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EXISTENCE AND UNIQUENESS RESULTS OF SOLUTIONS FOR HATTAF-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATION TO EPIDEMIOLOGY

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for a class of fractional differential equations with new generalized Hattaf fractional derivative and time delay. An application from epidemiology is given to illustrate our main results.

Keywords: Hattaf fractional derivative; delayed fractional differential equation; fixed point theory.

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1. Introduction

Recently, the fractional differential equations (FDEs) play an important role in modeling and describing the memory and the hereditary properties of several material and dynamics of various phenomena. The existence and uniqueness of such equations have been investigated by many authors. For instance, Katugampola [1] derived the existence and uniqueness results for a class

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of FDEs with a generalized Caputo-Katugampola derivative introduced in [2]. In 2018, Janaki et al. [3] studied the existence results for a class of FDEs with delay and generalized Caputo-Katugampola derivative with time delay. In 2020, Karakoç [4] investigated the existence and uniqueness of the solutions of FDEs with time delay and Hilfer fractional derivative which generalizes the Riemann-Liouville fractional derivative.

The fractional derivatives used in the above works have singular kernels. In this paper, we study the existence and uniqueness of solutions for FDEs with delay involving the new generalized Hattaf fractional (GHF) derivative [5] which covers the Caputo-Fabrizio fractional derivative [6], the Atangana-Baleanu fractional derivative [7], and the weighted Atangana-Baleanu fractional derivative [8]. For the existence and uniqueness of FDEs without delay and GHF derivative was recently studied in [9].

The rest of this paper is outlined as follows. Section 2 is devoted to some interesting preliminaries needed to the elaboration of this work. Section 3 deals with the existence and uniqueness results for FDEs with delay and GHF derivative. Finally, the paper ends with an application in order to illustrate our main results.

2. Preliminaries

In this section, we give the necessary definitions and results that are needed for the proof of the main results.

Definition 2.1. *Let $\alpha \in [0,1)$, $\beta, \gamma > 0$, and $f \in H^1(a,b)$. We define the GHF derivative of order α in Caputo sense of the function $f(t)$ with respect to the weight function $w(t)$ as follows* [5],

$$ {}^{C}D_{a,t,w}^{\alpha,\beta,\gamma} f(t) = \frac{N(\alpha)}{1-\alpha}\frac{1}{w(t)}\int_a^t E_\beta[-\mu_\alpha (t-\tau)^\gamma]\frac{d}{d\tau}(wf)(\tau)d\tau, \tag{1}$$

where $w \in C^1(a,b)$, $w, w' > 0$ on [a,b], $N(\alpha)$ is a normalization function obeying $N(0) = N(1) = 1$, $\mu_\alpha = \dfrac{\alpha}{1-\alpha}$ and $E_\beta(t) = \sum_{k=0}^{+\infty}\dfrac{t^k}{\Gamma(\beta k+1)}$ is the Mittag-Leffler function of parameter β.

The GHF derivative introduced in the above definition generalizes and extends many special cases. In the fact, when $w(t) = 1$ and $\beta = \gamma = 1$, we get the Caputo-Fabrizio fractional derivative

[6] given by

$$^{C}D_{a,t,1}^{\alpha,1,1}f(t) = \frac{N(\alpha)}{1-\alpha}\int_a^t \exp[-\mu_\alpha(t-\tau)]f'(\tau)d\tau.$$

We obtain the Atangana-Baleanu fractional derivative [7] when $w(t) = 1$ and $\beta = \gamma = \alpha$, equation (1) is given by

$$^{C}D_{a,t,1}^{\alpha,\alpha,\alpha}f(t) = \frac{N(\alpha)}{1-\alpha}\int_a^t E_\alpha[-\mu_\alpha(t-\tau)^\alpha]f'(\tau)d\tau.$$

For $\beta = \gamma = \alpha$, we get the weighted Atangana–Baleanu fractional derivative [8] given by

$$^{C}D_{a,t,w}^{\alpha,\alpha,\alpha}f(t) = \frac{N(\alpha)}{1-\alpha}\frac{1}{w(t)}\int_a^t E_\alpha[-\mu_\alpha(t-\tau)^\alpha]\frac{d}{d\tau}(wf)(\tau)d\tau.$$

For simplicity, we denote $^{C}D_{a,t,w}^{\alpha,\beta,\beta}$ by $\mathscr{D}_{a,w}^{\alpha,\beta}$. By [5], the generalized fractional integral associated to $\mathscr{D}_{a,w}^{\alpha,\beta}$ is given by the following definition.

Definition 2.2. [5] *The generalized fractional integral operator associated to* $\mathscr{D}_{a,w}^{\alpha,\beta}$ *is defined by*

$$\mathscr{I}_{a,w}^{\alpha,\beta}f(t) = \frac{1-\alpha}{N(\alpha)}f(t) + \frac{\alpha}{N(\alpha)}\,{}^{RL}\mathscr{I}_{a,w}^{\beta}f(t), \tag{2}$$

where ${}^{RL}\mathscr{I}_{a,w}^{\beta}$ *is the standard weighted Riemann-Liouville fractional integral of order* β *defined by*

$$^{RL}\mathscr{I}_{a,w}^{\beta}f(t) = \frac{1}{\Gamma(\beta)}\frac{1}{w(t)}\int_a^t (t-\tau)^{\beta-1}w(\tau)f(\tau)d\tau. \tag{3}$$

Now, we recall an important theorem that we will need in the following. This theorem extends the Newton-Leibniz formula introduced in [10, 11].

Theorem 2.3. [12] *Let* $\alpha \in [0,1)$, $\beta > 0$ *and* $f \in H^1(a,b)$. *Then we have the following properties:*

$$\mathscr{I}_{a,w}^{\alpha,\beta}\left(\mathscr{D}_{a,w}^{\alpha,\beta}f\right)(t) = f(t) - \frac{w(a)f(a)}{w(t)}, \tag{4}$$

and

$$\mathscr{D}_{a,w}^{\alpha,\beta}\left(\mathscr{I}_{a,w}^{\alpha,\beta}f\right)(t) = f(t) - \frac{w(a)f(a)}{w(t)}. \tag{5}$$

On the other hand, to study the existence and uniqueness of solutions we need to the following lemma [13].

Lemma 2.4. *Let E be a Banach space, C be a convex subset of E, and U be an open in C with $0 \in U$. Then each compact map $F : \overline{U} \to C$ has at least one of the following properties:*

(i): *F has a fixed point,*

(ii): *there is a $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda F(u)$.*

3. Existence and Uniqueness Results

In this section, we study the existence and uniqueness of solutions for the Hattaf-type fractional differential equations given by

$$\begin{cases} \mathscr{D}^{\alpha,\beta}_{0,w} x(t) = f(t, x_t), & t \in [0,T], \\ x(t) = \phi(t), & t \in [-r,0], \end{cases} \tag{6}$$

where $f : [0,T] \times \mathscr{C}([-r,0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ and $\phi \in \mathscr{C}([-r,0], \mathbb{R}^n)$. For any function x defined on $[-r,T]$ and any $t \in [0,T]$, we denote by x_t the element of $\mathscr{C}([-r,0], \mathbb{R}^n)$ and is defined by

$$x_t(\theta) = x(t+\theta),\ \theta \in [-r,0].$$

Here, $\mathscr{C}([-r,T], \mathbb{R}^n)$ is the Banach space of continuous functions mapping from $[-r,T]$ into $\mathbb{R}^n$.

Definition 3.1. *A function $x \in \mathscr{C}([-r,T], \mathbb{R}^n)$ is said to be a solution of (6), if x satisfies the equation $\mathscr{D}^{\alpha,\beta}_{0,w} x(t) = f(t, x_t)$ on $[0,T]$ and the condition $x(t) = \phi(t)$ on $[-r,0]$.*

Theorem 3.2. *Let $f : [0,T] \times \mathscr{C}([-r,0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$. Assume that*
($\mathscr{H}_1$) there exists a constant $L > 0$ such that $\| f(t,x_1) - f(t,x_2) \| \leq L \| x_1 - x_2 \|$, for all $t \in [0,T]$ and $x_1, x_2 \in \mathscr{C}([-r,0], \mathbb{R}^n)$. If $L \left(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)} \right) < 1$, then there exists a unique solution for equation (6) on the interval $[-r,T]$.

Proof. Transform (6) into a fixed point problem. So, consider the following operator $A : \mathscr{C}([-r,T], \mathbb{R}^n) \to \mathscr{C}([-r,T], \mathbb{R}^n)$ defined by

$$A(x)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \frac{w(0)\phi(0)}{w(t)} + \frac{1-\alpha}{N(\alpha)} f(t, x(t)) + B(x)(t), & \text{if } t \in [0,T], \end{cases} \tag{7}$$

where

$$B(x)(t) = \frac{\alpha}{N(\alpha)\Gamma(\beta)w(t)} \int_0^t (t-s)^{\beta-1} w(s) f(s, x_s)\, ds.$$

Let $x,y \in \mathscr{C}([-r,T],\mathbb{R}^n)$. For all $t \in [0,T]$, we have

$$\begin{aligned}
|A(x)(t)-A(y)(t)| &\leq \frac{1-\alpha}{N(\alpha)}|f(t,x(t))-f(t,y(t))| \\
&\quad +\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|f(s,x_s)-f(s,y_s)|ds \\
&\leq \frac{1-\alpha}{N(\alpha)}L\|x-y\|+\frac{\alpha L}{N(\alpha)\Gamma(\beta)}\|x-y\|\int_0^t (t-s)^{\beta-1}ds \\
&\leq L\left(\frac{1-\alpha}{N(\alpha)}+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\right)\|x-y\|.
\end{aligned}$$

Hence,

$$\|A(x)-A(y)\| \leq L\left(\frac{1-\alpha}{N(\alpha)}+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\right)\|x-y\|.$$

Since $L\left(\frac{1-\alpha}{N(\alpha)}+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\right)<1$, we conclude that A is a contraction mapping. It follows from Banach contraction mapping principle that (6) has a unique solution. □

Theorem 3.3. *Assume that the following hypotheses holds:*

($\mathscr{H}_2$) $f:[0,T]\times\mathscr{C}([-r,0],\mathbb{R}^n)\to\mathbb{R}^n$ is a continuous function;

($\mathscr{H}_3$) there exist a continuous nondecreasing function $g:[0,\infty)\to(0,\infty)$ and a function $h\in\mathscr{C}([0,T],\mathbb{R}^+)$ such that

$$\|f(t,x)\|\leq h(t)g(\|x\|),\ \textit{for all}\ (t,x)\in[0,T]\times\mathscr{C}([-r,0],\mathbb{R}^n);$$

($\mathscr{H}_4$) there exists a constant $\nu>0$ such that

$$\frac{\nu}{\phi(0)+\left(\frac{1-\alpha}{N(\alpha)}+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\right)g(\|\nu\|)\|h\|}>1.$$

Then the equation (6) has at least one solution on $[-r,T]$.

Proof. Consider the same operator $A:\mathscr{C}([-r,T],\mathbb{R}^n)\to\mathscr{C}([-r,T],\mathbb{R}^n)$ defined by (7). First, we well prove that the operator A is compact through the following three steps.

Step 1: We need to prove that the operator A is continuous. In fact, let $\{x_n\}$ be a sequence such that $x_n \to x$ in $\mathscr{C}([-r,T],\mathbb{R}^n)$, and let $M>0$ such that $\|x_n\| \leq M$. Then

$$\begin{aligned}|A(x_n)(t)-A(x)(t)| &\leq \frac{1-\alpha}{N(\alpha)}|f(t,x_n(t))-f(t,x(t))| \\ &+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|f(s,x_{ns})-f(s,x_s)|\,\mathrm{d}s \\ &\leq \frac{1-\alpha}{N(\alpha)}\|f(.,x_n)-f(.,x)\| \\ &+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\|f(.,x_n)-f(.,x)\|\int_0^t (t-s)^{\beta-1}\ \mathrm{d}s \\ &\leq (\frac{1-\alpha}{N(\alpha)}+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)})\|f(.,x_n)-f(.,x)\|.\end{aligned}$$

Thus,

$$\|A(x_n)-A(x)\| \leq (\frac{1-\alpha}{N(\alpha)}+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)})\|f(.,x_n)-f(.,x)\|.$$

Since f is continuous function, we deduce that $\lim\limits_{n\to\infty}\|A(x_n)-A(x)\|=0$, which implies that the operator A is continuous.

Step 2: Let show that the operator A maps bounded sets into bounded sets in $C([-r,T],\mathbb{R}^n)$. For any $\mu>0$, we define the closed ball of radius μ in $C([-r,T],\mathbb{R}^n)$ by

$$B_\mu=\{x\in C([-r,T],\mathbb{R}^n):\|x\|\leq\mu\}.$$

Let $x\in B_\mu$. For all $t\in[0,T]$ and according to $(\mathscr{H}_3)$, we have

$$\begin{aligned}|A(x)(t)| &\leq \phi(0)+\frac{1-\alpha}{N(\alpha)}|f(t,x(t))| \\ &+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|f(s,x_s)|\mathrm{d}s \\ &\leq \phi(0)+\frac{1-\alpha}{N(\alpha)}g(\|x\|)\|h\|+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}g(\|x\|)\|h\| \\ &\leq \phi(0)+\left(\frac{1-\alpha}{N(\alpha)}+\frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\right)g(\mu)\|h\|.\end{aligned}$$

Hence, A is bounded.

Step 3: It remains to demonstrate that the operator A maps bounded sets into equicontinuous sets of $C([-r,T],\mathbb{R}^n)$. Let $t_1,t_2\in[0,T]$ with $t_1<t_2$, B_μ be a bounded set of $\mathscr{C}([-r,T],\mathbb{R}^n)$ as

in Step 2, and let $x \in B_\mu$. Then

$$\begin{aligned}
|A(x)(t_2) - A(x)(t_1)| \leq & \frac{1-\alpha}{N(\alpha)} |f(t_2, x(t_2)) - f(t_1, x(t_1))| \\
& + \frac{\alpha}{N(\alpha)\Gamma(\beta)} \int_0^{t_1} \left| (t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \right| |f(s, x_s)| \, \mathrm{d}s \\
& + \frac{\alpha}{N(\alpha)\Gamma(\beta)} \int_{t_1}^{t_2} |t_2 - s|^{\beta-1} |f(s, x_s)| \, \mathrm{d}s \\
& \leq \frac{1-\alpha}{N(\alpha)} |f(t_2, x(t_2)) - f(t_1, x(t_1))| \\
& + \frac{\alpha}{N(\alpha)\Gamma(\beta)} g(\mu) \|h\| \int_0^{t_1} \left| (t_2 - s)^{\beta-1} - (t_1 - s) |^{\beta-1} \right| \mathrm{d}s \\
& + \frac{\alpha}{N(\alpha)\Gamma(\beta)} g(\mu) \|h\| \int_{t_1}^{t_2} |t_2 - s|^{\beta-1} \, \mathrm{d}s.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious. In consequence of steps 1 to 3, it follows by the Arzela-Ascoli theorem that $A : C([-r, T], \mathbb{R}^n) \to C([-r, T], \mathbb{R}^n)$ is compact.

On the other hand, we prove that there exists an open set $U \subseteq C([-r, T], \mathbb{R}^n)$ with $x \neq \rho A(x)$ for $\rho \in (0, 1)$ and $x \in \partial U$. Let $x \in C([-r, T], \mathbb{R}^n)$ and $x = \rho A(x)$ for some $0 < \rho < 1$. For all $t \in [0, T]$, we have

$$\begin{aligned}
x(t) = & \rho \left(\frac{w(0)\phi(0)}{w(t)} + \frac{1-\alpha}{N(\alpha)} f(t, x(t)) \right. \\
& \left. + \frac{\alpha}{N(\alpha)\Gamma(\beta)w(t)} \int_0^t (t - s)^{\beta-1} w(s) f(s, x_s) \, \mathrm{d}s \right).
\end{aligned}$$

It follows from $(\mathscr{H}_3)$, that

$$\|x(t)\| \leq \phi(0) + \left(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)} \right) g(\|x\|) \|h\|.$$

Thus,

$$\frac{\|x\|}{\phi(0) + \left(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)} \right) g(\|x\|) \|h\|} \leq 1.$$

According to $(\mathscr{H}_4)$, there exists a ν such that $\|x\| \neq \nu$. Let us assume,

$$U = \{x \in C([-r, T], \mathbb{R}^n) : \|x\| < \nu\}.$$

Note that the operator $A : \overline{U} \to C([-r,b],\mathbb{R}^n)$ is compact. From the choice of U, there is no $x \in \partial U$ such that $x = \rho A(x)$ for some $\rho \in (0,1)$. Therefore, it follows from Lemma 2.4 that A has a fixed point $x \in \overline{U}$, which is a solution of (6). This ends the proof. □

4. APPLICATION

In this section, we give an example arising from epidemiology in order to illustrate our main results. So, we consider the following delayed FDE model with GHF derivative:

$$
\begin{cases}
\mathscr{D}_{0,1}^{\alpha,\beta} S(t) &= \Lambda - \sigma S(t) - \frac{\kappa S(t)I(t)}{1+\varepsilon_1 S(t)+\varepsilon_2 I(t)+\varepsilon_3 S(t)I(t)},\\
\mathscr{D}_{0,1}^{\alpha,\beta} I(t) &= \frac{\kappa S(t-\tau)I(t-\tau)e^{-\sigma\tau}}{1+\varepsilon_1 S(t-\tau)+\varepsilon_2 I(t-\tau)+\varepsilon_3 S(t-\tau)I(t-\tau)} - (\sigma+d+\delta)I(t),\\
\mathscr{D}_{0,1}^{\alpha,\beta} R(t) &= \delta I(t) - \sigma R(t),
\end{cases} \tag{8}
$$

where $S(t), I(t)$ and $R(t)$ are the susceptible, infected and recovered individuals at time t, respectively. Λ is the recruitment rate of the population, σ is the natural death rate of the population, d is the death rate due to disease, δ is the recovery rate of the infective individual. The term $\frac{\kappa SI}{1+\varepsilon_1 S+\varepsilon_2 I+\varepsilon_3 SI}$, where $\varepsilon_1,\varepsilon_2,\varepsilon_3 \geq 0$ are constants and $\kappa > 0$ is the infection coefficient, represents the Hattaf specific incidence rate [14], which models the disease transmission process. The time delay τ is the incubation period. The term $e^{-\sigma\tau}$ is the probability of surviving from time $t-\tau$ to time t.

For epidemiological reasons, we assume that the initial conditions of (8) satisfy

$$
S(\theta) = \Phi_1(\theta), I(t) = \Phi_2(\theta), R(t) = \Phi_3(\theta),\ \theta \in [-\tau,0]. \tag{9}
$$

Let $\varphi = (\varphi_1,\varphi_2,\varphi_3)^T \in \mathscr{C}([-\tau,0],\mathbb{R}^3)$. We define

$$
f = (f_1,f_2,f_3) : \mathscr{C}([-\tau,0],\mathbb{R}^3) \to \mathbb{R}^3
$$

by

$$
\begin{cases}
f_1(\varphi) &= \Lambda - \sigma\varphi_1(0) - \dfrac{\kappa\varphi_1(0)\varphi_2(0)}{1+\varepsilon_1\varphi_1(0)+\varepsilon_2\varphi_2(0)+\varepsilon_3\varphi_1(0)\varphi_2(0)},\\
f_2(\varphi) &= \dfrac{\kappa\varphi_1(-\tau)\varphi_2(-\tau)e^{-\sigma\tau}}{1+\varepsilon_1\varphi_1(-\tau)+\varepsilon_2\varphi_2(-\tau)+\varepsilon_3\varphi_1(-\tau)\varphi_2(-\tau)} - (\sigma+d+\delta)\varphi_2(0),\\
f_3(\varphi) &= \delta\varphi_2(0) - \sigma\varphi_3(0).
\end{cases} \tag{10}
$$

Then problem (8)-(9) can be rewritten as the following functional differential equation

$$\begin{cases} \mathscr{D}_{0,1}^{\alpha,\beta} x(t) = f(x_t), & t \in [0,T], \\ x(t) = \Phi(t), & t \in [-\tau, 0], \end{cases} \tag{11}$$

where $x = (S, I, R)^T$ and $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T$. Moreover, the function f satisfies the Lipschitz condition as

$$\| f(x_1) - f(x_2) \| \leq L \| x_1 - x_2 \| . \tag{12}$$

By applying Theorem 3.2, we conclude that problem (8)-(9) has a unique solution if $L\big(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha T^\beta}{N(\alpha)\Gamma(\beta+1)}\big) < 1$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] U.N. Katugampola, Existence and uniqueness results for a class of generalized fractional differential equations, ArXiv:1411.5229 [Math]. (2016). http://arxiv.org/abs/1411.5229.

[2] U.N. Katugampola, A new approach to generalized fractional derivatives, Bull. Math. Anal. App. 6 (2014), 1–15.

[3] M. Janaki, K. Kanagarajan, D. Vivek, Existence results for Katugampola-type fractional functional differential equations, Int. J. Math. And Appl. 6 (2018), 13–20.

[4] F. Karakoç, Existence and uniqueness for fractional order functional differential equations with Hilfer derivative, Differ. Equ. Appl. 12 (2020), 323–336. https://doi.org/10.7153/dea-2020-12-21.

[5] K. Hattaf, A new generalized definition of fractional derivative with non-singular kernel, Computation. 8 (2020), 49. https://doi.org/10.3390/computation8020049.

[6] A. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 1 (2015), 73-85.

[7] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, Therm Sci. 20 (2016), 763–769. https://doi.org/10.2298/TSCI160111018A.

[8] M. Al-Refai, On weighted Atangana–Baleanu fractional operators, Adv. Differ. Equ. 2020 (2020), 3. https://doi.org/10.1186/s13662-019-2471-z.

[9] Kh. Hattaf, A.A. Mohsen, H.F. Al-Husseiny, Gronwall inequality and existence of solutions for differential equations with generalized Hattaf fractional derivative, J. Math. Computer Sci. 27 (2022), 18–27. https://doi.org/10.22436/jmcs.027.01.02.

[10] J.D. Djida, A. Atangana, I. Area, Numerical computation of a fractional derivative with non-local and non-singular kernel, Math. Model. Nat. Phenom. 12 (2017), 4–13. https://doi.org/10.1051/mmnp/201712302.

[11] D. Baleanu, A. Fernandez, On some new properties of fractional derivatives with Mittag-Leffler kernel, Commun. Nonlinear Sci. Numer. Simul. 59 (2018), 444–462. https://doi.org/10.1016/j.cnsns.2017.12.003.

[12] K. Hattaf, On some properties of the new generalized fractional derivative with non-singular kernel, Math. Probl. Eng. 2021 (2021), 1580396. https://doi.org/10.1155/2021/1580396.

[13] A. Granas, J. Dugundji, Fixed point theory, Springer, New York, 2003.

[14] K. Hattaf, N. Yousfi, A. Tridane, Stability analysis of a virus dynamics model with general incidence rate and two delays, Appl. Math. Comput. 221 (2013), 514–521. https://doi.org/10.1016/j.amc.2013.07.005.