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REGIONAL OPTIMAL HARVESTING CONTROLS OF A SPATIOTEMPORAL PREY-PREDATOR THREE SPECIES FISHERY MODEL

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Abstract. In this paper, we propose another extension of the basic prey-predators model. We incorporate the spatial behavior of the fish population and a term of regional control to provide a realistic description of the prey effects of two predators. We present a study of regional optimal control strategies of a spatiotemporal prey-predator model. Based on an existing model, we add the Laplacian to describe the spatial mobility of its individual. Our main objective is to characterize two harvesting strategies of control, which allow users to increase prey density and decrease the density of predators and super predators to maintain a differential chain system and ensure sustainability. Firstly, by applying the semigroup theory, we investigate the existence of the solution and estimations of the unique strong global solution for the controlled system. Secondly, we prove the existence of a pair of controls and characterize the controls in terms of state and assistant functions according to Pontryagin's maximum principle. Finally, some numerical simulations for several cases to verify the theoretical analysis are obtained. Also, the results show the effectiveness of the controls if the harvesting strategies of the regional controls are applied simultaneously.

Keywords: spatiotemporal prey-predator model; environmental sustainability; fish harvesting; spatiotemporal prey-predator model; regional optimal control problem.

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1. INTRODUCTION

In recent years, the application of control in the field of fishing has become important. This is to preserve the natural and animal wealth that have seen a strong and continuous destruction due to human activities and technological developments. Due to their economic importance, these activities have increased over years. Mathematical modeling is a powerful tool that makes it easier for those in charge to choose the right strategies and also to test the effectiveness of these strategies before adopting any policies or taking any measures. In order to model this situation, modelling with prey predator system is the most suitable and most used. In the population dynamics of a prey predator system, the predator plays the role of an individual reproducing by feeding on a prey; however, the survival of the predators depends on the survival of the prey; the absence of the prey leads to the death and disappearance of the predators. Predator-prey systems present much different dynamical behavior such as steady states, oscillations and bifurcations. Recently, mathematicians and ecologists [3, 4, 5] have been studying the predator-prey system in population dynamics. The Lotka Volterra Model [6]-[7] is the basic and classical model. In this model there are two nonlinear differential equations, each expressing the progression of each species as well as the interaction between them. This model has been adopted and developed in several works, for example in [9] the authors have developed this model in a discrete case using the difference equations; in [8] the authors have generalized this model for several species; in [10] the authors have introduced an extension of this model taking into account the effect of the spatial factor by adding terms that describe the movement of fish populations. There are mainly three types of functional responses: Holling types I, II, and III, all originally suggested by Holling [6]. Type I occurs when there is a maximum linear situation in the density of prey consumed for each predator as the prey density decreases. Type II product happens when the response occurs at a decreasing rate toward a high value. Finally, type III occurs, when the response is sigmoid again approaching an upper asymptote. Many researchers have studied the Holling types of functional responses (see [11]).

RELATED WORKS

There are several works that have studied the dynamics of fish populations based on the lotka volterra model. For bioeconomic model see [15]. In [15] Kar and Chakraborty examined the dynamical behavior of a biological economic prey predator model where prey population is harvested using differential algebraic and bifurcation theory. In [14] the authors gave a mathematical analysis of a multi-site bio-economic fishing model, with the objective of maximizing the profit of the human-fisherman respecting the biological equilibrium. The same is true for Mchich et al. [17] where the authors have constructed a model of bioeconomic but generalized several areas where populations are mobilized; their mathematical study has allowed them to optimize fishing effort spatial distribution and the specification of a set of effective management measures. Several models have been established in a space and continous time, using predator-prey models based on reaction-diffusion [13, 18]. The 1-D spatial-temporal dynamics of a prey-predator system with Holling type-II functional response of the predator and logistic growth of the prey have been studied numerically by [13]. In [18], the author considered a ratio-dependent spatially expanded food chain model. He has given the spatial pattern formation via numerical simulation based on Hopf-bifurcation analysis. Thus, a complex reproduction of the model was observed by the dynamics of the system. Regarding the predator-prey models with spatio-temporal dynamic, many scientists concentrated on two dimensional maps (see [19, 20, 21]). In [19] the authors considered a predator-prey system that is discrete both in spatial and temporal and supposed that the prey was affected by a weak Allee effect, where the predators dynamics incorporate an intraspecific competition. In addition, they have considered a neighbourhood of the Turing-Hopf bifurcation in their system. In [12] the researchers examined the spatial pattern formation of a diffuse prey-predator system with ratio dependent functional response implicating the impact of intra-species competition among predators between two dimensional space. In [13] the researchers have given a generalized coupled map network model of ecosystem dynamics. Furthermore, they examined the spatio-temporal response of a prey-predator map, a model of host-parasitoid interactions, and competition between two species. More recently, a considerable amount of work has been done on studying prey predator model. For example, Weigang Zhou et al. [22] addressed the problem of bifurcation stabilization for a fractional predator- prey system with delay under disparate orders using a technique of hybrid control. In [24] the authors have performed a bifurcation control theoretical exploration predator-prey model of delayed fractional order according to the improved control feedback approach. In [23] the authors have discussed a theoretical analysis of bifurcation for a delayed fractional-order predator prey system considering different delay.

Statement of Problem

Many research studies have used the idea of regional control, a concept that was introduced and further developed by El Jai et al. (see [40]). However, to our knowledge, this approach has not yet been implemented in fish-population situations; its deliberations are limited to the theoretical concepts or physical situations. The present paper is set in this general background. The goal of this work is to investigate a spatiotemporal predator-prey model. We introduce a distributed regional control problem based on the variational method. We further provide a novel control application and present a structure for analytical spatial control. A multi objective optimization method is given that we take two harvesting regional controls.

The purpose of both of these two regional optimal control strategies is to have fishing fleets while fishing to ensure good catchability in the considered area and augmenting the function of harvesting to harvest super predators and the predators, which menace the prey, for preserving the environment and food chain.

The purpose of both of these two regional optimal control strategies is to optimize the harvest of fishing fleets by harvesting super predators and predators that threaten the preys so that the environment and food chain would be preserved. Our work is original not only because it considers the spatiotemporal character, but also in the assumption that our control strategy is only restricted to a sub-region ω of Ω. The choice of the sub-region ω is shown by a concern for cost optimization and by ω to be a geographically easy area to adopt as a spatial support for a possible maneuver or control.

We use a variational approach to solves the problem, i.e., we develop an adequate dual system and our optimal control is the feedback of the correspondent adjoint state. Then we present numerical simulations based on Forward/Backward Sweep Method (FSBM) iterative schema, a numerical way by which the optimality system is solved [25]. Numerical simulations demonstrate that the control impact is efficient if the two catchability regional strategies of control are employed concurrently. Further models of problems of optimal control and population dynamics can be seen in [26], [27], [28], [38], [37] and [41].

This paper is organized as follows : Section 2, focuses on the basic mathematical model and the associated optimal problem of control. In Section 3, we apply the semi-group theoretical results and certain known theorems of existence found in [34, 39] to deduce existence and uniqueness of a positive strong global solution of our system. First, we attach a problem that is truncated and whose existence can be established. A local solution of our system is obtained from the solution of the truncated problem. In addition, we illustrate the boundedness of the solution on its maximum definition set; therefore, it could be extended over the entire domain $[0,\ T]\times\Omega$. Section 4 is dedicated to the optimal control existence for our problem. In the following section, necessary optimality conditions are deduced. The dual system and the transversality conditions to obtain a characterization of optimal control are written. The numerical simulations corresponding to our problem of control are provided in Section 6. Our conclusion is presented in Section 7.

2. The Basic Mathematical Model

In this section, based on the article developed by El Bhih et al [26] and as a natural continuation of this work, we will present the basic model without control. This model describes the population density evolution of the fish population and is divided into three compartments: population of prey $x(t)$, population of predator $y(t)$ and population of super predator $z(t)$ at time t. More precisely, we have the following system of differential equations

$$\text{(1)} \qquad \left\{\begin{array}{rcl} \dfrac{dx}{dt} & = & rx\left(1-\dfrac{x}{K}\right)-\dfrac{\alpha xy}{a+x}-m\dfrac{x^2z}{b+x^2}, \\ \dfrac{dy}{dt} & = & \dfrac{\beta xy}{a+x}-d_1y-\dfrac{ny^2z}{b+y^2}-h(y), \\ \dfrac{dz}{dt} & = & n_1\dfrac{y^2z}{b+y^2}+m_1\dfrac{x^2z}{b+x^2}-d_2z-h(z), \end{array}\right.$$

with initial conditions $x(0)\geq 0, y(0)\geq 0, z(0)\geq 0$.

For the density of the prey $x(t)$: We assume that the prey population grows logically with carrying capacity K and constant growth rate r. This population decreases as a result of interactions with predators and super predators. Moreover, the prey population is a favorite food of predator population with a Holling type II functional response (also called Michaelis-Menten functional response) denoted by $\dfrac{\alpha xy}{a+x}$, and the super predator population feeds on the prey population with a Holling type III functional response which is denoted by $\dfrac{x^2z}{b+x^2}$.

For the density of predator $y(t)$: This population increases by feeding on prey population which represent with a Holling type II. On the other hand, the predator is predated by super predator population with Holling type III functional response which is denoted by $\dfrac{x^2z}{b+x^2}$ and decrease by natural death rate d_1. The term $h(y)$ denoted the harvesting function.

For the density of the super predators $z(t)$: This population grows as result of predating the prey, represented by $m_1\dfrac{x^2z}{b+x^2}$ and predating the predator with the term $m_1\dfrac{x^2z}{b+x^2}$. This population decreases due the natural death, with the rate d_2, and by harvesting $h(z)$.

We assume that the harvesting functions, $h(y)$ and $h(z)$, are given by $h(y) = qEy$ and $h(z) = qEz$. Which means that these functions are associated with predator and super predator populations with same catchability coefficient q [1], and E representing the fishing effort rate used to harvest the predator and super predator population. Note that the model (1), with only one harvest function on the super predator, was studied by Roy et al. [30].

The parameters used in this model are summarized in the follow table.

TABLE 1. List of all parameters of system (1)

Parameter	Physical interpretation
r	Intrinsic growth rate of prey.
K	Environmental carrying capacity of the prey.
α	Capture rate of the predator to prey.
m	Capture rate of the super predator to prey.
n	Capture rate of the super predator to predator.
a, b_1, b_2	Half-saturation constants.
d_1	Natural death rate of predator.
d_2	Natural death rate of super predator.
β	Predator consumption rate on prey.
n_1	Super predator consumption rate on predator.

Modeling fish-populations should take in consideration their mobility for mating, looking for food or seeking shelter [2]. Therefore, it is crucial to take spatial structuring into modeling fish-populations. In fact, spatial factor in several papers (see for example [29, 24]) . In order to describe the mobility of the fish-population, we introduce the ordinary spatial diffusion term in each compartment x, y and z.

Let Ω be a fixed and bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and η is the outward unit normal vector on the boundary. The time belongs to a finite interval $[0,T]$, $T > 0$, while X varies in a bounded domain Ω. Therefore, we have the following model

$$\left\{\begin{array}{rcl} \dfrac{\partial x}{\partial t} - \alpha_1 \Delta x & = & rx\left(1 - \dfrac{x}{K}\right) - \dfrac{\alpha x y}{a + x} - m\dfrac{x^2 z}{b + x^2}, \\ \dfrac{\partial y}{\partial t} - \alpha_2 \Delta y & = & \dfrac{\beta x y}{a + x} - d_1 y - \dfrac{n y^2 z}{b + y^2} - qEy \\ \dfrac{\partial z}{\partial t} - \alpha_3 \Delta z & = & n_1 \dfrac{y^2 z}{b + y^2} + m_1 \dfrac{x^2 z}{b + x^2} - d_2 z - qEz \end{array}\right. , \qquad (t, X) \in Q = [0,T] \times \Omega$$

The parameters α_1, α_2,and α_3 are positive coefficients of diffusion's of the prey population, predator population and super predator population, respectively. As we are dealing with population densities of species, we assume that initial condition are non-negative. Moreover, we assume that all parameters of model are also are non-negative.

We assume that the prey is in competition with each other. However, the two predators are not competing. The initial conditions and flux boundary conditions are given by

$$\frac{\partial x}{\partial \eta}=\frac{\partial y}{\partial \eta}=\frac{\partial z}{\partial \eta}=0, \qquad (t,X)\in \sum=[0,T]\times\partial\Omega$$

and

$$x(0,X)=x^0,\quad y(0,X)=y^0 \ and \quad z(0,X)=z^0, \qquad X\in\Omega$$

The main reason for our choice of no-flux boundary conditions is that we assume that fish population can enter or leave through the habitat boundary, i.e. we assume that the environment is isolated. We are also concerned with the self-organization of pattern, zero-flux conditions that does not involve any external input [30].

The model with controls. In this part, the strategy chosen is to use a regional fishing control effort that targets the sources area of the interaction of the species. In addition, we apply a harvesting function strategy in order to ensure environmental sustainability and maintain a differential chain system. Therefore, we introduce two regional control strategies $\chi_\omega(X)u_1(t,X)$ and $\chi_\omega(X)u_2(t,X)$, depended on time $t\in[0,T]$ and in the location $X\in\omega\subset\Omega$ (χ_ω is the characteristic function of w, the introduction of the characteristic function implies that the controls u_1, u_2 acts only on the small area $\omega\subset\Omega$).

The regional controls $\chi_\omega(X)u_1(t,X)$, $\chi_\omega(X)(u_2(t,X))$ represent the increase effort on catchability q to harvest the predators population (super predator population), while to minimize the number of predators (super predators) y (z) respectively, but with focusing on preservation of predators and super predators populations, which threaten the prey x overtime to maintain a differential chain system and ensure environmental sustainability.

With the new changes, our controlled system becomes as follows

$$(2)\qquad \begin{cases} \dfrac{\partial x}{\partial t}-\alpha_1\Delta x &= rx\left(1-\dfrac{x}{K}\right)-\dfrac{\alpha xy}{a+x}-m\dfrac{x^2z}{b+x^2}, \\ \dfrac{\partial y}{\partial t}-\alpha_2\Delta y &= \dfrac{\beta xy}{a+x}-d_1y-\dfrac{ny^2z}{b+y^2}-qE\chi_\omega(X)u_1(t,X)y, \\ \dfrac{\partial z}{\partial t}-\alpha_3\Delta z &= n_1\dfrac{y^2z}{b+y^2}+m_1\dfrac{x^2z}{b+x^2}-d_2z-qE\chi_\omega(X)u_2(t,X)z, \end{cases} \qquad (t,X)\in Q$$

with the initial conditions and no-flux boundary conditions are given by

$$\frac{\partial x}{\partial \eta} = \frac{\partial y}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0, \quad (t,X) \in \sum \tag{3}$$

$$x(0,X) = x^0, \quad y(0,X) = y^0 \text{ and } z(0,X) = z^0, \quad X \in \Omega. \tag{4}$$

We define our objective functional as

$$\begin{aligned} J((x,y,z);(u_1,u_2)) &= \int_0^T \int_\Omega \rho_1 \chi_\omega(X) x(t,X) + \rho_2 \chi_\omega(X) h(y(t,X)) + \rho_3 \chi_\omega(X) h(z(t,X)) dxdt \\ &- \frac{\eta_1}{2} \|u_1\|_{L^2([0,T]\times\omega)} - \frac{\eta_2}{2} \|u_2\|_{L^2([0,T]\times\omega).} \end{aligned} \tag{5}$$

subject to the state system given by $(2-4)$.

The objective is to minimize the number of predators y and the super predators z by maximizing the harvesting functions $h(y)$ and $h(z)$, increasing the number of prey in all space Ω, and minimizing the cost of implementing the controls by using possible minimal control variables u_i for $i = 1,2$. The constants ρ_1, ρ_2 , η_1, $\eta_2 > 0$ are weights representing the relative importance of each term in the objective functional J.

The square of the control variables $\|u_1\|^2_{L^2([0,T]\times\omega)}$ and $\|u_2\|^2_{L^2([0,T]\times\omega)}$ reflects the severity of the side effects of the mechanisms of fishing effort.

Our objective is to find control functions such that

$$J((x^*,y^*,z^*);(u_1^*,u_2^*)) = \max\left\{J((x,y,z);(u_1,u_2)), (u_1,u_2) \in U_{ad}\right\},$$

subject to system $(2-4)$,in the control set is defined as

$$U_{ad} = \{(u_1,u_2) \in L^\infty([0,T]\times\omega) \times L^\infty([0,T]\times\omega) \ / \ 0 \le u_1 \le u_1^{\max} \le 1 \text{ and } 0 \le u_2 \le u_2^{\max} \le 1\} \tag{6}$$

3. Existence of Global Solution

In this section, we investigate the existence of a strong global solution, positivity and boundedness of solutions of problem (2). Since this model describes the fish-population, it should remain non-negative and bounded.

Let $\tilde{y} = (y_1, y_2, y_3) = (x, y, z)$, $y^0 = (y_1^0, y_2^0, y_3^0) = (x_0, y_0, z_0)$ and $H(\Omega) = (L^2(\Omega))^3$.

We define also A the linear operator as follow

$$\begin{array}{rcl} A : D(A) \subset H(\Omega) & \longrightarrow & H(\Omega) \\ A\tilde{y} & \longrightarrow & (\Delta y_1, \Delta y_2, \Delta y_3), \end{array} \tag{7}$$

with

$$D(A) = \left\{ \tilde{y} = (y_1, y_2, y_3) \in \left(H^2(\Omega)\right)^3, \frac{\partial y_1}{\partial \eta} = \frac{\partial y_2}{\partial \eta} = \frac{\partial y_3}{\partial \eta} = 0, \quad a.e \ \ x \in \partial\Omega \right\} \tag{8}$$

If we consider the function $f(t, \tilde{y}(t)) = (f_1(t, \tilde{y}(t)), f_2(t, \tilde{y}(t)), f_3(t, \tilde{y}(t)))$ with

$$\left\{ \begin{array}{rcl} f_1(t, \tilde{y}(t)) & = & r y_1 \left(1 - \dfrac{y_1}{k}\right) - \dfrac{\alpha y_1 y_2}{a + y_1} - m \dfrac{y_1^2 y_3}{b + y_1^2} \\ f_2(t, \tilde{y}(t)) & = & \dfrac{\beta y_1 y_2}{a + y_1} - d_1 y_2 - n \dfrac{y_2^2 y_3}{b + y_2^2} - qE\chi_\omega(X) u_1 y_2 \\ f_3(t, \tilde{y}(t)) & = & n_1 \dfrac{y_2^2 y_3}{b + y_2^2} + m_1 \dfrac{y_1^2 y_3}{b + y_1^2} - d_2 y_3 - qE\chi_\omega(X) u_2 y_3 \end{array} \right.$$

and $N = y_1^0 + y_2^0 + y_3^0$.

Now let $h : [0, T] \times D(f) \longmapsto (L^2(\Omega))^3$ defined by $h = (h_1, h_2, h_3)$, where

$$D(f) = \left\{ \tilde{y} \in \left(L^2(\Omega)\right)^3, \quad f(t, \tilde{y}(\cdot)) \in \left(L^2(\Omega)\right)^3, \ \forall t \in [0, T] \right\}.$$

and for all $\tilde{y} \in D(f)$, for all $t \in [0, T]$, and a.e $x \in \Omega$

$$h(t, \tilde{y})(x) = (h_1(t, \tilde{y}), h_2(t, \tilde{y}), h_3(t, \tilde{y}))(x) = f(t, x, \tilde{y}(t))$$

Then the problem (2) can be rewritten in the space $H(\Omega)$ in the following form

$$\left\{ \begin{array}{rcl} \tilde{y}(0) & = & \tilde{y}^0 \in D(A) \\ \dfrac{\partial \tilde{y}}{\partial t} & = & A\tilde{y} + f(t, \tilde{y}(t)), \quad t \in [0, T]. \end{array} \right. \tag{9}$$

We denote by $W^{1,2}([0, T], H(\Omega))$ the space of all absoletely continous functions $\tilde{y} : [0, T] \mapsto H(\Omega)$ having the property that $\dfrac{\partial \tilde{y}}{\partial t} \in L^2(0, T, H(\Omega))$ and $L(T, \Omega) = L^2\left(0, T, H^2(\Omega)\right) \cap L^\infty\left(0, T, H^1(\Omega)\right)$.

Theorem 3.1. *Let Ω be a bounded domain from $\mathbb{R}^2$, with the boundary smooth enough $y_i^0 \geq 0$ on Ω $(for\ i = 1, 2, 3)$,*

$r, \alpha, m, q, \eta, \beta, \eta_1, m_1, a, b \geq 0$, $(u_1, u_2) \in (U_{ad})^2$ the problem $(2-4)$ has a unique (global)

strong solution

$\tilde{y} \in W^{1,2}([0,T], H(\Omega))$ *such that* $y_i \in L(T,\Omega) \cap L^{\infty}(\Omega)$ $(for\ i = 1,2,3)$. *In addition* y_1, y_2 *and* y_3 *are non negative and bounded uniformly in* $L^{\infty}(Q)$. *Furthermore there exists* $C > 0$ *(independent of* (u_1, u_2)*) for all* $t \in [0,T]$

$$\left\| \frac{\partial y_i}{\partial t} \right\|_{L^2(\Omega)} + \|y_i\|_{L^2(0,T,H^2(\Omega))} + \|y_i\|_{H^1(\Omega)} + \|y_i\|_{L^\infty(Q)} \leq C, \qquad for\ i = 1,2,3 \tag{10}$$

Proof. Notice that the function $f(t,y)$ defined in (9) is not Lipschitz continuous with respect to y, uniformly for $t \in [0,T]$. Therefore, we cannot apply Theorem (7.1) (see appendix) for our problem directly. To overcome this problem, we consider using the well-known technique of truncation function before making the use of Theorem (7.1). The proof of this theorem is rather long; thus, we split it into the following two steps.

Step 1: This step studies the local existence of positive solutions to system (2-4) in view of Theorem (7.1) (see appendix). We use a truncation procedure for h. For a fixed positive integer $k > 0$, let us define the function sets $D_1 = \{z/z > k\}$, $D_2 = \{z/|z| < k\}$,$D_3 = \{z/z < -k\}$ and consider the following auxiliary problem:

$$\begin{cases} \dfrac{\partial \tilde{y}^k}{\partial t} = A\tilde{y}^k + f^k\left(t, \tilde{y}^k(x,t)\right), & in\, Q, \\ \\ \tilde{y}^k(x,0) = y^0, & in\, \Omega, \end{cases}$$

where $f^k\left(t,\tilde{y}^k\right) = \left(f_1^k\left(t,\tilde{y}^k\right), f_2^k\left(t,\tilde{y}^k\right), f_3^k\left(t,\tilde{y}^k\right)\right)$. Here, for each index i, $f_i^k\left(t,\tilde{y}^k\right)$ are defined as follows:

$$f_i^k\left(t,\tilde{y}^k\right) = f_i\left(t, [y_1]_{Ds_1}, [y_2]_{Ds_2}, [y_3]_{Ds_3}\right)$$

where $[y_i]_{Ds_i}$ means that $y_i \in Ds_i$, and

$$[y_i]_{Ds_i} = \begin{cases} k & if\, s_i = 1, \\ y_i & if\, s_i = 2, \\ -k & if\, s_i = 3. \end{cases}$$

As the operator A defined in (7-8) is dissipating, self-adjoint and generates a C_0-semi-group of contractions on $H(\Omega)$ [34], it is clear that function $f^k\left(t,\tilde{y}^k\right)$ becomes Lipschitz continuous in

$\tilde{y}^k$ uniformly with respect to $t \in [0,T]$. Therefore, theorem (7.1) (see appendix) assures problem $(2-4)$ and admits a unique strong solution $\tilde{y}^k \in W^{1,2}([0,T], H(\Omega))$ with

$$y_1^k,\, y_2^k,\, y_3^k \in L^2\left(0,T;H^2(\Omega)\right) \tag{11}$$

Let us now prove the boundedness of $\tilde{y}^k$ on Q. Indeed, if we denote

$$M_k = \max\left\{ \left\| f_1^k \right\|_{L^\infty(Q)}, \|y_1^0\|_{L^\infty(\Omega)} \right\}$$

and $\{S(t), t \geq 0\}$ is the C_0-semi-group generated by the operator $B : D(B) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ where

$$By_1^k = \lambda_1 \Delta y_1^k$$

and

$$D(B) = \left\{ y_1^k \in H^2(\Omega),\ \frac{\partial y_1^k}{\partial \eta} = 0,\ a.e\ \partial\Omega \right\}.$$

it is obvious that function $Y_1^k(t,X) = y_1^k - M_k t - \|y_1^0\|_{L^\infty(\Omega)}$ satisfies the Cauchy problem

$$\begin{cases} \frac{\partial Y_1^k}{\partial t}(t,X) &=& \lambda_1 \Delta Y_1^k + f_1^k\left(t, \tilde{y}^k(t)\right) - M_k \quad t \in [0,T] \\ Y_1^k(0,x) &=& y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \end{cases}$$

The corresponding strong solution is

$$Y_1^k(t) = S(t)\left(y_1^0 - \|y_1^0\|_{L^\infty(\Omega)}\right) + \int_0^t S(t-s)\left(f_1^k\left(s, \tilde{y}^k(s)\right) - M_k\right) ds,$$

Since $y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \leq 0$ and $f_1^k\left(t, \tilde{y}^k(t)\right) - M_k \leq 0$, it follows that $Y_1^k(t,X) \leq 0,\ \forall (t,x) \in Q$. Moreover the function $W_1^k(t,X) = y_1^k - M_k t - \|y_1^0\|_{L^\infty(\Omega)}$ satisfies the Cauchy problem

$$\begin{cases} \frac{\partial W_1^k}{\partial t}(t,X) &=& \lambda_1 \Delta W_1^k + f_1^k\left(t, \tilde{y}^k(t)\right) + M_k \quad t \in [0,T] \\ W_1^k(0,X) &=& y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \end{cases}$$

The corresponding strong solution is

$$W_1^k(t) = S(t)\left(y_1^0 + \|y_1^0\|_{L^\infty(\Omega)}\right) + \int_0^t S(t-s)\left(f_1^k\left(s, \tilde{y}^k(s)\right) + M_k\right) ds,$$

Since $y_1^0 + \|y_1^0\|_{L^\infty(\Omega)} \geq 0$ and $f_1^k\left(t, \tilde{y}^k(t)\right) + M_k \geq 0$, it follows that $W_1^k(t,x) \geq 0,\ \forall (t,x) \in Q$. Then

$$\left| y_1^k(t,x) \right| \leq M_k t + \|y_1^0\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q$$

and analogously

$$\left|y_i^k(t,x)\right| \le M_k t + \left\|y_i^0\right\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q \text{ for } i = 2,3, \tag{12}$$

Thus we have proved that

$$y_i^k \in L^\infty(Q) \ (\forall (t,x) \in Q) \text{ for } i = 1,2,3. \tag{13}$$

By the first equation of (2) one obtains

$$\begin{aligned}
&\int_0^t \int_\Omega \left|\frac{\partial y_1^k}{\partial s}\right|^2 dsdx + \alpha_1^2 \int_0^t \int_\Omega \left|\Delta y_1^k\right|^2 dsdx - 2\alpha_1 \int_0^t \int_\Omega \frac{\partial y_1^k}{\partial s} \Delta y_1^k dsdx \\
= &\int_0^t \int_\Omega \left(r y_1^k \left(1 - \frac{y_1^k}{K}\right) - \frac{\alpha y_1^k y_2^k}{a + y_1^k} - m \frac{(y_1^k)^2 y_3^k}{b + (y_1^k)^2} \right)^2 dsdx.
\end{aligned}$$

Using the regularity of y_1^k and the Green's formula, we can write

$$2\int_\Omega \frac{\partial y_1^k}{\partial s} \Delta y_1^k dsdx = -\frac{\partial}{\partial s}\left(\int_\Omega \left|\Delta y_1^k\right|^2 dx\right) ds = -\int_\Omega \left|\nabla y_1^k\right|^2 dx + \int_\Omega \left|\nabla y_1^0\right|^2 dx$$

Then

$$\begin{aligned}
&\int_0^t \int_\Omega \left|\frac{\partial y_1^k}{\partial s}\right|^2 dsdx + \alpha_1^2 \int_0^t \int_\Omega \left|\Delta y_1^k\right|^2 dsdx + 2\alpha_1 \int_\Omega \left|\nabla y_1^k\right|^2 dx - 2\alpha_1 \int_\Omega \left|\nabla y_1^0\right|^2 dx \\
= &\int_0^t \int_\Omega \left(r y_1^k \left(1 - \frac{y_1^k}{K}\right) - \frac{\alpha y_1^k y_2^k}{a + y_1^k} - m \frac{(y_1^k)^2 y_3^k}{b + (y_1^k)^2} \right)^2 dsdx.
\end{aligned}$$

Since $\|y_i\|_{L^\infty(\Omega)}$ for $i = 1,2,3$ are bounded independently of (u_1, u_2) and $y_1^0 \in H^2(\Omega)$, we deduce that

$$y_1^k \in L^\infty\left(0, T, H^1(\Omega)\right) \tag{14}$$

We make use of (11), (13), and (14), in order to get $y_1^k \in L(T,\Omega) \cap L^\infty(Q)$ and concluding that the inequality in (10) holds for $i = 1$, similarly for $y_2^k$and y_3^k.

In order to show the nonnegativeness of y_i^k for $i=1,2,3$, we write the system (2) in the form

$$\begin{cases} \frac{\partial y_1^k}{\partial t} & = & \alpha_1 \Delta y_1^k + F_1(y_1^k,\, y_2^k,\, y_3^k) \\ \\ \frac{\partial y_2^k}{\partial t} & = & \alpha_2 \Delta y_2^k + F_2(y_1^k,\, y_2^k,\, y_3^k) \\ \\ \frac{\partial y_3^k}{\partial t} & = & \alpha_3 \Delta y_3^k + F_3(y_1^k,\, y_2^k,\, y_3^k) \end{cases}$$

It is obvious to see that the functions $F_1\left(\begin{array}{ccc} y_1^k, & y_2^k, & y_3^k \end{array}\right), F_2\left(\begin{array}{ccc} y_1^k, & y_2^k, & y_3^k \end{array}\right)$ and $F_3\left(\begin{array}{ccc} y_1^k, & y_2^k, & y_3^k \end{array}\right)$ are continuously differentiable satisfying

$$F_1\left(\begin{array}{ccc} 0, & y_2^k, & y_3^k \end{array}\right) = F_2\left(\begin{array}{ccc} y_1^k, & 0, & y_3^k \end{array}\right) = 0,$$

and

$$F_3\left(\begin{array}{ccc} y_1^k, & y_2^k, & 0 \end{array}\right) = m_1 \frac{(y_1^k)^2 y_2^k}{b+(y_1^k)^2} \geq 0 \text{ for all } y_1^k, y_2^k, y_3^k \geq 0.$$

Since initial values of system (2) are non-negative, we deduce that $y_1^k(t,x) \geq 0, y_2^k(t,x) \geq 0$, and $y_3^k(t,x) \geq 0, \forall (t,x) \in Q$ (see [31]).

Now we particularize $k>0$ large enough such that

$$M_k \theta + \left\| y_i^0 \right\|_{L^\infty(\Omega)} \leq k, \quad i=1,2,3, \quad \text{for some } \theta \in [0,T] \tag{15}$$

For example, we can take $k > 2max\left\{ \left\| y_i^0 \right\|_{L^\infty(\Omega)}, i=1,2,3 \right\}$. Let $\theta \in (0,T)$ be maximal with property (15). By (12)-(15), it is clear that $\left| y_i^k(t,x) \right| < k$, for $(t,x) \in [0,\theta] \times \Omega$ and $i=1,2,3$. So, $f^k(t,y_1,y_2,y_3)$ coincides with $f(t,y_1,y_2,y_3)$ for$(t,x) \in [0,\theta] \times \Omega$, and consequently $\tilde{y}^k = \left(y_1^k, y_2^k, y_3^k\right)$ is a local solution for $(2-4)$ defined on $[0,\theta] \times \Omega$.

We show now that $\tilde{y}$ is bounded on $(0,\theta) \times \Omega$ and thus, it is actually a global solution of $(2-4)$ defined on $(0,\theta) \times \Omega$.

We put $f(y_i) = \frac{y_i}{a+y_i}$, $g(y_i) = \frac{y_i^2}{a+y_i^2}$, and $l(y_1) = r\left(1 - \frac{y_1}{k}\right)$

We work under the following hypotheses on $f, g, l : \mathbb{R} \to \mathbb{R}$

$(H1)$ l is continous and bounded on $(0,\infty)$.

$(H2)$ f, g are continous and positive on $(0,\infty)$ and bounded on bounded sets.

In view of hypothesis $(H1)$, $(H2)$ by the first equation of (2), we deduce that

$$\frac{\partial y_1}{\partial t} \leq \alpha_1 \Delta y_1 + m y_1, \qquad t \in (0,\theta), \; x \in \Omega$$

Here $m > 0$ is an upper bound for l via $(H1)$. This leads to estimate

$$0 \leq y_1(t,x) \leq \exp(mt)\tilde{S}(t)\, y_1^0(x), \quad (t,x) \in (0,\theta) \times \Omega,$$

where $\left\{\tilde{S}(t), t \geq 0\right\}$ is the C_0-semigroup of contractions on $L^2(\Omega)$ generated by the operator $By_1 = \alpha_1 \Delta y_1$, with the domain

$$D(B) = \left\{ y_1 \in H^2(\Omega), \qquad \frac{\partial y_1}{\partial \eta} = 0, \quad \text{a.e } on \; \partial\Omega \right\}.$$

Therefore $\|y_1\|_{L^\infty((0,\theta)\times\Omega)} \leq m_1$ for some $m_1 > 0$ independent of N and of u. Next, we can easly deduce the boundedness of y_2 and y_3 on $(0,\theta) \times \Omega$. Consequently, y_i are defined on the whole set Q (and also positive and bounded). If we take the second power in (2), integrate over $[0,t] \times \Omega$ with $t \in [0,T]$, the claim (10) follows in view of Green's formula and of $(H1)$, $(H2)$. Thus (y_1, y_2, y_3) is a global positive strong solution of system $(2)-(4)$ and it satisfies (10). This completes the proof.

□

4. Existence of the Optimal Solution

In this section, we will prove the existence of an optimal control for the problem $(2-4)$ subject to reaction-diffusion system $(2-4)$ and $(u_1, u_2) \in (U_{ad})^2$. The most important result of this section is the theorem below.

Theorem 4.1. *Under the hypothesis of theorem* (7.1)*, the optimal control problem* $(2-4)$ *admits an optimal solution* $(\tilde{y}^*, (u_1^*, u_2^*))$.

Proof. From Theorem (7.1), we know that, u_1, u_2, y_1, y_2 and y_3 are bounded uniformly in $L^\infty(Q)$, J is a minimizing sequence such that

$$\lim_{n\to\infty} J(\tilde{y}^n, (u_1^n, u_2^n)) = \inf_{(u_1,u_2)\in(U_{ad})^2} J(\tilde{y}, (u_1, u_2))$$

where $\left(y_1^n, y_2^n, y_3^n\right)$ is the solution of system $(2-4)$ corresponding to the control $\left(u_1^n, u_2^n\right)$ for $n = 1, 2, \ldots$ that is

$$\left\{\begin{array}{lcl} \frac{\partial y_1^n}{\partial t} & = & \alpha_1 \Delta y_1^n + r y_1^n \left(1 - \frac{y_1^n}{K}\right) - \frac{\alpha y_1^n y_2^n}{a + y_1} - m \frac{\left(y_1^n\right)^2 y_3^n}{b + \left(y_1^n\right)^2}, \\ \frac{\partial y_2^n}{\partial t} & = & \alpha_2 \Delta y_2^n + \frac{\beta y_1^n y_2^n}{a + y_1^n} - d_1 y_2^n - \frac{n \left(y_2^n\right)^2 y_3^n}{b + \left(y_2^n\right)^2} - qE\chi_\omega(X) u_1^n(t,X) y_2^n, \\ \frac{\partial y_3^n}{\partial t} & = & \alpha_3 \Delta y_3^n + n_1 \frac{\left(y_2^n\right)^2 y_3^n}{b + \left(y_2^n\right)^2} + m_1 \frac{\left(y_1^n\right)^2 y_3^n}{b + \left(y_1^n\right)^2} - d_2 y_3^n - qE\chi_\omega(X) u_2^n(t,X) y_3^n, \end{array}\right. \tag{16}$$

$$\begin{array}{ll} \frac{\partial y_1^n}{\partial \eta} = \frac{\partial y_2^n}{\partial \eta} = \frac{\partial y_3^n}{\partial \eta} = 0 & (t,x) \in \Sigma\, (t,x) \in \Sigma \\ y_i^n(0,x) = y_i^0 \quad for \;\; i = 1,2,3 & x \in \Omega \end{array} \tag{17}$$

By theorem (7.1) using the estimate (10) of the solution y_i^n, there exists a constant $C > 0$ such that for all $n \geq 1, t \in [0,T]$

(18)

$$\|y_i^n\|_{L^\infty(Q)} \leq C, \quad \left\|\frac{\partial y_i^n}{\partial t}\right\|_{L^2(Q)} \leq C, \quad \|y_i^n\|_{L^2\left(0,T,H^2(\Omega)\right)} \leq C, \quad \|y_i^n\|_{H^1(\Omega)} \leq C, \;\; i = 1,2,3$$

$H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, so we deduce that $y_1^n(t)$ is compact in $L^2(\Omega)$. Let's show that $\left\{y_1^n(t),\;\; n \geq 1\right\}$ is equicontinous in $C\left([0,T] : L^2(\Omega)\right)$. As $\frac{\partial y_1^n}{\partial t}$is bounded in $L^2(Q)$, this implies that for all $s, t \in [0,T]$

$$\left|\int_\Omega (y_1^n)^2(t,x)\,dx - \int_\Omega (y_1^n)^2(s,x)\,dx\right| \leq K\;\; |t-s|$$

The Ascoli-Arzela Theorem (See a Chapter IV.5 in Brezis et al. [32]) implies that y_1^n is compact in $C\left([0,T] : L^2(\Omega)\right)$. Hence, selecting further sequences, if necessary, we have $y_1^n \longrightarrow y_1^*$ in $L^2(\Omega)$, uniformly with respect to t.

and analogously $y_i^n \longrightarrow y_i^*$ in $L^2(\Omega)$, uniformly with respect to t, for $i = 2,3$.

From the boundedness of Δy_i^n in $L^2(Q)$, which implies it is weakly convergent in $L^2(Q)$ on a subsequence denoted again Δy_i^n then for all distribution φ

$$\int_Q \varphi \Delta y_i^n = \int_Q y_i^n \Delta\varphi \rightarrow \int_Q y_i^* \Delta\varphi = \int_Q \varphi \Delta y_i^*$$

Which implies that $\Delta y_i^n \rightarrow \Delta y_i^*$ weakly in $L^2(Q), i = 1,2,3$

In addition estimates (18) lead to

$\frac{\partial y_i^n}{\partial t} \to \frac{\partial y_i^*}{\partial t}$ weakly in $L^2(Q)$, $i = 1,2,3$

$y_i^n \to y_i^*$ weakly in $L^2\left(0,T;H^2(\Omega)\right)$, $i = 1,2,3$

$y_i^n \to y_i^*$ weakly in $L^\infty\left(0,T;H^1(\Omega)\right)$, $i = 1,2,3$

We now show that $y_i^n y_j^n \to y_i^* y_j^*$ for $i = 1,2,3$ and $j = 1,2,3$.

Since u_1^n and u_2^n are bounded in $L^2([0,T]\times\omega)$ and in $L^2([0,T]\times\omega)$ respectively, we can assume that $\chi_\omega u_1^n \rightharpoonup \chi_\omega u_1^*$ weakly in $L^2([0,T]\times\omega)$ and $\chi_\omega u_2^n \rightharpoonup \chi_\omega u_2^*$ weakly in $L^2([0,T]\times\omega)$ on a subsequence denoted again u_1^n and u_2^n.

Since U_{ad} is a closed and convex set in $L^2(Q)$ it is weakly closed, so $(u_1^*,u_2^*) \in (U_{ad})^2$.

We now show that

$$\chi_\omega(x)\,u_1^n y_2^n \longrightarrow \chi_\omega(x)\,u_1^* y_2^* \text{ in } L^2([0,T]\times\omega) \text{ and } \chi_\omega(x)\,u_2^n y_3^n \longrightarrow \chi_\omega(x)\,u_2^* y_3^* \text{ in } L^2([0,T]\times\omega)$$

By writing

$$\chi_\omega(x)\,u_1^n y_2^n - \chi_\omega(x)\,u_1^* y_2^* = (y_2^n - y_2^*)\,\chi_\omega(x)\,u_1^n - (u_1^n - u_1^*)\,\chi_\omega(x)\,y_2^*$$

and

$$\chi_\omega(x)\,u_2^n y_3^n - \chi_\omega(x)\,u_2^* y_3^* = (y_3^n - y_3^*)\,\chi_\omega(x)\,u_2^n - (u_2^n - u_2^*)\,\chi_\omega(x)\,y_3^*$$

and making use of the convergences $y_{i+1}^n \longrightarrow y_{i+1}^*$ strongly in $L^2(Q)$ for $i = 1,2$, $\chi_\omega u_1^n \rightharpoonup \chi_\omega u_1^*$ in $L^2([0,T]\times\omega)$ and $\chi_\omega u_2^n \rightharpoonup \chi_\omega u_2^*$ in $L^2([0,T]\times\omega)$ we obtains that $\chi_\omega(x)\,u_1^n y_2^n \longrightarrow \chi_\omega(x)\,u_1^* y_2^*$ in $L^2([0,T]\times\omega)$ and $\chi_\omega(x)\,u_2^n y_3^n \longrightarrow \chi_\omega(x)\,u_2^* y_3^*$ in $L^2([0,T]\times\omega)$

By taking $n\to\infty$ in $((16-17))$, we obtain that y^* is a solution of $(2-4)$ corresponding to $(u_1^*,u_2^*) \in (U_{ad})^2$.

Therefore

$$\begin{aligned}
J(\tilde{y}^*,(u_1^*,u_2^*)) &= \rho_1 \textstyle\int_0^T\int_\Omega y_1^*(t;x)\,dxdt + \rho_2 \int_0^T\int_\Omega y_2^*(t;x)\,dxdt + \frac{\eta_1}{2}\|u_1^*\|^2_{L^2([0,T]\times\omega)} + \frac{\eta_2}{2}\|u_2^*\|_{L^2([0,T]\times\omega)} \\
&\le \liminf_{n\to\infty}\Big(\rho_1 \textstyle\int_0^T\int_\Omega y_1^n(t;x)\,dxdt + \rho_2\int_0^T\int_\Omega y_2^n(t;x)\,dxdt + \frac{\eta_1}{2}\|u_1^n\|^2_{L^2([0,T]\times\omega)} + \frac{\eta_2}{2}\|u_2^n\|^2_{L^2([0,T]\times\omega)}\Big) \\
&\le \lim_{n\to\infty}\Big(\rho_1 \textstyle\int_0^T\int_\Omega y_1^n(t;x)\,dxdt + \rho_2\int_0^T\int_\Omega y_2^n(t;x)\,dxdt + \frac{\eta_1}{2}\|u_1^n\|^2_{L^2([0,T]\times\omega)} + \frac{\eta_2}{2}\|u_2^n\|^2_{L^2([0,T]\times\omega)}\Big) \\
&\le \inf_{(u_1,u_2)\in(U_{ad})^2} J((u_1,u_2))
\end{aligned}$$

This shows that J attains its minimum at $(\tilde{y}^*,(u_1^*,u_2^*))$, we deduce that $(\tilde{y}^*,(u_1^*,u_2^*))$ verifies problem $(2-4)$ and minimizes the objective functional (5). The proof is complete. $\square$

5. Necessary Optimality Condition

Given $u=(u_1,u_2)^\top \in U_{ad}$ and $u^*=(u_1^*,u_2^*)^\top \in U_{ad}$, we aim, in this section, to give a characterization of the optimal control by proving the conditions of optimality to the problem $(2-6)$. First, we use the gateaux differentiability of the mapping $u \to \tilde{y}(u)$. For this purpose, denoting by $\tilde{y}^\varepsilon = \left(y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon\right) = (y_1,y_2,y_3)\left(u^\varepsilon\right)$ and $y^* = \left(y_1^*,y_2^*,y_3^*\right) = (y_1,y_2,y_3)\left(u^*\right)$ the solution of $(2-4)$ corresponding to u^ε and u^* respectively, where u^ε and u^* respectively, where $(\tilde{y}^*,u^*)$ is an optimal pair such that $u^\varepsilon = u^*+\varepsilon u \in U_{ad}$ (for $\varepsilon>0$ small) and $u\in U_{ad}$.
Let's define the matrix

$$H=\begin{pmatrix} f_1 & -\dfrac{\alpha y_1^*}{a+y_1^*} & -m\dfrac{(y_1^*)^2}{b+\left(y_1^*\right)^2} \\ \beta\dfrac{y_2^*}{a+y_1^*} & f_2 & -n\dfrac{(y_2^*)^2}{b+\left(y_2^*\right)^2} \\ m_1\dfrac{\left(y_1^n\right)^2 y_3^n}{b+\left(y_1^n\right)^2} & n_1\dfrac{y_2^* y_3^*}{b+\left(y_2^*\right)^2} & f_3 \end{pmatrix},$$

with

$$f_1 = r\left(1-\frac{y_1^*}{K}\right)-\frac{\alpha y_2^*}{a+y_1^*}-m\frac{y_1^* y_3^n}{b+\left(y_1^*\right)^2}-\frac{\alpha y_1^*}{a+y_1^*}$$

$$f_2 = \beta\frac{y_1^*}{a+y_1^*}-d_1 y_1^* - n\frac{y_2^* y_1^*}{b+\left(y_2^*\right)^2}-qEu_1^*\chi_\omega(x)$$

$$f_3 = n_1\frac{(y_2^*)^2}{b+\left(y_2^*\right)^2}+m_1\frac{(y_1^*)^2}{b+\left(y_1^*\right)^2}-d_2-qEu_2^*\chi_\omega(x)$$

and the matrix

$$G=\begin{pmatrix} 0 & 0 \\ -qEy_2^*\chi_\omega(x) & 0 \\ 0 & -qEy_3^*\chi_\omega(x) \end{pmatrix}.$$

We have the following theorem

Theorem 5.1. *The mapping $y: U_{ad} \to W^{1,2}\left([0,T], H(\Omega)\right)$ with $y_i \in L(T,\Omega)$ for $i=1,2,3$ is gateaux differentiable with respect to u^*. For all direction $u\in U_{ad}$, $y'\left(u^*\right)u = U$ is the unique* *solution in* $W^{1,2}([0,T],H(\Omega))$ *with* $U_i \in L(T,\Omega)$ *of the following equation*

$$\begin{cases} \frac{\partial U}{\partial t} & = & AU + HU + Gu \quad t \in [0,T] \\ \\ U(0,x) & = & 0 \end{cases} \tag{19}$$

Proof. We put $U_i^\varepsilon = \dfrac{y_i^\varepsilon - y_i^*}{\varepsilon}$ for $i = 1,2,3$ and we denote P^ε the system $(2-4)$ corresponding to u^ε and P^* the system $(2-4)$ corresponding to u^*, subtracting system P^ε from P^*. As in [16], authors supposed in their stability study of this model that the total population N does not vary in terms of x_T; y_T and z_T. Thus, we take into account the same condition to obtain

(20)

$$\begin{cases} \frac{\partial U_1^\varepsilon}{\partial t} & = & \alpha_1 \Delta U_1^\varepsilon + \left(r\left(1 - \frac{y_1^*}{K}\right) - \frac{\alpha y_2^*}{a + y_1^*} - m\frac{y_1^* y_3^n}{b + (y_1^*)^2} \right) U_1^\varepsilon - \frac{\alpha y_1^*}{a + y_1^*} U_2^\varepsilon - m\frac{(y_1^*)^2}{b + (y_1^*)^2} U_3^\varepsilon \\ \frac{\partial U_2^\varepsilon}{\partial t} & = & \alpha_2 \Delta U_2^\varepsilon + \beta \frac{y_2^*}{a + y_1^*} U_1^\varepsilon + \left(\beta \frac{y_1^*}{a + y_1^*} - d_1 y_1^* - n\frac{y_2^* y_1^*}{b + (y_2^*)^2} - qE\chi_\omega(x) u_1^* \right) U_2^\varepsilon - n\frac{(y_2^*)^2}{b + (y_2^*)^2} U_3^\varepsilon \\ \frac{\partial U_3^\varepsilon}{\partial t} & = & \alpha_3 \Delta U_3^\varepsilon + \left(m_1 \frac{(y_1^n)^2 y_3^n}{b + (y_1^n)^2} \right) U_1^\varepsilon + n_1 \frac{y_2^* y_3^*}{b + (y_2^*)^2} U_2^\varepsilon \\ & & + \left(n_1 \frac{(y_2^*)^2}{b + (y_2^*)^2} + m_1 \frac{(y_1^*)^2}{b + (y_1^*)^2} - d_2 - qE\chi_\omega(x) u_2^* \right) U_3^\varepsilon \end{cases}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial U_1^\varepsilon}{\partial \eta} = \frac{\partial U_2^\varepsilon}{\partial \eta} = \frac{\partial U_3^\varepsilon}{\partial \eta} = 0, \qquad (x,t) \in \sum = [0,T] \times \partial\Omega$$

$$U_i^\varepsilon(0,x) = 0 \qquad x \in \Omega, \quad \text{for } \; i = 1,2,3 \tag{21}$$

We prove that U_i^ε are bounded in $L^2(Q)$ uniformly with respect to ε. For this, we define

$$H^\varepsilon = \begin{pmatrix} f_1^\varepsilon & -\frac{\alpha y_1^\varepsilon}{a + y_1^\varepsilon} & -m\frac{\left(y_1^\varepsilon\right)^2}{b + \left(y_1^\varepsilon\right)^2} \\ \beta \frac{y_2^\varepsilon}{a + y_1^\varepsilon} & f_2^\varepsilon & -n\frac{\left(y_2^\varepsilon\right)^2}{b + \left(y_2^\varepsilon\right)^2} \\ m_1 \frac{\left(y_1^\varepsilon\right)^2 y_3^\varepsilon}{b + \left(y_1^\varepsilon\right)^2} & n_1 \frac{y_2^\varepsilon y_3^\varepsilon}{b + \left(y_2^\varepsilon\right)^2} & f_3^\varepsilon \end{pmatrix}$$

With

$$f_1^\varepsilon = r\left(1 - \frac{y_1^\varepsilon}{K}\right) - \frac{\alpha y_2^\varepsilon}{a + y_1^\varepsilon} - m\frac{y_1^\varepsilon y_3^n}{b + \left(y_1^\varepsilon\right)^2}$$

$$f_2^\varepsilon = \beta \frac{y_1^\varepsilon}{a+y_1^\varepsilon} - d_1 y_1^\varepsilon - n\frac{y_2^\varepsilon y_1^\varepsilon}{b+\left(y_2^\varepsilon\right)^2} - qE\chi_\omega(x)u_1^\varepsilon$$

$$f_3^\varepsilon = n_1 \frac{\left(y_2^\varepsilon\right)^2}{b+\left(y_2^\varepsilon\right)^2} + m_1 \frac{\left(y_1^\varepsilon\right)^2}{b+\left(y_1^\varepsilon\right)^2} - d_2 - qE\chi_\omega(x)u_2^\varepsilon$$

$$U^\varepsilon = \left(\begin{array}{ccc} U_1^\varepsilon, & U_2^\varepsilon, & U_3^\varepsilon \end{array} \right),$$

and

$$G = \left(\begin{array}{cc} 0 & 0 \\ -qE\chi_\omega(x)y_2^* & 0 \\ 0 & -qE\chi_\omega(x)y_3^* \end{array} \right).$$

Then, we can rewrite system (20) as

$$\left\{ \begin{array}{rcll} \frac{\partial U^\varepsilon}{\partial t} & = & AU^\varepsilon + H^\varepsilon U^\varepsilon + Gu & t \in [0,T] \\ U^\varepsilon(0,x) & = & 0 & \end{array} \right. \tag{22}$$

$(S(t), t \geq 0)$ is the semi-group generated by A, then the solution of (22) can be expressed as

$$U^\varepsilon(t) = \int_0^t S(t-s)H^\varepsilon(s)U^\varepsilon(s)ds + \int_0^t S(t-s)Gu(s)ds. \tag{23}$$

On the other hand the coefficients of the matrix H^ε are bounded uniformly with respect to ε, using Gronwall's inequality, we have

$$\|U_i^\varepsilon\| \leq \beta$$

where $\beta > 0$ $(i = 1,2,3)$. Then

$$\|y_i^\varepsilon - y_i^*\|_{L^2(Q)} = \varepsilon \|U_i^\varepsilon\|_{L^2(Q)}$$

Hence $y_i^\varepsilon \to y_i^*$ in $L^2(Q)$, $i = 1,2,3$. We put

$$H = \left(\begin{array}{ccc} f_1^* & -\frac{\alpha y_1^*}{a+y_1^*} & -m\frac{\left(y_1^*\right)^2}{b+\left(y_1^*\right)^2} \\ \beta\frac{y_2^*}{a+y_1^*} & f_2^* & -n\frac{\left(y_2^*\right)^2}{b+\left(y_2^*\right)^2} \\ m_1\frac{\left(y_1^n\right)^2 y_3^n}{b+\left(y_1^n\right)^2} & n_1\frac{y_2^* y_3^*}{b+\left(y_2^*\right)^2} & f_3^* \end{array} \right)$$

with

$$f_1^* = r\left(1-\frac{y_1^*}{K}\right) - \frac{\alpha y_2^*}{a+y_1^*} - m\frac{y_1^* y_3^n}{b+\left(y_1^*\right)^2}$$

$$f_2^* = \beta\frac{y_1^*}{a+y_1^*} - d_1 y_1^* - n\frac{y_2^* y_1^*}{b+\left(y_2^*\right)^2} - qE\chi_\omega(x)u_1^*$$

$$f_3^* = n_1\frac{(y_2^*)^2}{b+\left(y_2^*\right)^2} + m_1\frac{(y_1^*)^2}{b+\left(y_1^*\right)^2} - d_2 - qE\chi_\omega(x)u_2^*$$

and $U = (U_1, U_2, U_3)$. Hence, then system $(20-24)$ can be written in the form

$$\begin{cases} \frac{\partial U}{\partial t} &= AU + HU + Gu \quad t \in [0,T] \\ U(0) &= 0 \end{cases}$$

and its solution can be expressed as

$$U(t) = \int_0^t S(t-s)H(s)U(s)\,ds + \int_0^t S(t-s)Gu(s)\,ds. \tag{24}$$

By (25) and (24) one deduces that

$$U^\varepsilon(t) - U(t) = \int_0^t S(t-s)H^\varepsilon(s)\left(U^\varepsilon - U\right)(s)\,ds + U(s)\left(H^\varepsilon(s) - H(s)\right)ds.$$

Thus all the coefficients of the matrix H^ε tend to the corresponding coefficients of the matrix H in $L^2(Q)$.

By using of Gronwall's inequality, we derive that thus $U_i^\varepsilon \to U_i$ in $L^2(Q)$ as $\varepsilon \to 0$, for $i = 1,2,3$. The proof is complete. □

Let $p = (p_1, p_2, p_3)$ and $\rho = (0, \rho_1, 0)$ the adjoint variable, we can write the dual system associated to our problem

$$\begin{cases} -\frac{\partial p}{\partial t} - Ap - F^* p &= D^* D\rho \quad t \in [0,T] \\ p(T,x) &= 0 \\ \frac{\partial p}{\partial \eta} &= 0 \end{cases} \tag{25}$$

Let u^* be an optimal control of $(2-4)$, $\tilde{y}^* = \left(y_1^*, y_2^*, y_3^*\right)$ be the optimal state, D is the matrix defined by $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, D^* be the adjoint matrix associated to D, H^* be the adjoint matrix associated to H

Lemma 5.1. *Under hypothesis of theorem* (7.1)*, if* $(\tilde{y}^*, (u_1^*, u_2^*))$ *is an optimal pair, then there exists a unique strong solution* $p \in W^{1,2}\left([0,T], H(\Omega)\right)$ *to the system* (25) *with* $p_i \in L(T,\Omega)$ *for* $i = 1,2,3$.

Proof. Like in theorem (7.1), by making the change of variable $s = T - t$ and the change of functions $q_i(s,x) = p_i(T-s,x) = p_i(t,x), \quad (t,x) \in Q$, $i = 1,2,3$. We can easily prove the existence of the solution of this lemma. □

To obtain the necessary conditions for the optimal control problem, applying standard optimality techniques. By analyzing the objective functional and using relationships between the state and adjoint equations, we obtain the following characterization of the optimal control

Theorem 5.2. *Let* u^* *be an optimal control of* $(2-6)$ *and let* $y^* \in W^{1,2}\left([0,T], H(\Omega)\right)$ *with* $y_i^* \in L(T,\Omega)$ *for* $i = 1,2,3$ *be the optimal state, that is* y^* *is the solution to* $(2-4)$ *with the control* u^**. Then, there exists a unique solution* $p \in W^{1,2}\left([0,T], H(\Omega)\right)$ *with* $p_i \in L(T,\Omega)$ *of the linear problem*

$$\left\{\begin{array}{rcl} -\frac{\partial p}{\partial t} - Ap - F^*p & = & D^*\rho \quad t \in [0,T] \\ p(T,x) & = & 0 \\ \frac{\partial p}{\partial \eta} & = & 0 \end{array}\right. \tag{26}$$

(27)
$$u_1^* = \min\left(u_1^{\max}, \max\left(0, -\frac{qE\chi_\omega(x)p_2}{\eta_1} y_2^*\right)\right) \quad \textit{and} \quad u_2^* = \min\left(u_2^{\max}, \max\left(0, -\frac{qE\chi_\omega(x)p_3}{\eta_2} y_3^*\right)\right)$$

Proof. We suppose u^* is an optimal control and $\tilde{y}^* = \begin{pmatrix} y_1^*, & y_2^*, & y_3^* \end{pmatrix} = \begin{pmatrix} y_1, & y_2, & y_3 \end{pmatrix}(u^*)$ are the corresponding state variables. Consider $u^\varepsilon = u^* + \varepsilon h \in U_{ad}$ and corresponding state solution $\tilde{y}^{\varepsilon}=\begin{pmatrix} y_1^{\varepsilon}, & y_2^{\varepsilon}, & y_3^{\varepsilon}\end{pmatrix}=\begin{pmatrix} y_1, & y_2, & y_3\end{pmatrix}(u^{\varepsilon})$, we have

(28)

$$
\begin{aligned}
& J'(u^*)h \\
= \;& \lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}\left(J(u^{\varepsilon})-J(u^*)\right) \\
= \;& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\begin{array}{c} \rho_1\int_0^T\int_\Omega (y_2^{\varepsilon}-y_2^*)(t,x)\,dxdt+\rho_2\int_0^T\int_\Omega \left(y_3^{\varepsilon}-y_3^*\right)(t,x)\,dxdt \\ +\frac{\eta_1}{2}\int_0^T\int_\omega\left((u_1^{\varepsilon})^2-(u_1^*)^2\right)(t,x)\,dxdt+\frac{\eta_2}{2}\int_0^T\int_\omega\left((u_2^{\varepsilon})^2-(u_2^*)^2\right)(t,x)\,dxdt\end{array}\right] \\
= \;& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\begin{array}{c} \rho_1\int_0^T\int_\Omega\left(\dfrac{y_2^{\varepsilon}-y_2^*}{\varepsilon}\right)(t,x)\,dxdt+\rho_2\int_0^T\int_\Omega\left(\dfrac{y_3^{\varepsilon}-y_3^*}{\varepsilon}\right)(t,x)\,dxdt \\ +\dfrac{\eta_1}{2}\int_0^T\int_\omega\left(\varepsilon(h_1)^2+2h_1u_1^*\right)(t,x)\,dxdt+\dfrac{\eta_2}{2}\int_0^T\int_\omega\left(\varepsilon(h_2)^2+2h_2u_2^*\right)(t,x)\,dxdt\end{array}\right] \\
= \;& \rho_1\int_0^T\int_\Omega Y_2(t,x)\,dxdt+\rho_2\int_0^T\int_\Omega Y_3(t,x)\,dxdt+\eta_1\int_0^T\int_\omega (h_1u_1^*)(t,x)\,dxdt \\
& +\eta_2\int_0^T\int_\omega (h_2u_2^*)(t,x)\,dxdt \\
= \;& \int_0^T\langle D\rho, DY\rangle_{H(\Omega)}\,dt+\int_0^T\langle \eta u^*, h\rangle_{(L^2(\Omega))^2}\,dt
\end{aligned}
$$

with $\eta u^*=\begin{pmatrix}\eta_1u^*\\ \eta_2u^*\end{pmatrix}$. We use (19) and $(26-27)$, we have

$$
\begin{aligned}
\int_0^T\langle D\rho, DY\rangle_{H(\Omega)}\,dt &= \int_0^T\langle D^*D\rho, Y\rangle_{H(\Omega)}\,dt \\
&= \int_0^T\left\langle -\frac{\partial p}{\partial t}-Ap-F^*p, Y\right\rangle_{H(\Omega)}dt \\
&= \int_0^T\left\langle p, \frac{\partial Y}{\partial t}-AY-FY\right\rangle_{H(\Omega)}dt \\
&= \int_0^T\langle p, Gh\rangle_{H(\Omega)}\,dt \\
&= \int_0^T\langle G^*p, h\rangle_{(L^2(\Omega))^2}\,dt
\end{aligned}
$$

Since J is gateaux differentiable at $u^*=\begin{pmatrix}u_1^*\\ u_2^*\end{pmatrix}$ and U_{ad} is convex, as the minimum of the objective functional is atteined at u^* it is seen that $J'(u^*)(v-u^*)\geq 0 \quad$ for all $\; v\in U_{ad}$.

We take $h=v-u^*$ and we use $(28-29)$ then

$$
J'(u^*)(v-u^*)=\int_0^T\langle G^*p+\eta u^*, v-u^*\rangle_{(L^2(\Omega))^2}\,dt.
$$

We conclude that $J'(u^*)(v-u^*) \geq 0$ equivalent to $\int_0^T \langle G^* p + \eta u^*, v-u^* \rangle_{\left(L^2(\Omega)\right)^2} dt \geq 0$ for all $v \in U_{ad}$. By standard arguments varying v, we obtain

$$\eta u^* = -G^* p.$$

Then

$$u_1^* = \frac{-qEp_2}{\eta_1}\chi_\omega(x) y_2^* \qquad \text{and} \qquad u_2^* = \frac{-qEp_3}{\eta_1}\chi_\omega(x) y_3^*.$$

As $(u_1^*, u_2^*) \in U_{ad}$, we have

$$u_1^* = \min\left(u_1^{\max}, \max\left(0, -\frac{qEp_2}{\eta_1}\chi_\omega(x) y_2^*\right)\right) \text{ and } u_2^* = \min\left(u_2^{\max}, \max\left(0, -\frac{qEp_3}{\eta_2}\chi_\omega(x) y_3^*\right)\right)$$

□

6. Numerical Simulation

6.1. Numerical simulation without control. In order to show the effect of the spatial factor of the proposed model, we give a numerical simulation over a period of $t = 360$ days with figures (1-3). The values of the parameters used in these simulations as well as the size of the grid are presented in section 4 entitled Numerical simulations. The system is numerically solved using MATLAB, and we give a description of the numerical method used to resolve our dynamical system in section 4. In the subsection (6.2), we present numerical values and methods. Figure 1, 2, and 3 present numerical results of the density of prey, predator, and super predator. We consider two situations, in the first one the interaction of the prey, predator, and super predator starts from the corner (1) , and in the second one, it starts from the middle. The only difference between these two cases is that in the second situation, the predators, and super predators harvest the prey rapidly. This demonstrates the significance of the spatial concept that has been applied.

In figures $1-3$, we can clearly see that there is no significant effect until $t = 70$ days after the interaction of the prey, predators and super predators. The density of preys decreases sharply that may lead to their absence; there is a significant increase of the density of predators until $t = 70$ days, and just then we observe a remarkable decrease until t = 360 days. The density of super predators increases until $t = 210$ days and decreases later until $t = 360$ days.

Considering the results of these simulations motivates us to think about the definition of an appropriate control strategy. The strategy proposed in this paper is the introduction of two harvesting functions that aim to make fishing control efforts target the source area of the interaction of the species.

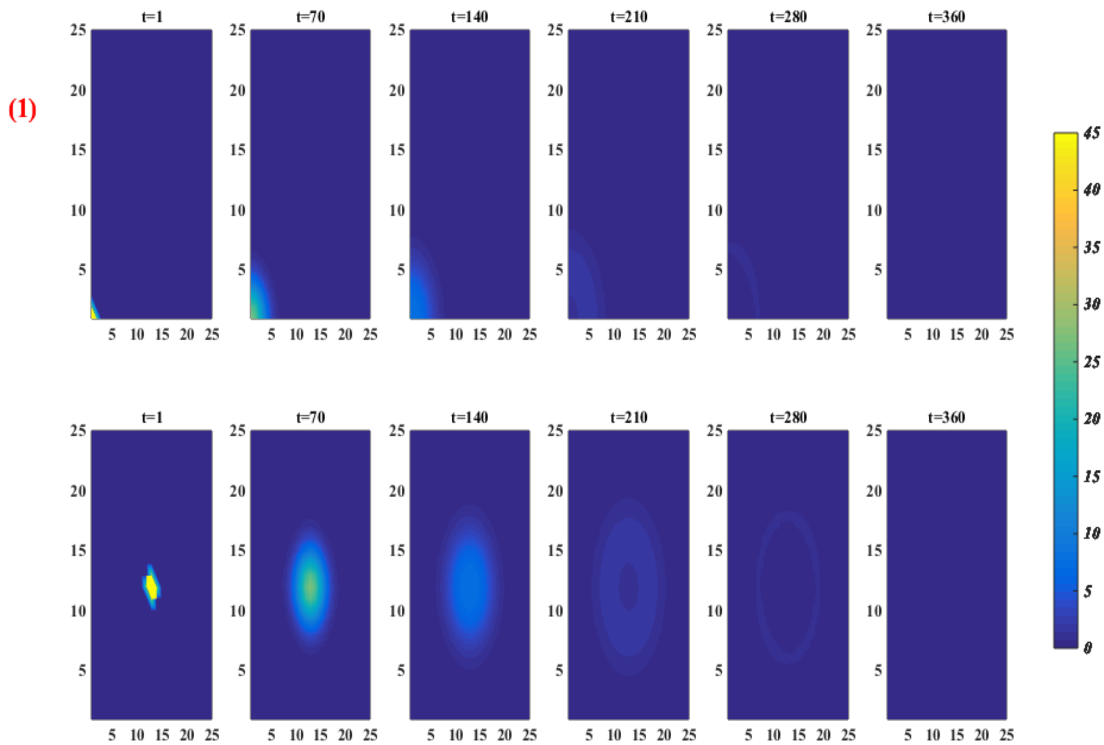

FIGURE 1. The density of prey behavior within Ω without control (1) the evolution of density of different fish population begins starts from the corner of the area. (2) the evolution of density of different fish population begins starts from the middle of the area

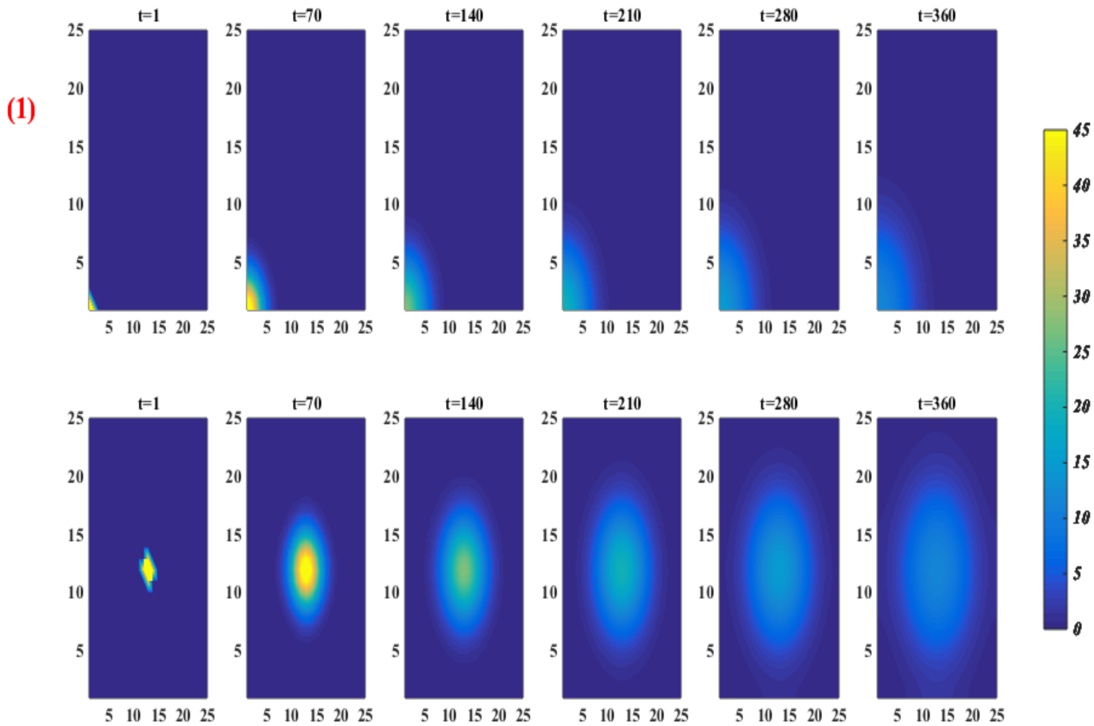

FIGURE 2. The density of predator behavior within Ω without control (1) the evolution of density of different fish population starts from the corner of the area. (2) The evolution of density of different fish population begins starts from the middle of the area

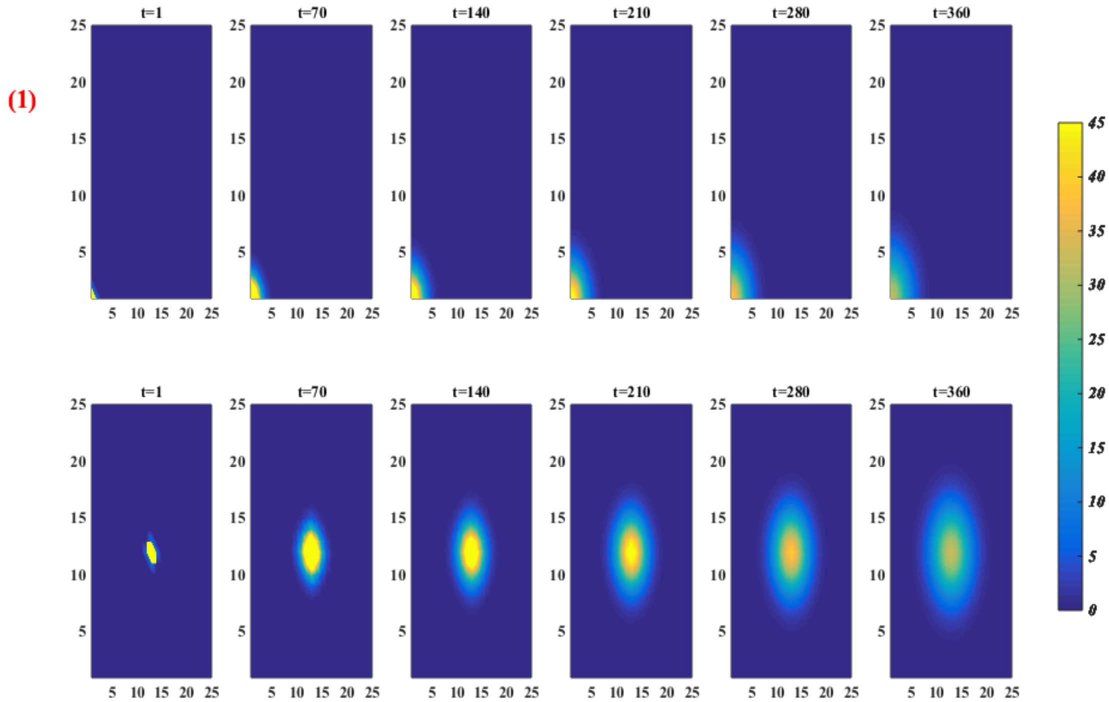

FIGURE 3. The density of super predator behavior within Ω without control: (1) the evolution of density of these species begins starts from the corner of the considered area. (2) The evolution of density of these species begins starts from the middle of the considered area.

6.2. Numerical simulation with controls. This section describes numerical simulations to illustrate our theoretical results and to show the performance of the strategy we have adopted in the framework to protect the prey population and preserve the sustainable ecosystem. The initial values in area 1 and area 2 are the same in the table 1. As far as the initial values, they are estimated from a statistical study [16].

Such formulation, a strategy of optimal regional control is obtained through resolving the optimality system, consisting of PDEs from the state variables and boundary conditions $(2-4)$ and adjoint equations with transversality conditions (25) by applying the forward-backward sweep method (FSBM) [33].

The equations of the states are solved by using a straightforward method in an iterative procedure using the explicit Euler to discretize the second order derivatives ∂x, ∂y, and ∂z. In addition, we use the explicit Euler second-order method where the initial control variables are divined in the initiation of the iterative method and then, the adjoint equations are backwards solved in time. Lastly, the variables of control are revealed from the adjoint solutions and the current state. The iterative process is then repeated until a tolerance criterion is reached. We assume that the initial densities of prey, predator, and super predator populations are estimated as $x_0 = 1050, y_0 = 920$ and $z_0 = 730$.. In order to illustrate the significance of this study and without losing its generality, the model is valid for all areas of the world, but the simulation will be limited to the study of a model that is composed of two areas (source of prey-predator interaction : Ω_1 where the evolution of density of the various fish populations begins with the lower left corner of Ω, and Ω_2 where the evolution begins from the center of Ω)

We consider a population total $N_0 = 2700$ in a $25km \times 25km$ rectangulary grid, we assume that prey-population are distributed homogeneously with 10 in each of $1km \times 1km$. Furthermore, the higher limits of the condition of optimality are regarded as $u_1^{max} = 0.7$ and $u_2^{max} = 0.8$, and in the objective function constant weight values are $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 10$. concerned do not provide more explanatory information, people can be left feeling that there is something wrong, which leads to a lot of gossips.

Notations	Value	Description($Units$)
r_1	1.9	Intrinsic growth rate of prey (day^{-1})
K	60	Environmental carriying capacity of prey (day^{-1})
α	1.5	Capture rate: of: the: predator: to: prey (day^{-1})
E	7.6×10^{-3}	fishing effort rate (day^{-1})
a	0.6	Half-saturation constants (day^{-1})
b	0.7	Half-saturation constants (day^{-1})
β	1.2	Predator's consumption rate on prey (day^{-1})
m_1	1.92	Super predator's consumption rate on prey (day^{-1})
q	2×10^{-4}	Catchability coefficient $(year^{-1})$
m	1.91	Capture rate: of: the: super: predator: to: prey (day^{-1})
n	0.83	Capture rate: of: the: super: predator: to: predator (day^{-1})
n_1	0.8	Super predator's consumption rate on predator $(year^{-1})$
d_1	3×10^{-4}	Natural death rate of predator (day^{-1})
d_2	1.2×10^{-4}	Natural death rate of super predator (day^{-1})
t	$[0, 360]$	time period (day)

TABLE 2. Initial conditions and parameters values

To provide an illustration and showing the influence of every control and its impact on the preservation of prey. We choose to adopt two cases. In the first case, we optimize simultaneously the regional catchability controls $\chi_\omega u_1(t,X)$ and $\chi_\omega u_2(t,X)$. However, in the second case, the regional fishing effort control $\chi_\omega u_2(t,X)$ is not optimized but held the constant while the regional fishing control $\chi_\omega u_1(t,X)$ is optimized. Figures (4-9) illustrate the found results found in the both cases for all scenarios.

(i) Case 1: applying both regional controls $\chi_\omega u_1(t,X)$ and $\chi_\omega u_2(t,X)$.

(ii) Case 2: applying only regional control $\chi_\omega u_1(t,X)$

CASE 1: APPLYING REGIONAL CONTROLS $\chi_\omega u_1$ AND $\chi_\omega u_2$

In this case we will give two scenarios. In the first scenario, we will limit ourselves to control only the area in the center, in this scenario we choose to targeted a circular area ω_1 which contains the source zone of the interaction of the species with a larger radius $R = 7$. However, the second scenario we controlling only the area in the corner where the targeted circular area ω_2 the radius is $R = 7$.

(i) First scenario: Controlling the center area ω_1 .

(ii) Second scenario: Controlling the corner of area ω_2.

First scenario: Applying controls u_1 and u_2 by making a good catchability by the fishing fleets applied to the predators and super predators in the center of the circular areas. In the figure (4-6), when we use our spatiotemporal control based on two controls. We admit that optimal strategies begin on days $t = 1$ which is the same day harvested the predators and super predators. We observe that the amount of prey decreases at a very slight rate over the 360-day period. In a lesser way compared to the results obtained in the figure (1-3), in which we note the disappearance of these prey. While there has been a slight increase in the density of predators and generalist predators during the 360 days. Which is very beneficial and reflects the importance of our control strategy.

Second scenario: Applying controls u_1 and u_2 by making good catchability by the fishing fleets applied to the predators and super predators in the corner of the circular areas. To realize this strategy in the figure (4-6), we investigate numerical results with two strategies of spatial-temporal control $(\chi_\omega(X)u_1(t,X),\, \chi_\omega(X)u_2(t,X))$ representing the fishing effort control of predators and super predators population. These controls are highly noticeable in maintaining the density of the prey and thus maintaining a different chain system. We see that in this case, the number of prey decreases at very law but in a way less than the first scenario in the center areas, however the number of predators and super predators we note a slight increase without reaching the same number in the first scenarios. the above-mentioned strategy has been proven to be efficient as a result of the use of these controls.

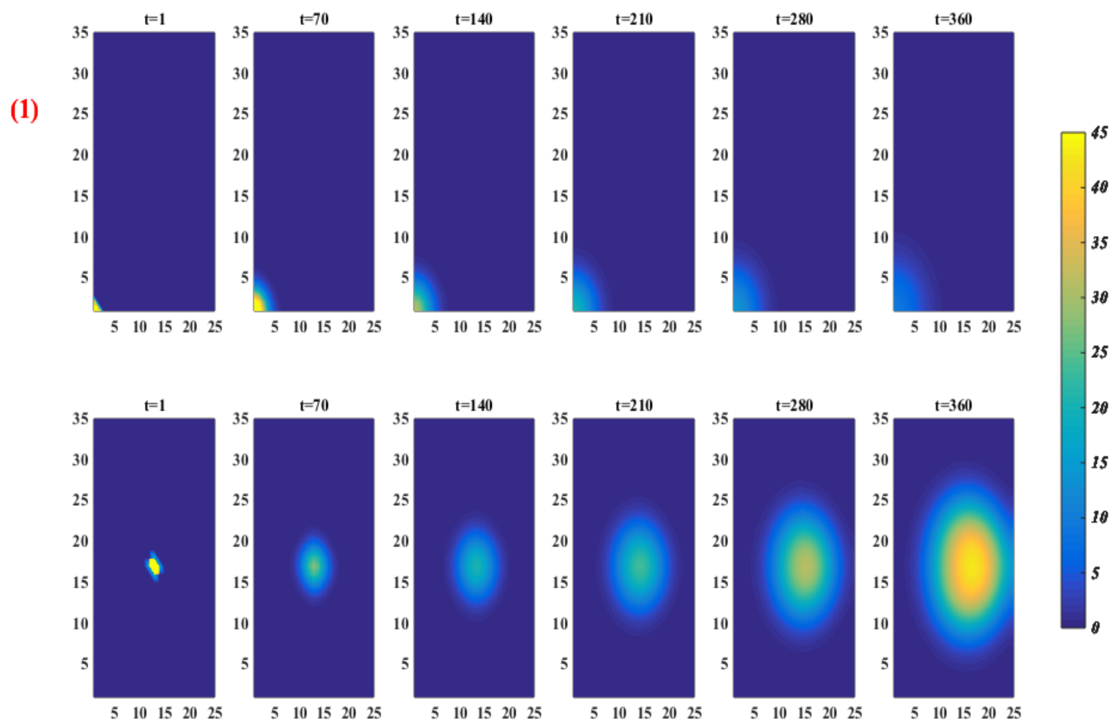

FIGURE 4. The density of prey behavior within Ω with control: (1) the evolution of density of these species begins starts from the corner of the controlled area ω_1. (2) the evolution of density of these species begins starts from the middle of the controlled area ω_2.

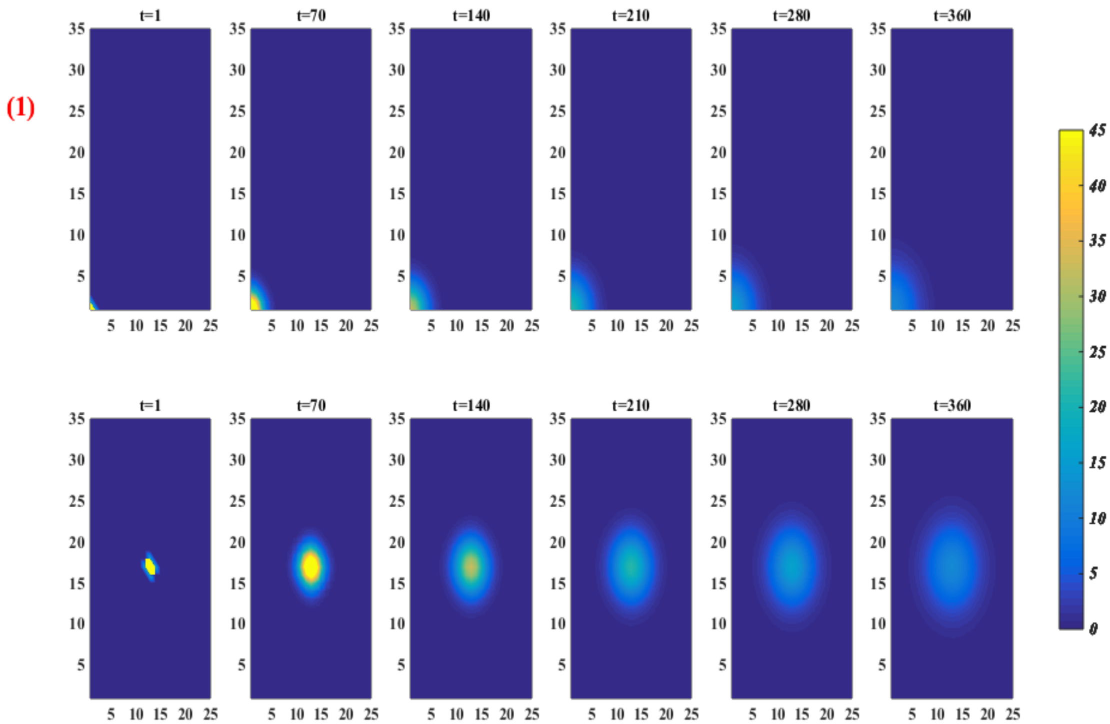

FIGURE 5. The density of predator behavior within Ω with control: (1) the evolution of density of these species begins starts from the corner of the controlled area ω_1. (2) the evolution of density of these species begins starts from the middle of the controlled area ω_2.

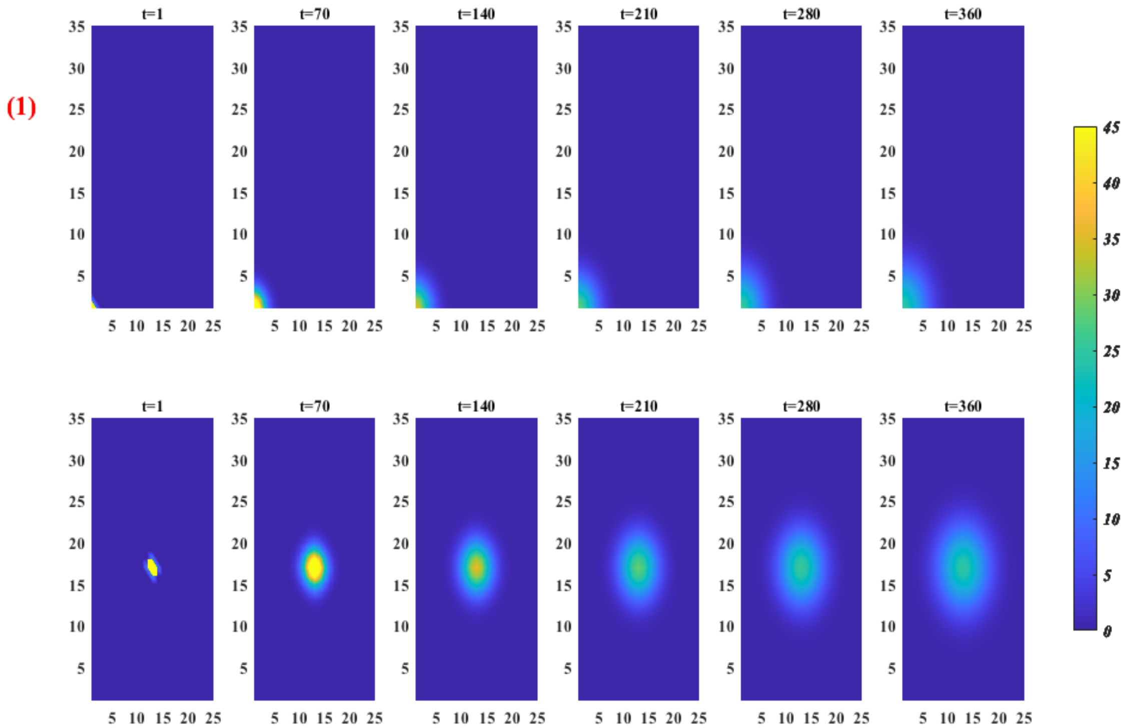

FIGURE 6. The density of super predator behavior within Ω with control: (1) the evolution of density of these species begins starts from the corner of the controlled area ω_1. (2) the evolution of density of these species begins starts from the middle of the controlled area ω_2.

CASE 2: APPLYING ONLY A REGIONAL CONTROL $\chi_{\omega} u_1$

In this subsection, in order to show the influence of the choice of the form of the area, we choose two controlled regions as a circular area ω_1 and ω_2 where ω_1 with radius is $R = 7$ when the interaction of the species starts from the low left corner, ω_2 with radius is $R = 7$ when the interaction of the species starts from the middle.

The effect of the control implemented on super predators ($u_2(t,X)$) is not being optimized in this case, whereas the impact of the control applying to predators ($u_1(t,X)$) is therefore optimized to observe its influence on the development of the different species (prey, predators and super predators). In Figures (7-9), we notice the efficiency of the simultaneous application strategies of spatial-temporel control $u_1(t,X)$, such controls are very remarkable in preserving the density of the density of prey in its absence and consequently in keeping a system of different chains. We can observe in the figure(1) after 360 days in the lack of the control the disappearance of the prey population. The prey population is harvested by predators and super predators (see figure 2 and 3). However, we can remark a decrease in the density of prey with their continuation in the center and corner of the considerd areas in the presence of the control. There is also an increase in population density of predators and super predators during 360 days. This implies that the holistic approach of the intervention strategies is the most effective way to preserve the prey.

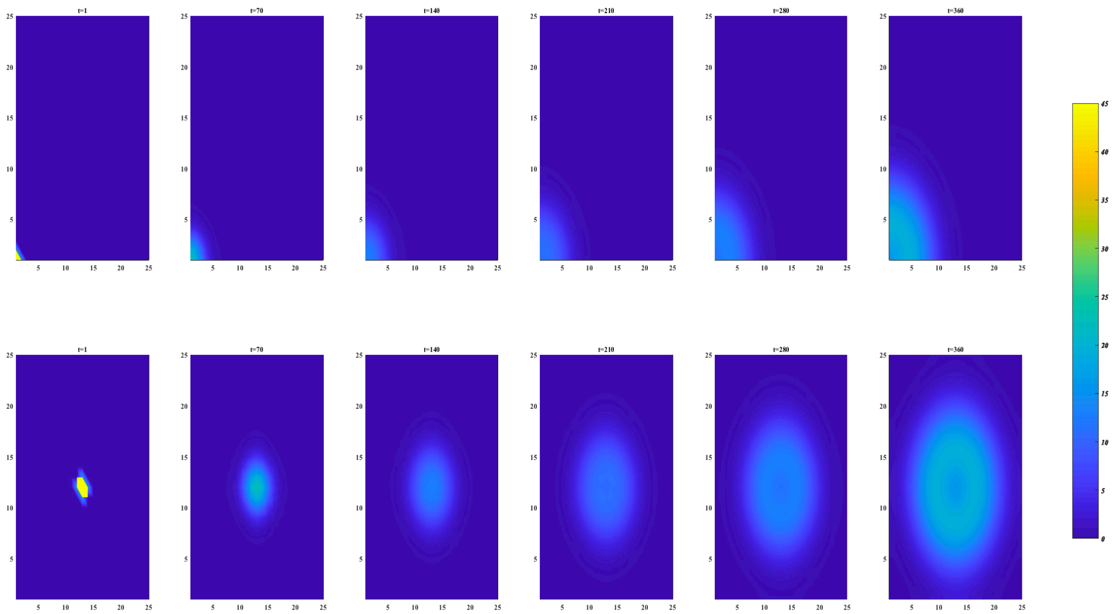

FIGURE 7. The density of prey behavior within Ω with control u_1: (1) the evolution of density of these species begins starts from the corner of the controlled area ω_1. (2) the evolution of density of these species begins starts from the middle of the controlled area ω_2.

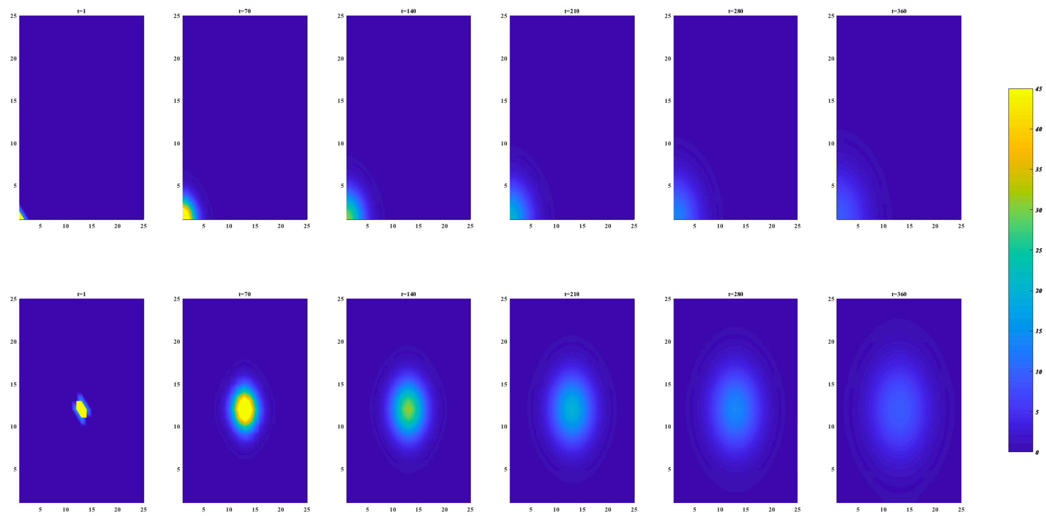

FIGURE 8. The density of predator behavior within Ω with control u_1:(1) the evolution of density of these species begins starts from the corner of the controlled area ω_1. (2) the evolution of density of these species begins starts from the middle of the controlled area ω_2.

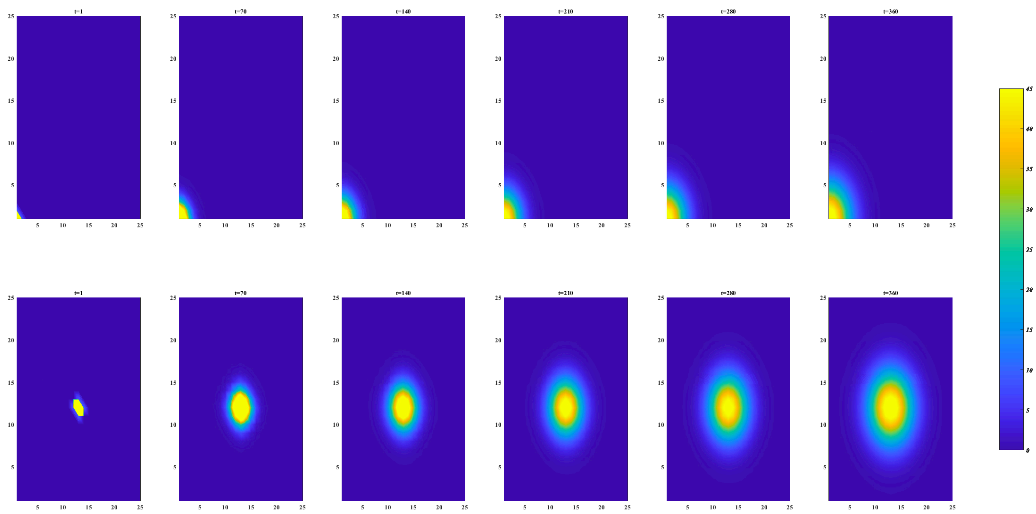

FIGURE 9. The density of super predator behavior within Ω with control u_1: (1) the evolution of density of these species begins starts from the corner of the controlled area ω_1. (2) the evolution of density of these species begins starts from the middle of the controlled area ω_2.

7. Conclusion

In this paper, we studied a prey-predator model that describes the population dynamics of the interaction of three species: two predators and one prey. Our model is a Spatio-temporal model described by a system of parabolic partial differential equations.

This work is more important in marine ecology because irrational management of marine resources can even lead to the disappearance of some species, which is an attack on the ecology and its stability and discredits an entire regional economy. It induces a natural imbalance whose consequences have been, among others, the total disappearance of some species, the distraction of biomass, the distraction of vital reefs for marine species, the enlargement of toxic zones, as well as the undesirable growth of specific predators.

To contribute to this topic, we developed a regional control strategy to maximize prey density and minimize predator density to ensure the sustainability of the marine population while ensuring optimal harvests. From a practical standpoint, population control of predators and prey cannot be in a larger spatial domain due to the chance of logistical resources.

The most significant novelty of our work is not only to consider the Spatio-temporal character but also to assume that our control strategy is limited to a subarea ω of Ω. The choice of the sub-area ω is reflected by a concern for cost optimization and by ω as a geographically accessible area to adopt as spatial support for a possible maneuver or control. This approach has not yet been applied in fish population systems; their deliberations are limited to theoretical concepts.

To achieve this goal, we introduced two regional harvest control strategies for a Spatio-temporal prey-predator model representing the correct catchability to harvest the predator and super-predator population. Theoretically, we have shown the global existence and the uniqueness of the strong solution of the controlled system and the subsistence of an optimal couple of controls. We have derived an optimality system and used optimal control techniques to characterize the controls, a characterization of the optimal regional controls is obtained in terms of assistant functions and state. To solve the optimality conditions, we used the forward-backward scanning method (FSBM). Numerical simulations demonstrate that the control impact is efficient if the two harvesting strategies of the regional controls are applied simultaneously, which allowed us to compare and see the difference between each scenario in a concrete way. The objective of this work has been accomplished, the performance of our strategy has been proven, which allows us to create an ecological balance and preserve the ecosystem.

As a natural continuation of this work we are studying the following problem.

Problem 1. We investigate the same problem presented in (2) but in this study we treat the case when the sub-domains ω of Ω are different, more precisely we deal with the following situations

- Case 1: $\omega_1 \neq \omega_2$ and $\omega_1 \cup \omega_2 = \Omega$ with $\omega_1 \cap \omega_2 = \emptyset$.
- Case 2: $\omega_1 = \Omega$ and $\omega_2 = \emptyset$.
- Case 3: $\omega_1 = \emptyset$ and $\omega_2 = \Omega$.

Problem 2. We investigate the same problem presented in (2) but in this study we treat the case when the sub-domains ω_n, $n = 1,2,...,\infty$ of Ω are described as follows

- $\omega_n = C(0, \frac{1}{n})$ where $n = 1$, $n = 2$, $n = 3$, ..., $n = \infty$ with $C(0, \frac{1}{n})$ is a circular domain of center 0 and radius $\frac{1}{n}$.

APPENDIX

First recall a general existence result which we use in the sequel (Proposition $1.2, p.175$, [35]; see also [34], [36]). Consider the initial value problem

$$\left\{ \begin{array}{rcl} \frac{\partial z}{\partial t} & = & Az(t) + f(t, z(t)), \quad t \in [0, T] \\ z(0) & = & z_0 \end{array} \right. \tag{29}$$

where A is a linear operator defined on a Banach space X, with the domain $D(A)$ and $f : [0,T] \times X \to X$ is a given function. If X is a Hilbert space endowed with the scalar product $(\cdots, \cdots)$, then the linear operator A is called dissipative if $(Az, z) \leq 0$, $(\forall z \in D(A))$.

Theorem 7.1. *X be a real Banach space, $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of linear contractions $S(t), t \geq 0$ on X, and $f : [0,T] \times X \to X$ be a function measurable in t and Lipschitz continuous in $x \in X$, uniformly with respect to $t \in [0,T]$.*

(i) *If* $z_0 \in X$*, then problem* (29) *admits a unique mild solution, i.e. a function* $z \in C([0,T],X)$ *which verifies the equality* $z(t) = S(t)z_0 + \int_0^t S(t-s)\, f(s,z(s))\, ds, (\forall t \in [0,T])$.

(ii) *If* X *is a Hilbert space,* A *is self-adjoint and dissipative on* X *and* $z_0 \in D(A)$*, then the mild solution is in fact a strong solution and* $z \in W^{1,2}([0,T];X) \cap L^2(0,T;D(A))$.

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DATA AVAILABILITY

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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