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A STOCHASTIC HYBRID DIFFERENTIAL EQUATION MODEL: ANALYSIS AND APPLICATION

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Abstract. Infectious diseases represent a real challenge to humanity and a true challenge for researchers to propose relevant solutions in order to reduce the number of infected individuals. Mathematical modeling of infectious diseases represents a better way to understand and control the spread of epidemics. In this work, we propose a stochastic *SIQS* epidemic model with a nonlinear incidence function and Markov switching. Firstly, we present our proposed stochastic model and its parameters. Secondly, we show the global existence and uniqueness of the positive solution. Then, we show a sufficient condition for the extinction of disease. Finally, we give numerical simulation to enrich our analytical results.

Keywords: epidemic model; Markov chain; persistence; extinction.

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1. INTRODUCTION AND PRELIMINARY

The mathematical study of the biological system represents a more important theme in the field of mathematical biology and draws the attention of several authors (see for example,

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[1, 2, 3, 4, 5, 6]). Khan et al. [19] proposed a heroin epidemic model in partial differential equations form, including age-dependent susceptibility and recovery-age. They employed some control optimal theoretical background to determine the existence of optimal control variables that minimize their objective function. To express the effect of memory on the dynamics of a generalized SIR epidemic model, Das and Samanta in [20] have used the modeling by the fractional-order differential equations. Kumar and Goel in [21], formulated an epidemic model that included a delay and a treatment function, and they gave a detailed analysis of the proposed system.

Stochastic systems represent an additional degree of realism compared with deterministic systems and may reveal how random noise affects the population system (see, [4, 22, 23, 24]). The following system represents the stochastic *SIQS* epidemic model with withe noise:

$$\left\{\begin{array}{l} dS(t) = [\chi - \lambda S\Psi(I) - \mu S + \gamma I + \theta Q]\,dt - \eta S\Psi(I)dM_B(t), \\ \\ dI(t) = [\lambda S\Psi(I) - (\mu + \rho + \vartheta + \gamma)I]\,dt + \eta S\Psi(I)dM_B(t), \\ \\ dQ(t) = [\vartheta I - (\mu + \rho + \theta)Q]\,dt, \end{array}\right. \tag{1}$$

where $S(t)$, $I(t)$ and $Q(t)$ represents the number of susceptible, infected, and quarantined populations at time t, respectively. The parameters χ represent the recruitment rate of susceptible corresponding to births and immigration. μ is the natural death rate. ϑ denotes the transfer rate of infectious individuals from the infective class (I) to the quarantine class (Q). ρ represents the disease-related death rate constant in class (I) and (Q). Parameters γ and θ represent the rates at which individuals recover and return to class (S) from class (I) and class (Q), respectively. M_B is a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets), and η represents the intensity of M_B. The function Ψ satisfies the following conditions:

- $\Psi(y)$ is non-negative twice continuously differentiable such that $\Psi(0) = 0$. In addition, the function $\frac{\Psi(y)}{y}$ is monotonically decreasing on $[0, \infty)$.

Note that the above conditions implies that:

$$\frac{\Psi(y)}{y} < \Psi'(0), \forall y > 0.$$

We inform that in the case where $\eta = 0$ and $\lambda S\Psi(I) = \lambda SI$, we find the corresponding deterministic model of (1), presented in [7] by Herbert et al. Then, according to the theoretical part in [7], the basic reproduction number of the system (1) is

$$\mathscr{R}_0 = \frac{\beta\chi}{\mu(\mu+\rho+\vartheta+\gamma)}.$$

Moreover, if $\mathscr{R}_0 \leq 1$, then the corresponding deterministic model of (1) has only the disease-free equilibrium $E_0\left(\frac{\chi}{\mu}, 0, 0\right)$ which is globally asymptotical stable, and if $\mathscr{R}_0 > 1$, E_0 becomes unstable and there exists a global asymptotically stable endemic equilibrium $E_*(S^\star, I^\star, Q^\star)$, with

$$S^\star = \frac{\chi}{\mu\mathscr{R}_0},\ I^\star = \frac{\chi - \frac{\chi}{\mathscr{R}_0}}{(\mu+\rho)\left(1+\frac{\vartheta}{\mu+\rho+\theta}\right)},\ Q^\star = \frac{\vartheta I^\star}{\mu+\rho+\theta}.$$

The incidence rate of the disease represents an essential aspect in the modeling of infectious diseases. It's defined as the number of new cases per unit of time. In the literature review, many works have employed the bilinear incidence rate or standard incidence rate to model the transmission of epidemics. However, the bilinear incidence rate βSI or the standard incidence rate $\beta SI/N$, (where N represents the total population number) is not preferable to describe several situations of epidemic evolution. Take as an example the situation in which the population is saturated. In this case, the number of infectious diseases in the population is so large, thus the incidence rate depends non-linearly to (I). To describe the saturated population, Capasso and Serio in [8] have proposed the saturated incidence rate defined by the quantity $\beta SI/(1+aI)$, (where a is a positive constant measures the psychological or inhibitory effect of the population). There are other forms of nonlinear incidence (see, Table 1) and each represents some advantages in modeling the dynamics of epidemics.

As is known, the population system fluctuates around a stable average value. But, in reality, population systems may expose to change abruptly (due to; climate change, nutrition problems, factors social), which driving the system to switch between different environmental regimes.

In addition, the classic stochastic systems (with white noise) cannot describe the fact that the population may suffer from environmental changes. Consequently, to express the switching between two or more regimes of environment, we use another random noise named (Telegraph noise or colored noise) [14, 15]. Usually, the switching between various environmental regimes is memoryless, and the waiting time for the next switch follows the exponential distribution [16]. Then, the regime-switching can be modeled by a continuous-time Markov chain $m(t)$ taking values in a finite state space $\mathbb{L} = \{1, 2, ..., d\}$. Hence, in this paper, we propose the following stochastic model with nonlinear incidence and telegraphic noises

$$
\left\{
\begin{array}{l}
dS = [\chi(m(t)) - \lambda(m(t))S\Psi(I) - \mu(m(t))S + \gamma(m(t))I + \theta(m(t))Q]\,dt \\
\qquad -\eta(m(t))S\Psi(I)dM_B(t), \\
dI = [\lambda(m(t))S\Psi(I) - (\mu(m(t)) + \rho(m(t)) + \vartheta(m(t)) + \gamma(m(t)))I]\,dt \\
\qquad +\eta(m(t))S\Psi(I)dM_B(t), \\
dQ = [\vartheta(m(t))I - (\mu(m(t)) + \rho(m(t)) + \theta(m(t)))Q]\,dt.
\end{array}
\right. \tag{2}
$$

In this paper, the Markov chain $\{m(t)\}_{t\geq 0}$ is supposed independent of the Brownian motion $M_B(t)$ and defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with infinitesimal generator $\Phi = (\phi_{uv})_{1\leq u,v\leq d}$ given, for $\delta > 0$, by

$$
P\left(r(t+\delta) = v \mid r(t) = u\right) = \left\{
\begin{array}{ll}
\phi_{uv}\delta + o(\delta) & \text{if } u \neq v, \\
1 + \phi_{uu}\delta + o(\delta) & \text{if } u = v.
\end{array}
\right.
$$

Here, ϕ_{uv} is the transition rate from u to v and $\phi_{uv} \geq 0$ if $u \neq v$, while

$$
\phi_{uu} = -\sum_{u\neq v} \phi_{uv}.
$$

Suppose that the Markov chain $r(t)$ is independent of the Brownian motion $B(.)$ and it is irreducible. Under this condition, the Markov chain has a unique stationary distribution

$\pi = (\pi_1, ..., \pi_d)$, which can be determined by solving the linear equation $\pi\Phi = 0$, subject to $\sum_{i=1}^{d} \pi_i = 1$, and $\pi_i > 0$, $\forall i \in \mathbb{L}$. Then, for any vector $h = (h(1), ..., h(d))^T$, let $\hat{h} = \min_{i\in\mathbb{L}} \{h(i)\}$ and $\check{h} = \max_{i\in\mathbb{L}} \{h(i)\}$.

We consider the following stochastic system

$$d\vartheta(t) = f(t, \vartheta(t), r(t))dt + g(t, \vartheta(t), r(t))dB(t), \tag{3}$$

where $B(t)$ is a d-dimensional standard Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Denote by $C^{1,2}(\mathbb{R}^n \times \mathbb{L}; \mathbb{R}_+)$ the family of all nonnegative functions $\mathscr{H}$ defined on $\mathbb{R}^n \times \mathbb{L}$ such that they are continuously twice differentiable in ϑ. The operator $\mathscr{L}\mathscr{H}$ associated with (3) is defined as follows

$$\begin{aligned}\mathscr{L}\mathscr{H}(\vartheta, i) &= \mathscr{H}_t(t, \vartheta, i) + \mathscr{H}_\vartheta(t, \vartheta, i) f(t, \vartheta, i) + \frac{1}{2} g^T(t, \vartheta, i) \mathscr{H}_{\vartheta,\vartheta}(t, \vartheta, i) g(t, \vartheta, i) \\ &\quad + \sum_{j\in\mathbb{L}} \phi_{ij} \mathscr{H}(t, \vartheta, j),\end{aligned}$$

where $\mathscr{H}_\vartheta$ and $\mathscr{H}_{\vartheta,\vartheta}$ represents the gradient and Hessian of $\mathscr{H}$, and T is the transpose of a matrix.

By Itô's formula, if $\mathscr{H}$ defined on $\mathbb{R}^n \times \mathbb{L}$, we have

$$d\mathscr{H}(\vartheta(t), i) = \mathscr{L}\mathscr{H}(\vartheta(t), i)dt + \mathscr{H}_\vartheta(\vartheta(t), i) g(\vartheta(t), i) dB(t).$$

The purpose of this paper is to investigate the extinction of diseases in the model (2). By proposing the threshold that includes the noises terms, we establish sufficient conditions for the extinction of diseases. Moreover, our model (2) can be used to represent the impact of environmental switching on disease transmission. The rest of this article is organized in the following. In Section 2, we establish that the system (2) is mathematically and biologically well-posed by showing the global existence, positivity, and boundlessness of the solution. In Section 3, we show sufficient conditions for the extinction of the diseases. In Section 4, we give numerical simulations to validate our analytic results.

TABLE 1. Presentation of some nonlinear incidence rates

Incidence	Expression	Reference
Half-saturated incidence	$\frac{\beta SI}{H+I}$	[9]
Holling-type II incidence rate	$\frac{\beta SI}{m+S}$	[10]
Beddington-DeAngelis functional response	$\frac{\beta SI}{1+k_1S+k_2I}$	[11]
Crowley-Martin functional response	$\frac{\beta SI}{1+k_1S+k_2I+k_1k_2SI}$	[12]
Incidence with media coverage effect	$\beta_1-\beta_2\frac{I}{I+m}$	[13]

2. EXISTENCE OF SOLUTION

Our model describes the dynamics of a biological system. Thus, the solution of the system (2) must be positive, bounded, and unique. For this let

$$T(t) \quad = \quad S(t)+I(t)+Q(t).$$

For any $i \in \mathbb{L}$, we have

$$\begin{aligned}
& (\mu(m(t))+\rho(m(t)))\left[\frac{\hat{\chi}}{\check{\mu}+\check{\rho}}-T(t)\right]dt \\
\leq \quad & [\chi(m(t))-(\mu(m(t))+\rho(m(t)))T(t)]\,dt \\
\leq \quad & dT(t) \\
\leq \quad & [\chi(m(t))-\mu(m(t))T(t)]\,dt \\
\leq \quad & \chi(m(t))\left[\frac{\check{\chi}}{\hat{\mu}}-T(t)\right].
\end{aligned}$$

Hence, we have

$$\frac{d\left(\frac{\check{\chi}}{\hat{\mu}}-T(t)\right)}{dt}+\mu(m(t))\left(\frac{\check{\chi}}{\hat{\mu}}-T(t)\right) \quad \geq \quad 0,$$

and

$$\frac{d\left(\frac{\check{\chi}}{\hat{\mu}+\hat{\rho}}-T(t)\right)}{dt}+(\mu(m(t))+\rho(m(t)))\left(\frac{\check{\chi}}{\hat{\mu}+\check{\rho}}-T(t)\right) \quad \leq \quad 0.$$

Therefore

$$\frac{\check{\chi}}{\hat{\mu}} - T(t) \;\geq\; \left[\frac{\check{\chi}}{\hat{\mu}} - T(0)\right] \exp\left\{-\int_0^t \mu(m(s))ds\right\},$$

and

$$\frac{\check{\chi}}{\hat{\mu}+\hat{\rho}} - T(t) \;\geq\; \left[\frac{\check{\chi}}{\hat{\mu}+\hat{\rho}} - T(0)\right] \exp\left\{-\int_0^t (\mu(m(s)) + \rho(m(s)))ds\right\}.$$

Since $\frac{\check{\chi}}{\hat{\mu}+\hat{\rho}} \leq T(0) \leq \frac{\check{\chi}}{\hat{\mu}}$, thus

$$\frac{\check{\chi}}{\hat{\mu}+\hat{\rho}} \leq \;\; T(t) \;\; \leq \frac{\check{\chi}}{\hat{\mu}}.$$

Let

$$\Gamma \;=\; \left\{(S(t),I(t),Q(t)) \in \mathbb{R}^3_+ : \frac{\check{\chi}}{\hat{\mu}+\hat{\rho}} \leq T(t) \leq \frac{\check{\chi}}{\hat{\mu}}\right\}$$

Theorem 2.1. *For any given initial value $(S(0),I(0),Q(0)) \in \Gamma$, there exists a unique solution $(S(t),I(t),Q(t)) \in \Gamma$ of system (2) on $t > 0$ and the solution will remain in Γ with probability 1.*

Proof. Since the coefficients of stochastic system (2) are locally Lipschitz continuous, then their exist a unique local solution of system (2) on $t \in [0,\tau_e)$ for any given initial value $(S(0),I(0),Q(0)) \in \Gamma$ (τ_e is the explosion time, for more detail see, [17]). In order to prove that the solution is global, we need to show that $\tau_e = \infty \; a.s.$. Next, we define the stopping time τ_k for each $k > 0$ by

$$\tau_k \;=\; \inf\{t \in [0,\tau_e) : S(t) \leq k, \text{ or } I(t) \leq k, \text{ or } Q(t) \leq k\}.$$

Throughout this paper setting $\inf\emptyset = \infty$, with $\emptyset$ is the empty set. Obviously, τ_k increase when k tends towards zero. Let $\tau_0 = \lim_{k\to 0} \tau_k$, in addition $\tau_0 \leq \tau_e \; a.s.$ Thus, if we show that $\tau_0 = \infty \; a.s.$, then $\tau_e = \infty \; a.s.$, this implies that the solution $(S(t),I(t),Q(t))$ the solution belongs to Γ almost surely for all $t \geq 0$. We suppose that this assertion is false then there exist a pair of constants $C > 0$ and $\varepsilon \in (0,1)$ such that $\mathbb{P}\{\tau_0 \leq C\} > \varepsilon$. Define a C^2-function W, by the expression

$$W(S,I,Q) \;=\; \ln(S(t)I(t)Q(t)).$$

Using the generalised Itô's formula, we have

$$\begin{aligned} dW(S,I,Q) &= \left[\frac{\chi(m(t)}{S}-\lambda(m(t))\Psi(I)-\mu(m(t)+\gamma(m(t))\frac{I}{S}+\theta(m(t))\frac{Q}{S}\right. \\ &+\lambda(m(t))\frac{S\Psi(I)}{I}-\mu(m(t))-\rho(m(t))-\vartheta(m(t))-\gamma(m(t)) \\ &+\vartheta(m(t))\frac{I}{Q}-\mu(m(t))-\rho(m(t))-\theta(m(t))+0.5\eta^2(m(t))\Psi^2(I) \\ &\left.+0.5\eta^2(m(t))\frac{S^2\Psi^2(I)}{I^2}\right]dt+\left[\eta(m(t))\frac{S\Psi(I)}{I}-\eta(m(t))\Psi(I)\right]dM_B(t). \end{aligned}$$

Then, we have

$$\begin{aligned} W(S,I,Q) &\geq \int_0^t [-\lambda(m(s))\Psi(I)-3\mu(m(s)-2\rho(m(s))-\vartheta(m(s))-\gamma(m(s)) \\ &\quad -\theta(m(s))]\,ds+\int_0^t\left[\eta(m(s))\frac{S\Psi(I)}{I}-\eta(m(s))\Psi(I)\right]dM_B(s) \\ &\geq \int_0^t\left[-\check{\lambda}\Psi(I)-3\check{\mu}-2\check{\rho}-\check{\vartheta}-\check{\gamma}-\check{\theta}\right]ds \\ &\quad +\int_0^t\left[\eta(m(s))\frac{S\Psi(I)}{I}-\eta(m(s))\Psi(I)\right]dM_B(s). \end{aligned} \tag{4}$$

Since some components of $(S(\tau_k),I(\tau_k),Q(\tau_k))$ equal k, then,

$$\lim_{k\to 0} W(S(\tau_k),I(\tau_k),Q(\tau_k))=-\infty.$$

Letting $t\to\tau_k$ in (4), we obtain

$$\begin{aligned} &\int_0^t\left[-\check{\lambda}\Psi(I)-3\check{\mu}-2\check{\rho}-\check{\vartheta}-\check{\gamma}-\check{\theta}\right]ds \\ &+\int_0^t\left[\eta(m(s))\frac{S\Psi(I)}{I}-\eta(m(s))\Psi(I)\right]dM_B(s)>-\infty. \end{aligned}$$

Letting k tend to zero, leads to the contradiction with our assumption. The proof is complete.

□

3. EXTINCTION

In this section, we present sufficient conditions for the extinction of the disease. For this, we introduce the following threshold of our stochastic SIQS epidemic model (2) under switching

as follows:

$$\mathscr{R}_{st} = \frac{\sum_{i=1}^{d} \pi_i \lambda(i) \frac{\chi}{\mu} \Psi'(0)}{\sum_{i=1}^{d} \pi_i \left((\mu(i) + \rho(i) + \vartheta(i) + \gamma(i)) + \frac{\eta^2(i)}{2} \left(\frac{\chi}{\mu} \Psi'(0) \right)^2 \right)}$$

Theorem 3.1. *Assume that $(S(t), I(t), Q(t))$ be the solution of model (2) with any initial value $(S(0), I(0), Q(0)) \in \Gamma$. We have*

(a) If $\mathscr{R}_{st} < 1$, then

$$\begin{aligned} & \limsup_{t\to\infty} \frac{\ln I(t)}{t} \\ \leq & \sum_{i=1}^{d} \pi_i \left((\mu(i) + \rho(i) + \vartheta(i) + \gamma(i)) + \frac{\eta^2(i)}{2} \left(\frac{\chi}{\mu} \Psi'(0) \right)^2 \right) [\mathscr{R}_{st} - 1] < 0 \ a.s. \end{aligned}$$

i.e. the infected $I(t)$ go to extinction with probability one.
(b) If $\sum_{i=1}^{d} \pi_i \frac{\lambda^2(m(t))}{2\eta^2(m(t))} \leq \sum_{i=1}^{d} \pi_i (\mu(m(t)) + \rho(m(t)) + \vartheta(m(t)) + \gamma(m(t)))$, then the disease in system (2) dies out with probability one.

Proof. (a) By virtue of Itô's formula, we obtain

$$\begin{aligned} d\ln I(t) &= \frac{1}{I} \{ [\lambda(m(t)) S\Psi(I) - (\mu(m(t)) + \rho(m(t)) + \vartheta(m(t)) + \gamma(m(t)))I] \\ &\quad - \frac{\eta^2(m(t))}{2} \left(S\frac{\Psi(I)}{I} \right)^2 \Big\} dt + \eta(m(t)) \frac{S\Psi(I)}{I} dM_B(t) \\ &= \Big\{ \lambda(m(t)) S\frac{\Psi(I)}{I} - (\mu(m(t)) + \rho(m(t)) + \vartheta(m(t)) + \gamma(m(t))) \\ &\quad - \frac{\eta^2(m(t))}{2} \left(S\frac{\Psi(I)}{I} \right)^2 \Big\} dt + \eta(m(t)) \frac{S\Psi(I)}{I} dM_B(t). \end{aligned} \tag{5}$$

Since the function g is increasing over the interval $\left[0, \frac{\lambda(i)}{\eta^2(i)}\right]$ and $S \in \Gamma$, then

$$\begin{aligned} d\ln I(t) &\leq \Big\{ \lambda(m(t)) \frac{\chi}{\mu} \Psi'(0) - (\mu(m(t)) + \rho(m(t)) + \vartheta(m(t)) + \gamma(m(t))) \\ &\quad - \frac{\eta^2(m(t))}{2} \left(\frac{\chi}{\mu} \Psi'(0) \right)^2 \Big\} dt + \eta(m(t)) \frac{S\Psi(I)}{I} dM_B(t). \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\ln I(t)}{t} &\leq \frac{\ln I(0)}{t}+\frac{1}{t}\int_0^t\left\{\lambda(m(s))\frac{\chi}{\mu}\Psi'(0)-(\mu(m(s))+\rho(m(s))+\vartheta(m(s))+\gamma(m(s)))\right.\\ &\left.-\frac{\eta^2(m(s))}{2}\left(\frac{\chi}{\mu}\Psi'(0)\right)^2\right\}ds+\frac{\mathscr{M}(t)}{t}, \qquad (6)\end{aligned}$$

where $\mathscr{M}(t)=\int_0^t\eta(m(s))S(s)\frac{\Psi(I(s))}{I(s)}dM_B(s)$ is local continuous martingale with $\mathscr{M}(0)=0$ and $\lim_{t\to\infty}\frac{\mathscr{M}(t)}{t}=0$ $a.s.$ Also, by the ergodic property of the Markov chain, we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left\{\lambda(m(s))\frac{\chi}{\mu}\Psi'(0)-(\mu(m(s))+\rho(m(s))+\vartheta(m(s))+\gamma(m(s)))\right.\\ &\left.-\frac{\eta^2(m(s))}{2}\left(\frac{\chi}{\mu}\Psi'(0)\right)^2\right\}ds\\ =&\sum_{i=1}^d\pi_i\left\{\lambda(i)\frac{\chi}{\mu}\Psi'(0)-(\mu(i)+\rho(i)+\vartheta(i)+\gamma(i))-\frac{\eta^2(i)}{2}\left(\frac{\chi}{\mu}\Psi'(0)\right)^2\right\}.\end{aligned}$$

By taking the limit superior on both sides of expression (6), we obtain

$$\begin{aligned}&\limsup_{t\to\infty}\frac{\ln I(t)}{t}\\ \leq&\sum_{i=1}^d\pi_i\left\{\lambda(i)\frac{\chi}{\mu}\Psi'(0)-(\mu(i)+\rho(i)+\vartheta(i)+\gamma(i))-\frac{\eta^2(i)}{2}\left(\frac{\chi}{\mu}\Psi'(0)\right)^2\right\}\ a.s.\end{aligned}$$

Hence

$$\begin{aligned}&\limsup_{t\to\infty}\frac{\ln I(t)}{t}\\ \leq&\sum_{i=1}^d\pi_i\left((\mu(i)+\rho(i)+\vartheta(i)+\gamma(i))+\frac{\eta^2(i)}{2}\left(\frac{\chi}{\mu}\Psi'(0)\right)^2\right)[\mathscr{R}_{st}-1]<0\ a.s.\end{aligned}$$

Thus, we conclude that $\lim_{t\to\infty}I(t)=0$ $a.s.$

(b) If , in view of (5), we get

$$\begin{aligned}d\ln I(t) &= \left\{\frac{\lambda^2(m(t))}{2\eta^2(m(t))}-(\mu(m(t))+\rho(m(t))+\vartheta(m(t))+\gamma(m(t)))\right.\\ &\left.-\frac{\eta^2(m(t))}{2}\left(S\frac{\Psi(I)}{I}-\frac{\lambda(m(t))}{\eta^2(m(t))}\right)^2\right\}dt+\eta(m(t))\frac{S\Psi(I)}{I}dM_B(t).\end{aligned}$$

Then

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq \sum_{i=1}^{d}\pi_i\frac{\lambda^2(m(t))}{2\eta^2(m(t))}-\sum_{i=1}^{d}\pi_i\left(\mu(m(t))+\rho(m(t))+\vartheta(m(t))+\gamma(m(t))\right),$$

(8)

which implies

$$\lim_{t\to\infty} I(t) = 0\ a.s.$$

□

Remark 3.1. *The condition $\mathscr{R}_{st}<1$ implies the extinction of disease.*

4. Numerical Examples

In this section, we illustrate our theoretical result with the help of numerical simulations. For this, we use the Milstein method (see, [18]), and we consider the Markov chain $\{m(t)\}_{t\geq 0}$ taking values on the state space $\mathbb{L}=1,2$, with a generator defined by:

$$\Phi=\begin{pmatrix}-1 & 1\\ 2 & -2\end{pmatrix}.$$

and the stationary distribution $\pi=(2/3,1/3)$. Figure 1 demonstrates the path of the Markov chain $\{m(t)\}_{t\geq 0}$. Then, we take the parameters in our stochastic system (2) as follows:

$$\chi(1)=1,\lambda(1)=0.05,\mu(1)=0.1,\gamma(1)=0.2,\theta(1)=0.1,\rho(1)=0.1,\vartheta(1)=0.2,\alpha 3=0.1,$$

$$\chi(2)=1,\lambda(2)=0.05,\mu(2)=0.1,\gamma(2)=0.2,\theta(2)=0.1,\rho(2)=0.1,\vartheta(2)=0.2,\alpha 3=0.1,$$

$$\eta(1)=0.1,\eta(2)=0.2$$

By simple computation we get

$$\mathscr{R}_{st} = \frac{\sum_{i=1}^{d}\pi_i\lambda(i)\frac{\chi}{\mu}\Psi'(0)}{\sum_{i=1}^{d}\pi_i\left((\mu(i)+\rho(i)+\vartheta(i)+\gamma(i))+\frac{\eta^2(i)}{2}\left(\frac{\chi}{\mu}\Psi'(0)\right)^2\right)} =< 1.$$

Therefore, from Theorem 3.1. the epidemic in the system (2) will disappear, Figure 2 support this result. To see the noise effects on the model 2, we simulate the model 2 with two groups of

noise values; (i) $\eta(1) = 0.1$, $\eta(2) = 0.12$, (ii) $\eta(1) = 0.3$, $\eta(2) = 0.31$. Figure 3.1. shows the power of environmental noise to stop the propagation of the epidemic in the population. When the noise value increases the epidemic will disappear quickly from the population.

5. CONCLUSION

In this work, we have proposed a stochastic SIS epidemic model with Markovian switching. Then, we have proved the global existence of the positive solution. Also, we have given a threshold value of our stochastic switched system that is used to determine the extinction of disease in the population. Exactly,

- If $\mathscr{R}_{st} < 1$, then

$$\begin{aligned} &\limsup_{t\to\infty} \frac{\ln I(t)}{t} \\ \leq\ & \sum_{i=1}^{d} \pi_i \left((\mu(i) + \rho(i) + \vartheta(i) + \gamma(i)) + \frac{\eta^2(i)}{2} \left(\frac{\chi}{\mu} \Psi'(0) \right)^2 \right) [\mathscr{R}_{st} - 1] < 0 \ a.s. \end{aligned}$$

  This means that $I(t)$ will extinct with probability one.
- If $\sum_{i=1}^{d} \pi_i \frac{\lambda^2(m(t))}{2\eta^2(m(t))} \leq \sum_{i=1}^{d} \pi_i (\mu(m(t)) + \rho(m(t)) + \vartheta(m(t)) + \gamma(m(t)))$, then the disease in system (2) dies out with probability one.

From our analytical and numerical study, we found that the integration of random noises in the deterministic system makes them more realistic and can control more rapidly the spread of the epidemic in the population (see, Figures 2 and 3). In perspective, we want to make the model (2) more realistic. For this, we will integrate the Lévy jumps into the system to represent the massive and grave events caused by natural disasters and pandemic, etc.

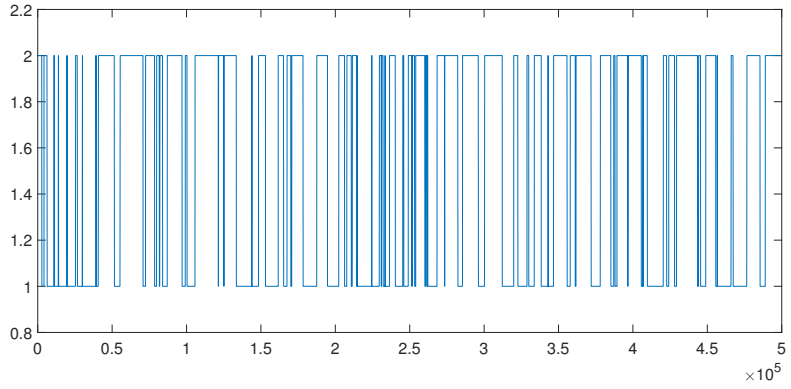

FIGURE 1. Path of Markov chain $\{m(t)\}_{t\geq 0}$.

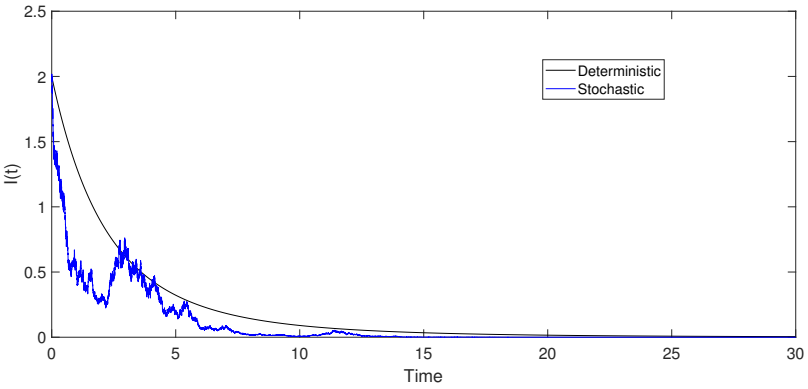

FIGURE 2. Path of $I(t)$ with $\eta(1) = 0.1$ and $\eta(2) = 0.2$.

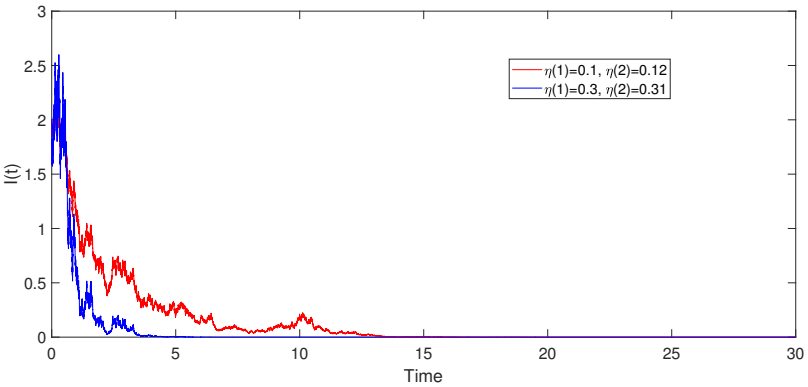

FIGURE 3. Path of $I(t)$ with different noise value.

DATA AVAILABILITY

The data used to support the findings of this study are included within the article.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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