Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2022, 2022:42
https://doi.org/10.28919/cmbn/7376
ISSN: 2052-2541

THE EFFECT OF FEAR ON THE DYNAMICS OF TWO COMPETING PREY-ONE PREDATOR SYSTEM INVOLVING INTRA-SPECIFIC COMPETITION

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Abstract: A three-species food web model consisting of two competing prey – one predator with fear is created mathematically in the current work. Intra-specific competition within the predator's population, as well as a modified Holling type II functional response, are used. The study's goal is to look at the role of fear and intra-specific competition. Following a discussion of the solution's existence and uniqueness, other dynamical features of the solution, such as stability, persistence, and local bifurcation, were studied. Ultimately, the system is studied numerically with Matlab to fully understand global dynamics and the impact of altering parameter values. Different dynamical behaviors are discovered, such as stable point, stable line, and bi-stability.

Keywords: food-web; fear; intra-specific competition; stability; persistence; bifurcation.

2010 AMS Subject Classification: 92D40, 34D23, 37G10.

1. INTRODUCTION

The prey-predator relationship is one of the most essential instruments in the ecological system. Due to its worldwide occurrence and relevance, the interactive features among predators and their

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Received March 21, 2022

prey have long been considered and will remain to be one of the core topics across both biology and computational ecology. Mathematics has had a huge impact in recent decades as a mechanism for describing and comprehending biological processes. As a result, biologists have presented mathematicians with a variety of difficult issues, resulting in advancements in the field of nonlinear differential equations. This form of the differential equation has long been significant in the study of theoretical population structure, and it will likely remain so in the future. One of the most well-known uses of mathematics in biology is the building of system dynamics models for species interactions. Because the effect of indirect predation is greater than that of direct predation, the victim's fear of the predator is also crucial. In recent years, several biologists, experimentalists, and theorists have investigated the implications of supplying extra food to predators in prey-predator systems, see [1] and the references therein. As a result of the implementation and use of analytical methodologies, as well as the expansion of computational power, our knowledge of these models has grown. Despite tremendous advances in the prey-predator theory, several hard mathematical and ecological problems remain unsolved. Experimenters and theoreticians face additional challenges as the complexity of differential equations and dimensions grows.

Freedman and Waltman [2] investigated three-level food webs with two competing predators feeding on a sole prey as well as a single predator preying on competing prey species. They help to ensure the system's long-term viability. Predation and competition are frequently thought to be major factors affecting species' coexistence in ecological systems [3-4]. Previous research [5–8] has largely focused on two species, making it difficult to explain how fear affects predation rates when multiple species are present. Most studies also ignore the impact of fear on predation rates. Predator species' indirect influence on prey species has a higher impact than direct killing, according to current field studies. As a result, the current research examines predator anxiety and how it affects the behavior of competing prey species as well as predation rates. The competition factor, according to [9-10], indicates a situation in which the environment has few resources and both populations compete for survival. Several academics have recently looked into prey-predator models with two predators competing for prey [11-13].

In a constrained resource setting, Firdiansyah and Nurhidayati [14] developed a prey-predator model with two predators eating a single prey. They described the feeding process using two types of functional responses (Holling types I and II). Furthermore, the fear effect is regarded as an indirect influence generated by both predators in their model. Manna et al. [15] investigated a three-species food web model that included two competitive prey interactions with a generalist predator that feeds on both. They looked at how population distributions were affected by random dispersal of all three species and nonlocal intra-specific competition for two prey species. Maghool and Naji [16] investigate the effects of predation anxiety on the behavior of a three-species food chain. Because each prey in the system has anti-predator properties, the authors used the Sokol-Howell kind of functional response. As seen, the model is capable of displaying complicated dynamics, including chaos.

With these researches in mind, certain prey-predator models have been built in order to obtain more realistic models that correlate to actual natural settings. For example, prey-predator models with the Allee effect have been considered in [17-18]; Mondal et al. [19] recently explored the impact of providing additional food to a predator in a delayed prey-predator scenario under the influence of fear. Prey-predator models with fear have been extensively investigated in [20-22]; prey-predator models with fear and refuge have been studied in [23-24], and the effect of fear and time delay on prey-predator dynamics have been discussed in [25-27]. However, [27] investigates the prey-predator paradigm in the presence of fear and group defense.

A modified Holling type II functional response is examined to reflect the interaction between two prey and predator in this research, which is based on a three-species food web model with two competing prey and one predator. Fear, as well as intra-specific competition, are taken into account. The following is a summary of the paper's outline: The model, as well as its dimensionlessness, are detailed in Section 2. The equilibrium points and their current state are described in Section 3. The topic of local stability is discussed in section 4. Section 5 discusses the model's persistence, whereas Section 6 specifies the Basin of attractions for equilibrium points. Section 7 discusses the local bifurcation. The simulation of the model is performed in section 8. Finally, in the concluding section, the concluding and discussion are supplied.

2. THE MATHEMATICAL FORMULATION

In this section, an ecological model based on a three-species food web with two prey and one predator is developed and mathematically formulated in the form:

$$\begin{aligned} \frac{dX}{dT} &= r_1X - a_1X^2 - b_1XY - c_1g_1(X,Y)Z, \\ \frac{dY}{dT} &= r_2Y - a_2Y^2 - b_2XY - c_2g_2(X,Y)Z, \\ \frac{dZ}{dT} &= \big(e_1c_1g_1(X,Y) + e_2c_2g_2(X,Y)\big)Z - a_3Z^2 - dZ, \end{aligned} \tag{1}$$

where $X(t)$, $Y(t)$, and $Z(t)$ denote the population size of the first prey, second prey, and predator at time t respectively. As per the updated Holling type-II functional response, the two contending preys are expected to develop logistically while being consumed by a predator. The predator's diet is entirely dependent on these two competing prey, and in the absence of them, it decays exponentially. Furthermore, predator species are thought to have intra-specific competition. On the other hand, it is well recognized that the predator has an indirect impact on the prey population by inducing fear and changing the prey's behavior. In fact, the prey population increases awareness, reduces foraging activity, sacrifices higher intake zones and feeds in safer locations, adjusts the reproductive cycle, and so on owing to fear of predation. As a result, the impact of predator-induced anxiety is added to the suggested model, resulting in the model (2):

$$\begin{aligned} \frac{dX}{dT} &= \frac{r_1X}{(1+n_1Z)} - a_1X^2 - b_1XY - \frac{c_1XZ}{1+q_1X+q_2Y}, \\ \frac{dY}{dT} &= \frac{r_2Y}{(1+n_2Z)} - a_2Y^2 - b_2XY - \frac{c_2YZ}{1+q_1X+q_2Y}, \\ \frac{dZ}{dT} &= \frac{(e_1c_1X+e_2c_2Y)Z}{1+q_1X+q_2Y} - a_3Z^2 - dZ, \end{aligned} \tag{2}$$

where $X(0) \geq 0, Y(0) \geq 0,\ Z(t) \geq 0$, and $g_1(X,Y) = \frac{X}{1+q_1X+q_2Y}$, and $g_2(X,Y) = \frac{Y}{1+q_1X+q_2Y}$. While, the first and second prey's fear functions are represented by $\frac{1}{(1+n_1Z)}$ and $\frac{1}{(1+n_2Z)}$, respectively. In model (2), all of the parameters are supposed for being positive and are described in table (1) below.

Table 1: Explanation of parameters.

Parameter	Description
r_1, r_2	The net growth rates of the first and second prey respectively.
a_1, a_2, a_3	Intra-specific competition rates of the first prey, second prey, and predator respectively.
b_1, b_2	The intensity of intra-specific competition between the two preys.
$c_1,\ c_3$	The prey consumption rates.
$e_1,\ e_1$	Rates at which meal from the 1st and 2nd prey is converted.
n_1, n_2	Fear levels for the 1st and 2nd prey respectively.
$q_1,\ q_1$	Environmental protection rates of the 1st and 2nd prey respectively.
d	The natural mortality rate of a predator.

The fact that system (2) has 16 parameters makes analysis challenging. As a result, using the nondimensional variables and parameters indicated below, the set of parameters is reduced to 12, providing the model (3).

$$x = \frac{a_1 X}{r_1}, y = \frac{b_1 Y}{r_1}, z = \frac{c_1 Z}{r_1}, t = r_1 T,\ m_1 = \frac{n_1 r_1}{c_1},\quad m_2 = \frac{q_1 r_1}{a_1}, m_3 = \frac{q_2 r_1}{b_1}, m_4 = \frac{r_2}{r_1}$$

$$m_5 = \frac{n_2 r_1}{c_1}, m_6 = \frac{a_2}{b_1}, m_7 = \frac{b_2}{a_1},\ m_8 = \frac{c_2}{c_1}, m_9 = \frac{e_1 c_1}{a_1}, m_{10} = \frac{e_2 c_2}{b_1}, m_{11} = \frac{a_3}{c_1}, m_{12} = \frac{d}{r_1}.$$

As a result, the nondimensional system that relates to a system (2) is defined as follows:

$$\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+m_1 z} - x - y - \frac{z}{1+m_2 x + m_3 y}\right] = x f_1(x,y,z), \\ \frac{dy}{dt} &= y\left[\frac{m_4}{1+m_5 z} - m_6 y - m_7 x - \frac{m_8 z}{1+m_2 x + m_3 y}\right] = y f_2(x,y,z), \\ \frac{dz}{dt} &= z\left[\frac{m_9 x + m_{10} y}{1+m_2 x + m_3 y} - m_{11} z - m_{12}\right] = z f_3(x,y,z). \end{aligned} \tag{3}$$

The interactivity functions are defined on $\mathbb{R}_+^{\ 3} = \{(x,y,z): x(t) \geq 0, y(t) \geq 0, z(t) \geq 0\}$. Furthermore, the interactive functions in the system (3) are Lipschitzian functions since they are having continuous partial derivatives. As a result, there is a system (3) solution that is unique.

Theorem 1: With initial conditions falling in the $\mathbb{R}_+^{\ 3}$, all solutions of system (3) are uniformly bounded.

Proof. Consider any solution of the system (3) with an initial condition $(x_0, y_0, z_0) \in \mathbb{R}_+^{\ 3}$. From the first equation of the system (3), it is obtained that:

$$\frac{dx}{dt} \leq x - x^2.$$

Hence by solving this differential inequality it is obtained that $x \leq 1$ as $t \to \infty$. Similarly, from the second equation of the system (3), it's observed that:

$$\frac{dy}{dt} \leq m_4 y - m_6 y^2,$$

which gives that $y \leq \frac{m_4}{m_6}$ as $t \to \infty$.

Look to the function $Q(t) = c_1 x(t) + c_2 y(t) + z(t)$, then

$$\frac{dQ}{dt} = c_1 \left[\frac{x}{1+m_1 z} - x^2 - xy - \frac{xz}{1+m_2 x+m_3 y}\right] + c_2 \left[\frac{m_4 y}{1+m_5 z} - m_6 y^2 - m_7 xy - \frac{m_8 zy}{1+m_2 x+m_3 y}\right] + \left[\frac{m_9 x+m_{10} y}{1+m_2 x+m_3 y} z - m_{11} z^2 - m_{12} z\right].$$

So, by choosing $c_1 = m_9$, and $c_2 = \frac{m_{10}}{m_8}$, it is obtained that:

$$\frac{dQ}{dt} \leq 2m_9 x + \frac{2m_{10} m_4}{m_8} y - \delta Q \leq 2\left(m_9 + \frac{m_{10} m_4^2}{m_6 m_8}\right) - \delta Q,$$

where $\delta = \{1, m_4, m_{12}\}$. Therefore, direct computation gives, for $t \to \infty$, that:

$$Q(t) \leq \frac{2\left(m_9 + \frac{m_{10} m_4^2}{m_6 m_8}\right)}{\delta}.$$

As a result, all of the solutions in the following region are uniformly bounded.

$$\left\{(x, y, z) \in \mathbb{R}_+{}^3, 0 \leq x(t) \leq 1, 0 \leq y(t) \leq \frac{m_4}{m_6}, 0 < m_9 x(t) + \frac{m_{10}}{m_8} y(t) + z(t) \leq \frac{2(m_9 m_6 m_8 + m_{10} m_4^2)}{\delta m_6 m_8}\right\}.$$

3. Existence of Equilibrium Points

The system (3) has a maximum of seven non-negative equilibrium points, the form of which is given below, along with their existence requirements.

The vanishing equilibrium point (VEP), denoted by $\varepsilon_0 = (0,0,0)$ exists at all times.

The first axial equilibrium point (FAEP), denoted by $\varepsilon_1 = (1,0,0)$ exists at all times.

The second axial equilibrium point (SAEP), represented by $\varepsilon_2 = (0,\ y_{**}, 0)$, where $y_{**} = \frac{m_4}{m_6}$, exists at all times.

The predator-free equilibrium point (PFEP), represented by $\varepsilon_3 = (\bar{x}, \bar{y}, 0)$, where $\bar{x} = \frac{m_4 - m_6}{m_7 - m_6}$ and $\bar{y} = \frac{m_7 - m_4}{m_7 - m_6}$, exists if and only if one of the following requirements true:

$$\left.\begin{array}{c} m_6 < m_4 < m_7 \\ or \\ m_7 < m_4 < m_6 \end{array}\right\} \tag{4}$$

However, it is observed there is a line $(x + y = 1)$ of PFEP when $m_6 = m_4 = m_7$.

The 1st prey free equilibrium point (FPYFEP) is denoted by $\varepsilon_4 = (0, \bar{\bar{y}}, \bar{\bar{z}})$, where $\bar{\bar{y}}$ is obtained by the fourth-order polynomial equation's positive root:

$$\sigma_1 y^4 + \sigma_2 y^3 + \sigma_3 y^2 + \sigma_4 y + \sigma_5 = 0, \tag{5a}$$

where:

$$\sigma_1 = (m_{10} - m_3 m_{12}) m_3{}^2 m_5 m_6 m_{11} + m_3{}^3 m_6 m_{11}{}^2,$$

$$\sigma_2 = m_3 m_{11}(2m_5 m_6 m_{10} - m_3{}^2 m_4 m_{11}) + 3m_3{}^2 m_6 m_{11}(m_{11} - m_5 m_{12}),$$

$$\sigma_3 = m_{10} m_5 m_8 (m_{10} - 2m_{12} m_3) + m_{12} m_3{}^2 m_8 (m_{12} m_5 - m_{11}) + m_5 m_6 m_{11}(m_{10} - 3m_{12} m_3) + m_{10} m_3 m_8 m_{11} + 3m_3 m_{11}{}^2 (m_6 - m_3 m_4),$$

$$\sigma_4 = -2m_{12} m_5 m_8 (m_{10} - m_3 m_{12}) + m_6 m_{11}(m_{11} - m_{12} m_5) + m_8 m_{11}(m_{10} - 2m_3 m_{12}) - 3m_3 m_4 m_{11}{}^2,$$

$$\sigma_5 = m_{12} m_8 (m_5 m_{12} - m_{11}) - m_4 m_{11}{}^2.$$

While, $\bar{\bar{z}}$ is given by:

$$\bar{\bar{z}} = \frac{(m_{10} - m_3 m_{12})\bar{\bar{y}} - m_{12}}{m_{11}(1 + m_3 \bar{\bar{y}})}. \tag{5b}$$

If the following adequate conditions are satisfied, the FPYFEP exists uniquely in the first quadrant of the yz −plane.

$$0 < \frac{m_{12}}{(m_{10} - m_3 m_{12})} < \bar{\bar{y}}, \tag{6a}$$

with one of the conditions listed below

$$\left.\begin{array}{c} \sigma_2 > 0, \sigma_3 > 0, \sigma_4 > 0, \sigma_5 < 0 \\ \sigma_2 < 0, \sigma_3 < 0, \sigma_4 < 0, \sigma_5 < 0 \\ \sigma_2 > 0, \sigma_4 < 0, \sigma_5 < 0 \end{array}\right\}. \tag{6b}$$

The 2nd prey free equilibrium point (SPYFEP) is represented by $\varepsilon_5 = (\hat{x}, 0, \hat{z})$, where $\hat{x}$ is provided by a positive root of the fourth-order polynomial equation:

$$\rho_1 x^4 + \rho_2 x^3 + \rho_3 x^2 + \rho_4 x + \rho_5 = 0, \tag{7a}$$

where:

$$\rho_1 = (m_9 - m_2 m_{12}) {m_2}^2 m_1 m_{11} + {m_2}^3 {m_{11}}^2,$$

$$\rho_2 = m_2 m_{11}(2 m_1 m_9 - {m_2}^2 m_{11}) + 3 {m_2}^2 m_{11}(m_{11} - m_1 m_{12}),$$

$$\rho_3 = m_9 m_1 (m_9 - 2 m_2 m_{12}) + m_{12} {m_2}^2 (m_{12} m_1 - m_{11}) + m_1 m_{11}(m_9 - 3 m_{12} m_2) + m_9 m_2 m_{11} + 3 m_2 {m_{11}}^2 (1 - m_2),$$

$$\rho_4 = -2 m_{12} m_1 (m_9 - m_2 m_{12}) + m_{11}(m_{11} - m_{12} m_1) + m_{11}(m_9 - 2 m_2 m_{12}) - 3 m_2 {m_{11}}^2,$$

$$\rho_5 = m_{12}(m_1 m_{12} - m_{11}) - {m_{11}}^2.$$

While, $\hat{z}$ is given by:

$$\hat{z} = \frac{(m_9 - m_2 m_{12})\hat{x} - m_{12}}{m_{11}(1 + m_2 \hat{x})}. \tag{7b}$$

If the following adequate conditions are satisfied, the SPYFEP exists uniquely in the first quadrant of the xz −plane.

$$0 < \frac{m_{12}}{(m_9 - m_2 m_{12})} < \hat{x}, \tag{8a}$$

with one of the conditions listed below

$$\left.\begin{array}{c} \rho_2 > 0, \rho_3 > 0, \rho_4 > 0, \rho_5 < 0 \\ \rho_2 < 0, \rho_3 < 0, \rho_4 < 0, \rho_5 < 0 \\ \rho_2 > 0, \rho_4 < 0, \rho_5 < 0 \end{array}\right\}. \tag{8b}$$

If there is a single solution to the following set of algebraic equations, the coexistence equilibrium point (CEP) represented by $\varepsilon_6 = (x^*, y^*, z^*)$, arises uniquely in the interior of ${\mathbb{R}_+}^3$.

$$\begin{array}{l} f_1(x, y, z) = 0, \\ f_2(x, y, z) = 0\,, \\ f_3(x, y, z) = 0, \end{array} \tag{9}$$

where $f_i; i = 1,2,3$ are written in system (3). Straightforward computation shows that:

$$z^* = \frac{(m_9 - m_2 m_{12}) x^* + (m_{10} - m_3 m_{12}) y^* - m_{12}}{m_{11}(1 + m_2 x^* + m_3 y^*)}. \tag{10}$$

While, the point (x^*, y^*) represents a unique intersection point of the following two isoclines in the interior of the first quadrant of the xy −plane:

$$v_1(x,y) = \frac{1}{1+m_1\left(\frac{(m_9-m_2m_{12})x+(m_{10}-m_3m_{12})y-m_{12}}{m_{11}(1+m_2x+m_3y)}\right)} - x - y - \frac{\left(\frac{(m_9-m_2m_{12})x+(m_{10}-m_3m_{12})y-m_{12}}{m_{11}(1+m_2x+m_3y)}\right)}{1+m_2x+m_3y} = 0. \tag{11a}$$

$$v_2(x,y) = \frac{m_4}{1+m_5\left(\frac{(m_9-m_2m_{12})x+(m_{10}-m_3m_{12})y-m_{12}}{m_{11}(1+m_2x+m_3y)}\right)} - m_6y - m_7x - \frac{m_8\left(\frac{(m_9-m_2m_{12})x+(m_{10}-m_3m_{12})y-m_{12}}{m_{11}(1+m_2x+m_3y)}\right)}{1+m_2x+m_3y} = 0. \tag{11b}$$

Clearly, as $y = 0$, the two isoclines become:

$$v_1(x,0) = \zeta_1x^4 + \zeta_2x^3 + \zeta_3x^2 + \zeta_4x + \zeta_5 = 0, \tag{12a}$$

$$v_2(x,0) = \zeta_6x^4 + \zeta_7x^3 + \zeta_8x^2 + \zeta_9x + \zeta_{10} = 0, \tag{12b}$$

where:

$$\zeta_1 = m_{11}{m_2}^3(m_{11} - m_1m_{12}) + m_1m_{11}{m_2}^2m_9,$$

$$\zeta_2 = 3m_{11}{m_2}^2(m_{11} - m_1m_{12}) - {m_{11}}^2{m_2}^3 + 2m_1m_{11}m_2m_9,$$

$$\zeta_3 = 3{m_{11}}^2m_2 - 3m_1m_{11}m_{12}m_2 - 3{m_{11}}^2{m_2}^2 - m_{11}m_{12}{m_2}^2 + m_1{m_{12}}^2{m_2}^2 + m_1m_{11}m_9 + m_{11}m_2m_9 - 2m_1m_{12}m_2m_9 + m_1{m_9}^2$$

$$\zeta_4 = m_{11}(m_{11} - m_1m_{12}) - 3{m_{11}}^2m_2 - 2m_{12}m_2(m_{11} - m_1m_{12}) + m_9(m_{11} - 2m_1m_{12})$$

$$\zeta_5 = -{m_{11}}^2 - m_{12}(m_{11} - m_1m_{12})$$

$$\zeta_6 = {m_2}^3m_7m_{11}(m_{11} - m_5m_{12}) + m_{11}{m_2}^2m_5m_7m_9,$$

$$\zeta_7 = -{m_{11}}^2{m_2}^3m_4 + 3{m_2}^2m_7m_{11}(m_{11} - m_5m_{12}) + 2m_{11}m_2m_5m_7m_9$$

$$\zeta_8 = -3{m_{11}}^2{m_2}^2m_4 + 3m_2m_7m_{11}(m_{11} - m_5m_{12}) - m_{12}{m_2}^2m_8(m_{11} - m_5m_{12}) + m_{11}m_5m_7m_9 + m_{11}m_2m_8m_9 - 2m_{12}m_2m_5m_8m_9 + m_5m_8{m_9}^2$$

$$\zeta_9 = -3{m_{11}}^2m_2m_4 + m_{11}m_7(m_{11} - m_{12}m_5) - 2m_2m_8m_{12}(m_{11} - m_5m_{12}) + m_8m_9(m_{11} - 2m_5m_{12})$$

$$\zeta_{10} = -{m_{11}}^2m_4 - m_{12}m_8(m_{11} - m_5m_{12})$$

According to the polynomial equations (12a) and (12b), each one has a unique positive root designated by x_1 and x_2, if and only if the following sufficient conditions are met:

$$\left.\begin{array}{r} m_{11} > m_1 m_{12} \\ \zeta_2 > 0 \\ \zeta_4 < 0 \end{array}\right\}. \tag{13a}$$

$$\left.\begin{array}{r} m_{11} > m_5 m_{12} \\ \zeta_7 > 0 \\ \zeta_9 < 0 \end{array}\right\}. \tag{13b}$$

Keeping the above in mind, the CEP exists uniquely if in addition to conditions (13a) and (13b) the following sufficient conditions are met:

$$x_1 < x_2. \tag{14a}$$

$$\frac{dy}{dx} = -\left(\frac{\partial v_1(x,y)}{\partial x}\right)/\left(\frac{\partial v_1(x,y)}{\partial y}\right) > 0. \tag{14b}$$

$$\frac{dy}{dx} = -\left(\frac{\partial v_2(x,y)}{\partial x}\right)/\left(\frac{\partial v_2(x,y)}{\partial y}\right) < 0. \tag{14c}$$

$$(m_9 - m_2 m_{12})x^* + (m_{10} - m_3 m_{12})y^* > m_{12}. \tag{14d}$$

3. LOCAL STABILITY

In this section, the local behavior of the above equilibrium points is explored by determining the system's (3) Jacobian matrix at the point (x, y, z):

$$J = \begin{pmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{pmatrix} = (a_{ij})_{3\times 3}, \tag{15}$$

where:

$$a_{11} = x(-1 + \frac{m_2 z}{(1+m_2x+m_3y)^2}) + \frac{1}{(1+m_1 z)} - x - y - \frac{z}{(1+m_2x+m_3y)},$$

$$a_{12} = -x + \frac{m_3 x z}{(1+m_2x+m_3y)^2},$$

$$a_{13} = -\frac{m_1 x}{(1+m_1 z)^2} - \frac{x}{1+m_2x+m_3y},$$

$$a_{21} = -m_7 y + \frac{m_2 m_8 y z}{(1+m_2x+m_3y)^2},$$

$$a_{22} = y\left(-m_6 + \frac{m_3 m_8 z}{(1+m_2x+m_3y)^2}\right) + \frac{m_4}{1+m_5 z} - m_6 y - m_7 x - \frac{m_8 z}{1+m_2x+m_3y},$$

$$a_{23} = y\left(-\frac{m_4 m_5}{(1+m_5 z)^2} - \frac{m_8}{1+m_2x+m_3y}\right),$$

$$a_{31} = \left(\frac{m_9+(m_3m_9-m_2m_{10})y}{(1+m_2x+m_3y)^2}\right)z\ ,$$

$$a_{32} = \left(\frac{m_{10}+(m_2m_{10}-m_3m_9)x}{(1+m_2x+m_3y)^2}\right)z\ ,$$

$$a_{33} = -2m_{11}z + \frac{m_9x+m_{10}\,y}{1+m_2x+m_3y} - m_{12}\ ,$$

Therefore, the Jacobian matrix at VEP is:

$$J_{\varepsilon_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_4 & 0 \\ 0 & 0 & -m_{12} \end{pmatrix}. \tag{16}$$

Then the eigenvalues are given by $\lambda_{01} = 1,\ \lambda_{02} = m_4,\ \lambda_{03} = -m_{12}$. Hence the VEP ($\varepsilon_0$) is a saddle point.

The Jacobian matrix at FAEP, can be calculated as:

$$J_{\varepsilon_1} = \begin{pmatrix} -1 & -1 & -m_1 - 1 \\ 0 & m_4 - m_7 & 0 \\ 0 & 0 & -m_{12} + \frac{m_9}{1+m_2} \end{pmatrix}. \tag{17}$$

Then the eigenvalues of J_{ε_1} are given by $\lambda_{11} = -1,\ \lambda_{12} = m_4 - m_7, \lambda_{13} = -m_{12} + \frac{m_9}{1+m_2}$. Hence, the equilibrium point ε_1 is locally asymptotically stable if and only if the following conditions are met.

$$m_4 < m_7 \tag{18a}$$

$$\frac{m_9}{1+m_2} < m_{12} \tag{18b}$$

The Jacobian matrix at the SAEP is calculated as:

$$J_{\varepsilon_2} = \begin{pmatrix} 1 - y_{**} & 0 & 0 \\ -m_7y_{**} & -m_6\,y_{**} & -m_4m_5\,y_{**} - \frac{m_8\,y_{**}}{(1+m_3\,y_{**})} \\ 0 & 0 & \frac{m_{10}y_{**}}{1+m_3\,y_{**}} - m_{12} \end{pmatrix}. \tag{19}$$

Therefore, the eigenvalues of J_{ε_2} are given by $\lambda_{21} = 1 - y_{**},\ \lambda_{22} = -m_6\,y_{**}$, and $\lambda_{53} = \frac{m_{10}y_{**}}{1+m_3\,y_{**}} - m_{12}$. As a result, if and only if the following conditions are met, the point ε_2 is locally asymptotically stable:

$$m_6 < m_4. \tag{20a}$$

$$(m_{10} - m_3m_{12})m_4 < m_6m_{12}. \tag{20b}$$

The Jacobian matrix at the PFEP is calculated as:

$$J_{\varepsilon_3} = \begin{pmatrix} -\bar{x} & -\bar{x} & -m_1\bar{x} - \frac{\bar{x}}{1+m_2\bar{x}+m_3\bar{y}} \\ -m_7\bar{y} & -m_6\bar{y} & -m_4m_5\bar{y} - \frac{m_8\bar{y}}{1+m_2\bar{x}+m_3\bar{y}} \\ 0 & 0 & \frac{m_9\bar{x}+m_{10}\bar{y}}{1+m_2\bar{x}+m_3\bar{y}} - m_{12} \end{pmatrix}. \quad (21)$$

As a result, the characteristic equation of J_{ε_3} can be formulated as:

$$[\lambda^2 + (\bar{x} + m_6\bar{y})\lambda + (m_6 - m_7)\bar{x}\,\bar{y}]\left[\frac{m_9\bar{x}+m_{10}\bar{y}}{1+m_2\bar{x}+m_3\bar{y}} - m_{12} - \lambda\right] = 0. \quad (22)$$

Accordingly, the eigenvalues of J_{ε_3} are computed by:

$$\lambda_{31} = \frac{-(\bar{x}+m_6\bar{y})+\sqrt{(\bar{x}+m_6\bar{y})^2-4(m_6-m_7)\bar{x}\bar{y}}}{2},$$

$$\lambda_{32} = \frac{-(\bar{x}+m_6\bar{y})-\sqrt{(\bar{x}+m_6\bar{y})^2-4(m_6-m_7)\bar{x}\bar{y}}}{2},$$

$$\lambda_{33} = \frac{m_9\bar{x}+m_{10}\bar{y}}{1+m_2\bar{x}+m_3\bar{y}} - m_{12}.$$

As a result the PFEP, given by ε_3, is locally asymptotically stable provided the following requirements are met:

$$m_7 < m_6. \quad (23a)$$

$$\frac{m_9\bar{x}+m_{10}\bar{y}}{1+m_2\bar{x}+m_3\bar{y}} < m_{12}. \quad (23b)$$

Note that, because condition (23a) satisfies one of the existing criteria given by Eq. (4), the equilibrium point ε_3 will be asymptotically stable for any initial points in the first quadrant of the xy −plane, and it will be a saddle point otherwise.

The Jacobian matrix at FPYFEP can be determined by:

$$J_{\varepsilon_4} = \begin{pmatrix} \frac{1}{(1+m_1\bar{\bar{z}})} - \bar{\bar{y}} - \frac{\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})} & 0 & 0 \\ -m_7\bar{\bar{y}} + \frac{m_2m_8\bar{\bar{y}}\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^2}, & -m_6\bar{\bar{y}} + \frac{m_3m_8\bar{\bar{z}}\bar{\bar{y}}}{(1+m_3\bar{\bar{y}})^2} & -\bar{\bar{y}}\left(\frac{m_4m_5}{(1+m_5\bar{\bar{z}})^2} + \frac{m_8}{1+m_3\bar{\bar{y}}}\right) \\ \left(\frac{m_9+(m_3m_9-m_2m_{10})\bar{\bar{y}}}{(1+m_3\bar{\bar{y}})^2}\right)\bar{\bar{z}} & \frac{m_{10}\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^2} & -m_{11}\bar{\bar{z}} \end{pmatrix} = [b_{ij}]_{3\times 3}. \quad (24)$$

So the characteristic equation of J_{ε_4} can be represented as:

$$[\lambda^2 - (b_{22} + b_{33})\lambda - (b_{22}b_{33} - b_{23}b_{32})][b_{22} - \lambda] = 0 \quad (25)$$

Then the eigenvalues of J_{ε_4} are given by:

$$\lambda_{41}=\frac{1}{(1+m_1\bar{\bar{z}})}-\bar{\bar{y}}-\frac{\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})},$$

$$\lambda_{42}=\frac{(b_{22}+b_{33})+\sqrt{(b_{22}+b_{33})^2-4(b_{22}b_{33}-b_{23}b_{32})}}{2},$$

$$\lambda_{43}=\frac{(b_{22}+b_{33})-\sqrt{(b_{22}+b_{33})^2-4(b_{22}b_{33}-b_{23}b_{32})}}{2}.$$

Hence, the equilibrium point ε_4 is locally asymptotically stable if and only if the following sufficient requirements are met:

$$\frac{1}{(1+m_1\bar{\bar{z}})}<\bar{\bar{y}}+\frac{\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})}. \tag{26a}$$

$$\frac{m_3m_8\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^2}<m_6. \tag{26b}$$

The Jacobian matrix at SPYFEP is determined by:

$$J_{\varepsilon_5}=\begin{bmatrix} -\hat{x}+\frac{m_2\hat{x}\hat{z}}{(1+m_2\hat{x})^2} & \hat{x}+\frac{m_3\hat{x}\hat{z}}{(1+m_2\hat{x})^2} & -\frac{m_1\hat{x}}{(1+m_1\hat{z})^2}-\frac{\hat{x}}{1+m_2\hat{x}} \\ 0 & \frac{m_4}{1+m_5\hat{z}}-m_7\hat{x}-\frac{m_8\hat{z}}{1+m_2\hat{x}} & 0 \\ \frac{m_9\hat{z}}{(1+m_2\hat{x})^2} & \left(\frac{m_{10}+(m_2m_{10}-m_3m_9)\hat{x}}{(1+m_2\hat{x})^2}\right)\hat{z} & -m_{11}\hat{z} \end{bmatrix}=[c_{ij}]_{3\times3}. \tag{27}$$

Thus the characteristic equation of J_{ε_5} is formulated as:

$$(c_{22}-\lambda)\ [\lambda^2-(c_{11}+c_{33})\lambda+(c_{11}c_{33}-c_{13}c_{31})]=0. \tag{28}$$

Accordingly, the eigenvalues of J_{ε_5} can be written as:

$$\lambda_{41}=\frac{m_4}{1+m_5\hat{z}}-m_7\hat{x}-\frac{m_8\hat{z}}{1+m_2\hat{x}},$$

$$\lambda_{42}=\frac{(c_{11}+c_{33})+\sqrt{(c_{11}+c_{33})^2-4(c_{11}c_{33}-c_{13}c_{31})}}{2},$$

$$\lambda_{43}=\frac{(c_{11}+c_{33})-\sqrt{(c_{11}+c_{33})^2-4(c_{11}c_{33}-c_{13}c_{31})}}{2}.$$

As a result, if and only if the following sufficient conditions are met, the SPYFEP is locally asymptotically stable:

$$\frac{m_4}{1+m_5\hat{z}}<m_7\hat{x}+\frac{m_8\hat{z}}{1+m_2\hat{x}}. \tag{29a}$$

$$\frac{m_2\hat{z}}{(1+m_2\hat{x})^2}<1. \tag{29b}$$

Finally, the following theorem established sufficient conditions to ensure CEP's local stability.

Theorem (2): The CEP is locally asymptotically stable provided that the following sufficient requirements are met

$$\max\{m_2, m_3\} < \frac{(1+m_2x^*+m_3y^*)^2}{z^*} \tag{30a}$$

$$\frac{m_7}{m_2} < \frac{m_8z^*}{(1+m_2x^*+m_3y^*)^2} < \frac{m_6}{m_3} \tag{30b}$$

$$\frac{m_2m_{10}y^*}{1+m_3y^*} < m_9 \tag{30c}$$

$$\frac{m_3m_9x^*}{1+m_2x^*} < m_{10}, \tag{30d}$$

$$-d_{13}d_{21}d_{32} < d_{12}d_{23}d_{31} < d_{13}d_{22}d_{31}, \tag{30e}$$

where $d_{ij}; i,j = 1,2,3$ are the Jacobian elements that given in the proof.

Proof. Substituting the CEP in the general Jacobian matrix given by Eq. (15) yields that:

$$J_{\varepsilon_6} = [d_{ij}]_{3\times3}, \tag{31}$$

where:

$$d_{11} = -x^* + \frac{m_2x^*z^*}{(1+m_2x^*+m_3y^*)^2},$$

$$d_{12} = -x^* + \frac{m_3x^*z^*}{(1+m_2x^*+m_3y^*)^2},$$

$$d_{13} = -x^*\left(\frac{m_1}{(1+m_1z^*)^2} + \frac{1}{1+m_2x^*+m_3y^*}\right),$$

$$d_{21} = -m_7y^* + \frac{m_2m_8y^*z^*}{(1+m_2x^*+m_3y^*)^2},$$

$$d_{22} = y^*\left(-m_6 + \frac{m_3m_8z^*}{(1+m_2x^*+m_3y^*)^2}\right),$$

$$d_{23} = -y^*\left(\frac{m_4m_5}{(1+m_5z^*)^2} + \frac{m_8z^*}{1+m_2x^*+m_3y^*}\right),$$

$$d_{31} = \left(\frac{m_9+(m_3m_9-m_2m_{10})y^*}{(1+m_2x^*+m_3y^*)^2}\right)z^*,$$

$$d_{32} = \left(\frac{m_{10}+(m_2m_{10}-m_3m_9)x^*}{(1+m_2x^*+m_3y^*)^2}\right)z^*,$$

$$d_{33} = -m_{11}z^*.$$

Hence the characteristic equation of J_{ε_6} is represented as:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{32}$$

where:

$$A_1 = -(d_{11} + d_{22} + d_{33}),$$

$$A_2 = d_{11}d_{33} + d_{11}d_{22} + d_{22}d_{33} - d_{13}d_{31} - d_{23}d_{32} - d_{12}d_{21},$$
$$A_3 = d_{22}d_{13}d_{31} - d_{12}d_{23}d_{31} + d_{21}(d_{12}d_{33} - d_{13}d_{32}) - d_{11}(d_{22}d_{33} - d_{23}d_{32}),$$

with

$$\Delta = A_1A_2 - A_3 = -(d_{11} + d_{22})[d_{11}d_{22} - d_{12}d_{21}] - (d_{11} + d_{33})[d_{11}d_{33} - d_{13}d_{31}] \\ -(d_{22} + d_{33})[d_{22}d_{33} - d_{23}d_{32}] - 2d_{11}d_{22}d_{33} + d_{13}d_{21}d_{32} + d_{12}d_{23}d_{31}.$$

The characteristic equation (32) possesses three negative real part eigenvalues, according to the Routh-Hurwitz criterion, if and only if $A_1 > 0$, $A_3 > 0$, and $\Delta > 0$. Simple computation showed that the Routh-Hurwitz criterion is fulfilled and the proof is complete if the conditions (30a)-(30e) are met.

5. PERSISTENCE

The persistence of the system (3) is explored in this section; the system (3) persists if the system's trajectory that starts at a positive initial point, does not approach an omega limit set on the domain's boundary planes.

The system (3) contains three subsystems that are located in the xy −plane, xz −plane, and yz −plane, and can be expressed as follows:

$$\left.\begin{aligned} \frac{dx}{dt} &= x[1 - x - y] = \delta_1(x,y), \\ \frac{dy}{dt} &= y[m_4 - m_6y - m_7x] = \delta_2(x,y). \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+m_1z} - x - \frac{z}{1+m_2x}\right] = \delta_3(x,z), \\ \frac{dz}{dt} &= z\left[\frac{m_9x}{1+m_2x} - m_{11}z - m_{12}\right] = \delta_4(x,z). \end{aligned}\right\} \tag{34}$$

And

$$\left.\begin{aligned} \frac{dy}{dt} &= y\left[\frac{m_4}{1+m_5z} - m_6y - \frac{m_8z}{1+m_3y}\right] = \delta_5(y,z), \\ \frac{dz}{dt} &= z\left[\frac{m_{10}y}{1+m_3y} - m_{11}z - m_{12}\right] = \delta_6(y,z). \end{aligned}\right\} \tag{35}$$

The preceding subsystems (33), (34), and (35) have positive equilibrium points in the interior of boundary planes xy −plane, xz −plane and yz −plane, respectively, which coincide with those in the corresponding planes of the system (3). The Bendixson–Dulac theorem is now applied to determine the possibility of periodic dynamics in boundary planes.

Now, consider the following functions $\beta_1\,(x,y)=\frac{1}{xy}$, $\beta_2\,(x,z)=\frac{1}{xz}$, and $\beta_3\,(y,z)=\frac{1}{yz}$. Clearly these functions are positive and is C^1 function in interior of first quadrants of $xy-$plane, $xz-$plane and $yz-$plane respectively. Moreover, it is obvious that:

$$\wp_1(x,y)=\frac{\partial}{\partial x}(\beta_1\cdot\delta_1)+\frac{\partial}{\partial y}(\beta_1\cdot\delta_2)=-\left(\frac{1}{y}+\frac{m_6}{x}\right).$$

$$\wp_2(x,z)=\frac{\partial}{\partial x}(\beta_2\cdot\delta_3)+\frac{\partial}{\partial z}(\beta_2\cdot\delta_4)=-\left(\frac{1}{z}-\frac{m_2}{(1+m_2x)^2}+\frac{m_{11}}{x}\right).$$

$$\wp_3(y,z)=\frac{\partial}{\partial y}(\beta_3\cdot\delta_5)+\frac{\partial}{\partial z}(\beta_3\cdot\delta_6)=-\left(\frac{m_6}{z}-\frac{m_3m_8}{(1+m_3y)^2}+\frac{m_{11}}{y}\right).$$

Therefore, $\wp_1(x,y)$ has the same sign ($\neq 0$) almost everywhere in a simply connected region of the $xy-$plane. However, $\wp_2(x,z)$, and $\wp_3(y,z)$ have the same sign ($\neq 0$) almost everywhere in a simply connected region of the $xz-$plane and $yz-$plane, respectively, provided the following sufficient requirements are met.

$$\frac{m_2}{(1+m_2x)^2}<\frac{1}{z}+\frac{m_{11}}{x}. \tag{36}$$

$$\frac{m_3m_8}{(1+m_3y)^2}<\frac{m_6}{z}+\frac{m_{11}}{y}. \tag{37}$$

Consequently, the persistence requirements of the system (3) are built in the following theorem.

Theorem (3): If the border planes do not have periodic dynamics, the system (3) is uniformly persistent as long as the below conditions are met.   .

$$m_7<m_4<m_6, \tag{38a}$$

$$m_{12}<min\left\{\frac{m_9}{1+m_2},\frac{m_4m_{10}}{m_6+m_3m_4}\right\}, \tag{38b}$$

$$m_{12}<\frac{m_9\bar{x}+m_{10}\bar{y}}{1+m_2\bar{x}+m_3\bar{y}}, \tag{38c}$$

$$\bar{\bar{y}}+\frac{\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})}<\frac{1}{(1+m_1\bar{\bar{z}})}, \tag{38d}$$

$$m_7\hat{x}+\frac{m_8\hat{z}}{1+m_2\hat{x}}<\frac{m_4}{1+m_5\hat{z}}. \tag{38e}$$

Proof. Define $\Im(x,y,z)=x^{p_1}y^{p_2}z^{p_3}$, where p_1,p_2,p_3 are positive constants. It is clear that, $\Im(x,y,z)>0$ for each $(x,y,z)\in Int\,\mathbb{R}_+{}^3$, and $\Im(x,y,z)=0$ if x,y, or z approaches zero.

Therefore, direct computation gives:

$$\mathbb{Q}(x,y,z)=\frac{\Im'(x,y,z)}{\Im(x,y,z)}=p_1f_1+p_2f_2+p_3f_3,$$

where $f_i; i=1,2,3$, are mentioned in the system (3).

The proof is now satisfied according to the average Lyapunov technique if and only if $\mathbb{Q}(x,y,z)>0$ for each boundary equilibrium points.

Now, since

$$\mathbb{Q}(x,y,z)=p_1\left[\frac{1}{1+m_1z}-x-y-\frac{z}{1+m_2x+m_3y}\right]$$
$$+p_2\left[\frac{m_4}{1+m_5z}-m_6y-m_7x-\frac{m_8z}{1+m_2x+m_3y}\right]$$
$$+p_3\left[\frac{m_9x+m_{10}y}{1+m_2x+m_3y}-m_{11}z-m_{12}\right].$$

Then, we have that

$$\mathbb{Q}(\varepsilon_0)=p_1[1]+p_2[m_4]+p_3[-m_{12}].$$

Obviously, $\mathbb{Q}(\varepsilon_0)>0$ is produced by choosing random positive values for p_1 and p_2 that are sufficiently greater than p_3.

$$\mathbb{Q}(\varepsilon_1)=p_2[m_4-m_7]+p_3\left[\frac{m_9}{1+m_2}-m_{12}\right].$$

Note that, $\mathbb{Q}(\varepsilon_1)>0$ is produced due to the conditions (38a) and (38b).

$$\mathbb{Q}(\varepsilon_2)=p_1[1-y_{**}]+p_3\left[\frac{m_{10}y_{**}}{1+m_3y_{**}}-m_{12}\right]$$

Similarly, $\mathbb{Q}(\varepsilon_2)>0$ is produced due to the conditions (38a) and (38b).

$$\mathbb{Q}(\varepsilon_3)=p_3\left[\frac{m_9\bar{x}+m_{10}\bar{y}}{1+m_2\bar{x}+m_3\bar{y}}-m_{12}\right].$$

From the condition (38c), It is obtained that $\mathbb{Q}(\varepsilon_3)>0$ for any positive constant p_3.

$$\mathbb{Q}(\varepsilon_4)=p_1\left[\frac{1}{1+m_1\bar{\bar{z}}}-\bar{\bar{y}}-\frac{\bar{\bar{z}}}{1+m_3\bar{\bar{y}}}\right]$$

According to the condition (38d), it is clear that, $\mathbb{Q}(\varepsilon_4)>0$ for any positive constant p_1.

Finally, we have:

$$\mathbb{Q}(\varepsilon_5)=p_2\left[\frac{m_4}{1+m_5\hat{z}}-m_7\hat{x}-\frac{m_8\hat{z}}{1+m_2\hat{x}}\right]$$

Hence, $\mathbb{Q}(\varepsilon_5)>0$ for any positive constant p_2, due to condition (38e). Thus the system (3) is uniformly persistent, and the proof is done.

6. BASIN OF ATTRACTION

The basin of attraction of each asymptotic stable equilibrium point is determined in this section. Furthermore, if and only if their basin of attraction equals the interior of $\mathbb{R}^3_+$, the equilibrium point is said to be globally asymptotic stable.

Theorem (4): If the FAEP is locally asymptotically stable, then it is globally asymptotically stable if and only if the following condition is met.

$$m_9(m_1+1) < m_{12}. \tag{39}$$

Proof. Consider the real-valued function $\psi_1(x,y,z) = m_9(x-1-\ln x) + \frac{m_{10}}{m_8}y + z$.

It is clear the function $\psi_1(x,y,z)$ satisfies that $\psi_1(1,0,0) = 0$, while $\psi_1(x,y,z) > 0$, for all values in the region $\{(x,y,z) \in \mathbb{R}^3_+ : x > 0, y \geq 0, z \geq 0;\ (x,y,z) \neq (1,0,0)\}$. After that, using algebraic manipulation, you get:

$$\frac{d\psi_1}{dt} \leq -\frac{m_9}{1+m_1 z}(x-1)^2 - \left(m_9 + \frac{m_7\, m_{10}}{m_8}\right)(x-1)y - [\, m_{12} - (m_1+1)m_9]z.$$

Under the system's boundedness theorem and the provided condition, it is observed that, $\frac{d\psi_1}{dt} < 0$. As a result, the derivative, $\frac{d\psi_1}{dt}$, is a negative definite, and then the FAEP is globally asymptotically stable.

Theorem (5): Assume that the SAEP is asymptotically stable locally, then their basin of attraction satisfies the following requirements:

$$\frac{m_6 m_9 + m_4 m_7}{m_6(m_9+m_7)} < y \tag{40a}$$

$$(m_4 m_5 + m_8)\frac{m_4}{m_6} < m_{12} \tag{40b}$$

Proof: Define that

$$\psi_2(x,y,z) = x + \left(y - y_{**} - y_{**}\ln\left(\frac{y}{y_{**}}\right)\right) + z$$

It is clear the function ψ_2 satisfies that $\psi_2(0, y_{**}, 0) = 0$, while $\psi_2(x,y,z) > 0$, for each values belongs to $\{(x,y,z) \in \mathbb{R}^3_+ : x \geq 0, y > 0, z \geq 0;\ (x,y,z) \neq (0, y_{**}, 0)\}$. As a result of some algebraic manipulation, the following is obtained:

$$\frac{d\psi_2}{dt} \leq -x[(m_9+m_7)y-(m_9+m_7y_{**})]$$
$$-m_9x^2-\frac{m_4m_5yz}{(1+m_5z)}-[m_{12}-(m_4m_5+m_8)y_{**}]z$$
$$-m_6(y-y_{**})^2-\frac{(m_8-m_{10})yz}{(1+m_2x+m_3y)}-m_{11}z^2.$$

According the conditions (40a)-(40b), it is observed that, the derivative $\frac{d\psi_2}{dt}$ is negative definite. Then, the basin of attraction of the SAEP satisfies the given requirements.

Theorem (6): If the PFEP is locally asymptotically stable, then it is globally asymptotically stable if and only if the following requirements are satisfied:

$$(1+m_7)^2 < 4\,m_6 \tag{41a}$$

$$(1+m_1)\bar{x}+(m_4m_5+m_8)\bar{y} < m_{12} \tag{41b}$$

Proof. Consider the function

$$\psi_3(x,y,z)=\left(x-\bar{x}-\bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right)+\left(y-\bar{y}-\bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right)+z.$$

It is easy to verify that $\psi_3(x,y,z)$ satisfies that is $\psi_3(\bar{x},\bar{y},0)=0$, while $\psi_3(x,y,z)>0$, for all values belongs to $\{(x,y,z)\in\mathbb{R}_+^3: x>0, y>0, z\geq 0;\ (x,y,z)\neq(\bar{x},\bar{y},0)\}$. As a result of some algebraic manipulation with the use of given conditions, the following is obtained:

$$\frac{d\psi_3}{dt} \leq -\left[(x-\bar{x})-\sqrt{m_6}(y-\bar{y})\right]^2-[m_{12}-(1+m_1)\bar{x}-(m_4m_5+m_8)\bar{y}]z$$

So the derivative $\frac{d\psi_3}{dt}$ is negative definite in the interior of $\mathbb{R}_+^3$ and hence the PFEP is globally asymptotically stable.

Theorem (7): Assume that the FPYFEP is asymptotically stable locally, then their basin of attraction satisfies the following conditions:

$$\frac{m_3m_8\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})} < m_6 \tag{42a}$$

$$m_9+m_7\bar{\bar{y}} < \frac{m_9\bar{\bar{z}}}{\left(1+m_2+m_3\frac{m_4}{m_6}\right)}, \tag{42b}$$

$$\frac{m_2m_8\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})} < m_7+m_9, \tag{42c}$$

$$p_{23}^2 < 4p_{22}m_{11}, \tag{42d}$$

where

$$p_{22} = m_6 - \frac{m_3 m_8 \bar{\bar{z}}}{(1+m_3\bar{\bar{y}})},\ p_{23} = \frac{m_4 m_5}{(1+m_5 z)(1+m_5\bar{\bar{z}})} + \frac{(m_8-m_{10})+m_3 m_8 \bar{\bar{y}}}{(1+m_2 x+m_3 z)(1+m_3\bar{\bar{y}})}.$$

Proof: Let

$$\psi_4(x,y,z) = m_9 x + \left(y - \bar{\bar{y}} - \bar{\bar{y}}\ln\left(\frac{y}{\bar{\bar{y}}}\right)\right) + \left(z - \bar{\bar{z}} - \bar{\bar{z}}\ln\left(\frac{z}{\bar{\bar{z}}}\right)\right).$$

It is obvious that ψ_4 verifies that is $\psi_4(0,\bar{\bar{y}},\bar{\bar{z}}) = 0$, while $\psi_4(x,y,z) > 0$, for all values in $\{(x,y,z) \in \mathbb{R}^3_+ : x \geq 0, y > 0, z > 0;\ (x,y,z) \neq (0,\bar{\bar{y}},\bar{\bar{z}})\}$. As a result of some algebraic manipulation, the following is obtained:

$$\frac{d\psi_4}{dt} \leq -\mathrm{p}_{22}(y-\bar{\bar{y}})^2 - p_{23}(y-\bar{\bar{y}})(z-\bar{\bar{z}}) - m_{11}(z-\bar{\bar{z}})^2$$
$$-\left[\frac{m_9\bar{\bar{z}}}{\left(1+m_2+m_3\frac{m_4}{m_6}\right)} - (m_9+m_7\bar{\bar{y}})\right]x - \left[(m_7+m_9) - \frac{m_2 m_8\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})}\right]xy$$
$$-\frac{m_2(m_8-m_{10})\bar{\bar{y}}\bar{\bar{z}}}{(1+m_2 x+m_3 z)(1+m_3\bar{\bar{y}})}x.$$

Consequently, by the use of the given conditions, it's obtained that:

$$\frac{d\psi_4}{dt} \leq -\left[\sqrt{p_{22}}\,(y-\bar{\bar{y}}) + \sqrt{m_{11}}(z-\bar{\bar{z}})\right]^2$$
$$-\left[\frac{m_9\bar{\bar{z}}}{\left(1+m_2+m_3\frac{m_4}{m_6}\right)} - (m_9+m_7\bar{\bar{y}})\right]x$$
$$-\frac{m_2(m_8-m_{10})\bar{\bar{y}}\bar{\bar{z}}}{(1+m_2 x+m_3 z)(1+m_3\bar{\bar{y}})}x.$$

It is obtained that, due to the conditions (42a)-(42d), the derivative $\frac{d\psi_4}{dt}$ is negative definite. Therefore, the FPYFEP has a basin of attraction satisfies the given condition.

Theorem (8): Suppose that the SPYFEP is asymptotically stable locally, then their basin of attraction satisfies the following conditions:

$$\frac{m_2\hat{z}}{(1+m_2\hat{x})} < 1, \tag{43a}$$

$$\frac{m_3\hat{z}}{(1+m_2\hat{x})} < 1 + m_7, \tag{43b}$$

$$m_4 + \hat{x} < \frac{m_{10}\hat{z}}{\left(1+m_2+m_3\frac{m_4}{m_6}\right)}, \tag{43c}$$

$$q_{13}^2 < 4\mathrm{q}_{11}\mathrm{m}_{11}, \tag{43d}$$

where,

$$q_{11} = 1 - \frac{m_2\hat{z}}{(1+m_2\hat{x})}, \text{ and } q_{13} = \frac{m_1}{(1+m_1 z)(1+m_1\hat{z})} + \frac{(1-m_9)+m_2\hat{x}}{(1+m_2x+m_3y)(1+m_2\hat{x})}.$$

Proof: Assume that

$$\psi_5(x,y,z) = (x - \hat{x} - \hat{x}\ln\left(\frac{x}{\hat{x}}\right) + y + \left(z - \hat{z} - \hat{z}\ln\left(\frac{z}{\hat{z}}\right)\right).$$

So ψ_5 verifies that $\psi_5(\hat{x},0,\hat{z}) = 0$, while $\psi_5(x,y,z) > 0$, for all values belong to $\{(x,y,z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z > 0;\ (x,y,z) \neq (\hat{x},0,\hat{z})\}$. As a result of some algebraic manipulation, the following is obtained:

$$\begin{aligned}\frac{d\psi_5}{dt} \leq &-q_{11}(x-\hat{x})^2 - q_{13}(x-\hat{x})(z-\hat{z}) - m_{11}(z-\hat{z})^2 \\ &- \left[\frac{m_{10}\hat{z}}{\left(1+m_2+m_3\frac{m_4}{m_6}\right)} - (m_4+\hat{x})\right] y - \left[(1+m_7) - \frac{m_3\hat{z}}{(1+m_2\hat{x})}\right] xy \\ &- \frac{m_3(1-m_9)\hat{x}\hat{z}}{(1+m_2x+m_3y)(1+m_2\hat{x})} y - \frac{(m_8-m_{10})yz}{(1+m_2x+m_3y)}.\end{aligned}$$

Consequently, by the use of the given conditions, it’s obtained that:

$$\begin{aligned}\frac{d\psi_5}{dt} \leq &-\left[\sqrt{q_{11}}\,(x-\hat{x}) + \sqrt{m_{11}}(z-\hat{z})\right]^2 \\ &- \left[\frac{m_{10}\hat{z}}{\left(1+m_2+m_3\frac{m_4}{m_6}\right)} - (m_4+\hat{x})\right] y \\ &- \frac{m_3(1-m_9)\hat{x}\hat{z}}{(1+m_2x+m_3y)(1+m_2\hat{x})} y.\end{aligned}$$

Therefore, due to the conditions (43a)-(43d), the derivative $\frac{d\psi_5}{dt}$ is negative definite, and then the SPYFEP has a basin of attraction satisfies the given condition.

Theorem (9): Suppose that the CEP is asymptotically stable locally, then their basin of attraction satisfies the following conditions:

$${l_{12}}^2 < l_{11}l_{22}, \tag{44a}$$

$${l_{13}}^2 < l_{11}\,m_{11}, \tag{44b}$$

$${l_{23}}^2 < m_{11}l_{22}, \tag{44c}$$

$$\frac{m_2z^*}{(1+m_2x^*+m_3y^*)} < 1, \tag{44d}$$

$$m_3m_8z^* < m_6(1+m_2x^*+m_3y^*), \tag{44e}$$

where

$$l_{11} = 1 - \frac{m_2 z^*}{(1+m_2x^*+m_3y^*)},$$

$$l_{12} = 1 + m_7 - \frac{(m_3+m_2m_8)z^*}{(1+m_2x+m_3y)(1+m_2x^*+m_3y^*)},$$

$$l_{23} = \frac{m_4 m_5}{(1+m_5z)(1+m_5z^*)} + \frac{m_8(1+m_2x^*+m_3y^*)-[m_{10}+x^*(m_2m_{10}-m_3m_9]}{(1+m_2x+m_3y)(1+m_2x^*+m_3y^*)},$$

$$l_{13} = \frac{m_1}{(1+m_1z)(1+m_1z^*)} + \frac{(1+m_2x^*+m_3y^*)-[m_9+y^*(m_3m_9-m_2m_{10})]}{(1+m_2x+m_3y)(1+m_2x^*+m_3y^*)}.$$

$$l_{22} = m_6 - \frac{m_3m_8z^*}{(1+m_2x^*+m_3y^*)}.$$

Proof: Define the function:

$$\psi_6(x,y,z) = \left(x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right) + \left(y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right) + \left(z - z^* - z^* \ln\left(\frac{z}{z^*}\right)\right).$$

It is clear that, ψ_6 verifies that is $\psi_6(x^*, y^*, z^*) = 0$, while $\psi_6(x, y, z) > 0$, for all values in $\{(x, y, z) \in \mathbb{R}^3_+ : x > 0, y > 0, z > 0;\ (x, y, z) \neq (x^*, y^*, z^*)\}$. As a result of some algebraic manipulation, the following is obtained:

$$\frac{d\psi_6}{dt} = -\frac{l_{11}}{2}(x-x^*)^2 - l_{12}(x-x^*)(y-y^*) - \frac{l_{22}}{2}(y-y^*)^2$$
$$-l_{13}(x-x^*)(z-z^*) - l_{23}(y-y^*)(z-z^*) - \frac{m_{11}}{2}(z-z^*)^2,$$

Consequently, using (44a)-(44e) gives

$$\frac{d\psi_6}{dt} \leq -\frac{1}{2}\left[\sqrt{l_{11}}\,(x-x^*) + \sqrt{l_{22}}(y-y^*)\right]^2$$
$$-\frac{1}{2}\left[\sqrt{l_{22}}\,(y-y^*) + \sqrt{m_{11}}(z-z^*)\,\right]^2$$
$$-\frac{1}{2}\left[\sqrt{l_{11}}\,(x-x^*) - \sqrt{m_{11}}(z-z^*)\right]^2.$$

Note that, $\frac{d\psi_6}{dt}$ is clearly negative definite. As a result, the CEP has an attractive basin that satisfies the specified conditions.

7. Bifurcation Analysis

This section examines the effect of modifying the model parameters on the system's dynamical behavior (3) using Sotomayor's theorem for local bifurcation. Remember that a non-hyperbolic equilibrium point in a dynamical system is a required but not sufficient condition for a bifurcation to occur. As a result, the value that renders the equilibrium point a non-hyperbolic point is chosen as a candidate bifurcation parameter. Rewrite the system (3) in the following format:

$$\frac{dX}{dt} = F(X),\ X = (x, y, z)^T,\ F = (xf_1, yf_2, zf_3)^T. \quad (45)$$

The second directional derivative of the system (3) can also be calculated as follows:

$$D^2F(X, \mu)(\mathcal{L}, \mathcal{L}) = [\gamma_{i1}]_{3x1}, \quad (46)$$

where $\mathcal{L} = (\ell_1, \ell_2, \ell_3)^T$ be any non-zero vector, with

$$\begin{aligned}
\gamma_{11} = & -2\left[1 - \frac{m_2 z(1+m_3 y)}{(1+m_2 x+m_3 y)^3}\right]{\ell_1}^2 - 2\left[1 - \frac{m_3 z(1-m_2 x+m_3 y)}{(1+m_2 x+m_3 y)^3}\right]\ell_1\ell_2 \\
& - \frac{2m_3^2 xz}{(1+m_2 x+m_3 y)^3}{\ell_2}^2 - 2\left(\frac{m_1}{(1+m_1 z)^2} + \frac{(1+m_3 y)}{(1+m_2 x+m_3 y)^2}\right)\ell_1\ell_3 \\
& + \frac{2\,m_3 x}{(1+m_2 x+m_3 y)^2}\ell_2\ell_3 + \frac{2m_1^2 x}{(1+m_1 z)^3}{\ell_3}^2 \\
\gamma_{21} = & -2\frac{m_2^2 m_8 yz}{(1+m_2 x+m_3 y)^3}{\ell_1}^2 - 2\left(m_7 - \frac{m_2 m_8 z(1+m_2 x-m_3 y)}{(1+m_2 x+m_3 y)^3}\right)\ell_1\ell_2 + 2\frac{m_2 m_8 y}{(1+m_2 x+m_3 y)^2}\ell_1\ell_3 \\
& + 2\frac{m_4 m_5^2 y}{(1+m_5 z)^3}{\ell_3}^2 - 2\left[m_6 - \frac{m_3 m_8 z(1+m_2 x)}{(1+m_2 x+m_3 y)^3}\right]{\ell_2}^2 \\
& - 2\left[\frac{m_4 m_5}{(1+m_5 z)^2} + \frac{m_8(1+m_2 x)}{(1+m_2 x+m_3 y)^2}\right]\ell_2\ell_3, \\
\gamma_{31} = & -\frac{2m_2 z[m_9+(m_3 m_9-m_2 m_{10})y]}{(1+m_2 x+m_3 y)^3}{\ell_1}^2 - 2\frac{m_3[m_{10}z+(m_2 m_{10}-m_3 m_9)x]}{(1+m_2 x+m_3 y)^3}{\ell_2}^2 \\
& - 2\frac{(m_2 x-m_3 y)(m_2 m_{10}-m_3 m_9)z+(m_2 m_{10}+m_3 m_9)z}{(1+m_2 x+m_3 y)^3}\ell_1\ell_2 - 2m_{11}{\ell_3}^2 \\
& + 2\frac{m_{10}+(m_2 m_{10}-m_3 m_9)x}{(1+m_2 x+m_3 y)^2}\ell_2\ell_3 + 2\frac{m_9+(m_3 m_9-m_2 m_{10})y}{(1+m_2 x+m_3 y)^2}\ell_1\ell_3.
\end{aligned}$$

The theorems that follow analyze the potential of local bifurcation in the system based on the above calculation (3).

Theorem (10): If the condition (18a) is met, then a transcritical bifurcation of the system (3) at the FAEP happens when the parameter m_{12} passes over the value $m_{12}^* = \frac{m_9}{1+m_2}$.

Proof: At FAEP with m_{12}^*, the Jacobian matrix of the system (3) is expressed as:

$$J_1 = J(\varepsilon_1, m_{12}^*) = \begin{pmatrix} -1 & -1 & -m_1 - 1 \\ 0 & m_4 - m_7 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this matrix, two of the eigenvalues have negative real portions, while the third is zero and denoted ${\lambda_{13}}^* = 0$.Thus FAEP is a non-hyperbolic point at m_{12}^*.

Let $\mathcal{L}_1 = (\ell_{11}, \ell_{12}, \ell_{13})^T$ be the eigenvector conjugate with the eigenvalue ${\lambda_{12}}^* = 0$.

Thus, $J_1\mathcal{L}_1 = 0$, gives that $\mathcal{L}_1 = (-(m_1 + 1)\ell_{13}, 0, \ell_{13})^T$, and $\ell_{13} \neq 0$ is any real number.

Now, let $\Theta_1 = (\vartheta_{11}, \vartheta_{12}, \vartheta_{13})^T$ represents the eigenvector conjugate with the eigenvalue $\lambda_{12}{}^* = 0$, of the matrix $J_1{}^T$.

Thus, $J_1{}^T\Theta_1 = 0$ gives that $\Theta_1 = (0,0,\vartheta_{13})^T$, where $\vartheta_{13} \neq 0$ is any real number. Following Sotomayor's theorem, gives that:

$$\frac{\partial F}{\partial m_{12}} = F_{m_{12}}(X, m_{12}) = (0, 0\,, -\mathrm{z})^T \implies \frac{\partial F}{\partial m_{12}} = F_{m_{12}}(\varepsilon_1, m_{12}^*) = (0, 0\,, 0)^T.$$

Therefore, $\Theta_1{}^T F_{m_{12}}(\varepsilon_1, m_{12}^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$DF_{m_{12}}(X, m_{12}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \implies DF_{m_{12}}(\varepsilon_1, m_{12}^*)\mathcal{L}_1 = (0, 0, -\ell_{13})^T.$$

Then, $\Theta_1{}^T DF_{m_{12}}(\varepsilon_1, m_{12}^*)\mathcal{L}_1 = -\ell_{13}\vartheta_{13} \neq 0$.

Also, by using equation (46), it is obtained that

$$D^2F(\varepsilon_1, m_{12}^*)(\mathcal{L}_1, \mathcal{L}_1) = \begin{pmatrix} -2\left(1 - \frac{1}{(1+m_2)^2}\right)(m_1 + 1)\ell_{13}{}^2 + 2m_1^2\ell_{13}{}^2 \\ 0 \\ -2m_{11}\ell_{13}{}^2 - 2\frac{m_9}{(1+m_2)^2}\,(m_1 + 1)\ell_{13}{}^2 \end{pmatrix}$$

Accordingly, the following is obtained:

$$\Theta_1{}^T D^2F(\varepsilon_1, m_{12}^*)(\mathcal{L}_1, \mathcal{L}_1) = -2\left[m_{11} + \frac{m_9(m_1+1)}{(1+m_2)^2}\right]\ell_{13}{}^2\vartheta_{13} \neq 0.$$

Hence a transcrtical bifurcation take place.

Theorem (11): If the condition (20a) is met, then a transcritical bifurcation of the system (3) at the SAEP happens if m_{12} passes through the value $m_{12}^{**} = \frac{m_{10}y_{**}}{1+m_3\,y_{**}}$.

Proof: The Jacobian matrix at $(\varepsilon_2, m_{12}^{**})$ is determined by:

$$J_2 = J(\varepsilon_2, m_{12}^{**}) = \begin{pmatrix} 1 - y_{**} & 0 & 0 \\ -m_7 y_{**} & -m_6\, y_{**} & -m_4 m_5\, y_{**} - \frac{m_8\, y_{**}}{(1+m_3\, y_{**})} \\ 0 & 0 & 0 \end{pmatrix},$$

Obviously, under condition (20a), two of the eigenvalues have negative real portions, while the third is zero and denoted $\lambda_{23}{}^{**} = 0$.Thus SAEP is a non-hyperbolic point at m_{12}^{**}.

Let $\mathcal{L}_2 = (\ell_{21}, \ell_{22}, \ell_{23})^T$ be the eigenvector conjugate with the eigenvalue $\lambda_{23}{}^{**} = 0$.

Hence, $J_2\mathcal{L}_2 = 0$, gives that $\mathcal{L}_2 = (0, H\ell_{23}, \ell_{23})^T$, where $H = -\frac{m_4 m_5(1+m_3\, y_{**})+m_8}{m_6(1+m_3\, y_{**})} < 0$, and $\ell_{23} \neq 0$ is any real number.

Now, let $\Theta_2 = (\vartheta_{21}, \vartheta_{22}, \vartheta_{23})^T$ represents the eigenvector conjugate with the eigenvalue $\lambda_{23}{}^{**} = 0$ of the matrix $J_2{}^T$.

Thus, $J_2{}^T\Theta_2 = 0$ gives that $\Theta_2 = (0,0,\vartheta_{23})^T$, where $\vartheta_{23} \neq 0$ is any real number.

Now, since:

$$\frac{\partial F}{\partial m_{12}} = F_{m_{12}}(X, m_{12}) = (0, 0\,, -\mathrm{z})^T \Rightarrow \frac{\partial F}{\partial m_{12}} = F_{m_{12}}(\varepsilon_2, m_{12}^{**}) = (0, 0\,, 0)^T.$$

Therefore, $\Theta_2{}^T F_{m_{12}}(\varepsilon_2, m_{12}^{**}) = 0$, hence the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$DF_{m_{12}}(X, m_{12}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow DF_{m_{12}}(\varepsilon_2, m_{12}^{**})\mathcal{L}_5 = (0,0,-\ell_{23})^T.$$

Then, $\Theta_2{}^T DF_{m_{12}}(\varepsilon_5, m_{12}^{**})\mathcal{L}_5 = -\vartheta_{23}\ell_{23} \neq 0.$

Also, by using equation (46), it is obtained that:

$$D^2F(\varepsilon_2, m_{12}^{**})(\mathcal{L}_2, \mathcal{L}_2) = \begin{pmatrix} 0 \\ 2m_4 m_5^2 \mathrm{y}_{**}\ell_{23}{}^2 - 2m_6 H^2\ell_{23}{}^2 - 2\left[m_4 m_5 + \frac{m_8}{(1+m_3\mathrm{y}_{**})^2}\right] H\ell_{23}{}^2 \\ -2m_{11}\ell_{23}{}^2 + 2\frac{m_{10}}{(1+m_3\mathrm{y}_{**})^2} H\ell_{23}{}^2 \end{pmatrix}.$$

Accordingly, the following is obtained:

$$\Theta_2{}^T D^2F(\varepsilon_2, m_{12}^{**})(\mathcal{L}_2, \mathcal{L}_2) = 2\left[-m_{11} + \frac{m_{10}}{(1+m_3\mathrm{y}_{**})^2} H\right]\vartheta_{23}\ell_{23}{}^2 \neq 0.$$

Hence, in the sense of Sotomayor, a transcrtical bifurcation take place.

Theorem (12): If the condition (23a) is met, then a transcritical bifurcation of the system (3) at the PFEP happens when the parameter m_{12} passes over the value $\bar{m}_{12} = \frac{m_9\bar{x}+m_{10}\bar{y}}{1+m_2\bar{x}+m_3\bar{y}}$, if and only if the following condition is satisfied.

$$\bar{\gamma}_{31} \neq 0, \tag{47}$$

where $\bar{\gamma}_{31}$ is computed in the proof.

Proof: The Jacobian matrix at $(\varepsilon_2, \bar{m}_{12}\ )$ can be represented by:

$$J_3 = J(\varepsilon_2, \overline{m}_{12}\ ) = \begin{pmatrix} -\bar{x} & -\bar{x} & -\ m_1\bar{x} - \frac{1}{1+m_2\bar{x}+m_3\bar{y}} \\ -m_7\bar{y} & -m_6\bar{y} & -m_4m_5\bar{y} - \frac{m_8\bar{y}}{1+m_2\bar{x}+m_3\bar{y}} \\ 0 & 0 & 0 \end{pmatrix} = \left(a_{ij}\right)_{3\times 3},$$

Due to the existence of a zero eigenvalue, say $\bar{\lambda}_{33} = 0$, PFEP becomes a non-hyperbolic point at $m_{12} = \overline{m}_{12}$, whereas the other two eigenvalues have negative real portions under the condition (23a).

Let $\mathcal{L}_3 = (\ell_{31}, \ell_{32}, \ell_{33})^T$ be the eigenvector conjugate with the eigenvalue $\bar{\lambda}_{33} = 0$.

Thus $J_3\mathcal{L}_3 = 0$, gives that:

$$\mathcal{L}_3 = (\mathrm{A}_1\ell_{33}, \mathrm{A}_2\ell_{33}, \ell_{33})^T,$$

where $\mathrm{A}_1 = \frac{a_{12}a_{23}-a_{22}a_{13}}{a_{11}a_{22}-a_{21}a_{12}}$, $\mathrm{A}_2 = \frac{a_{21}a_{13}-a_{11}a_{23}}{a_{11}a_{22}-a_{21}a_{12}}$, and $\ell_{33} \neq 0$ is any real number.

Now, let $\Theta_3 = (\vartheta_{31}, \vartheta_{32}, \vartheta_{33})^T$ represents the eigenvector conjugate with the eigenvalue $\bar{\lambda}_{33} = 0$ of the matrix $J_3{}^T$.

Thus, $J_3{}^T\Theta_3 = 0$ gives that $\Theta_3 = (0,0,\vartheta_{33})^T$, where $\vartheta_{33} \neq 0$ is any real number. Now, since:

$$\frac{\partial F}{\partial m_{12}} = F_{m_{12}}(X, m_{12}) = (0,0\,, -\mathrm{z})^T \ \Rrightarrow\ \frac{\partial F}{\partial m_{12}} = F_{m_{12}}(\varepsilon_3, \overline{m}_{12}) = (0,0\,,0)^T.$$

Therefore, $\Theta_3{}^T F_{m_{12}}(\varepsilon_3, \overline{m}_{12}) = 0$, then the first requirement for the transcritical bifurcation is met.

Moreover, since

$$DF_{m_{12}}(X, m_{12}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rrightarrow DF_{m_{12}}(\varepsilon_3, \overline{m}_{12})\mathcal{L}_3 = (0,0,-\ell_{33})^T.$$

Then, $\Theta_3{}^T DF_{m_{12}}(\varepsilon_3, \overline{m}_{12})\mathcal{L}_3 = -\vartheta_{23}\ell_{33} \neq 0.$

Also, by using equation (46), it is obtained that:

$$D^2F(\varepsilon_3, \overline{m}_{12})(\mathcal{L}_3, \mathcal{L}_3) = [\bar{\gamma}_{i1}]_{3\times 1},$$

where

$$\bar{\gamma}_{11} = -2(\mathrm{A}_1\ell_{33})^2 - 2\,\mathrm{A}_1\mathrm{A}_2\ell_{33}{}^2 - 2\left(m_1 + \frac{(1+m_3\bar{y})}{(1+m_2\bar{x}+m_3\bar{y})^2}\right)\mathrm{A}_1\ell_{33}{}^2 + \frac{2\,m_3\bar{x}}{(1+m_2\bar{x}+m_3\bar{y})^2}\,\mathrm{A}_2\ell_{33}{}^2 + 2m_1^2\bar{x}\ell_{33}{}^2,$$

$$\bar{\gamma}_{21} = -2m_7\,\mathrm{A}_1\mathrm{A}_2{\ell_{33}}^2 + 2\frac{m_2m_8\bar{y}}{(1+m_2\bar{x}+m_3\bar{y})^2}\,\mathrm{A}_1{\ell_{33}}^2 + 2m_4m_5^2\bar{y}{\ell_{33}}^2 - 2m_6(\mathrm{A}_2\ell_{33})^2 -2\left[m_4m_5 + \frac{m_8(1+m_2\bar{x})}{(1+m_2\bar{x}+m_3\bar{y})^2}\right]\mathrm{A}_2{\ell_{33}}^2,$$

$$\bar{\gamma}_{31} = -2\,\frac{m_3(m_2m_{10}-m_3m_9)\bar{x}}{(1+m_2\bar{x}+m_3\bar{y})^3}\,(\mathrm{A}_2\ell_{33})^2 + 2\frac{m_{10}+(m_2m_{10}-m_3m_9)\bar{x}}{(1+m_2\bar{x}+m_3\bar{y})^2}\mathrm{A}_2{\ell_{33}}^2 - 2m_{11}{\ell_{33}}^2 + 2\frac{m_9+(m_3m_9-m_2m_{10})\bar{y}}{(1+m_2\bar{x}+m_3\bar{y})^2}\,\mathrm{A}_1{\ell_{33}}^2.$$

Therefore, condition (47) yields that:

$$\Theta_3{}^T D^2F(\varepsilon_3,\bar{m}_{12})(\mathcal{L}_3,\mathcal{L}_3) = \vartheta_{23}\bar{\gamma}_{31} \neq 0.$$

Hence a transcrtical bifurcation take place.

Theorem (13): If the condition (26b) is met, then a transcritical bifurcation of the system (3) at the FPYFEP happens when the parameter m_1 passes over the value $\bar{\bar{m}}_1 = \frac{(1+m_3\bar{\bar{y}})(1-\bar{\bar{y}})-\bar{\bar{z}}}{\bar{\bar{z}}(\bar{\bar{y}}(1+m_3\bar{\bar{y}})+\bar{\bar{z}})}$, if and only if the following condition is satisfied.

$$\bar{\bar{\gamma}}_{11} \neq 0, \tag{48}$$

where $\bar{\bar{\gamma}}_{11}$ is computed through the proof.

Proof: At the FPYFEP the Jacobian matrix, with $m_1 = \bar{\bar{m}}_1$, can be written as:

$$J_4 = J(m_1 = \bar{\bar{m}}_1) = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

where $b_{ij}; i = 2,3. j = 1,2,3$ are written in the Jacobian matrix that given by (24).

Hence the determinant of the matrix J_4 is equal to zero. Therefore two eigenvalues of J_4 with negative real portions are existing under the condition (26b), while the third eigenvalue is given by $\bar{\bar{\lambda}}_{41} = 0$, and hence the FPYFEP is a non-hyperbolic point.

Let $\mathcal{L}_4 = (\ell_{41},\ell_{42},\ell_{43})^T$ be the eigenvector conjugate with the eigenvalue $\bar{\bar{\lambda}}_{41} = 0$.

Thus, $J_4\mathcal{L}_4 = 0$, gives that $\mathcal{L}_4 = (\ell_{41},\mathrm{H}_1\ell_{41},\mathrm{H}_2\ell_{41})^T$, where $\mathrm{H}_1 = \frac{b_{23}b_{31}-b_{21}b_{33}}{b_{22}b_{33}-b_{23}b_{32}}$, $\mathrm{H}_2 = \frac{b_{21}b_{32}-b_{22}b_{31}}{b_{22}b_{33}-b_{23}b_{32}}$, and $\ell_{41} \neq 0$ is any real number.

Now, let $\Theta_4 = (\vartheta_{41},\vartheta_{42},\vartheta_{43})^T$ denotes the eigenvector conjugate with the eigenvalue $\bar{\bar{\lambda}}_{41} = 0$

of the matrix J_4^T.

Thus, $J_4^T\Theta_4 = 0$ gives that $\Theta_4 = (\vartheta_{41}, 0,0,)^T$, where $\vartheta_{41} \neq 0$ is any real number.

Moreover, it is observed that:

$$\frac{\partial F}{\partial m_1} = F_{m_1}(X, m_1) = (\frac{-xz}{(1+m_1 z)^2}, 0, ,0)^T \implies F_{m_1}(\varepsilon_4, \bar{\bar{m}}_1) = (0,0,0)^T.$$

Therefore, it is obtained that $\Theta_4^T F_{m_1}(\varepsilon_4, \bar{\bar{m}}_1) = 0$, which means the first requirement for the transcritical bifurcation is met. Moreover, since

$$DF_{m_1}(X, m_1) = \begin{pmatrix} \frac{-z}{(1+m_1 z)^2} & 0 & \frac{-x(1-m_1 z)}{(1+m_1 z)^3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow DF_{m_1}(\varepsilon_4, \bar{\bar{m}}_1)\mathcal{L}_4 = \left(\frac{-\bar{\bar{z}}}{(1+\bar{\bar{m}}_1\bar{\bar{z}})^2}\ell_{41}, 0, 0\right)^T.$$

Consequently, we obtain that:

$$\Theta_4^T DF_{m_1}(\varepsilon_4, \bar{\bar{m}}_1)\mathcal{L}_4 = \frac{-\bar{\bar{z}}}{(1+\bar{\bar{m}}_1\bar{\bar{z}})^2}\ell_{41}\vartheta_{41} \neq 0.$$

Now, by using equation (46), it is obtained that:

$$D^2F(\varepsilon_4, \bar{\bar{m}}_1)(\mathcal{L}_4, \mathcal{L}_4) = [\bar{\bar{\gamma}}_{i1}]_{3\times 1},$$

where:

$$\bar{\bar{\gamma}}_{11} = -2\left[1 - \frac{m_2\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^2}\right]\ell_{41}{}^2 - 2\left[1 - \frac{m_3\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^2}\right] \mathrm{H}_1\ell_{41}{}^2$$
$$-2\left(\frac{\bar{\bar{m}}_1}{(1+\bar{\bar{m}}_1\bar{\bar{z}})^2} + \frac{1}{(1+m_3\bar{\bar{y}})}\right)\mathrm{H}_2\ell_{41}{}^2$$

$$\bar{\bar{\gamma}}_{21} = -2\frac{m_2^2 m_8\bar{\bar{y}}\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^3}\ell_{41}{}^2 - 2\left(m_7 - \frac{m_2 m_8\bar{\bar{z}}(1-m_3\bar{\bar{y}})}{(1+m_3\bar{\bar{y}})^3}\right)\mathrm{H}_1\ell_{41}{}^2$$
$$+2\frac{m_2 m_8\bar{\bar{y}}}{(1+m_3\bar{\bar{y}})^2}\mathrm{H}_2\ell_{41}{}^2 + 2\frac{m_4 m_5^2\bar{\bar{y}}}{(1+m_5\bar{\bar{z}})^3}\ell_3{}^2 - 2\left[m_6 - \frac{m_3 m_8\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^3}\right]\ell_2{}^2$$
$$-2\left[\frac{m_4 m_5}{(1+m_5\bar{\bar{z}})^2} + \frac{m_8}{(1+m_3\bar{\bar{y}})^2}\right]\mathrm{H}_1\mathrm{H}_2\ell_{41}{}^2,$$

$$\bar{\bar{\gamma}}_{31} = -\frac{2m_2\bar{\bar{z}}[m_9 + (m_3 m_9 - m_2 m_{10})\bar{\bar{y}}]}{(1+m_3\bar{\bar{y}})^3}\ell_{41}{}^2 - 2\frac{m_3 m_{10}\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^3}\mathrm{H}_1{}^2\ell_{41}{}^2$$
$$+2\frac{m_2 m_{10}(1+m_3\bar{\bar{y}})\bar{\bar{z}} + m_3 m_9(1-m_3\bar{\bar{y}})\bar{\bar{z}}}{(1+m_3\bar{\bar{y}})^3}\mathrm{H}_1\ell_{41}{}^2 - 2m_{11}\mathrm{H}_2{}^2\ell_{41}{}^2$$
$$+2\frac{m_{10}}{(1+m_3\bar{\bar{y}})^2}\mathrm{H}_1\mathrm{H}_2\ell_{41}{}^2 + 2\frac{m_9 + (m_3 m_9 - m_2 m_{10})\bar{\bar{y}}}{(1+m_3\bar{\bar{y}})^2}\mathrm{H}_2\ell_{41}{}^2.$$

Therefore, using condition (48) yields that

$$\Theta_4{}^T D^2 F(\varepsilon_4, \bar{\bar{m}}_1)(\mathcal{L}_4, \mathcal{L}_4) = \vartheta_{41}\bar{\bar{\gamma}}_{11} \neq 0$$

Hence a transcrtical bifurcation take place.

Theorem (14): If the condition (29b) is met, then a transcritical bifurcation of the system (3) at the SPYFEP happens when the parameter m_7 passes over the value $\widehat{m}_7 = \frac{1}{\hat{x}}\left[\frac{m_4}{(1+m_5\hat{z})} - \frac{m_8\hat{z}}{(1+m_2\hat{x})}\right]$, if and only if the following condition is satisfied.

$$\hat{\gamma}_{21} \neq 0, \tag{49}$$

where $\hat{\gamma}_{21}$ is computed in the proof.

Proof: The Jacobian matrix at $(\varepsilon_7, \widehat{m}_7)$ can be written as:

$$J_5 = J(\varepsilon_5, \widehat{m}_7) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$

where $c_{ij}; i = 1,3. j = 1,2,3$ are the Jacobian matrix elements that given by (27).

Hence the determinant of the matrix J_5 is equal to zero. Therefore it has two eigenvalues with negative real portions under the condition (29b), while the third eigenvalues is $\hat{\lambda}_{51} = 0$, and hence the SPYFEP is a non-hyperbolic point.

Let $\mathcal{L}_5 = (\ell_{51}, \ell_{52}, \ell_{53})^T$ be the eigenvector conjugate with the eigenvalue $\hat{\lambda}_{51} = 0$.

Thus, $J_5\,\mathcal{L}_5 = 0$, gives that $\mathcal{L}_5 = (D_1\ell_{52}, \ell_{52}, D_2\ell_{52})^T$, where $D_1 = \frac{c_{13}c_{32}-c_{12}c_{33}}{c_{11}c_{33}-c_{13}c_{31}}$, $D_2 = \frac{c_{12}c_{31}-c_{11}c_{32}}{c_{11}c_{33}-c_{13}c_{31}}$, and $\ell_{52} \neq 0$ is any real number.

Now, let $\Theta_5 = (\vartheta_{51}, \vartheta_{52}, \vartheta_{53})^T$ denotes to the eigenvector conjugate with the eigenvalue $\hat{\lambda}_{51} = 0$, of the matrix $J_5{}^T$.

Thus, $J_5{}^T\Theta_5 = 0$ gives that $\Theta_5 = (0, \vartheta_{52}, 0)^T$, where $\vartheta_{52} \neq 0$ is any real number.

Now, since:

$$\frac{\partial F}{\partial m_7} = F_{m_7}(X, m_7) = (0, -xy, 0)^T \Rightarrow \frac{\partial F}{\partial m_7} = F_{m_7}(\varepsilon_5, \widehat{m}_7) = (0, 0, 0)^T.$$

Therefore, $\Theta_5{}^T F_{m_7}(\varepsilon_5, \widehat{m}_7) = 0$, hence the system (3) has no saddle-node bifurcation. Moreover, since

$$DF_{m_7}(X, m_7) = \begin{pmatrix} 0 & 0 & 0 \\ -y & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow DF_{m_7}(\varepsilon_5, \widehat{m}_7)\mathcal{L}_5 = (0, \hat{x}\ell_{52}, 0)^T.$$

Then, $\Theta_5{}^T DF_{m_7}(\varepsilon_5, \widehat{m}_7)\mathcal{L}_5 = \hat{x}\ell_{52}\vartheta_{52} \neq 0$.

Also, by using equation (46), it is obtained that:

$$D^2F(\varepsilon_5, \widehat{m}_7)(\mathcal{L}_5, \mathcal{L}_5) = [\hat{\gamma}_{i1}]_{3\times 1},$$

where,

$$\hat{\gamma}_{11} = -2\left[1 - \frac{m_2\hat{z}}{(1+m_2\hat{x})^3}\right](D_1\ell_{52})^2 - 2\left[1 - \frac{m_3\hat{z}(1-m_2\hat{x})}{(1+m_2\hat{x})^3}\right] D_1\ell_{52}{}^2 - \frac{2m_3^2\hat{x}\hat{z}}{(1+m_2\hat{x})^3}\ell_{52}{}^2$$
$$-2\left[\frac{m_1}{(1+m_1\hat{z})^2} + \frac{1}{(1+m_2\hat{x})^2}\right] D_1D_2\ell_{52}{}^2 + \frac{2\,m_3\hat{x}}{(1+m_2\hat{x})^2} D_2\ell_{52}{}^2 + \frac{2m_1^2\hat{x}}{(1+m_1\hat{z})^3}(D_2\ell_{52})^2 .$$

$$\hat{\gamma}_{21} = -2\left[\widehat{m}_7 - \frac{m_2m_8\hat{z}}{(1+m_2\hat{x})^2}\right] D_1\ell_{52}{}^2 - 2\left[m_6 - \frac{m_3m_8\hat{z}}{(1+m_2\hat{x})^2}\right]\ell_{52}{}^2 - 2\left[\frac{m_4m_5}{(1+m_5\hat{z})^2} + \frac{m_8}{(1+m_2\hat{x})}\right] D_2\ell_{52}{}^2,$$

$$\hat{\gamma}_{31} = -\frac{2m_2m_9\hat{z}}{(1+m_2\hat{x})^3}(D_1\ell_{52})^2 - 2\,\frac{m_3[m_{10}\hat{z} + (m_2m_{10} - m_3m_9)\hat{x}]}{(1+m_2\hat{x})^3}\,\ell_{52}{}^2$$
$$-2\,\frac{m_2m_{10}(1+m_2\hat{x})\hat{z} + m_3m_9(1-m_2\hat{x})\hat{z}}{(1+m_2\hat{x})^3}\,D_1\ell_{52}{}^2 - 2m_{11}(D_2\ell_{52})^2$$
$$+2\,\frac{m_{10} + (m_2m_{10} - m_3m_9)\hat{x}}{(1+m_2\hat{x})^2} D_2\ell_{52}{}^2 + 2\,\frac{m_9}{(1+m_2\hat{x})^2}\,D_1D_2\ell_{52}{}^2.$$

Then, using conditions (49) yields:

$$\Theta_5{}^T D^2F(\varepsilon_5, \widehat{m}_7)(\mathcal{L}_5, \mathcal{L}_5) = \vartheta_{42}\hat{\gamma}_{21} \neq 0.$$

Hence, in the sense of Sotomayor, a transcrtical bifurcation take place.

Theorem (15): If the conditions (30a)-(30d) are met, then a saddle-node bifurcation of the system (3) at the CEP happens when the parameter m_{11} passes over the value $m_{11}^* = \frac{d_{12}d_{23}d_{31} - d_{13}d_{22}.d_{31} - d_{32}(d_{11}d_{22} - d_{13}d_{21})}{z^*(d_{11}d_{22} - d_{12}d_{21})}$, if and only if the following condition is satisfied.

$$B_3\gamma_{11}{}^* + B_4\gamma_{21}{}^* + \gamma_{31}{}^* \neq 0, \tag{50}$$

where the symbols of condition (50) are computed in the proof.

Proof: The Jacobian matrix at CEP with $m_{11} = m_{11}^*$ can be written as:

$$J_6 = J(\varepsilon_6, m_{11}^*) = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33}{}^* \end{pmatrix},$$

where the elements $d_{ij}, i, j = 1,2,3$ are given in the Jacobian matrix (31) with $d_{33}{}^* = d_{33}(m_{11}^*)$. Direct computation shows that the determinant of J_6 is equal to zero. Hence the matrix J_6 has a zero eigenvalue given by $\lambda_{63}{}^{*}=0$, with two negative real portions eigenvalues $\lambda_{61,62}=\frac{-A_1\pm\sqrt{A_1{}^2-4A_2}}{2}$, where $A_1>0$ and $A_2>0$ are given in Eq. (32). Therefore, the CEP is a non-hyperbolic point.

Let $\mathcal{L}_6=(\ell_{61},\ell_{62},\ell_{63})^T$ be the eigenvector conjugate with the eigenvalue $\lambda_{63}{}^{*}=0$.

Thus, $J_6\,\mathcal{L}_6=0$, gives that $\mathcal{L}_6=(B_1\ell_{63},\mathrm{B}_2\ell_{63},\ell_{63})^T$, where $\mathrm{B}_1=\frac{d_{12}d_{23}-d_{13}d_{22}}{d_{11}d_{22}-d_{12}d_{21}}$, $\mathrm{B}_2=\frac{d_{13}d_{21}-d_{11}d_{23}}{d_{11}d_{22}-d_{12}d_{21}}<0$, and $\ell_{63}\neq 0$ is any real number.

Now, let $\Theta_6=(\vartheta_{61},\vartheta_{62},\vartheta_{63})^T$ denotes to the eigenvector conjugate with the eigenvalue $\lambda_{63}{}^{*}=0$, of the matrix $J_6{}^T$.

Thus, $J_6{}^T\Theta_6=0$ gives that $\Theta_6=(B_3\vartheta_{63},B_4\vartheta_{63},\vartheta_{63})^T$, where $B_3=\frac{d_{21}d_{32}-d_{22}d_{31}}{d_{11}d_{22}-d_{21}d_{12}}$, $B_4=\frac{d_{12}d_{31}-d_{11}d_{32}}{d_{11}d_{22}-d_{21}d_{12}}<0$, with $\vartheta_{63}\neq 0$ is any real number.

Moreover, it is observed that:

$$\frac{\partial F}{\partial m_{11}}=F_{m_{11}}(X,m_{11})=(0,0,-z^2)^T\implies F_{m_{11}}(\varepsilon_6,m_{11}^*)=(0,0,-z^{*2})^T.$$

Therefore, $\Theta_6{}^T F_{m_{11}}(\varepsilon_6,m_{11}^*)=-\vartheta_{63}z^{*2}\neq 0$, Hence the first condition of a saddle node bifurcation is met.

Moreover, by using equation (46), it is obtained that:

$$D^2F(\varepsilon_6,m_{11}^*)(\mathcal{L}_6,\mathcal{L}_6)=[\gamma_{i1}{}^{*}]_{3\times1},$$

where

$$\begin{aligned}\gamma_{11}{}^{*}=&-2\left[1-\frac{m_2z^*(1+m_3y^*)}{(1+m_2x^*+m_3y^*)^3}\right](B_1\ell_{63})^2-2\left[1-\frac{m_3z^*(1-m_2x^*+m_3y^*)}{(1+m_2x^*+m_3y^*)^3}\right]B_1B_2\ell_{63}{}^2\\&-\frac{2m_3^2x^*z^*}{(1+m_2x^*+m_3y^*)^3}(B_2\ell_{63})^2-2\left(\frac{m_1}{(1+m_1z^*)^2}+\frac{(1+m_3y^*)}{(1+m_2x^*+m_3y^*)^2}\right)B_1\ell_{63}{}^2\\&+\frac{2\,m_3x^*}{(1+m_2x^*+m_3y^*)^2}B_2\ell_{63}{}^2+\frac{2m_1^2x^*}{(1+m_1z)^3}\ell_{63}{}^2,\\ \gamma_{21}{}^{*}=&-2\frac{m_2^2m_8y^*z^*}{(1+m_2x^*+m_3y^*)^3}(B_1\ell_{63})^2-2\left(m_7-\frac{m_2m_8z^*(1+m_2x^*-m_3y^*)}{(1+m_2x^*+m_3y^*)^3}\right)B_1B_2\ell_{63}{}^2\\&+2\frac{m_2m_8y^*}{(1+m_2x^*+m_3y^*)^2}B_1\ell_{63}{}^2-2\left[m_6-\frac{m_3m_8z^*(1+m_2x^*)}{(1+m_2x^*+m_3y^*)^3}\right](B_2\ell_{63})^2\\&+2\frac{m_4m_5^2y^*}{(1+m_5z)^3}\ell_{63}{}^2-2\left[\frac{m_4m_5}{(1+m_5z^*)^2}+\frac{m_8(1+m_2x^*)}{(1+m_2x^*+m_3y^*)^2}\right]B_2\ell_{63}{}^2,\end{aligned}$$

$$\gamma_{31}{}^{*} = -\frac{2m_2 z^*[m_9+(m_3m_9-m_2m_{10})y^*]}{(1+m_2x^*+m_3y^*)^3}(B_1\ell_{63})^2 - 2\,\frac{m_3[m_{10}z^*+(m_2m_{10}-m_3m_9)x^*]}{(1+m_2x^*+m_3y^*)^3}\,(B_2\ell_{63})^2$$
$$-2\,\frac{(m_2x^*-m_3y^*)(m_2m_{10}-m_3m_9)z^*+(m_2m_{10}+m_3m_9)z^*}{(1+m_2x^*+m_3y^*)^3}\,B_1B_2\ell_{63}{}^2 - 2m_{11}^*\ell_{63}{}^2$$
$$+2\,\frac{m_{10}+(m_2m_{10}-m_3m_9)x^*}{(1+m_2x^*+m_3y^*)^2}B_2\ell_{63}{}^2 + 2\,\frac{m_9+(m_3m_9-m_2m_{10})y^*}{(1+m_2x^*+m_3y^*)^2}\,B_1\ell_{63}{}^2.$$

Therefore, using the condition (50), it is obtained that

$$[B_3\gamma_{11}{}^{*} + B_4\gamma_{21}{}^{*} + \gamma_{31}{}^{*}]\vartheta_{63} \neq 0$$

Then a saddle-node bifurcation take place.

As a parameter reaches a critical point, the Hopf bifurcation refers to the birth or death of a periodic solution from equilibrium at a local level. A Hopf bifurcation occurs when a complex conjugate pair of eigenvalues of the linearised flow at a given position becomes fully imaginary, according to the Poincare-Andronov-Hopf bifurcation theorem. This means that a Hopf bifurcation can only occur in two-dimensional systems or higher. The restrictions that guarantee a Hopf bifurcation at the CEP are presented in below theorem.

Theorem (16): Assume that the requirements (30a)-(30d) are met, as well as the following:

$$d_{12}d_{23}d_{31} < \min\{d_{13}d_{22}d_{31}, -d_{13}d_{21}d_{32}\}, \tag{51a}$$

$$d_{12}(d_{11}+d_{22}) + d_{13}d_{32} > 0, \tag{51b}$$

$$\left(A_1(m_7^*)A_2(m_7^*)\right)' < A_3{}'(m_7^*), \tag{51c}$$

where, d_{ij} and A_i $for\ i,j = 1,2,3$ are respectively the elements of J_{ε_6} that given by Eq. (31) and the coefficients of the characteristic equation that given by Eq. (32). Then, as the parameter m_7 passes through the value m_7^*, system (3) experiences a Hopf bifurcation at the CEP, where

$$m_7^* = \frac{\varpi}{y^*[d_{12}(d_{11}+d_{22})+d_{13}d_{32}]} + \frac{m_2m_8z^*}{(1+m_2x^*+m_3y^*)^2},$$

with

$$\varpi = -(d_{11}+d_{22})d_{11}d_{22} - (d_{11}+d_{33})[d_{11}d_{33}-d_{13}d_{31}]$$
$$-(d_{22}+d_{33})[d_{22}d_{33}-d_{23}d_{32}] - 2d_{11}d_{22}d_{33} + d_{12}d_{23}d_{31}.$$

Proof: According to the form of $\Delta = A_1A_2 - A_3$ that given in Eq. (32), it is easy to verify that $\Delta = 0$ at $m_7 = m_7^*$, where $m_7^* > 0$ if the sufficient conditions (51a)-(51b) are met. Therefore, it is gotten that $A_1(m_7^*)A_2(m_7^*) = A_3(m_7^*)$. Consequently, Eq. (32) at $m_7 = m_7^*$ becomes

$$P_3(\lambda) = (\lambda + A_1)(\lambda^2 + A_2) = 0; \tag{52}$$

where A_1, and A_2 under the conditions (30a)-(30d) are positive. Now, simple calculation steps give that the Eq. (52) has the following roots

$$\lambda_1 = -A_1 \text{ and } \lambda_{2,3} = \pm i\sqrt{A_2}.$$

Note that, when $m_7 = m_7^*$, the first condition of the Hopf bifurcation is satisfied, and then two pure imaginary complex conjugate eigenvalues are arise. These complex conjugate eigenvalues in the vicinity of m_7^* are adopted the form $\lambda_{2,3} = \delta_1(m_7) \pm i\delta_2(m_7)$. As a result, in Eq. (32), substitute $\lambda = \delta_1(m_7) + i\delta_2(m_7)$, and then take the derivative with regard to the bifurcation parameter m_7. After comparing the two sides of the appearing equation and equating their real and imaginary components, the following result is obtained:

$$\begin{aligned} \Pi_1(m_7){\delta_1}'(m_7) + \Pi_2(m_7){\delta_2}'(m_7) &= -\Pi_3(m_7), \\ \Pi_2(m_7){\delta_1}'(m_7) + \Pi_1(m_7){\delta_2}'(m_7) &= -\Pi_4(m_7), \end{aligned} \tag{53}$$

where:

$$\Pi_1(m_7) = 3{\delta_1}^2(m_7) + 2A_1(m_7)\delta_1(m_7) + A_2(m_7) - 3{\delta_2}^2(m_7),$$

$$\Pi_2(m_7) = 6\delta_1(m_7)\delta_2(m_7) + 2A_1(m_7)\delta_2(m_7),$$

$$\Pi_3(m_7) = {\delta_1}^2(m_7){A_1}'(m_7) + {A_2}'(m_7)\delta_1(m_7) + {A_3}'(m_7) - {A_1}'(m_7){\delta_2}^2(m_7),$$

$$\Pi_4(m_7) = 2\delta_1(m_7)\delta_2(m_7){A_1}'(m_7) + {A_2}'(m_7)\delta_2(m_7).$$

Solving the linear system (53), gives that

$$\begin{aligned} {\delta_1}'(m_7) &= \frac{d\delta_1(m_7)}{dm_7} = -\frac{\Pi_3(m_7)\Pi_1(m_7)+\Pi_4(m_7)\Pi_2(m_7)}{[\Pi_1(m_7)]^2+[\Pi_2(m_7)]^2}, \\ {\delta_2}'(m_7) &= -\frac{\Pi_4(m_7)\Pi_1(m_7)-\Pi_3(m_7)\Pi_2(m_7)}{[\Pi_1(m_7)]^2+[\Pi_2(m_7)]^2}. \end{aligned} \tag{54}$$

Obviously, we have that $\delta_1(m_7^*) = 0$ and $\delta_2(m_7^*) = \sqrt{A_2(m_7^*)}$, then the coefficients of Eq. (53) at $m_7 = m_7^*$ become:

$$\Pi_1(m_7^*) = -2A_2(m_7^*),$$

$$\Pi_2(m_7^*) = 2A_1(m_7^*)\sqrt{A_2(m_7^*)},$$

$$\Pi_3(m_7^*) = {A_3}'(m_7^*) - {A_1}'(m_7^*)A_2(m_7^*),$$

$$\Pi_4(m_7^*) = A_2{}'(m_7^*)\sqrt{A_2(m_7^*)}.$$

Therefore, direct computation gives that

$$\Pi_3(m_7^*)\Pi_1(m_7^*) + \Pi_4(m_7^*)\Pi_2(m_7^*) = -2A_2(m_7^*)\left[A_3{}'(m_7^*) - \left(A_1(m_7^*)A_2(m_7^*)\right)'\right].$$

Hence, the transversality condition $\delta_1{}'(m_7^*) > 0$ is satisfied under the condition (51c). As a result, system (3) experiences Hopf bifurcation at $m_7 = m_7^*$.

8. Numerical Simulation

In this part, Matlab is used to solve system (3) numerically. To understand the global dynamics of the system (3) and the implications of varying their parameters, multiple hypothetical sets of parameter values with different initial points are employed. All of the findings are given in the form of phase portraits and time series. The sets of parameters used in this study are given in the table (2).

Table 2: Hypothetical sets of parameters:

parameters	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
Set (1)	0.4	4	3	0.9	0.5	0.9	0.8	1	0.1	0.75	0.2	0.1
Set (2)	0.5	0.2	0.4	1	0.5	0.2	1.5	1	0.5	0.5	0.1	0.01

The trajectories of system (3) approach asymptotically to the CEP for the set (1), regardless of whether the fear exists or not, as shown in Figures 1b and 1a, respectively. However, raising the first prey's fear rate or decreasing the second prey's fear rate induces extinction in the first prey, and the trajectories of system (3) approach FPYFEP asymptotically, as shown in Figures 1c and 1d, respectively.

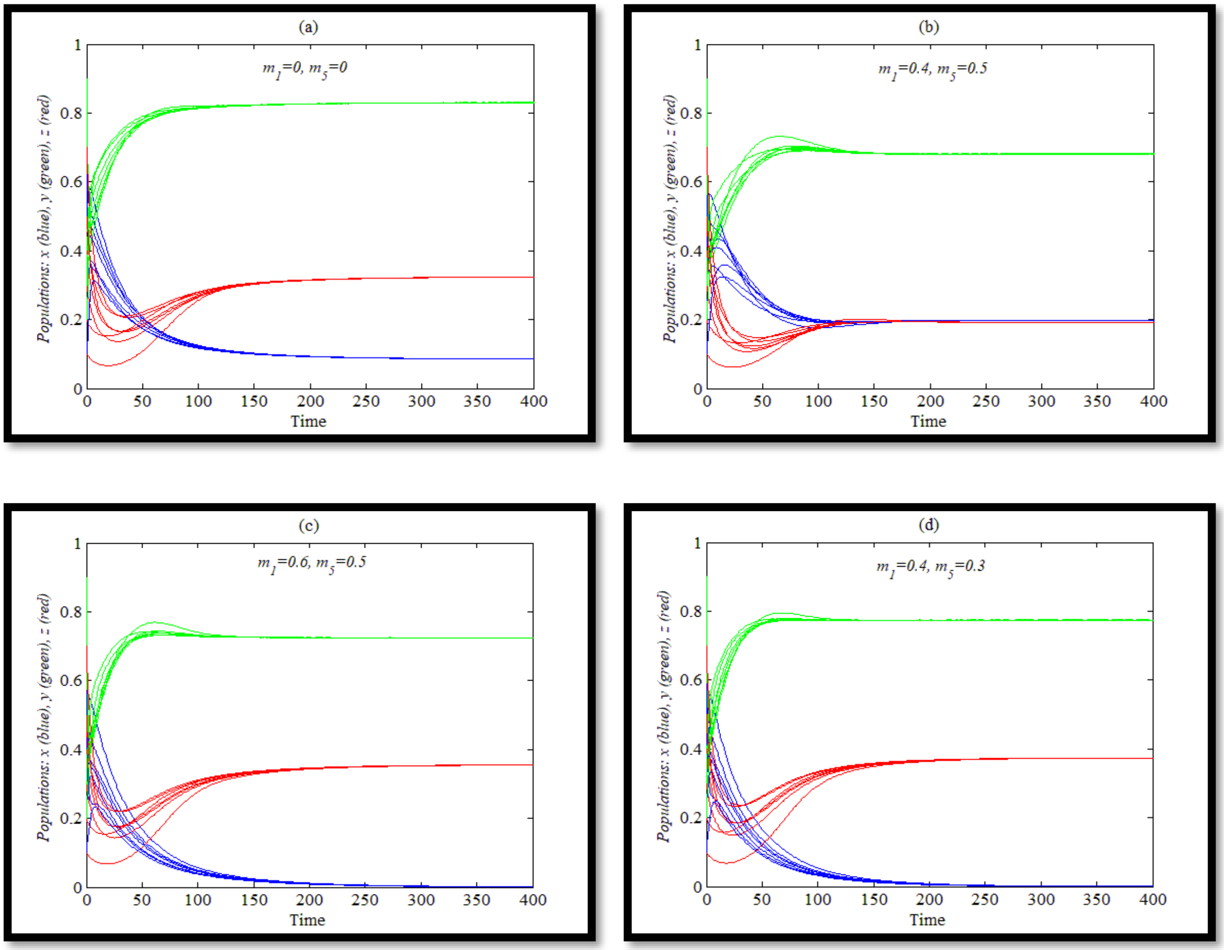

Fig. 1: Time series of system (3) trajectories utilizing set (1) of data starting at various initial positions. (a) Trajectories are approaching CEP globally at $(0.08,0.83,0.32)$. (b) Trajectories are approaching CEP globally at $(0.197,0.68,0.192)$. (c) Trajectories are approaching FPYFEP globally at $(0,0.72,0.35)$. (d) Trajectories are approaching FPYFEP globally at $(0,0.77,0.37)$.

It is observed that, rising the value of m_3 in the range $m_3 > 6.5$ leads to approaching to SAEP as shown in Figure 2, for the exemplary value of $m_3 = 10$.

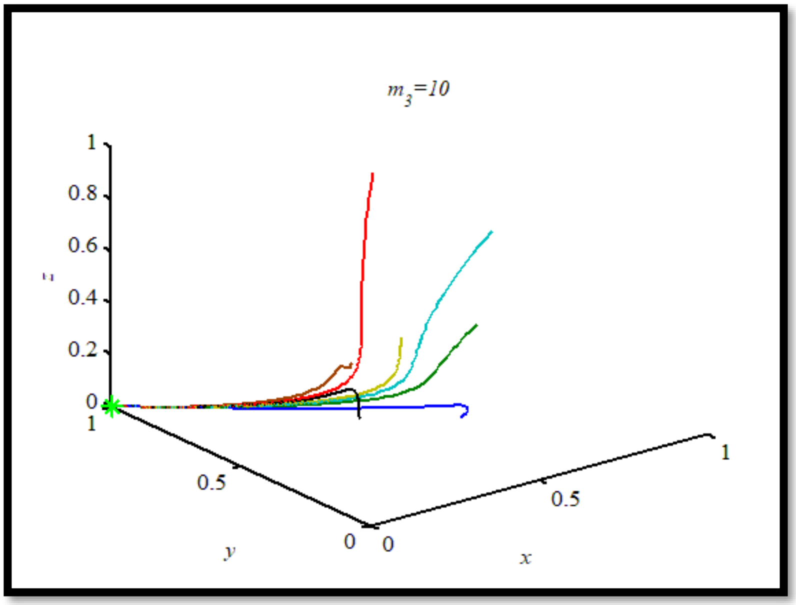

Fig. 2: Phase portrait of system (3) utilizing set (1) of data starting at various initial positions that shows the global stability of $\varepsilon_2 = (0,1,0)$.

Note that, while condition (20a) is not satisfied and the system has a zero eigenvalue when employing set (1) of data, the SAEP is globally asymptotically stable because condition (20a) is satisfied (20b). Now, varying the parameter m_4 in the range $m_4 < 0.8$ leads to approach FAEP, while for the range $m_4 > 1$, the system approaches to FPYFEP as presented in Figures 3a, 3b with 3c respectively for exemplary valuesof m_4.

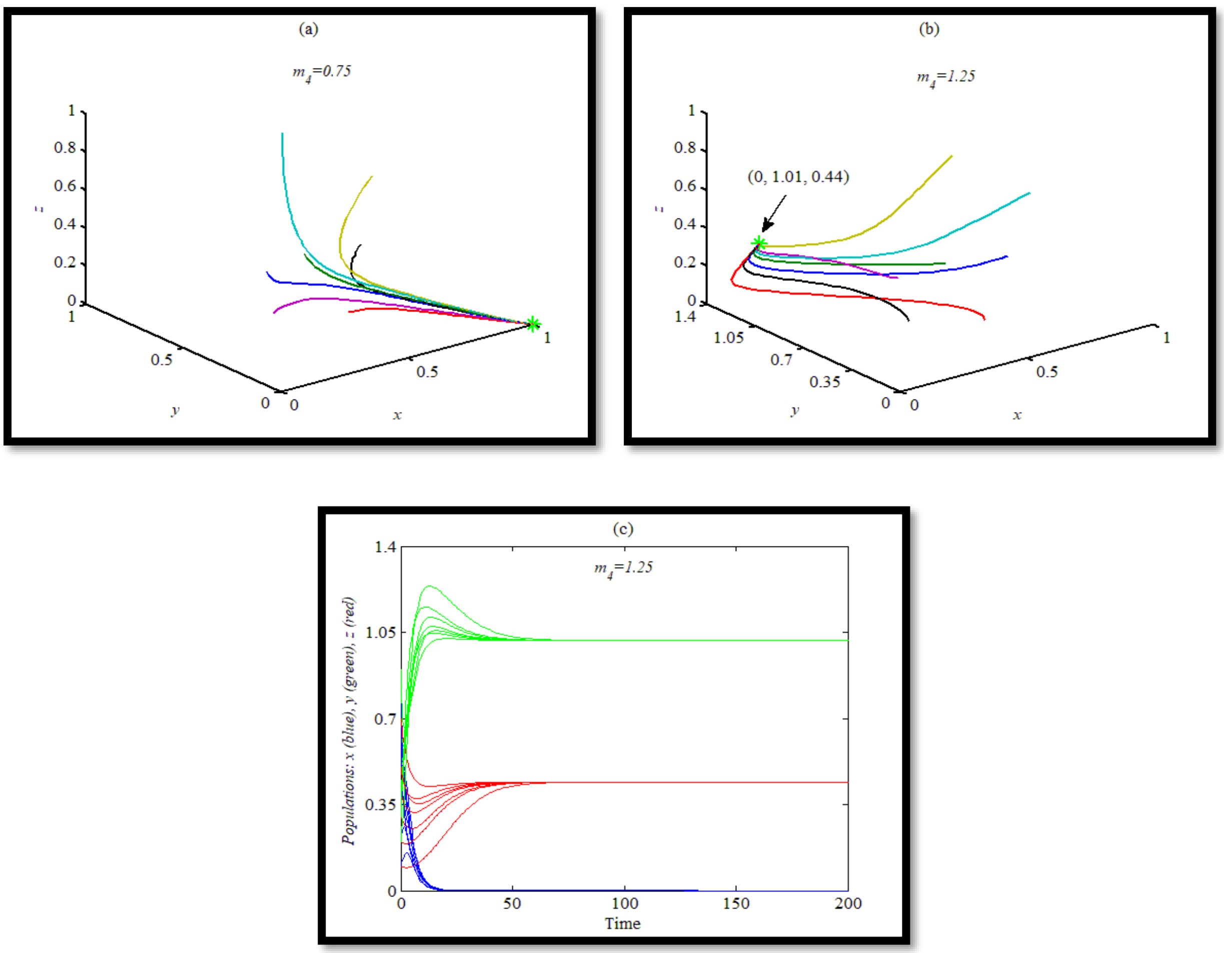

Fig. 3: The trajectories of system (3) utilizing set (1) of data starting at various initial positions. (a) Phase portrait that shows the global stability of $\varepsilon_1 = (1,0,0)$ using $m_4 = 0.75$. (b) Phase portrait that shows the global stability of $\varepsilon_4 = (0,1.01,0.44)$ using $m_4 = 1.25$. (c) Time series for the trajectories given in Fig. 3b.

Figures 4a-4b and 4c-4d, show the transferring of the trajectories of system (3) between the FPYFEP and PFEP as the parameter m_6 transfers between the ranges $m_6 < 0.8$ and $m_6 > 0.9$, respectively. However, Figures 5a-5b and 6a-6b, demonstrate the existence of stable line of equilibrium point $x + y = 1$ in the xy −plane and the asymptotic stability of the system (3) at FAEP when $m_7 = 0.9$ and $m_7 > 0.9$, respectively.

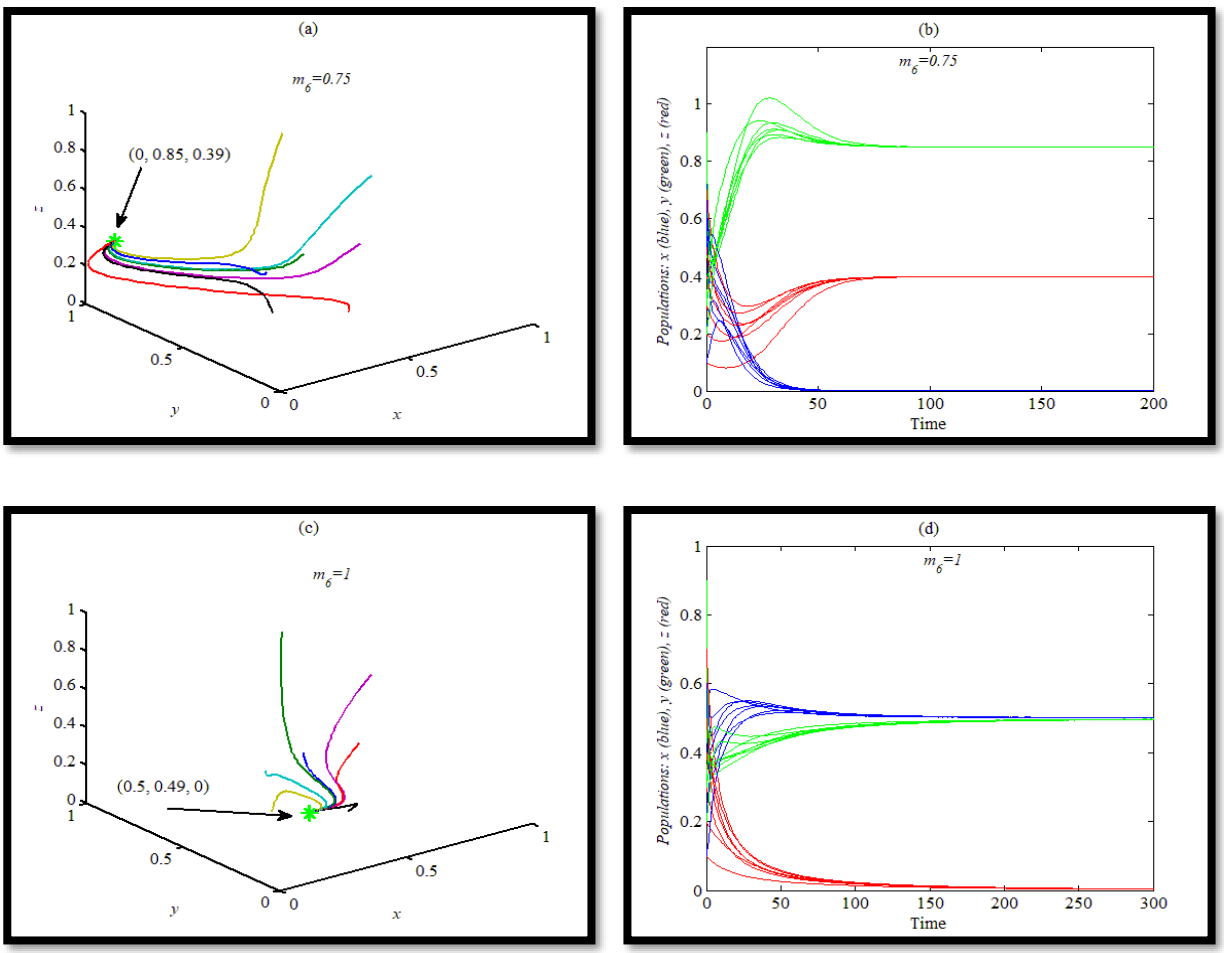

Fig. 4: The trajectories of system (3) utilizing set (1) of data starting at various initial positions. (a) Phase portrait that shows the global stability of $\varepsilon_4 = (0,0.85,0.39)$ using $m_6 = 0.75$. (b) Time series for the trajectories given in Fig. 4a. (c) Phase portrait that shows the global stability of $\varepsilon_6 = (0.5,0.49,0)$ using $m_6 = 1$. (d)) Time series for the trajectories given in Fig. 4c.

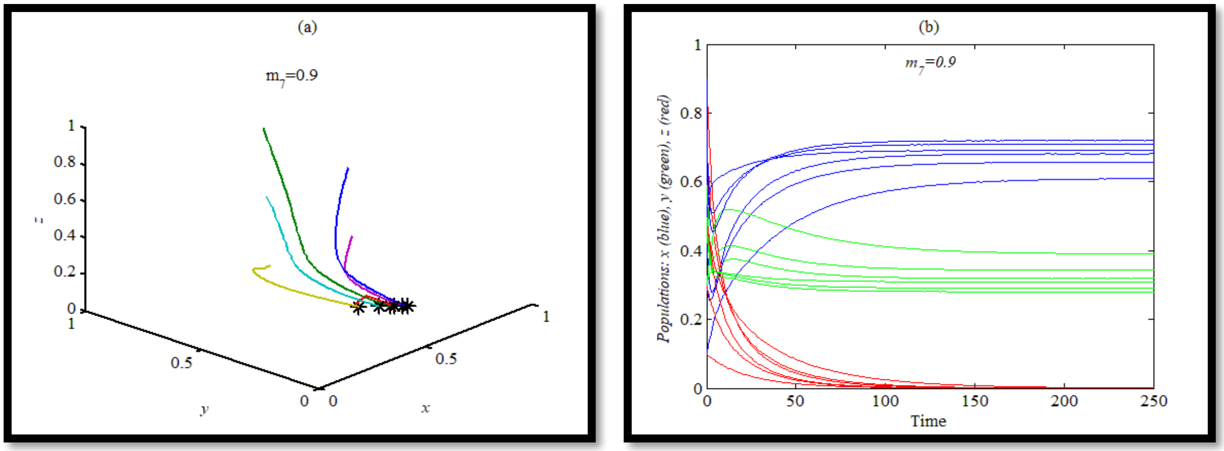

Fig. 5: The trajectories of system (3) utilizing set (1) of data starting at various initial positions. (a) Phase portrait that shows the existence of stable line in xy −plane using $m_7 = 0.9$. (b) Time series for the trajectories given in Fig. 5a.

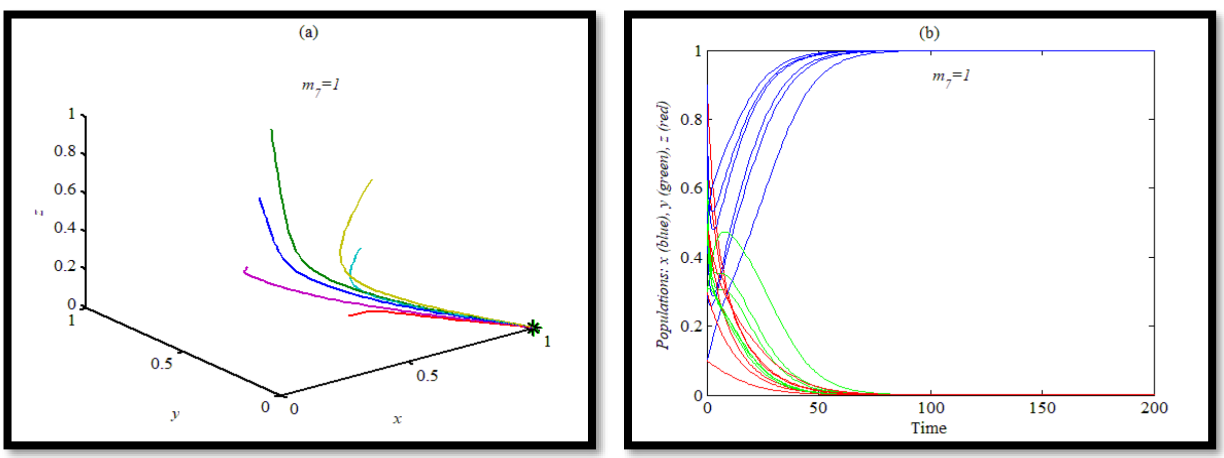

Fig. 6: The trajectories of system (3) utilizing set (1) of data starting at various initial positions. (a) Phase portrait that shows the global stability of $\varepsilon_1 = (1,0,0)$ using $m_7 = 1$. (b) Time series for the trajectories given in Fig. 6a.

Now, for the parameters m_8 and m_9 in the ranges $m_8 < 0.6$ and $m_9 > 1.4$, it is obtained that the system (3) approaches asymptotically to FPYFEP and SPYFEP as shown in Figures 7a-7b and 8a-8b, respectively.

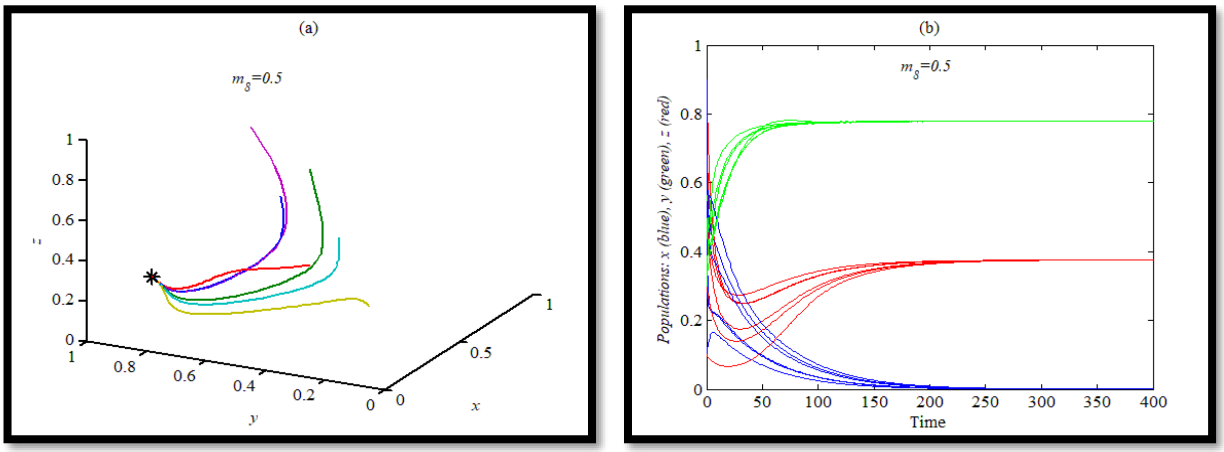

Fig. 7: The trajectories of system (3) utilizing set (1) of data starting at various initial positions. (a) Phase portrait that shows the global stability of $\varepsilon_4 = (0,0.77,0.37)$ using $m_8 = 0.5$. (b) Time series for the trajectories given in Fig. 7a.

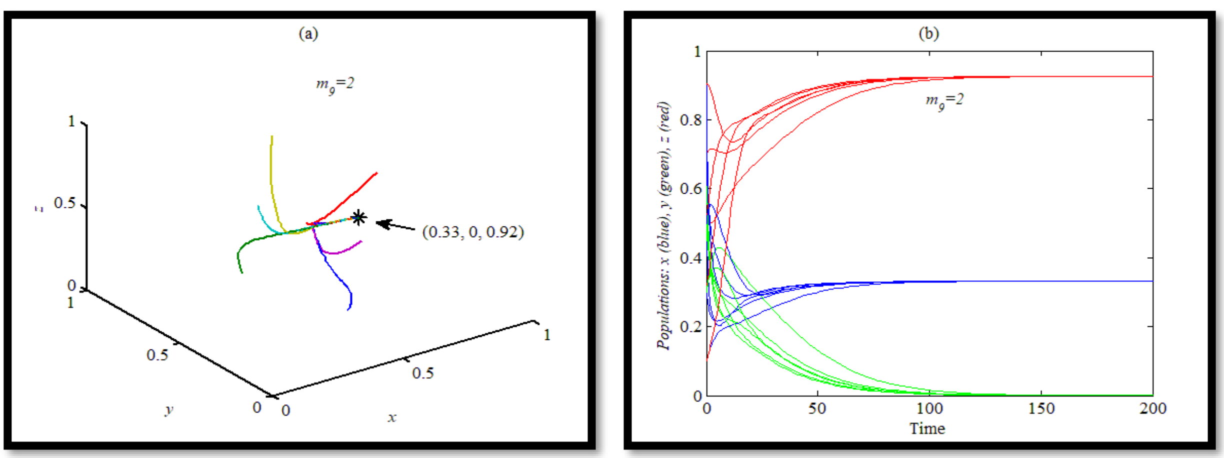

Fig. 8: The trajectories of system (3) utilizing set (1) of data starting at various initial positions. (a) Phase portrait that shows the global stability of $\varepsilon_5 = (0.33,0,0.92)$ using $m_9 = 2$. (b) Time series for the trajectories given in Fig. 8a.

The trajectories of system (3) approach asymptotically to the SAEP when the parameter w_{10} is varied in the range $w_{10} \leq 0.4$, as shown in Figure 9. When the parameter w_{12} is increased above the value of 0.2, the result is similar to that of w_{10}. While increasing the value of w_{11}, the value of the predator and the first prey gradually reduces as well, and the trajectories eventually converge to the SAEP.

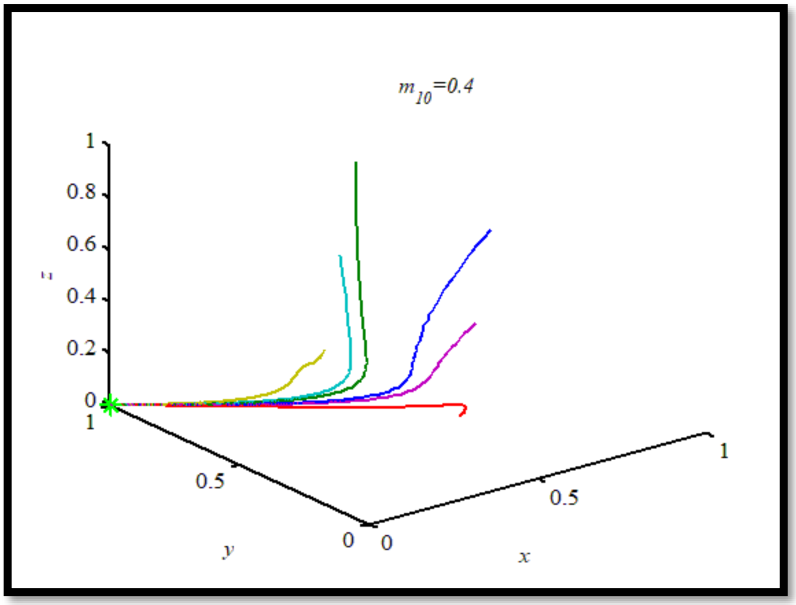

Fig. 9: Phase portrait of system (3) utilizing set (1) of data starting at various initial positions that shows the global stability of $\varepsilon_2 = (0,1,0)$, when $w_{10} = 0.4$.

With the preceding in mind, the dynamics of the system (3) are numerically studied utilizing set (2) of data. The goal is to demonstrate that when alternative hypothetical sets of data are used, different sorts of dynamical behavior can be created. However, the theoretical conclusions hold true for a variety of data sets.

It is observed that, system (3) undergoes a bi-stability behavior between the FPYFEP and SPYFEP for the set (2) of data, regardless of whether the fear exists or not, as shown in Figures 10b and 10a, respectively.

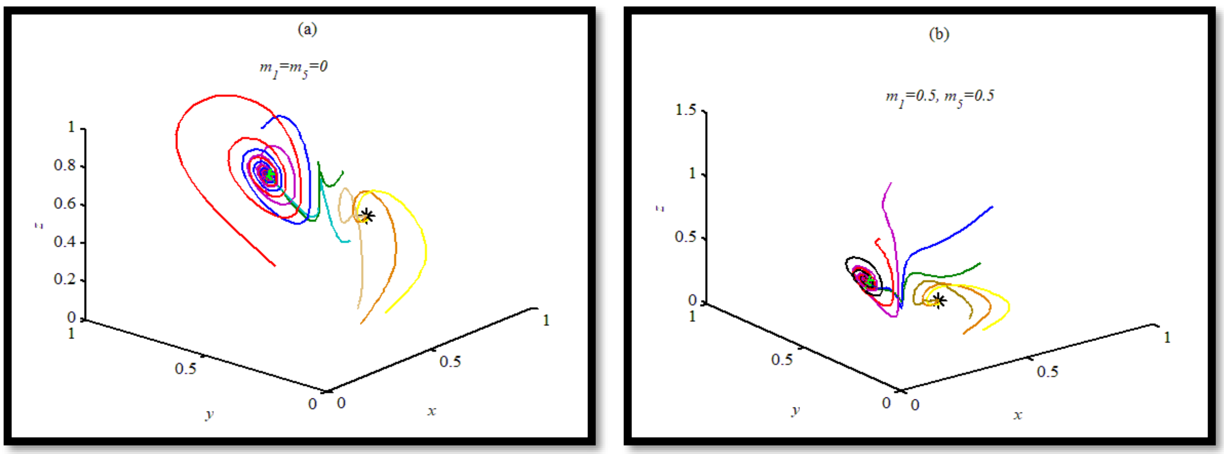

Fig. 10: Phase portrait of system (3) utilizing set (2) of data starting at various initial positions that shows bi-stability behavior. (a) For $m_1 = m_5 = 0$, bi-stability between $\varepsilon_4 = (0,0.25,1.04)$ and $\varepsilon_5 = (0.19,0,0.83)$. (b) For $m_1 = 0.5, m_5 = 0.5$, bi-stability between $\varepsilon_4 = (0,0.18,0.74)$ and $\varepsilon_5 = (0.15,0,0.62)$.

Although the existence of fear does not prevent the bi-stability behavior, it is reduced the populations size due to hiding as anti-predator behavior.

However, the presence of fear in either the first or second prey causes extinction in either the first or second prey, and the trajectories of system (3) approach FPYFEP or SPYFEP asymptotically, as shown in Figures 11a-11b and 11c-11d, respectively.

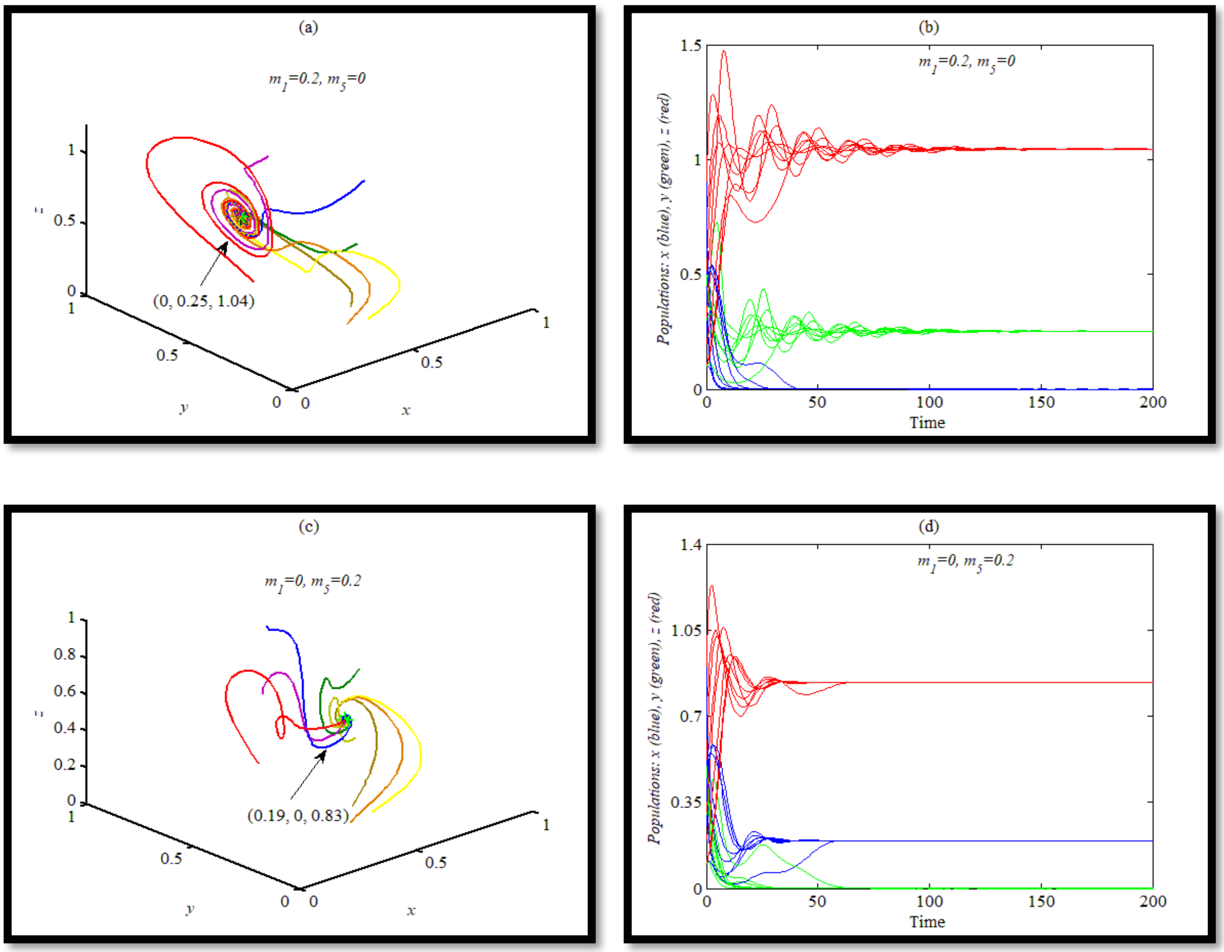

Fig. 11: The trajectories of system (3) utilizing set (2) of data starting at various initial positions. (a) Phase portrait that shows the global stability of $\varepsilon_4 = (0,0.25,1.04)$ using $m_1 = 0.2, m_5 = 0$. (b) Time series for the trajectories given in Fig. 11a. (c) Phase portrait that shows the global stability of $\varepsilon_5 = (0.19,0,0.83)$ using $m_1 = 0, m_5 = 0.2$. (d) Time series for the trajectories given in Fig. 11c.

9. DISCUSSION AND CONCLUSION

In this study, in the presence of fear, an ecological model based on a three-species food web with two competing prey and one predator is built. The solution's dynamical properties (stability, persistence, and bifurcation) are studied theoretically. To understand the global dynamics and impacts of modifying the system parameters, numerical simulation of the proposed system is performed using two alternative sets of hypothetical parameter values. The following findings have been presented.

1. The system (3) exhibits a variety of dynamical behaviors depending on the parameter values, including globally asymptotically stable CEP, stable line, bi-stability behavior, periodic, and even chaotic behavior.
2. The system approaches a CEP for an appropriate range of fear rate values in both competing preys. While increasing the fear rate in one of the two competing preys above a certain value causes extinction in that prey due to lack of food and the trajectories approach asymptotically to the opposite planar equilibrium point (see Figure 1).
3. Because the second prey is a stronger competitor than the first prey and represents a preferred food for the predator, rising the environmental safety rate associated with the second prey causes persistence to be lost, and the system's trajectories approach asymptotically to SAEP, as shown in Figure 2.
4. Due to extinction in the predator and first prey as a result of the competitive exclusion principle, decreasing the growth rate of the second prey pushes the trajectories of the system (3) to approach the SAEP, as shown in Figure 3. While increasing the value of this parameter above a certain threshold causes extinction in the first prey, the system's (3) trajectories approach the FPYFEP asymptotically.
5. As illustrated in Figure 4, lowering the intra-specific competition of the second prey below a certain threshold induces extinction in the first prey due to the winning of the second prey in the competition process, and the trajectories then approach the FPYFEP. However, raising the value of this parameter causes extinction in predator species due to the predator's

heavy dependence on the second prey for feeding, hence the system's (3) trajectories approach the PFEP.

6. Equating the parameters of growth rate, intra-specific competition, and inter-specific competition of the second prey with one another results in a stable line of PFEPs, as shown in Figure 5. Furthermore, according to set (1) of parameter values, increasing the value of inter-specific competition of second prey causes extinction in second prey and then predator due to the predator's feeding dependency on the second prey. As a result, the system (3)'s trajectories approach FAEP asymptotically, as seen in Figure 6.
7. Reduce the predator's attack rate on the second prey or increase the predator's conversion rate from the first prey yield to the approaching of the trajectories of system (3) to FPYFEP and SPYFEP, respectively, as shown in Figures 7 and 8.
8. Extinction in predator species is caused by lowering the predator's conversion rate from the second prey below a specific value (as shown in Figure 9) or increasing the predator's intra-specific competition, and thus the system's (3) trajectories approach the SAEP. This is due to the fact that the competitive exclusion principle leads to extinction in the first prey as well.
9. For the set (2) of parameter values, Figure 10 shows that system (3) undergoes a bi-stability behavior between the FPYFEP and SPYFEP, regardless of whether the fear exists or not, which indicates the complex dynamics of the system (3).
10. Finally, The increase in fear rate in either the first or second prey causes extinction in either the first or second prey, which stops the bi-stability behavior, and thus the trajectories of system (3) approach FPYFEP or SPYFEP asymptotically, as seen in Figure 11.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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