $0<\alpha\leq 1$ *we have*

$$D^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f'(s)}{(t-s)^{\alpha}}ds. \tag{2.4}$$

Property 2.4. [28] *The Laplace transform* $\mathscr{L}$ *of a Caputo fractional derivative of order* α *of f, satisfies*

$$\begin{aligned}\mathscr{L}\left[D^{\alpha}f(t)\right]&=\mathscr{L}\left[\phi_{m-\alpha}*D^{m}f(t)\right],\\&=\mathscr{L}\left[\phi_{m-\alpha}(t)\right]\mathscr{L}\left[D^{m}f(t)\right]=s^{\alpha-m}\mathscr{L}\left[D^{m}f(t)\right],\end{aligned}$$

where $\alpha\in\mathrm{C}$, $\mathrm{Re}(\alpha)>0$ *and* $m-1<\mathrm{Re}(\alpha)\leq m$*.*
As a result,

$$\mathscr{L}\left[D^{\alpha}f(t)\right]=s^{\alpha}F(s)-\sum_{k=0}^{m-1}f^{(k)}(0)s^{\alpha-k-1}. \tag{2.5}$$

With F is the Laplace transform of function f*.*

Definition 2.5. [29] *For any* $\alpha>0$*, the function* E_{α} *defined by* $E_{\alpha}(t)=\sum_{j=0}^{+\infty}\frac{t^j}{\Gamma(\alpha j+1)}$ *is called the Mittag-Leffler function.*

Let $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ with $n\geq 1$, consider the fractional order system

$$\begin{cases}D^{\alpha}x(t)=f(t,x(t))\\x(0)=x_0\end{cases},\text{where}\quad 0<\alpha\leq 1\quad\text{and}\quad x_0\in\mathbb{R}^n \tag{2.6}$$

Definition 2.6. *A point E is an equilibrium point of the fractional dynamic system* (2.6)*, if and only if* $f(t,E)=0$*. The fractional dynamic system* (2.6) *has the same equilibrium points as the integer-order system.*

Theorem 2.7. [30] *Assume that* f *satisfies the following conditions*
1. $f(t,x)$ *and* $\frac{\partial f}{\partial x}$ *are continuous with respect* $x\in\mathbb{R}^n$*;*
2. $\|f(t,x)\|\leq\omega+\lambda\|x\|\quad\forall x\in\mathbb{R}^n$*, for almost every* $t\in\mathbb{R}$ *and for all* $x\in\mathbb{R}^d$*, where* ω *and* λ *are two positive constants.*
Then, there exists an unique solution on $[0,+\infty)$ *solving the system* (2.6)*.*

3. Mathematical Model

The fractional order two-strain SEIR model with general incidence rates is presented in this section. And the system of equations is as follows

$$(3.1)\qquad \begin{cases} D^{\alpha}S(t) = A - f_1(S,I_1)I_1(t) - f_2(S,I_2)I_2(t) - \mu S(t), \\ D^{\alpha}E_1(t) = f_1(S,I_1)I_1(t) - (\beta_1+\mu)E_1(t), \\ D^{\alpha}E_2(t) = f_2(S,I_2)I_2(t) - (\beta_2+\mu)E_2(t), \\ D^{\alpha}I_1(t) = \beta_1 E_1(t) - (\lambda_1+\mu)I_1(t), \\ D^{\alpha}I_2(t) = \beta_2 E_2(t) - (\lambda_2+\mu)I_2(t), \\ D^{\alpha}R(t) = \lambda_1 I_1(t) + \lambda_2 I_2(t) - \mu R(t), \end{cases}$$

with the following non-negative initial conditions

$$(3.2)\qquad \begin{array}{lll} S(0)\geq 0, & E_1(0)\geq 0, & E_2(0)\geq 0, \\ I_1(0)\geq 0, & I_2(0)\geq 0, & R(0)\geq 0, \end{array}$$

and $N(t) = S(t)+E_1(t)+E_2(t)+I_1(t)+I_2(t)+R(t)$ denotes the total population at time $t \geq 0$. The biological description of the model parameters is given in table 1

The general incidence functions $f_1(S,I_1)$ and $f_2(S,I_2)$ stand for the infection transmission rates for strain 1 and strain 2, respectively. The incidence functions $f_1(S,I_1)$ and $f_2(S,I_2)$ are assumed to be continuously differential in the interior of $\mathbb{R}^2_+$ and satisfy the same properties as in [4, 31, 32].

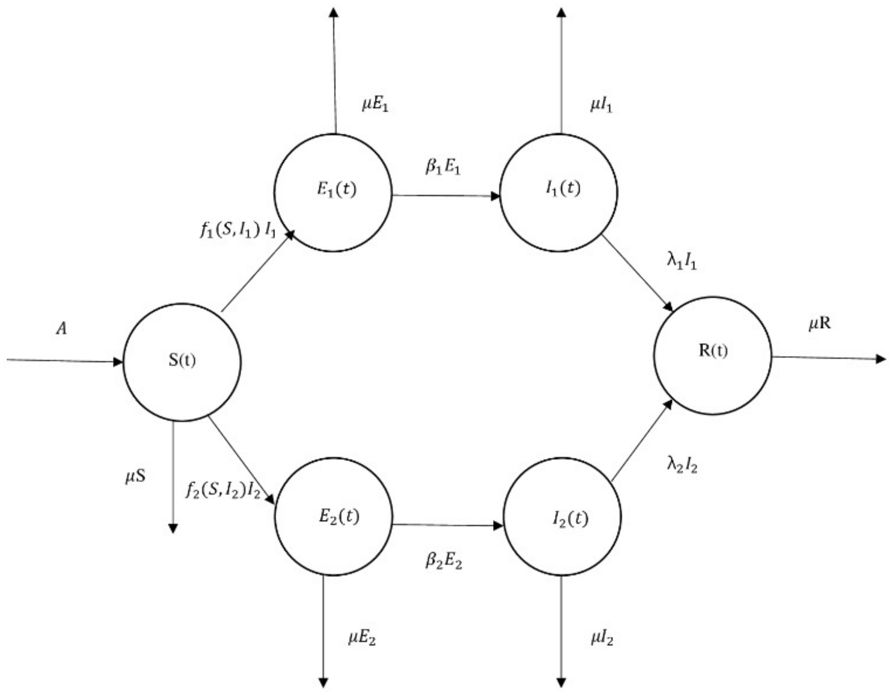

FIGURE 1. Fractional SEIR two-strain model

TABLE 1. **Description of the model parameters.**

Parameters of the system (3.1)	Description
A	Recruitment rate
β_i	Latency rate of strain i, i=1, 2
λ_i	Transfer rate from infected I_i to recovered, i=1, 2
μ	Death rate

We assume that $f_1(S,I_1)$ and $f_2(S,I_2)$ satisfy the following conditions

$$(3.3)\qquad \begin{cases} f_1(0,I_1) = f_2(0,I_2) = 0, \quad \text{for all } I_i \geq 0, \quad i=1,2, \quad (H_1) \\ \frac{\partial f_i}{\partial S}(S,I_i) > 0, \quad \forall S>0, \quad \forall I_i \geq 0, \quad i=1,2, \quad (H_2) \\ \frac{\partial f_i}{\partial I_i}(S,I_i) \leq 0, \quad \forall S \geq 0, \quad \forall I_i \geq 0, \quad i=1,2. \quad (H_3) \end{cases}$$

The properties (H_1), (H_2) and (H_3) are biologically verified by several classical incidence rates $f_i(i=1,2)$.

4. Positivity and Boundedness of Solution

Theorem 4.1. *Under the initial conditions, the fractional order system has a unique positive solution on $\mathbb{R}^6_+$. Moreover the closed set $\Omega = \{(S, E_1, E_2, I_1, I_2, R) \in \mathbb{R}^6_+ / N(t) \leq \frac{A}{\mu} + N(0)\}$ is positively invariant.*

Proof. Let $X(t) = (S(t), E_1(t), E_2(t), I_1(t), I_2(t), R(t))^T$ be in $\mathbb{R}^6_+$. The system (3.1) can be reformulated as follows: $D^\alpha X(t) = F(X(t))$, where

$$F(X) = \begin{pmatrix} A - f_1(S,I_1)I_1(t) - f_2(S,I_2)I_2(t) - \mu S(t), \\ f_1(S,I_1)I_1(t) - (\beta_1+\mu)E_1(t), \\ f_2(S,I_2)I_2(t) - (\beta_2+\mu)E_2(t), \\ \beta_1 E_1(t) - (\lambda_1+\mu)I_1(t), \\ \beta_2 E_2(t) - (\lambda_2+\mu)I_2(t), \\ \lambda_1 I_1(t) + \lambda_2 I_2(t) - \mu R(t). \end{pmatrix}, \tag{4.1}$$

It is obvious that F satisfies the first condition of Theorem 2.7.

To prove the second one, we denote

$$B = \begin{pmatrix} A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\beta_1+\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\beta_2+\mu) & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & -(\lambda_1+\mu) & 0 & 0 \\ 0 & 0 & \beta_2 & 0 & -(\lambda_2+\mu) & 0 \\ 0 & 0 & 0 & \lambda_1 & \lambda_2 & -\mu \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & -f_1(S,I_1) & -f_2(S,I_2) & 0 \\ 0 & 0 & 0 & f_1(S,I_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & f_2(S,I_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Then, we have

$$F\left(X(t)\right) = N_1X(t) + N_2X(t) + B.$$

Thus,

$$\|F(X)\| = \|N_1X(t)+N_2X(t)+B\| \leq \|B\| + \|N_1\|\|X(t)\| + \|N_2\|\|X(t)\| \leq w + (\|N_1\| + \|N_2\|)\|X(t)\|.$$

We conclude that

$$\|F(X)\| \leq w + \lambda\|X(t)\| \quad \text{where,} \quad w = \|B\| \quad \text{and } \lambda = (\|N_1\| + \|N_2\|). \tag{4.2}$$

By Theorem 2.7, the system (3.1) has a unique solution on $[0,+\infty[$.

For positively, we have

$$\begin{cases} D^{\alpha}S(t)|_{S=0} = A \geq 0, \\ D^{\alpha}E_1(t)|_{E_1=0} = f_1(S,I_1)I_1(t) \geq 0, \\ D^{\alpha}E_2(t)|_{E_2=0} = f_2(S,I_2)I_2(t) \geq 0, \\ D^{\alpha}I_1(t)|_{I_1=0} = \beta_1E_1(t) \geq 0, \\ D^{\alpha}I_2(t)|_{I_2=0} = \beta_2E_2(t) \geq 0, \\ D^{\alpha}R(t)|_{R=0} = \lambda_1I_1(t) + \lambda_2I_2(t) \geq 0. \end{cases} \tag{4.3}$$

Hence, under initial conditions the solution of (3.1) remains non-negative for all $t \geq 0$. To establish the boundedness of solution, we have $N(t) = S(t) + E_1(t) + E_2(t) + I_1(t) + I_2(t) + R(t)$, hence,

$D^{\alpha}N(t) = D^{\alpha}S + D^{\alpha}E_1 + D^{\alpha}E_2 + D^{\alpha}I_1 + D^{\alpha}I_2 + D^{\alpha}R$,

then,

$D^{\alpha}N(t) = A - \mu N(t)$,

hence,

$N(t) \leq N(0)E_{\alpha}(-\mu t^{\alpha})) + \frac{A}{\mu}(1 - E_{\alpha}(-\mu t^{\alpha}))$.

Since, $0 \leq E_{\alpha}(-\mu t^{\alpha})) \leq 1$, we obtain $N(t) \leq \frac{A}{\mu} + N(0)$ and the closed set

$\Omega = \{(S, E_1, E_2, I_1, I_2, R) \in \mathbb{R}_{+}^{6} / N(t) \leq \frac{A}{\mu} + N(0)\}$ is positively invariant. □

The first equations of the system (3.1) do not depend on the R and the population is constant, then we can omit the last equation of the system (3.1). So the problem can be reduced to

$$
\begin{cases}
D^{\alpha}S(t) = A - f_1(S, I_1)I_1(t) - f_2(S, I_2)I_2(t) - \mu S(t), \\
D^{\alpha}E_1(t) = f_1(S, I_1)I_1(t) - (\beta_1 + \mu)E_1(t), \\
D^{\alpha}E_2(t) = f_2(S, I_2)I_2(t) - (\beta_2 + \mu)E_2(t), \\
D^{\alpha}I_1(t) = \beta_1 E_1(t) - (\lambda_1 + \mu)I_1(t), \\
D^{\alpha}I_2(t) = \beta_2 E_2(t) - (\lambda_2 + \mu)I_2(t).
\end{cases} \tag{4.4}
$$

5. Equilibrium Points and Basic Reproduction Number

The study of the dynamics of an epidemic model is based on the threshold value of the basic reproduction number noted by $\mathscr{R}_0$.

In the model (3.1), the basic reproduction number $\mathscr{R}_0$ is obtained like in [4] and is given below $\mathscr{R}_0 = \max(\mathscr{R}_0^1, \mathscr{R}_0^2)$, where

$$
\mathscr{R}_0^1 = \frac{f_1\left(\frac{A}{\mu}, 0\right)\beta_1}{(\beta_1 + \mu)(\lambda_1 + \mu)} \quad \text{and} \quad \mathscr{R}_0^2 = \frac{f_2\left(\frac{A}{\mu}, 0\right)\beta_2}{(\beta_2 + \mu)(\lambda_2 + \mu)}.
$$

Let us

$$
d_1 = \beta_1 + \mu, \quad d_2 = \beta_2 + \mu, \quad d_3 = \lambda_1 + \mu, \quad d_4 = \lambda_2 + \mu.
$$

Then,

$$
\mathscr{R}_0^1 = \frac{f_1\left(\frac{A}{\mu}, 0\right)\beta_1}{d_1 d_3} \quad \text{and} \quad \mathscr{R}_0^2 = \frac{f_2\left(\frac{A}{\mu}, 0\right)\beta_2}{d_2 d_4}.
$$

By using the Definition 2.6, we calculate the equilibrium points of the system (4.4) by setting the right-hand of the system (4.4) equal to zero.

The equilibrium points are obtained as follows

$$
\begin{cases}
D^{\alpha}S(t) = A - f_1(S,I_1)I_1(t) - f_2(S,I_2)I_2(t) - \mu S(t) = 0, \\
D^{\alpha}E_1(t) = f_1(S,I_1)I_1(t) - d_1E_1(t) = 0, \\
D^{\alpha}E_2(t) = f_2(S,I_2)I_2(t) - d_2E_2(t) = 0, \\
D^{\alpha}I_1(t) = \beta_1E_1(t) - d_3I_1(t) = 0, \\
D^{\alpha}I_2(t) = \beta_2E_2(t) - d_4I_2(t) = 0.
\end{cases}
\tag{5.1}
$$

The steady states are [4]

- The disease free equilibrium noted $\overline{E}_f$=($\frac{A}{\mu}$,0,0,0,0) which exists when $\mathscr{R}_0 < 1$.
- The strain 1 endemic equilibrium noted $\overline{E}_{s_1} = \left(S_1^*, \frac{1}{d_1}\left(A - \mu S_1^*\right), 0, \frac{\beta_1}{d_1 d_3}\left(A - \mu S_1^*\right), 0\right)$ which exists if $\mathscr{R}_0^1 > 1$ and $S_1^* \in [0, \frac{A}{\mu}]$.
- The strain 2 endemic equilibrium noted $\overline{E}_{s_2} = \left(S_2^*, 0, \frac{1}{d_2}\left(A - \mu S_2^*\right), 0, \frac{\beta_2}{d_2 d_4}\left(A - \mu S_2^*\right)\right)$ which exists if $\mathscr{R}_0^2 > 1$ and $S_2^* \in [0, \frac{A}{\mu}]$.
- The total endemic equilibrium noted $\overline{E}_t = \left(S_t^*, E_{1,t}^*, E_{2,t}^*, I_{1,t}^*, I_{2,t}^*\right)$ exists when $\min(\mathscr{R}_0^1, \mathscr{R}_0^2) > 1$ and $A \geq \frac{f_1\left(\frac{A}{\mu},0\right)}{\mathscr{R}_0^1} I_{1,t}^* + \frac{f_2\left(\frac{A}{\mu},0\right)}{\mathscr{R}_0^2} I_{2,t}^*$ and where
  $E_{1,t}^* = \frac{d_3}{\beta_1} I_{1,t}^*$,
  $E_{2,t}^* = \frac{d_4}{\beta_2} I_{2,t}^*$,
  $S_t^* = \frac{1}{\mu}\left(A - \frac{f_1\left(\frac{A}{\mu},0\right)}{\mathscr{R}_0^1} I_{1,t}^* - \frac{f_2\left(\frac{A}{\mu},0\right)}{\mathscr{R}_0^2} I_{2,t}^*\right)$.

In the next section, we discuss the local and the global asymptotic stability of each equilibrium points.

6. STABILITY OF FRACTIONAL-ORDER SYSTEMS

6.1. Local stability analysis of $\overline{E}_f$,$\overline{E}_{s_1}$,$\overline{E}_{s_2}$,$\overline{E}_t$.

Theorem 6.1. [33] *The autonomous system: $D^{\alpha}x(t) = Gx(t)$ with $x(t_0) = x_0$ is asymptotically stable if and only if $|\arg(\mathrm{spec}(G))| > \frac{\alpha\pi}{2}$, where $\alpha \in [0,1)$, arg(.) is the principal argument of a given complex number and* spec(G) *is the spectrum (set of all eigenvalues) of G.*

Theorem 6.2. *$\overline{E}_f$ is locally asymptotically stable if* $\max(\mathscr{R}_0^1, \mathscr{R}_0^2) < 1$.

Proof. The Jacobien matrix for the system (4.4) evaluated at $\overline{E}_f$ is

$$J\left(\overline{E}_f\right) = \begin{pmatrix} -\mu & 0 & 0 & -f_1\left(\frac{A}{\mu},0\right) & -f_2(\frac{A}{\mu},0) \\ 0 & -d_1 & 0 & f_1(\frac{A}{\mu},0) & 0 \\ 0 & 0 & -d_2 & 0 & f_2(\frac{A}{\mu},0) \\ 0 & \beta_1 & 0 & -d_3 & 0 \\ 0 & 0 & \beta_2 & 0 & -d_4 \end{pmatrix}$$

One of the eigenvalues of $J(\overline{E}_f)$ is $\lambda_1 = -\mu < 0$.
Then, we consider the following matrix

$$P_1 = \begin{pmatrix} -d_1-\lambda & 0 & f_1(\frac{A}{\mu},0) & 0 \\ 0 & -d_2-\lambda & 0 & f_2(\frac{A}{\mu},0) \\ \beta_1 & 0 & -d_3-\lambda & 0 \\ 0 & \beta_2 & 0 & -d_4-\lambda \end{pmatrix} \tag{6.1}$$

The characteristic equation of P_1 can be written as

$$\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0, \tag{6.2}$$

where

$$b_3 = d_1 + d_3 + d_4 + d_2 > 0,$$

$$b_2 = d_2d_4\left(1-\mathscr{R}_0^2\right) + d_1d_3\left(1-\mathscr{R}_0^1\right) + (d_1+d_3)(d_4+d_2) > 0,$$

$$b_1 = d_2d_4(d_1+d_3)\left(1-\mathscr{R}_0^2\right) + d_1d_3(d_4+d_2)\left(1-\mathscr{R}_0^1\right) > 0,$$

$$b_0 = d_1d_2d_3d_4\left(1-\mathscr{R}_0^1-\mathscr{R}_0^2+\mathscr{R}_0^1\mathscr{R}_0^2\right) > 0.$$

If $\max(\mathscr{R}_0^1, \mathscr{R}_0^2) < 1$, we have

$$b_3 > 0, \quad b_2 > 0, \quad b_1 > 0, \quad b_0 > 0,$$

$$b_2 b_3 > b_1, \quad b_1(b_2 b_3 - b_1) > b_0 {b_3}^2.$$

Then by the FR-H all the eigenvalues of the P_1 matrix have negative real parts[34, 35]. Thus by the Theorem 6.1, $\overline{E}_f$ is locally asymptotically stable when $\max(\mathscr{R}_0^1, \mathscr{R}_0^2) < 1$. □

Let $J\left(\overline{E}_{s_1}\right)$ be the Jacobien matrix of the system (4.4) evaluated at $\overline{E}_{s_1}$

(6.3)
$$J\left(\overline{E}_{s_1}\right) = \begin{pmatrix} -\frac{\partial f_1(S_1^*, I_{1,s_1}^*) I_{1,s_1}^*}{\partial S} - \mu & 0 & 0 & -\frac{\partial f_1(S_1^*, I_{1,s_1}^*) I_{1,s_1}^*}{\partial I_1} - f_1(S_1^*, I_{1,s_1}^*) & -f_2(S_1^*, 0) \\ \frac{\partial f_1(S_1^*, I_{1,s_1}^*) I_{1,s_1}^*}{\partial S} & -d_1 & 0 & \frac{\partial f_1(S_1^*, I_{1,s_1}^*) I_{1,s_1}^*}{\partial I_1} + f_1(S_1^*, I_{1,s_1}^*) & 0 \\ 0 & 0 & -d_2 & 0 & f_2(S_1^*, 0) \\ 0 & \beta_1 & 0 & -d_3 & 0 \\ 0 & 0 & \beta_2 & 0 & -d_4 \end{pmatrix}$$

and $h_1(x)$ the characteristic equation given by

$$h_1(x) = x^5 + B_4 x^4 + B_3 x^3 + B_2 x^2 + B_1 x + B_0, \tag{6.4}$$

where, we noted $\mathscr{R}^2 = \frac{f_2(S_1^*, 0)\beta_2}{d_2 d_4}$ and

$$B_4 = d_1 + d_3 + d_4 + d_2 + \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu,$$

$$\begin{aligned} B_3 = {} & \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu\right)(d_1 + d_3 + d_4 + d_2) + d_1 d_3 + (d_1 + d_3)(d_4 + d_2) \\ & - \beta_1 \left(f_1(S_1^*, I_{1,s_1}^*) + \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial I_1} I_{1,s_1}^*\right) + d_4 d_2 \left(1 - \mathscr{R}^2\right), \end{aligned}$$

$$\begin{aligned} B_2 = {} & d_1 d_3 (d_4 + d_2) + \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu\right)((d_1 + d_3)(d_4 + d_2) + d_1 d_3) \\ & + d_4 d_2 \left(d_1 + d_3 + \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu\right)\left(1 - \mathscr{R}^2\right) \\ & - \beta_1 \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial I_1} I_{1,s_1}^* + f_1(S_1^*, I_{1,s_1}^*)\right)(d_4 + d_2 + \mu), \end{aligned}$$

$$
\begin{aligned}
B_1 = & \left[d_4 d_2 \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu \right) (d_1 + d_3) + d_1 d_3 d_4 d_2 \right] \left(1 - \mathscr{R}^2\right) \\
& + \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu \right) (d_4 + d_2) \left[d_1 d_3 - \beta_1 \left(f_1(S_1^*, I_{1,s_1}^*) + \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial I_1} I_{1,s_1}^* \right) \right] \\
& - \beta_1 d_4 d_2 \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial I_1} I_{1,s_1}^* + f_1(S_1^*, I_{1,s_1}^*) \right) \left(1 - \mathscr{R}^2\right) \\
& + \beta_1 (d_4 + d_2) \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial I_1} I_{1,s_1}^* + f_1(S_1^*, I_{1,s_1}^*) \right), \\
B_0 = & \, d_1 d_3 d_4 d_2 \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu \right) \left(1 - \mathscr{R}^2\right) \\
& - \beta_1 d_4 d_2 \left(f_1(S_1^*, I_{1,s_1}^*) + \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial I_1} I_{1,s_1}^* \right) \left(\frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* + \mu \right) \left(1 - \mathscr{R}^2\right) \\
& + \beta_1 d_4 d_2 \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial S} I_{1,s_1}^* \left(f_1(S_1^*, I_{1,s_1}^*) + \frac{\partial f_1(S_1^*, I_{1,s_1}^*)}{\partial I_1} I_{1,s_1}^* \right) \left(1 - \mathscr{R}^2\right).
\end{aligned}
$$

We obtain the following theorem

Theorem 6.3. *If $\mathscr{R}_0^2 < 1 < \mathscr{R}_0^1$ and B_i defined above for $(i = 0, 1, 2, 3, 4)$ satisfy the condition*

$$
\begin{cases}
B_4 B_3 - B_2 > 0, \\
B_2 (B_3 B_4 - B_2) - {B_4}^2 B_1 > 0, \\
(B_3 B_4 - B_2)(B_1 B_2 - B_0 B_3) - (B_1 B_4 - B_0)^2) > 0, \\
B_0 > 0,
\end{cases}
\tag{6.5}
$$

then $\overline{E}_{s_1}$ is locally asymptotically stable.

Proof. From the hypothesis (H_2), it is simple to check that B_4 is positive.
By the FR-H and under the condition (6.5) all roots of $h_1(x)$ have negative real parts [34, 35].
Then from Theorem 6.1, $\overline{E}_{s_1}$ is locally asymptotically stable. □

Let $J\left(\overline{E}_{s_2}\right)$ be the Jacobien matrix of the system (4.4) evaluated at $\overline{E}_{s_2}$

(6.6)
$$J\left(\overline{E}_{s_2}\right)=\begin{pmatrix} -\frac{\partial f_2(S_2^*,I_{2,s_2}^*)I_{2,s_2}^*}{\partial S}-\mu & 0 & 0 & -f_1(S_2^*,0) & -\frac{\partial f_2(S_2^*,I_{2,s_2}^*)I_{2,s_2}^*}{\partial I_2}-f_2(S_2^*,I_{2,s_2}^*) \\ 0 & -d_1 & 0 & f_1(S_2^*,0) & 0 \\ \frac{\partial f_2(S_2^*,I_{2,s_2}^*)I_{2,s_2}^*}{\partial S} & 0 & -d_2 & 0 & \frac{\partial f_2(S_2^*,I_{2,s_2}^*)I_{2,s_2}^*}{\partial I_2}+f_2(S_2^*,I_{2,s_2}^*) \\ 0 & \beta_1 & 0 & -d_3 & 0 \\ 0 & 0 & \beta_2 & 0 & -d_4 \end{pmatrix}$$

and $h_2(x)$ the characteristic equation given by

$$h_2(x)=x^5+C_4x^4+C_3x^3+C_2x^2+C_1x+C_0, \tag{6.7}$$

where, we noted $\mathscr{R}^1<\mathscr{R}_0^1<1$ and

$$\begin{aligned}
C_4 &= d_1+d_3+d_4+d_2+\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu,\\
C_3 &= \left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu\right)(d_1+d_3+d_4+d_2)+d_2d_4+(d_1+d_3)(d_4+d_2)\\
&\quad -\beta_2\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial I_2}I_{2,s_2}^*+f_2(S_2^*,I_{2,s_2}^*)\right)+d_1d_3\left(1-\mathscr{R}^1\right),
\end{aligned}$$

$$\begin{aligned}
C_2 &= d_2d_4\left(d_1+d_3\right)+\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu\right)\left((d_1+d_3)(d_4+d_2)+d_2d_4\right)\\
&\quad +d_1d_3\left(d_4+d_2+\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu\right)\left(1-\mathscr{R}^1\right)\\
&\quad -\beta_2\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial I_2}I_{2,s_2}^*+f_2(S_2^*,I_{2,s_2}^*)\right)\left(d_1+d_3+\mu\right),\\
C_1 &= \left[d_1d_3\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu\right)(d_2+d_4)+d_1d_3d_4d_2\right]\left(1-\mathscr{R}^1\right)\\
&\quad +\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu\right)(d_1+d_3)\left[d_4d_2-\beta_2\left(f_2(S_2^*,I_{2,s_2}^*)+\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial I_2}I_{2,s_2}^*\right)\right]\\
&\quad -\beta_2d_1d_3\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial I_2}I_{2,s_2}^*+f_2(S_2^*,I_{2,s_2}^*)\right)\left(1-\mathscr{R}^1\right)
\end{aligned}$$

$$+\beta_2(d_1+d_3)\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial I_2}I_{2,s_2}^*+f_2(S_2^*,I_{2,s_2}^*)\right),$$

$$\begin{aligned}
C_0 &= d_1d_3d_4d_2\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu\right)\left(1-\mathscr{R}^1\right)\\
&\quad -\beta_2d_1d_3\left(f_2(S_2^*,I_{2,s_2}^*)+\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial I_2}I_{2,s_2}^*\right)\left(\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*+\mu\right)\left(1-\mathscr{R}^1\right)\\
&\quad +\beta_2d_1d_3\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial S}I_{2,s_2}^*\left(f_2(S_2^*,I_{2,s_2}^*)+\frac{\partial f_2(S_2^*,I_{2,s_2}^*)}{\partial I_2}I_{2,s_2}^*\right)\left(1-\mathscr{R}^1\right).
\end{aligned}$$

and $\mathscr{R}^1=\frac{f_1(S_2^*,0)\beta_1}{d_1d_3}$.

By the same calculus as above, we obtain the following theorem

Theorem 6.4. *If $\mathscr{R}_0^1<1<\mathscr{R}_0^2$ and C_i defined above for $i=0,1,2,4$ satisfy the condition*

$$\begin{cases}
C_4C_3-C_2>0,\\
C_2\left(C_3C_4-C_2\right)-C_4{}^2C_1>0,\\
\left(C_3C_4-C_2\right)\left(C_1C_2-C_0C_3\right)-\left(C_1C_4-C_0\right)^2)>0,\\
C_0>0,
\end{cases}\tag{6.8}$$

then $\overline{E}_{s_2}$ is locally asymptotically stable .

Proof. From the hypothesis (H_2), it is simple to check that C_4 is positive.

By the FR-H and under the condition (6.8) all roots of $h_2(x)$ have negative real parts [34, 35].

Then from Theorem 6.1, $\overline{E}_{s_2}$ is locally asymptotically stable. □

Let $J\left(\overline{E}_t\right)$ be the Jacobien matrix of the system (4.4) evaluated at $\overline{E}_t$

$$J\left(\overline{E}_t\right)=\begin{pmatrix}
-\frac{\partial f_1(S_t^*,I_{1,s_t}^*)I_{1,s_t}^*}{\partial S}-\frac{\partial f_2(S_t^*,I_{2,s_t}^*)I_{2,s_t}^*}{\partial S}-\mu & 0 & 0 & -\frac{\partial f_1(S_t^*,I_{1,s_t}^*)I_{1,s_t}^*}{\partial I_1}-f_1(S_t^*,I_{1,s_t}^*) & -\frac{\partial f_2(S_t^*,I_{2,s_t}^*)I_{2,s_t}^*}{\partial I_2}-f_2(S_t^*,I_{2,s_t}^*)\\
\frac{\partial f_1(S_t^*,I_{1,s_t}^*)I_{1,s_t}^*}{\partial S} & -d_1 & 0 & \frac{\partial f_1(S_t^*,I_{1,s_t}^*)I_{1,s_t}^*}{\partial I_1}+f_1(S_t^*,I_{1,s_t}^*) & 0\\
\frac{\partial f_2(S_t^*,I_{2,s_t}^*)I_{2,s_t}^*}{\partial S} & 0 & -d_2 & 0 & \frac{\partial f_2(S_t^*,I_{2,s_t}^*)I_{2,s_t}^*}{\partial I_2}+f_2(S_2^*,I_{2,s_2}^*)\\
0 & \beta_1 & 0 & -d_3 & 0\\
0 & 0 & \beta_2 & 0 & -d_4
\end{pmatrix}\tag{6.9}$$

and $h_3(x)$ the characteristic equation given by

$$h_3(x)=x^5+F_4x^4+F_3x^3+F_2x^2+F_1x+F_0,\tag{6.10}$$

where

$$F_4 = d_1 + d_3 + d_2 + d_4 + \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^* + \mu\right),$$

$$\begin{aligned} F_3 &= d_2 d_4 + d_1 d_3 + (d_1 + d_3)(d_2 + d_4) \\ &+ (d_1 + d_3 + d_2 + d_4)\left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^* + \mu\right) \\ &- \beta_1 \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial I_1} I_{1,t}^* + f_1(S_t^*, I_{1,t}^*)\right) - \beta_2 \left(\frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial I_2} I_{2,t}^* + f_2(S_t^*, I_{2,t}^*)\right), \end{aligned}$$

$$\begin{aligned} F_2 &= d_1 d_3 (d_2 + d_4) + (d_1 + d_3) d_2 d_4 + \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^* + \mu\right) \\ &(d_1 d_3 + d_2 d_4 + (d_1 + d_3)(d_2 + d_4)) \\ &- \beta_1 \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^* + \mu\right)\left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial I_1} I_{1,t}^* + f_1(S_t^*, I_{1,t}^*)\right) \\ &+ \beta_1 \frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial I_1} I_{1,t}^* + f_1(S_t^*, I_{1,t}^*)\right) + \beta_2 \frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^* \left(\frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial I_2} I_{2,t}^* + f_2(S_t^*, I_{2,t}^*)\right) \\ &- \beta_2 \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^* + \mu\right)\left(\frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial I_2} I_{2,t}^* + f_2(S_t^*, I_{2,t}^*)\right) \\ &- \beta_2 (d_1 + d_3)\left(\frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial I_2} I_{2,t}^* + f_2(S_t^*, I_{2,t}^*)\right) - \beta_1 (d_2 + d_4)\left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial I_1} I_{1,t}^* + f_1(S_t^*, I_{1,t}^*)\right), \end{aligned}$$

$$\begin{aligned} F_1 &= d_1 d_3 d_4 d_2 + \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_2)}{\partial S} I_{2,t}^* + \mu\right)(d_1 d_3 (d_2 + d_4) + (d_1 + d_3) d_2 d_4) \\ &- \beta_1 \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial I_1} I_{1,t}^* + f_1(S_t^*, I_{1,t}^*)\right) \\ &\left[d_2 d_4 - (d_2 + d_4)\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + (d_2 + d_4)(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_2)}{\partial S} I_{2,t}^* + \mu)\right] \\ &- \beta_2 \left(\frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial I_2} I_{2,t}^* + f_2(S_t^*, I_{2,t}^*)\right) \\ &\left[d_1 d_3 + (d_1 + d_3)(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial S} I_{1,t}^* + \frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^* + \mu) - (d_1 + d_3)\frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial S} I_{2,t}^*\right] \\ &+ \beta_1 \beta_2 \left(\frac{\partial f_1(S_t^*, I_{1,t}^*)}{\partial I_1} I_{1,t}^* + f_1(S_t^*, I_{1,t}^*)\right)\left(\frac{\partial f_2(S_t^*, I_{2,t}^*)}{\partial I_2} I_{2,t}^* + f_2(S_t^*, I_{2,t}^*)\right), \end{aligned}$$

$$
\begin{aligned}
F_0 &= d_1d_3d_4d_2\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial S}I_{1,t}^* + \frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial S}I_{2,t}^* + \mu\right) \\
&- \beta_1 d_2 d_4\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial S}I_{1,t}^* + \frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial S}I_{2,t}^* + \mu\right)\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial I_1}I_{1,t}^* + f_1(S_t^*,I_{1,t}^*)\right) \\
&- \beta_2 d_1 d_3\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial S}I_{1,t}^* + \frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial S}I_{2,t}^* + \mu\right)\left(\frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial I_2}I_{2,t}^* + f_2(S_t^*,I_{2,t}^*)\right) \\
&+ \beta_1 d_2 d_4\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial S}I_{1,t}^*\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial I_1}I_{1,t}^* + f_1(S_t^*,I_{1,t}^*)\right) \\
&+ \beta_2 d_1 d_3\frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial S}I_{2,t}^*\left(\frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial I_2}I_{2,t}^* + f_2(S_t^*,I_{2,t}^*)\right) \\
&- \beta_1\beta_2\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial I_1}I_{1,t}^* + f_1(S_t^*,I_{1,t}^*)\right)\left(\frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial I_2}I_{2,t}^* + f_2(S_t^*,I_{2,t}^*)\right) \\
&\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial S}I_{1,t}^* + \frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial S}I_{2,t}^*\right) \\
&+ \beta_1\beta_2\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial S}I_{1,t}^* + \frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial S}I_{2,t}^* + \mu\right)\left(\frac{\partial f_1(S_t^*,I_{1,t}^*)}{\partial I_1}I_{1,t}^* + f_1(S_t^*,I_{1,t}^*)\right) \\
&\left(\frac{\partial f_2(S_t^*,I_{2,t}^*)}{\partial I_2}I_{2,t}^* + f_2(S_t^*,I_{2,t}^*)\right).
\end{aligned}
$$

By the same calculus as above, we obtain the following theorem

Theorem 6.5. *If* $\min(\mathscr{R}_0^1,\mathscr{R}_0^2) < 1$ *and* F_i *defined above for* $(i = 0,1,2,4)$ *satisfy the condition*

$$
\begin{cases}
F_4F_3 - F_2 > 0, \\
F_2\left(F_3F_4 - F_2\right) - {F_4}^2F_1 > 0, \\
\left(F_3F_4 - F_2\right)\left(F_1F_2 - F_0F_3\right) - \left(F_1F_4 - F_0\right)^2) > 0, \\
F_0 > 0,
\end{cases}
\tag{6.11}
$$

then $\overline{E}_t$ *is locally asymptotically stable .*

Proof. From the hypothesis (H_2), it is simple to check that F_4 is positive.
By the FR-H and under the condition (6.11) all roots of $h_3(x)$ have negative real parts [34, 35].
Then from Theorem 6.1, $\overline{E}_t$ is locally asymptotically stable. □

6.2. Global stability analysis of $\overline{E}_f$,$\overline{E}_{s_1}$,$\overline{E}_{s_2}$,$\overline{E}_t$.

To prove the global stability of the equilibrium points, $\overline{E}_f$, $\overline{E}_{s_1}$, $\overline{E}_{s_2}$ and $\overline{E}_t$, we need to introduce some results for the fractional systems and, we use the extend Barbalat's Lemma with the Lyapunov function to a fractional-order nonlinear system.

Proposition 6.6. [36, 37] *If $W : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function on $[t_0,\infty)$ and $J^{\alpha}|W|^p \leqslant M$ for all $t > t_0$, with $\alpha \in (0,1)$, p and M are two positive constants. Then $W(t) \to 0$ as $t \to \infty$.*

Lemma 6.7. *The solutions S,E_1,E_2,I_1,I_2 of system are uniformly continuous functions on $[0,+\infty[$.*

Proof. Each equation in system (4.4) can be written as follows

$$D^{\alpha}x_i(t) = g_i(x_i(t)), \tag{6.12}$$

where $x_1(t) = S(t)$, $x_2(t) = E_1(t)$, $x_3(t) = E_2(t)$, $x_4(t) = I_1(t)$, $x_5(t) = I_2(t)$,
and $g_1(x_1(t)) = D^{\alpha}S(t)$, $g_2(x_2(t)) = D^{\alpha}E_1(t)$, $g_3(x_3(t)) = D^{\alpha}E_2(t)$, $g_4(x_4(t)) = D^{\alpha}I_1(t)$, $g_5(x_5(t)) = D^{\alpha}I_2(t)$.
The solution of (6.12) satisfies the following integral equation

$$x_i(t) = x_i(0) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g_i(x_i(\tau))d\tau, \tag{6.13}$$

and for t, $s \in [0,+\infty[$ with $t \leq s$, we have

$$\begin{aligned}
\|x_i(s) - x_i(t)\| &\leq \left\| \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} g_i(x_i(\tau))d\tau - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g_i(x_i(\tau))d\tau \right\| \\
&\leq \left\| \int_0^t \left(\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right) g_i(x_i(\tau))d\tau + \int_t^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} g_i(x_i(\tau))d\tau \right\| \\
&\leq \frac{M}{\Gamma(\alpha)} \int_0^t \left[(s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1} \right] d\tau + \frac{M}{\Gamma(\alpha)} \int_t^s (s-\tau)^{\alpha-1} d\tau,
\end{aligned}$$

where $M = \max(\|g_i\|)$, for $1 \leq i \leq 4$.

By the change of variable $t-\tau = u_1$, $s-\tau = u_2$ and the inequality $(\xi^{\alpha}-\eta^{\alpha}) \leq (\xi-\eta)^{\alpha}$ for all $\xi, \eta > 0$, we conclude that

$$\begin{aligned}
\|x_i(t) - x_i(s)\| &\leq \frac{M}{\Gamma(\alpha)}\left(\int_{s-t}^{s} u_2^{\alpha-1} du_2 - \int_0^t u_1^{\alpha-1} du_1\right) + \frac{M}{\Gamma(\alpha)}\left(\int_0^{s-t} u_2^{\alpha-1} du_2\right) \\
&= \frac{M}{\Gamma(\alpha)}\left(\frac{u_2^{\alpha}}{\alpha}\Big|_{s-t}^{s} - \frac{u_1^{\alpha}}{\alpha}\Big|_0^{t} + \frac{u_2^{\alpha}}{\alpha}\Big|_0^{s-t}\right) \\
&= \frac{M}{\Gamma(\alpha)}\left(s^{\alpha} - (s-t)^{\alpha} - t^{\alpha} + (s-t)^{\alpha}\right) \\
&= \frac{M}{\Gamma(\alpha)}\left(s^{\alpha} - t^{\alpha}\right) \leq \frac{M}{\Gamma(\alpha)}(s-t)^{\alpha}.
\end{aligned}$$

Hence S, E_1, E_2, I_1 and I_2 are Hölder continuous functions.

Then they are uniformly continuous on $[0, +\infty[$. □

In (2.6), we noted by $\mathscr{L}f$ the Laplace transform of f. We recalled the classical result of Laplace transform theory.

Theorem 6.8. [38]*Assume that $\mathscr{L}f$ doses not have any singularities in the closed right half-plane $\{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda)\}$, except for possibly a simple pole at the origin. Then, $\lim_{t\to\infty} f(t) = \lim_{\lambda\to 0^+} \lambda(\mathscr{L}f)(\lambda)$.*

6.2.1. *Global stability of disease-free equilibrium.*

Theorem 6.9. *For $\max(\mathscr{R}_0^1, \mathscr{R}_0^2) \leq 1$, the disease free equilibrium point $\overline{E}_f$ is globally asymptotically stable on the interior of Ω.*

Proof. Let $V : \{(S, E_1, E_2, I_1, I_2) \in \Omega : S > 0\} \to \mathbb{R}$, a Lyapunov function defined in the following

$$V(S, E_1, E_2, I_1, I_2) = S - S_0^* - \int_{S_0^*}^{S} \frac{f_1(S_0^*, 0)}{f_1(X, 0)} dX + E_1 + E_2 + \frac{d_1}{\beta_1} I_1 + \frac{d_2}{\beta_2} I_2. \tag{6.14}$$

From [25], we have

$$D^{\alpha}V \leq D^{\alpha}S(t) - \frac{f_1(S_0^*, 0)}{f_1(S, 0)} D^{\alpha}S(t) + D^{\alpha}E_1 + D^{\alpha}E_2 + \frac{d_1}{\beta_1} D^{\alpha}I_1 + \frac{d_2}{\beta_2} D^{\alpha}I_2, \tag{6.15}$$

we obtain

$$D^{\alpha}V \leq -H(t), \tag{6.16}$$

where $H(t) = -\mu S_0^*\left(1-\frac{S}{S_0^*}\right)\left(1-\frac{f_1(S_0^*,0)}{f_1(S,0)}\right) - \frac{d_1 d_3}{\beta_1} I_1\left(\mathscr{R}_0^1 - 1\right) - \frac{d_2 d_4}{\beta_2} I_2\left(\frac{f_1(S_0^*,0)}{f_1(S,0)}\frac{f_2(S,I_2)}{f_2(S_0^*,0)}\mathscr{R}_0^2 - 1\right)$,

which is a defined positive function if $\max(\mathscr{R}_0^1,\mathscr{R}_0^2) \leq 1$ [4].

From (6.16), we conclude that $V(t) \leq V(0) - J^\alpha H(t)$.

Hence, $J^\alpha H(t) + V(t) \leq V(0)$.

Thus, $J^\alpha S_0^*\left(\frac{S}{S_0^*} - 1\right) \leq C$ where C=V(0).

From Lemma 6.7, $S_0^*\left(1-\frac{S}{S_0^*}\right)$ is uniform continuous and by Proposition 6.6, we deduced that $S_0^*\left(\frac{S}{S_0^*}-1\right) \to 0$ as $t \to \infty$.

Similarly we have $I_1 \to 0$ and $I_2 \to 0$ as $t \to \infty$.

From system (4.4) and using the Laplace transform, we obtain

$$\begin{aligned}
\mathscr{L}(D^\alpha E_1) &= \mathscr{L}\left(f_1\left(S,I_1\right)I_1\right) - d_1\mathscr{L}(E_1),\\
\lambda^\alpha\mathscr{L}(E_1) - \lambda^{\alpha-1}E_1(0) &= \mathscr{L}\left(f_1\left(S,I_1\right)I_1\right) - d_1\mathscr{L}(E_1),\\
\lambda^\alpha\mathscr{L}(E_1) + d_1\mathscr{L}(E_1) &= \mathscr{L}(f_1(S,I_1)I_1) + \lambda^{\alpha-1}E_1(0),\\
\mathscr{L}(E_1) &= \frac{\mathscr{L}(f_1(S,I_1)I_1)}{\lambda^\alpha + d_1} + \frac{\lambda^{\alpha-1}}{\lambda^\alpha + d_1}E_1(0).
\end{aligned}$$

This implies, by using Theorem 6.8 that

$$\begin{aligned}
\lim_{\lambda\to 0^+}\lambda\mathscr{L}(E_1) &= \lim_{\lambda\to 0^+}\frac{\mathscr{L}(\lambda f_1(S,I_1)I_1)}{\lambda^\alpha + d_1} + \lim_{\lambda\to 0^+}\frac{\lambda^\alpha}{\lambda^\alpha + d_1}E_1(0),\\
&= \frac{f_1(S_0^*,I_1^*)}{d_1}I_1^* = E_1^*.
\end{aligned}$$

Using Theorem 6.8 again, we get

$$\lim_{t\to\infty} E_1 = E_1^* = 0.$$

By the same calculation as the previously, we have

$$\lim_{t\to\infty} E_2 = E_2^* = 0.$$

Then, we conclude that

$S \to S_0^*$ as $t \to \infty$, $E_1 \to 0$ as $t \to \infty$, $E_2 \to 0$ as $t \to \infty$, $I_1 \to 0$ as $t \to \infty$, $I_2 \to 0$ as $t \to \infty$.

Therefore, $\lim\limits_{t\to\infty}(S,E_1,E_2,I_1,I_2) = (\frac{A}{\mu},0,0,0,0)$ independently of the initial data in the interior of Ω. This shows that $\overline{E}_f$ is globally asymptotically stable in the interior of Ω. □

6.2.2. *Global stability of strain 1 endemic equilibrium.* For the global stability of $\overline{E}_{s_1}$, we assume that the function f_1 satisfies the condition in the following

$$(H_4)\qquad \left(1-\frac{f_1(S,I_1)}{f_1(S,I^*_{1,s_1})}\right)\left(\frac{f_1(S,I^*_{1,s_1})}{f_1(S,I_1)}-\frac{I_1}{I^*_{1,s_1}}\right)\leq 0,\quad \forall S,I_1>0.$$

Theorem 6.10. *For $\mathscr{R}_0^2\leq 1<\mathscr{R}_0^1$, then the strain 1 endemic equilibrium is globally asymptotically stable on the interior of Ω.*

Proof. Let us define the following Lyapunov function $L_1:\{(S,E_1,E_2,I_1,I_2)\in\Omega : S,E_1,I_1>0\}\to\mathbb{R}$ as

$$\begin{aligned}L_1(S,E_1,E_2,I_1,I_2)=S-S_1^*-\int_{S_1^*}^{S}\frac{f_1(S_1^*,I^*_{1,s_1})}{f_1(X,I^*_{1,s_1})}dX+E_1^*\left(\frac{E_1}{E^*_{1,s_1}}-\ln\left(\frac{E_1}{E^*_{1,s_1}}\right)-1\right)+E_2\\ +\frac{d_1}{\beta_1}I^*_{1,s_1}\left(\frac{I_1}{I^*_{1,s_1}}-\ln\left(\frac{I_1}{I^*_{1,s_1}}\right)-1\right)+\frac{d_2}{\beta_2}I_2.\end{aligned}\tag{6.17}$$

From [25], we obtain

$$D^\alpha L_1\leq D^\alpha S\left(1-\frac{f_1\left(S_1^*,I^*_{1,s_1}\right)}{f_1\left(S,I^*_{1,s_1}\right)}\right)+D^\alpha E_1\left(1-\frac{E^*_{1,s_1}}{E_1}\right)+D^\alpha E_2+\frac{d_1}{\beta_1}D^\alpha I_1\left(1-\frac{I^*_{1,s_1}}{I_1}\right)+\frac{d_2}{\beta_2}D^\alpha I_2.\tag{6.18}$$

By the same calculations as above and under hypothesis (H_1-H_4), we have

$$D^\alpha L_1\leq -K(t),\tag{6.19}$$

where

$$\begin{aligned}K(t)=(\mu S-\mu S_1^*)\left(1-\frac{f_1(S_1^*,I^*_{1,s_1})}{f_1(S,I^*_{1,s_1})}\right)-d_1E^*_{1,s_1}\left(4-\frac{d_1E^*_{1,s_1}}{f_1(S,I^*_{1,s_1})I^*_{1,s_1}}-\frac{f_1(S,I_1)I_1}{d_1E_1}-\frac{I^*_{1,s_1}E_1}{I_1E^*_{1,s_1}}-\frac{f_1(S,I^*_{1,s_1})}{f_1(S,I_1)}\right)\\ -d_1E^*_{1,s_1}\left(1-\frac{f_1(S,I_1)}{f_1\left(S,I^*_{1,s_1}\right)}\right)\left(\frac{f_1\left(S,I^*_{1,s_1}\right)}{f_1(S,I_1)}-\frac{I_1}{I^*_{1,s_1}}\right)-\frac{d_2d_4}{\beta_2}I_2\left(\frac{f_1(S_1^*,I^*_{1,s_1})}{f_1(S,I^*_{1,s_1})}\mathscr{R}_0^2-1\right),\end{aligned}$$

which is defined and positive function, if $\mathscr{R}_0^2 \leq 1 < \mathscr{R}_0^1$ [4].

Then

(6.20)

$$K(t) \geq (\mu S - \mu S_1^*)\left(1 - \frac{f_1(S_1^*, I_{1,s_1}^*)}{f_1(S, I_{1,s_1}^*)}\right) - d_1 E_{1,s_1}^* \left(4 - \frac{d_1 E_{1,s_1}^*}{f_1(S, I_{1,s_1}^*) I_{1,s_1}^*} - \frac{f_1(S, I_1) I_1}{d_1 E_1} - \frac{I_{1,s_1}^* E_1}{I_1 E_{1,s_1}^*} - \frac{f_1(S, I_{1,s_1}^*)}{f_1(S, I_1)}\right)$$
$$- d_1 E_{1,s_1}^* \left(1 - \frac{f_1(S, I_1)}{f_1\left(S, I_{1,s_1}^*\right)}\right)\left(1 - \frac{I_1}{I_{1,s_1}^*}\right) - \frac{d_2 d_4}{\beta_2} I_2 \left(\frac{f(S_1^*, I_{1,s_1}^*)}{f(S, I_{1,s_1}^*)} \mathscr{R}_0^2 - 1\right).$$

From (6.19), we conclude that $L_1(t) \leq L_1(0) - J^\alpha K(t)$.

Hence, $J^\alpha K(t) + L_1(t) \leq L_1(0) = C'$.

This implies by (6.20), $J^\alpha(\mu S - \mu S_1^*) \leq C'$, $J^\alpha(\frac{I_1}{I_{1,s_1}^*} - 1) \leq C'$ and $J^\alpha I_2 \leq C'$.

Using Lemma 6.7, we obtain the uniform continuity for $\mu S - \mu S_1^*$, $\frac{I_1}{I_{1,s_1}^*} - 1$ and I_2.

From Proposition 6.6, we can write

$\mu S - \mu S_1^* \to 0$ as $t \to \infty$, $\frac{I_1}{I_{1,s_1}^*} - 1 \to 0$ as $t \to \infty$, $I_2 \to 0$ as $t \to \infty$.

As a result,

$S \to S_1^*$ as $t \to \infty$, $I_1 \to I_{1,s_1}^*$ as $t \to \infty$, $I_2 \to 0$ as $t \to \infty$.

From system (4.4) and using the Laplace transform, we obtain

$$\mathscr{L}(D^\alpha E_1) = \mathscr{L}\left(f_1(S, I_1) I_1\right) - d_1 \mathscr{L}(E_1),$$
$$\lambda^\alpha \mathscr{L}(E_1) - \lambda^{\alpha-1} E_1(0) = \mathscr{L}(f_1(S, I_1) I_1) - d_1 \mathscr{L}(E_1),$$
$$\lambda^\alpha \mathscr{L}(E_1) + d_1 \mathscr{L}(E_1) = \mathscr{L}(f_1(S, I_1) I_1) + \lambda^{\alpha-1} E_1(0),$$
$$\mathscr{L}(E_1) = \frac{\mathscr{L}(f_1(S, I_1) I_1)}{\lambda^\alpha + d_1} + \frac{\lambda^{\alpha-1}}{\lambda^\alpha + d_1} E_1(0).$$

This implies, by using Theorem 6.8 that

$$\lim_{\lambda \to 0^+} \lambda \mathscr{L}(E_1) = \lim_{\lambda \to 0^+} \frac{\mathscr{L}(\lambda f_1(S, I_1) I_1)}{\lambda^\alpha + d_1} + \lim_{\lambda \to 0^+} \frac{\lambda^\alpha}{\lambda^\alpha + d_1} E_1(0),$$
$$= \frac{f_1(S_1^*, I_{1,s_1}^*)}{d_1} I_{1,s_1}^* = E_{1,s_1}^*.$$

Using theorem 6.8 again, we get

$$\lim_{t \to \infty} E_1 = E_{1,s_1}^*.$$

As previously, we obtain

$$\lim_{t \to \infty} E_2 = E_{2,s_2}^* = 0.$$

Therefore, $\lim_{t\to\infty}(S,E_1,E_2,I_1,I_2) = (S_1^*, E_{1,s_1}^*, 0, I_{1,s_1}^*, 0)$, independently of the initial data in the interior of Ω. This shows that $\overline{E}_{s_1}$ is globally asymptotically stable in the interior of Ω. $\square$

6.2.3. *Global stability of strain 2 endemic equilibrium.* For the global stability of $\overline{E}_{s_2}$, we assume that the function f_2 satisfies the condition

$$(H_5) \qquad \left(1 - \frac{f_2(S,I_2)}{f_2\left(S, I_{2,s_2}^*\right)}\right)\left(\frac{f_2\left(S, I_{2,s_2}^*\right)}{f_2(S,I_2)} - \frac{I_2}{I_{2,s_2}^*}\right) \leq 0, \quad \forall S, I_2 > 0.$$

Theorem 6.11. *For $\mathscr{R}_0^1 \leq 1 < \mathscr{R}_0^2$, then the strain 2 endemic equilibrium is globally asymptotically stable on the interior of Ω.*

Proof. Let us define a Lyapunov function $L_2 : \{(S,E_1,E_2,I_1,I_2) \in \Omega : S, E_2, I_2 > 0\} \to \mathbb{R}$ as

(6.21)
$$L_2(S,E_1,E_2,I_1,I_2) = S - S_2^* - \int_{S_2^*}^{S} \frac{f_2\left(S_2^*, I_{2,s_2}^*\right)}{f_2\left(X, I_{2,s_2}^*\right)} dX + E_1 + E_{2,s_2}^*\left(\frac{E_2}{E_{2,s_2}^*} - \ln\left(\frac{E_2}{E_{2,s_2}^*}\right) - 1\right)$$
$$+ \frac{d_1}{\beta_1} I_1 + \frac{d_2}{\beta_2} I_{2,s_2}^*\left(\frac{I_2}{I_{2,s_2}^*} - \ln\left(\frac{I_2}{I_{2,s_2}^*}\right) - 1\right).$$

From [25], we obtain

(6.22)
$$D^\alpha L_2 \leq D^\alpha S\left(1 - \frac{f_2\left(S_2^*, I_{2,s_2}^*\right)}{f_2\left(S, I_{2,s_2}^*\right)}\right) + D^\alpha E_2\left(1 - \frac{E_{2,s_2}^*}{E_2}\right) + D^\alpha E_1 + \frac{d_2}{\beta_2} D^\alpha I_2\left(1 - \frac{I_{2,s_2}^*}{I_2}\right) + \frac{d_1}{\beta_1} D^\alpha I_1.$$

By the same calculations as above and under hypothesis $(H_1 - H_3)$ and (H_5), we have

$$D^\alpha L_2 \leq -M(t), \tag{6.23}$$

where

(6.24)
$$M(t) = (\mu S - \mu S_2^*)\left(1 - \frac{f_2\left(S_2^*, I_{2,s_2}^*\right)}{f_2\left(S, I_{2,s_2}^*\right)}\right) - d_2 E_{2,s_2}^*\left(4 - \frac{d_2 E_{2,s_2}^*}{f_2\left(S, I_{2,s_2}^*\right) I_{2,s_2}^*} - \frac{f_2(S,I_2) I_2}{d_2 E_2} - \frac{I_{2,s_2}^* E_2}{I_2 E_{2,s_2}^*} - \frac{f_2\left(S, I_{2,s_2}^*\right)}{f_2(S,I_2)}\right)$$
$$- d_2 E_{2,s_2}^*\left(1 - \frac{f_2(S,I_2)}{f_2(S,I_{2,s_2}^*)}\right)\left(\frac{f_2(S,I_{2,s_2}^*)}{f_2(S,I_2)} - \frac{I_2}{I_{2,s_2}^*}\right) - \frac{d_1 d_3}{\beta_1} I_1\left(\frac{f_2\left(S_2^*, I_{2,s_2}^*\right)}{f_2\left(S, I_{2,s_2}^*\right)} \mathscr{R}_0^1 - 1\right).$$

which is defined and positive function, if $\mathscr{R}_0^1 \leq 1 < \mathscr{R}_0^2$ [4].

Then

(6.25)

$$M(t) \geq (\mu S-\mu S_2^*)\left(1-\frac{f_2\left(S_2^*,I_{2,s_2}^*\right)}{f_2\left(S,I_{2,s_2}^*\right)}\right)-d_2E_{2,s_2}^*\left(4-\frac{d_2E_{2,s_2}^*}{f_2\left(S,I_{2,s_2}^*\right)I_{2,s_2}^*}-\frac{f_2(S,I_2)I_2}{d_2E_2}-\frac{I_{2,s_2}^*E_2}{I_2E_{2,s_2}^*}-\frac{f_2\left(S,I_{2,s_2}^*\right)}{f_2(S,I_2)}\right)$$

$$-d_2E_{2,s_2}^*\left(1-\frac{f_2(S,I_2)}{f_2(S,I_{2,s_2}^*)}\right)\left(1-\frac{I_2}{I_{2,s_2}^*}\right)-\frac{d_1d_3}{\beta_1}I_1\left(\frac{f_2\left(S_2^*,I_{2,s_2}^*\right)}{f_2\left(S,I_{2,s_2}^*\right)}\mathscr{R}_0^1-1\right).$$

From (6.23), $L_2(t) \leq L_2(0)-J^{\alpha}M(t)$.

Hence, $J^{\alpha}M(t)+L_2(t) \leq L_2(0)=C''$.

Then by (6.25) $J^{\alpha}(\mu S-\mu S_2^*) \leqslant C''$, $J^{\alpha}I_1 \leqslant C''$ and $J^{\alpha}(\frac{I_2}{I_{2,s_2}^*}-1) \leqslant C''$.

By Lemma 6.7, we obtain the uniform continuity from $\mu S-\mu S_2^*$, I_1 and $\frac{I_2}{I_{2,s_2}^*}-1$.

From Proposition 6.6, we can write

$\mu S-\mu S_2^* \to 0$ as $t\to\infty$, $I_1 \to 0$ as $t\to\infty$, $\frac{I_2}{I_{2,s_2}^*}-1 \to 0$ as $t\to\infty$.

As a result,

$S\to S_2^*$ as $t\to\infty$, $I_1\to 0$ as $t\to\infty$, $I_2\to I_{2,s_2}^*$ as $t\to\infty$.

From system (4.4) and using the Laplace transform, we obtain

$$\mathscr{L}(D^{\alpha}E_2)=\mathscr{L}\left(f_2(S,I_2)I_2\right)-d_2\mathscr{L}(E_2),$$

$$\lambda^{\alpha}\mathscr{L}(E_2)-\lambda^{\alpha-1}E_2(0)=\mathscr{L}(f_2(S,I_2)I_2)-d_2\mathscr{L}(E_2),$$

$$\lambda^{\alpha}\mathscr{L}(E_2)+d_2\mathscr{L}(E_2)=\mathscr{L}(f_2(S,I_2)I_2)+\lambda^{\alpha-1}E_2(0),$$

$$\mathscr{L}(E_2)=\frac{\mathscr{L}(f_2(S,I_2)I_2)}{\lambda^{\alpha}+d_2}+\frac{\lambda^{\alpha-1}}{\lambda^{\alpha}+d_2}E_2(0).$$

This implies, by using theorem 6.8 that

$$\begin{aligned}\lim_{\lambda\to 0^+}\lambda\mathscr{L}(E_2) &= \lim_{\lambda\to 0^+}\frac{\mathscr{L}(\lambda f_2(S,I_2)I_2)}{\lambda^{\alpha}+d_2}+\lim_{\lambda\to 0^+}\frac{\lambda^{\alpha}}{\lambda^{\alpha}+d_2}E_2(0),\\ &= \frac{f_2(S_2^*,I_{2,s_2}^*)}{d_2}I_{2,s_2}^*=E_{2,s_2}^*.\end{aligned}$$

Using theorem 6.8 again, we get

$$\lim_{t\to\infty}E_1=E_{1,s_1}^*=0.$$

As previously, we obtain

$$\lim_{t\to\infty}E_2=E_{2,s_2}^*.$$

Therefore, $\lim_{t\to\infty}(S,E_1,E_2,I_1,I_2) = (S_2^*,0,E_{2,s_2}^*,0,I_{2,s_2}^*)$, independently of the initial data in the interior of Ω. This shows that $\overline{E}_{s_2}$ is globally asymptotically stable in the interior of Ω. □

6.2.4. *Global stability of strain total endemic equilibrium.* For the global stability of $\overline{E}_t$, we assume that the functions f_1 and f_2 satisfies the following conditions

$$(H_6)\qquad \begin{gathered}\left(1-\frac{f_i(S,I_i)}{f_i(S_t^*,I_{i,t}^*)}\frac{f_j(S_t^*,I_{j,t}^*)}{f_j(S,I_{j,t}^*)}\right)\left(\frac{f_i(S_t^*,I_{i,t}^*)}{f_i(S,I_i)}\frac{f_j(S,I_{j,t}^*)}{f_j(S_t^*,I_{j,t}^*)}-\frac{I_i}{I_{i,t}^*}\right)\leq 0,\\ (i,j)=(1,2)\quad \text{or}\quad (i,j)=(2,1).\end{gathered}$$

Theorem 6.12. *If* $\min(\mathscr{R}_0^1,\mathscr{R}_0^2)>1$*, then the total endemic equilibrium* $\overline{E}_t$ *is globally asymptotically stable on the interior of* Ω.

Proof. Let us define a Lyapunov function $L_3:\{(S,E_1,E_2,I_1,I_2)\in\Omega : S,E_1,E_2,I_1,I_2>0\}\to\mathbb{R}$ as

$$\begin{aligned}L_3(S,E_1,E_2,I_1,I_2) &= S-S_t^*-\int_{S_t^*}^{S}\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(X,I_{1,t}^*\right)}dX+E_{1,t}^*\left(\frac{E_1}{E_{1,t}^*}-\ln\left(\frac{E_1}{E_{1,t}^*}\right)-1\right)\\ &\quad +E_{2,t}^*\left(\frac{E_2}{E_{2,t}^*}-\ln\left(\frac{E_2}{E_{2,t}^*}\right)-1\right)+\frac{d_1}{\beta_1}I_{1,t}^*\left(\frac{I_1}{I_{1,t}^*}-\ln\left(\frac{I_1}{I_{1,t}^*}\right)-1\right)\\ &\quad +\frac{d_2}{\beta_2}I_{2,t}^*\left(\frac{I_2}{I_{2,t}^*}-\ln\left(\frac{I_2}{I_{2,t}^*}\right)-1\right).\end{aligned}\tag{6.26}$$

From [25], we obtain

$$\begin{aligned}D^\alpha L_3 &\leq D^\alpha S\left(1-\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(S,I_{1,t}^*\right)}\right)+D^\alpha E_1\left(1-\frac{E_{1,t}^*}{E_1}\right)+D^\alpha E_2\left(1-\frac{E_{2,t}^*}{E_2}\right)+\frac{d_1}{\beta_1}D^\alpha I_1\left(1-\frac{I_{1,t}^*}{I_1}\right)\\ &\quad +\frac{d_2}{\beta_2}D^\alpha I_2\left(1-\frac{I_{2,t}^*}{I_2}\right).\end{aligned}\tag{6.27}$$

As previously and under hypothesis (H_1-H_4) and (H_6), we have

$$D^\alpha L_3 \leq -P_1(t),\tag{6.28}$$

where

(6.29)

$$\begin{aligned}
P_1(t) = &(\mu S-\mu S_t^*)\left(1-\frac{f\left(S_t^*,I_{1,t}^*\right)}{f\left(S,I_{1,t}\right)}\right)-d_1E_{1,t}^*\left(4-\frac{d_1E_{1,t}^*}{f_1\left(S_t^*,I_{1,t}^*\right)I_{1,t}^*}-\frac{f_1\left(S,I_1\right)I_1}{d_1E_1}-\frac{I_{1,t}^*E_1}{I_1E_{1,t}^*}-\frac{f_1\left(S,I_{1,t}^*\right)}{f_1\left(S,I_1\right)}\right)\\
&-d_2E_{2,t}^*\left(4-\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(S,I_{1,t}^*\right)}-\frac{f_2\left(S,I_2\right)I_2}{d_2E_2}-\frac{I_{2,t}^*E_2}{I_2E_{2,t}^*}-\frac{d_2E_{2,t}^*f_1\left(S,I_{1,t}^*\right)}{f_2\left(S,I_2\right)f_1\left(S_t^*,I_{1,t}^*\right)I_{2,t}^*}\right)\\
&-d_1E_{1,t}^*\left(1-\frac{f_1\left(S,I_1\right)}{f_1\left(S,I_{1,t}^*\right)}\right)\left(\frac{f_1\left(S,I_{1,t}^*\right)}{f_1\left(S,I_1\right)}-\frac{I_1}{I_{1,t}^*}\right)\\
&-d_2E_{2,t}^*\left(1-\frac{f_2\left(S,I_2\right)}{f_2\left(S_t^*,I_{2,t}^*\right)}\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(S,I_{1,t}^*\right)}\right)\left(\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_2\right)}\frac{f_1\left(S,I_{1,t}^*\right)}{f_1\left(S_t^*,I_{1,t}^*\right)}-\frac{I_2}{I_{2,t}^*}\right).
\end{aligned}$$

which is defined and positive function, if $\min(\mathscr{R}_0^1,\mathscr{R}_0^2)<1$[4].

Then

(6.30)

$$\begin{aligned}
P_1(t) \geq &(\mu S-\mu S_t^*)\left(1-\frac{f\left(S_t^*,I_{1,t}^*\right)}{f\left(S,I_{1,t}\right)}\right)-d_1E_{1,t}^*\left(4-\frac{d_1E_{1,t}^*}{f_1\left(S_t^*,I_{1,t}^*\right)I_{1,t}^*}-\frac{f_1\left(S,I_1\right)I_1}{d_1E_1}-\frac{I_{1,t}^*E_1}{I_1E_{1,t}^*}-\frac{f_1\left(S,I_{1,t}^*\right)}{f_1\left(S,I_1\right)}\right)\\
&-d_2E_{2,t}^*\left(4-\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(S,I_{1,t}^*\right)}-\frac{f_2\left(S,I_2\right)I_2}{d_2E_2}-\frac{I_{2,t}^*E_2}{I_2E_{2,t}^*}-\frac{d_2E_{2,t}^*f_1\left(S,I_{1,t}^*\right)}{f_2\left(S,I_2\right)f_1\left(S_t^*,I_{1,t}^*\right)I_{2,t}^*}\right)\\
&-d_1E_{1,t}^*\left(1-\frac{f_1\left(S,I_1\right)}{f_1\left(S,I_{1,t}^*\right)}\right)\left(1-\frac{I_1}{I_{1,t}^*}\right)\\
&-d_2E_{2,t}^*\left(1-\frac{f_2\left(S,I_2\right)}{f_2\left(S_t^*,I_{2,t}^*\right)}\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(S,I_{1,t}^*\right)}\right)\left(\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_2\right)}\frac{f_1\left(S,I_{1,t}^*\right)}{f_1\left(S_t^*,I_{1,t}^*\right)}-\frac{I_2}{I_{2,t}^*}\right).
\end{aligned}$$

From (6.28), $L_3(t)\leq L_3(0)-J^{\alpha}P_1(t)$.

Hence $J^{\alpha}P_1(t)+L_3(t)\leq L_3(0)=C^*$.

Then by (6.30), $J^{\alpha}(\mu S-\mu S_t^*)\leq C^*$ and $J^{\alpha}(\frac{I_1}{I_{1,t}^*}-1)\leq C^*$.

By Lemma 6.7, we obtain the uniform continuity of $\mu S-\mu S_t^*$ and $(\frac{I_1}{I_{1,t}^*}-1)$.

From the Proposition 6.6, we can write

$(\mu S-\mu S_t^*)\to 0$ as $t\to\infty$ and $(I_1-I_{1,t}^*)\to 0$ as $t\to\infty$.

As result,

$S\to S_t^*$ as $t\to\infty$ and $I_1\to I_{1,t}^*$ as $t\to\infty$.

From system (4.4) and using the Laplace Transform, we obtain

$$\mathscr{L}\left(D^{\alpha}E_1\right)=\mathscr{L}(f_1\left(S,I_1\right)I_1)-d_1\mathscr{L}\left(E_1\right),$$

$$\lambda^{\alpha}\mathscr{L}\left(E_1\right)-\lambda^{\alpha-1}\mathscr{L}\left(E_1(0)\right)=\mathscr{L}(f_1\left(S,I_1\right)I_1)-d_1\mathscr{L}\left(E1\right),$$

$$\lambda^{\alpha}\mathscr{L}(E_1)+d_1\mathscr{L}\left(E_1\right)=\mathscr{L}(f_1\left(S,I_1\right)I_1)+\lambda^{\alpha-1}E_1(0),$$

$$\mathscr{L}\left(E_1\right)=\frac{\mathscr{L}(f_1\left(S,I_1\right)I_1)}{\lambda^{\alpha}+d_1}+\frac{\lambda^{\alpha-1}}{\lambda^{\alpha}+d_1}E_1(0).$$

This implies

$$\lim_{\lambda\to 0^+}\lambda\mathscr{L}(E_1)=\lim_{\lambda\to 0^+}\frac{\mathscr{L}(\lambda f_1\left(S,I_1\right)I_1)}{\lambda^{\alpha}+d_1}+\lim_{\lambda\to 0^+}\frac{\lambda^{\alpha}}{\lambda^{\alpha}+d_1}E_1(0),$$
$$=\frac{f_1\left(S_t^*,I_{1,t}^*\right)I_{1,t}^*}{d_1}=E_{1,t}^*.$$

Using theorem 6.8 again, we get $\lim\limits_{t\to\infty}E_1=E_{1,t}^*$.

In order to prove the convergence of I_2, we consider another Lyapunov function $L_4:\{(S,E_1,E_2,I_1,I_2)\in\Omega:S,E_1,E_2,I_1,I_2>0\}\to\mathbb{R}$ as

$$\begin{aligned}L_4(S,E_1,E_2,I_1,I_2)&=S-S_t^*-\int_{S_t^*}^{S}\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(X,I_{2,t}^*\right)}dX\\&+E_{1,t}^*\left(\frac{E_1}{E_{1,t}^*}-\ln\left(\frac{E_1}{E_{1,t}^*}\right)-1\right)+E_{2,t}^*\left(\frac{E_2}{E_{2,t}^*}-\ln\left(\frac{E_2}{E_{2,t}^*}\right)-1\right)\\&+\frac{d_1}{\beta_1}I_{1,t}^*\left(\frac{I_1}{I_{1,t}^*}-\ln\left(\frac{I_1}{I_{1,t}^*}\right)-1\right)+\frac{d_2}{\beta_2}I_{2,t}^*\left(\frac{I_2}{I_{2,t}^*}-\ln\left(\frac{I_2}{I_{2,t}^*}\right)-1\right).\end{aligned}\tag{6.31}$$

From [25], we obtain

$$\begin{aligned}D^{\alpha}L_4\le D^{\alpha}S\left(1-\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_{2,t}^*\right)}\right)+D^{\alpha}E_1\left(1-\frac{E_{1,t}^*}{E_1}\right)+D^{\alpha}E_2\left(1-\frac{E_{2,t}^*}{E_2}\right)\\+\frac{d_1}{\beta_1}D^{\alpha}I_1\left(1-\frac{I_{1,t}^*}{I_1}\right)+\frac{d_2}{\beta_2}D^{\alpha}I_2\left(1-\frac{I_{2,t}^*}{I_2}\right).\end{aligned}\tag{6.32}$$

As above and under hypothesis (H_1-H_3) and (H_5-H_6), we have

$$D^{\alpha}L_4\le -P_2(t),\tag{6.33}$$

where

(6.34)

$$
\begin{aligned}
P_2(t) = & -(\mu S-\mu S_t^*)\left(1-\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_{2,t}^*\right)}\right)+d_2E_{2,t}^*\left(4-\frac{d_2E_{2,t}^*}{f_2\left(S,I_{2,t}^*\right)I_{2,t}^*}-\frac{f_2\left(S,I_2\right)I_2}{bE_2}-\frac{I_{2,t}^*E_2}{I_2E_{2,t}^*}-\frac{f_2\left(S,I_{2,t}^*\right)}{f_2\left(S,I_2\right)}\right)\\
& +d_1E_{1,t}^*\left(4-\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_{2,t}^*\right)}-\frac{f_1\left(S,I_1\right)I_1}{d_1E_1}-\frac{I_{1,t}^*E_1}{I_1E_{1,t}^*}-\frac{d_1E_{1,t}^*f_2\left(S,I_{1,t}^*\right)}{f_1\left(S,I_1\right)f_2\left(S_t^*,I_{2,t}^*\right)I_{1,t}^*}\right)\\
& +d_2E_{2,t}^*\left(1-\frac{f_2\left(S,I_2\right)}{f_2\left(S,I_{2,t}^*\right)}\right)\left(\frac{f_2\left(S,I_{2,t}^*\right)}{f_2\left(S,I_2\right)}-\frac{I_2}{I_{2,t}^*}\right)\\
& +d_1E_{1,t}^*\left(1-\frac{f_1\left(S,I_1\right)}{f_1\left(S_t^*,I_{1,t}^*\right)}\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_{2,t}^*\right)}\right)\left(\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(S,I_1\right)}\frac{f_2\left(S,I_{2,t}^*\right)}{f_2\left(S_t^*,I_{2,t}^*\right)}-\frac{I_1}{I_{1,t}^*}\right),
\end{aligned}
$$

which is defined and positive function, if $\min(\mathscr{R}_0^1,\mathscr{R}_0^2)<1$.

Then,

(6.35)

$$
\begin{aligned}
P_2(t) \geq & (\mu S-\mu S_t^*)\left(1-\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_{2,t}^*\right)}\right)+d_2E_{2,t}^*\left(4-\frac{d_2E_{2,t}^*}{f_2\left(S,I_{2,t}^*\right)I_{2,t}^*}-\frac{f_2\left(S,I_2\right)I_2}{d_2E_2}-\frac{I_{2,t}^*E_2}{I_2E_{2,t}^*}-\frac{f_2\left(S,I_{2,t}^*\right)}{f_2\left(S,I_2\right)}\right)\\
& -d_1E_{1,t}^*\left(4-\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_{2,t}^*\right)}-\frac{f_1\left(S,I_1\right)I_1}{d_1E_1}-\frac{I_{1,t}^*E_1}{I_1E_{1,t}^*}-\frac{d_1E_{1,t}^*f_2\left(S,I_{1,t}^*\right)}{f_1\left(S,I_1\right)f_2\left(S_t^*,I_{2,t}^*\right)I_{1,t}^*}\right)\\
& -d_2E_{2,t}^*\left(1-\frac{f_2\left(S,I_2\right)}{f_2\left(S,I_{2,t}^*\right)}\right)\left(1-\frac{I_2}{I_{2,t}^*}\right)\\
& -d_1E_{1,t}^*\left(1-\frac{f_1\left(S,I_1\right)}{f_1\left(S_t^*,I_{1,t}^*\right)}\frac{f_2\left(S_t^*,I_{2,t}^*\right)}{f_2\left(S,I_{2,t}^*\right)}\right)\left(\frac{f_1\left(S_t^*,I_{1,t}^*\right)}{f_1\left(S,I_1\right)}\frac{f_2\left(S,I_{2,t}^*\right)}{f_2\left(S_t^*,I_{2,t}^*\right)}-\frac{I_1}{I_{1,t}^*}\right).
\end{aligned}
$$

From (6.33), we have $L_4(t)\leq L_4(0)-J^{\alpha}P_2(t)$.

Hence $J^{\alpha}P_2(t)+L_4(t)\leq L_4(0)=C^{**}$.

Then by (6.35), $J^{\alpha}(\frac{I_2}{I_{2,t}^*}-1)\leq C^{**}$.

By the Lemma 6.7, we obtain the uniform continuity of $(\frac{I_2}{I_{2,t}^*}-1)$.

From the Proposition 6.6, we can write

$(I_2-I_{2,t}^*)\to 0$ as $t\to\infty$.

As result,

$I_2\to I_{2,t}^*$ as $t\to\infty$.

From system (4.4) and using the Laplace Transform, we obtain

$$\begin{aligned}\mathscr{L}\left(D^{\alpha}E_2\right) &= \mathscr{L}\left(f_2\left(S,I_2\right)I_2\right) - d_2\mathscr{L}\left(E_2\right),\\ \lambda^{\alpha}\mathscr{L}\left(E_2\right) - \lambda^{\alpha-1}\mathscr{L}\left(E_2(0)\right) &= \mathscr{L}\left(f_2\left(S,I_2\right)I_2\right) - d_2\mathscr{L}\left(E_2\right),\\ \lambda^{\alpha}\mathscr{L}(E_2) + d_2\mathscr{L}\left(E_2\right) &= \mathscr{L}(f_2\left(S,I_2\right)I_2) + \lambda^{\alpha-1}\mathscr{L}E_2(0),\\ \mathscr{L}(E_2) &= \frac{\mathscr{L}(f_2\left(S,I_2\right)I_2)}{\lambda^{\alpha}+d_2} + \frac{\lambda^{\alpha-1}}{\lambda^{\alpha}+d_2}\mathscr{L}E_2(0).\end{aligned}$$

This implies

$$\begin{aligned}\lim_{\lambda\to 0^+}\lambda\mathscr{L}(E_2) &= \lim_{\lambda\to 0^+}\frac{\lambda\mathscr{L}(f_2\left(S,I_2\right)I_2)}{\lambda^{\alpha}+d_2} + \lim_{\lambda\to 0^+}\frac{\lambda^{\alpha}}{\lambda^{\alpha}+d_2}E_2(0),\\ &= \frac{f_2(S_t^*,I_{2,t}^*)I_{2,t}^*}{d_2} = E_{2,t}^*.\end{aligned}$$

Using theorem 6.8 again, we get

$$\lim_{t\to\infty}E_2 = E_{2,t}^*.$$

Therefore, $\lim\limits_{t\to\infty}(S,E_1,E_2,I_1,I_2) = (S_t^*,E_{1,t}^*,E_{2,t}^*,I_{1,t}^*,I_{2,t}^*)$ independently of the initial data in the interior of Ω. This shows that $\overline{E}_t$ is globally asymptotically stable in the interior of Ω. □

7. Numerical Result

This section presents some numerical simulations and discussions of the Caputo-derivative two strain SEIR model. The proposed fractional model is solved numerically using a generalized predictor-corrector of the Adams–Bashforth–Moulton method [38, 39, 40]. We have also used the Matlab code fde12.m designed by Garrappa (2011) for the fracPECE iterative scheme. The simulations are conducted with different values of the order of the fractional derivative $\alpha = 1;\ 0.9;\ 0.8;\ 0.7$. In order to compare the ordinary epidemic model (when $\alpha = 1$) and the fractional epidemic model (when $0 < \alpha < 1$), the same biological parameters values with the same initial conditions as [4] are chosen. For different values of $0 < \alpha \leq 1$ and $f_1(S,I_1) = \theta_1 S$ and $f_2(S,I_2) = \theta_2 S$, we illustrated the impact of α on the speed of convergence towards the different equilibrium states.

7.1. Disease-free equilibrium stability. To show the global asymptotic stability of the disease free point, we calculated the basic reproduction numbers of the two strains of our fractional epidemic model and we find that it is the same as [4]. So, we have $\mathscr{R}_0^1 = 0.6 < 1$ and $\mathscr{R}_0^2 = 0.5263 < 1$. Figure 2 proves that for all different α (when $0 < \alpha \leq 1$), the disease is eliminated and the solution converges towards the same disease free equilibrium point $(5,0,0,0,0,0)$ as [4]. Consequently the numerical results show that when α is smaller than one, S, E_1, E_2, I_1, I_2 and R present a slower convergence to the free state of disease, when E_1, E_2, I_1, I_2, R towards to 0. The convergence to the free equilibrium point, when $\max(\mathscr{R}_0^1, \mathscr{R}_0^2) \leq 1$, supported our theorem 6.9.

7.2. Strain 1 endemic equilibrium stability. In [4], the solution converges towards the $\overline{E}_{s_1}$, when $\mathscr{R}_0^1 = 11.111 > 1$ and $\mathscr{R}_0^2 = 0.3609 < 1$. Here, under the same values of the parameters as [4] and by varying the value of α $(0 < \alpha \leq 1)$, we show that the convergence of solutions towards $\overline{E}_{s_1} = (1.8, 1.0667, 0, 0.7111, 0, 1.4222)$ becomes slow when α becomes small. For example, when $\alpha = 0.7$ in figure 3, we remark a long period of infection with the strain 1 unlike when $\alpha > 0.7$. This explains the memory nature of the fractional model.

7.3. Strain 2 endemic equilibrium stability. Figure 4 explains the memory nature of the fractional model with the strain 2 like figure 3.

7.4. Total strains endemic equilibrium stability. The last endemic equilibrium is characterized by the persistence of the infection with the two strains, when the basic reproduction numbers $\mathscr{R}_0^1 > 1$ and $\mathscr{R}_0^2 > 1$. The figure 5 shows the same convergence of the solution as [4] towards $\overline{E}_{s_t} = (0.8167, 0.5196, 0.6757, 0.7422, 0.9652, 1.2806)$. We see, in figure 5 that, if $\alpha = 0.7$, there is a long period of infection with the two strains I_1 and I_2 unlike when $\alpha > 0.7$.

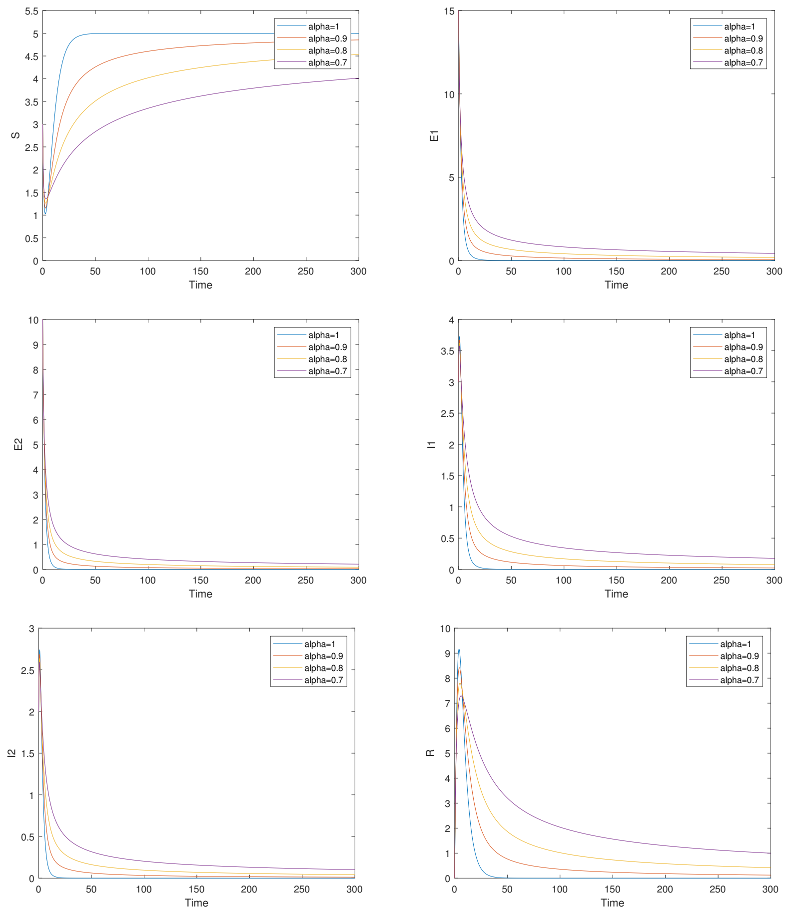

FIGURE 2. Time evolution of susceptible $S(t)$, the strain 1 latent individuals $E_1(t)$, the strain 2 latent individuals $E_2(t)$, the strain 1 infectious individuals $I_1(t)$, the strain 2 infectious individuals $I_2(t)$ and the recovered $R(t)$ illustrating the stability of the disease free equilibrium $\overline{E}_f$ with fractional order $\alpha \in \{0.7, 0.8, 0.9, 1\}$.

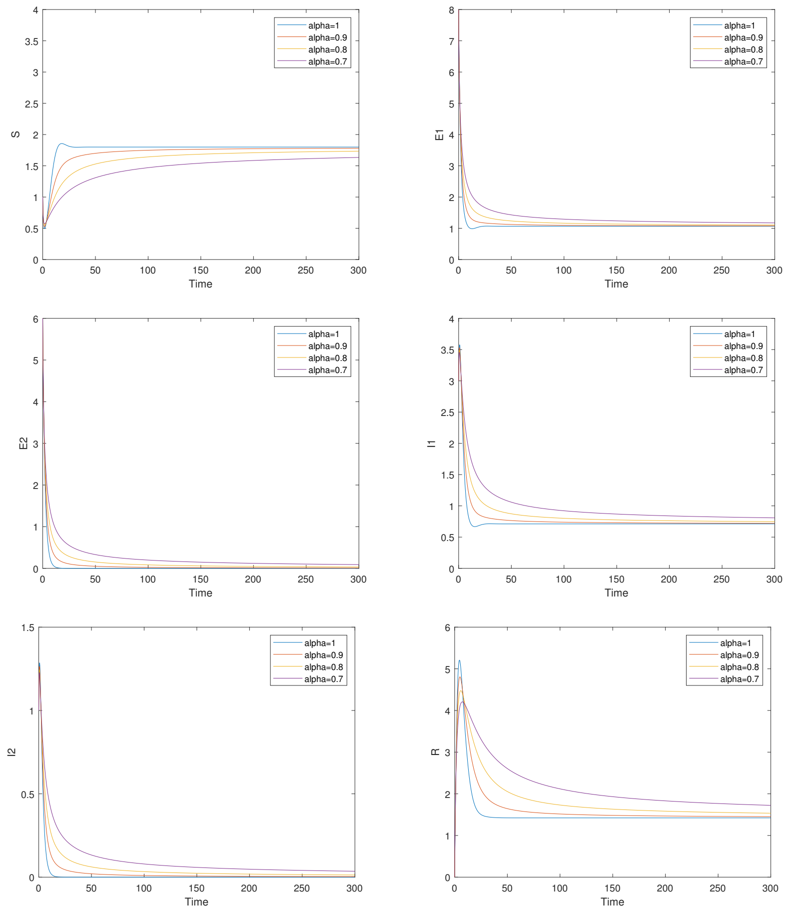

FIGURE 3. Time evolution of susceptible $S(t)$, the strain 1 latent individuals $E_1(t)$, the strain 2 latent individuals $E_2(t)$, the strain 1 infectious individuals $I_1(t)$, the strain 2 infectious individuals $I_2(t)$ and the recovered $R(t)$ illustrating the stability of the strain 1 endemic equilibrium $\overline{E}_{s_1}$ with fractional order $\alpha \in \{0.7, 0.8, 0.9, 1\}$.

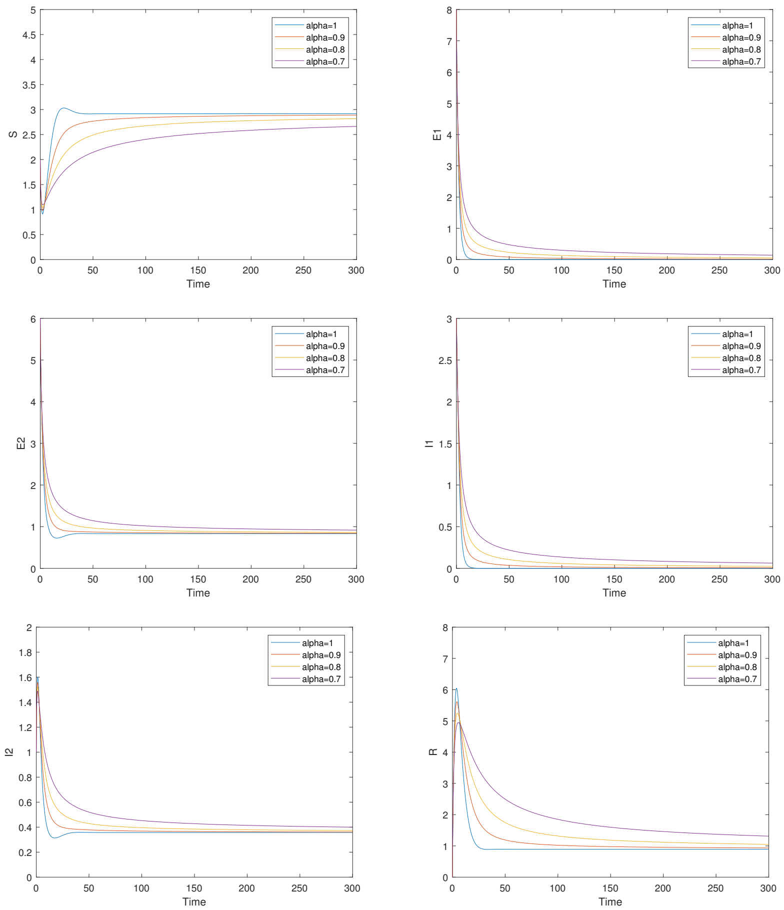

FIGURE 4. Time evolution of susceptible $S(t)$, the strain 1 latent individuals $E_1(t)$, the strain 2 latent individuals $E_2(t)$, the strain 1 infectious individuals $I_1(t)$, the strain 2 infectious individuals $I_2(t)$ and the recovered $R(t)$ illustrating the stability of the strain 2 endemic equilibrium $\overline{E}_{s_2}$ with fractional order $\alpha \in \{0.7, 0.8, 0.9, 1\}$.

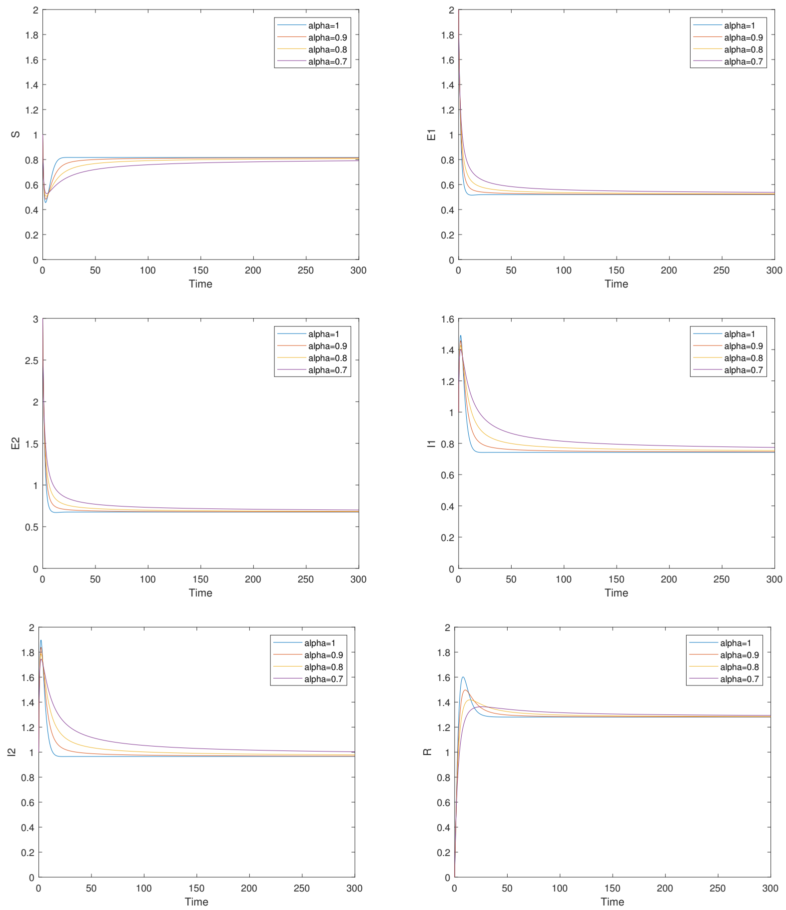

FIGURE 5. Time evolution of susceptible $S(t)$, the strain 1 latent individuals $E_1(t)$, the strain 2 latent individuals $E_2(t)$, the strain 1 infectious individuals $I_1(t)$, the strain 2 infectious individuals $I_2(t)$ and the recovered $R(t)$ illustrating the stability of the total endemic equilibrium $\overline{E}_{s_t}$ with fractional order $\alpha \in \{0.7, 0.8, 0.9, 1\}$.

8. Conclusion

In this work, we proposed a fractional order two-strain epidemic model with two general incidence rates. At the beginning some basic results of fractional-order system are recalled. The positivity and boundedness of solution are proved. Four equilibrium points and the basic reproduction rate are giving.

After that, the local stability analysis of equilibrium points is proved by applying the (FR-H) and the global stability of these equilibrium point is established by using (FB) and some fractional-order Lyapunov like lemma. The (FB) is a power tool for the asymptotic analysis of the fractional order dynamic.

In the end, some numerical simulations are giving to support our theoretical results and explain the impact of α in the model. Thus, our model fractional is more realistic than the ordinary model.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] I.A. Baba, E. Hincal, Global stability analysis of two-strain epidemic model with bilinear and non-monotone incidence rates, Eur. Phys. J. Plus. 132 (2017), 208. https://doi.org/10.1140/epjp/i2017-11476-x.

[2] D. Bentaleb, S. Amine, Lyapunov function and global stability for a two-strain SEIR model with bilinear and non-monotone incidence, Int. J. Biomath. 12 (2019), 1950021. https://doi.org/10.1142/s1793524519500219.

[3] A.J.N. May, E.J.A. Vales, Global dynamics of a two-strain flu model with a single vaccination and general incidence rate, ArXiv:2004.10713 [Math]. (2020). http://arxiv.org/abs/2004.10713.

[4] O. Khyar, K. Allali, Global dynamics of a multi-strain SEIR epidemic model with general incidence rates: application to COVID-19 pandemic, Nonlinear Dyn. 102 (2020), 489–509. https://doi.org/10.1007/s11071-020-05929-4.

[5] I. Ahmed, I.A. Baba, A. Yusuf, P. Kumam, W. Kumam, Analysis of Caputo fractional-order model for COVID-19 with lockdown, Adv. Differ. Equ. 2020 (2020), 394. https://doi.org/10.1186/s13662-020-02853-0.

[6] R.T. Alqahtani, Mathematical model of SIR epidemic system (COVID-19) with fractional derivative: stability and numerical analysis, Adv. Differ. Equ. 2021 (2021), 2. https://doi.org/10.1186/s13662-020-03192-w.

[7] Y. Yang, L. Xu, Stability of a fractional order SEIR model with general incidence, Appl. Math. Lett. 105 (2020), 106303. https://doi.org/10.1016/j.aml.2020.106303.

[8] M. Altaf Khan, S. Ullah, S. Ullah, M. Farhan, Fractional order SEIR model with generalized incidence rate, AIMS Math. 5 (2020), 2843–2857. https://doi.org/10.3934/math.2020182.

[9] P.A. Naik, M. Yavuz, S. Qureshi, J. Zu, S. Townley, Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan, Eur. Phys. J. Plus. 135 (2020), 795. https://doi.org/10.1140/epjp/s13360-020-00819-5.

[10] M.A. Bahloul, A. Chahid, T.-M. Laleg-Kirati, Fractional-order SEIQRDP model for simulating the dynamics of COVID-19 epidemic, IEEE Open J. Eng. Med. Biol. 1 (2020), 249–256. https://doi.org/10.1109/ojemb.2020.3019758.

[11] A. Mouaouine, A. Boukhouima, K. Hattaf, N. Yousfi, A fractional order SIR epidemic model with nonlinear incidence rate, Adv. Differ. Equ. 2018 (2018), 160. https://doi.org/10.1186/s13662-018-1613-z.

[12] L.C. Cardoso, R.F. Camargo, F.L.P. dos Santos, J.P.C. Dos Santos, Global stability analysis of a fractional differential system in hepatitis B, Chaos Solitons Fractals. 143 (2021), 110619. https://doi.org/10.1016/j.chaos.2020.110619.

[13] N.H. Sweilam, S.M. AL-Mekhlafi, Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives, J. Adv. Res. 7 (2016), 271–283. https://doi.org/10.1016/j.jare.2015.06.004.

[14] A. Yusuf, S. Qureshi, M. Inc, A.I. Aliyu, D. Baleanu, A.A. Shaikh, Two-strain epidemic model involving fractional derivative with Mittag-Leffler kernel, Chaos. 28 (2018), 123121. https://doi.org/10.1063/1.5074084.

[15] B. Kaymakamzade, E. Hincal, D. Amilo, A fractional-order two-strain epidemic model with two vaccinations, AIP Conf. Proc. 2325 (2021), 020048. https://doi.org/10.1063/5.0040309.

[16] B. Ghanbari, J.F. Gómez-Aguilar, Analysis of two avian influenza epidemic models involving fractal-fractional derivatives with power and Mittag-Leffler memories, Chaos. 29 (2019), 123113. https://doi.org/10.1063/1.5117285.

[17] Y. Li, F. Haq, K. Shah, M. Shahzad, G. ur Rahman, Numerical analysis of fractional order Pine wilt disease model with bilinear incident rate, J. Math. Computer Sci. 17 (2017), 420–428. https://doi.org/10.22436/jmcs.017.03.07.

[18] X. Wang, Z. Wang, X. Huang, Y. Li, Dynamic Analysis of a delayed fractional-order SIR model with saturated incidence and treatment functions, Int. J. Bifurcation Chaos. 28 (2018), 1850180. https://doi.org/10.1142/s0218127418501808.

[19] S.M. Simelane, P.G. Dlamini, A fractional order differential equation model for Hepatitis B virus with saturated incidence, Results Phys. 24 (2021), 104114. https://doi.org/10.1016/j.rinp.2021.104114.

[20] N. Ahmed, N. Shahid, Z. Iqbal, et al. Numerical modeling of seiqv epidemic model with saturated incidence rate, J. Appl. Environ. Biol. Sci. 8 (2018), 67-82.

[21] Swati, Nilam, Fractional order SIR epidemic model with Beddington–De Angelis incidence and Holling type II treatment rate for COVID-19, J. Appl. Math. Comput. (2022). https://doi.org/10.1007/s12190-021-01658-y.

[22] M. Naim, G. Benrhmach, F. Lahmidi, A. Namir, Local stability of a fractional order sis epidemic model with specific nonlinear incidence rate and time delay, Commun. Math. Biol. Neurosci. 2021 (2021), Article ID 33. https://doi.org/10.28919/cmbn/4409.

[23] S. Djillali, A. Atangana, A. Zeb, C. Park, Mathematical analysis of a fractional-order epidemic model with nonlinear incidence function, AIMS Math. 7 (2022), 2160–2175. https://doi.org/10.3934/math.2022123.

[24] M. Naim, F. Lahmidi, A. Namir, Stability analysis of a delayed fractional order sirs epidemic model with nonlinear incidence rate, Int. J. Appl. Math. 32 (2019), 733-745. https://doi.org/10.12732/ijam.v32i5.1.

[25] A. Boukhouima, E.M. Lotfi, M. Mahrouf, S. Rosa, D.F.M. Torres, N. Yousfi, Stability analysis and optimal control of a fractional HIV-AIDS epidemic model with memory and general incidence rate, Eur. Phys. J. Plus. 136 (2021), 103. https://doi.org/10.1140/epjp/s13360-020-01013-3.

[26] P.T. Karaji, N. Nyamoradi, Analysis of a fractional SIR model with general incidence function, Appl. Math. Lett. 108 (2020), 106499. https://doi.org/10.1016/j.aml.2020.106499.

[27] R. Gorenflo, F. Mainardi, Fractional calculus: Integral and differential equations of fractional order, ArXiv:0805.3823 [Cond-Mat, Physics:Math-Ph]. (2008). http://arxiv.org/abs/0805.3823.

[28] R.F. Camargo, E.C. de Oliveira, Cálculo fracionário, editora livraria da fısica, Sao Paulo, Brasil, 2015.

[29] I. Podlubny, Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic Press, San Diego, 1999.

[30] W. Lin, Global existence theory and chaos control of fractional differential equations, J. Math. Anal. Appl. 332 (2007), 709–726. https://doi.org/10.1016/j.jmaa.2006.10.040.

[31] K. Hattaf, M. Khabouze, N. Yousfi, Dynamics of a generalized viral infection model with adaptive immune response, Int. J. Dynam. Control. 3 (2014), 253–261. https://doi.org/10.1007/s40435-014-0130-5.

[32] K. Hattaf, N. Yousfi, A. Tridane, Mathematical analysis of a virus dynamics model with general incidence rate and cure rate, Nonlinear Anal.: Real World Appl. 13 (2012), 1866–1872. https://doi.org/10.1016/j.nonrwa.2011.12.015.

[33] D. Matignon, Stability results for fractional differential equations with applications to control processing, In: Computational Engineering in Systems Applications, vol. 2, 963–968, 1996.

[34] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, On some Routh–Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems, Phys. Lett. A. 358 (2006), 1–4. https://doi.org/10.1016/j.physleta.2006.04.087.

[35] A.E. Matouk, Stability conditions, hyperchaos and control in a novel fractional order hyperchaotic system, Phys. Lett. A. 373 (2009), 2166–2173. https://doi.org/10.1016/j.physleta.2009.04.032.

[36] R. Zhang, Y. Liu, A new Barbalat's lemma and Lyapunov stability theorem for fractional order systems, 2017 29th Chinese Control And Decision Conference (CCDC). (2017). https://doi.org/10.1109/ccdc.2017.7979143.

[37] F. Wang, Y. Yang, Fractional order barbalat's lemma and its applications in the stability of fractional order nonlinear systems, Math. Model. Anal. 22 (2017), 503-513.

[38] K. Diethelm, A.D. Freed, The fracpece subroutine for the numerical solution of differential equations of fractional order, Forschung und wissenschaftliches Rechnen, 1999 (1998), 57-71.

[39] A.R.M. Carvalho, C.M.A. Pinto, Non-integer order analysis of the impact of diabetes and resistant strains in a model for TB infection, Commun. Nonlinear Sci. Numer. Simul. 61 (2018), 104–126. https://doi.org/10.1016/j.cnsns.2018.01.012.

[40] K. Diethelm, N.J. Ford, A.D. Freed, Yu. Luchko, Algorithms for the fractional calculus: A selection of numerical methods, Computer Meth. Appl. Mech. Eng. 194 (2005), 743–773. https://doi.org/10.1016/j.cma.2004.06.006.