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THE INFLUENCE OF FEAR ON THE DYNAMICS OF A PREY-PREDATOR-SCAVENGER MODEL WITH QUADRATIC HARVESTING

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Abstract: A mathematical model of a Holling type -II food web comprising prey-predator-scavenger is created and investigated in this study. Fear and quadratic harvesting are discussed. The properties of the system's solution are described in detail. All of the potential equilibrium points have been identified. Analytical research is done on local and global stability, persistence, local bifurcation, and Hopf- bifurcation. Numerical simulation is often used to explore the system's global dynamical behavior as well as the effects of altering parameter values. The solution appears to approach either the asymptotic stable point or a Hopf-bifurcation. Furthermore, both fear and harvesting have a stabilizing effect on the system's behavior up to a certain point, and then extinction occurred.

Keywords: food-web; fear; harvesting; stability; local bifurcation; Hopf - bifurcation.

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1. INTRODUCTION

For ecologists and applied mathematics, the modeling study of predator-prey relationships is becoming the most important research issue. The basic predator-prey models of Lotka and Volterra

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are, in fact, mathematical ecology's cornerstones. Several scholars have explored the Lotka and Volterra models thoroughly since they were initially introduced in the early 1900s. Their models have been improved, resulting in more realistic models like [1-4], for example.

Nolting, Paullet, and Previte [5] investigated the effects of incorporating scavengers into a predator-prey system. They've discovered the condition that ensures scavengers' survival. Gupta and Chandra [6] explored the impacts of quadratic harvesting in a Lotka-Volterra type predator-prey system with scavenger species. They discovered a Hopf-bifurcation in the model around the co-existing equilibrium point. Furthermore, the model depicts a period-doubling path to chaos, which can be controlled and made stable by a sufficient quantity of predator harvesting. Later, Satar, and Naji [7] devised and studied a prey-predator-scavenger model with a Michaelis-Menten type of harvesting function. They discovered that the suggested model is quite sensitive to changes in parameter values, particularly those related to the scavenger, and exhibits several sorts of local bifurcation. On the other hand, a Lotka-Volterra food web model that includes scavengers, toxicants, and harvesting is explored by Satar and Naji[8]. They explained that the model is extremely sensitive to changes in parameter values and that it experiences transcritical bifurcation at several equilibrium points. In the literature, there are also few research studies [9-12] that look at predator-prey interactions in the existence of scavenger species.

Predation hazards are well known to have a negative impact on prey biomass and efficiency of the growth, and hence predators have an impact on the structure of food webs. Fear can cause prey to die, and it can also limit productivity, thus fear can be equally as important as predator killing in reducing prey populations [13–14]. Many prey-predator models have been proposed and thoroughly explored, in which the predator kills the prey or the predator's presence changes the behavior of the prey population due to predation fear [15-17]. The impact of prey fear and team defense versus predation on the dynamics of the food-web model was examined by Maghool and Naji [18]. They discovered that the fear factor acts as a system stabilizer up to a certain point, after which it causes the predator to become extinct. However, enhancing the prey's team defense causes predator species to become extinct. Fear has an effect on a stage structure prey-predator system with anti-predator behavior, according to Rahi, Kurnaz, and Naji [19]. Ibrahim, Bahlool, Satar, and Naji [20] suggested and investigated a prey-predator system with a Holling type II functional response, which combines predation fear with a predator-dependent prey's refuge. They discovered that the system is bistable between a limit cycle and a coexistence equilibrium point, however, the fear has a destabilizing effect on the system's dynamics.

Fear's effect on the dynamics of the prey-predator-scavenger model with quadratic refuge is addressed in this work. The formulation of the model is the subject of the next section. Local stability analysis is discussed in section three. The model's persistence criteria are provided in section (4). Section (5), on the other hand, is concerned with global stability. The local bifurcation was examined in Section (6). The Hopf bifurcation, on the other hand, is addressed in section (7). In section (8), there is a numerical simulation. The study's result is presented in section (9).

2. FORMULATION OF THE MODEL

This section deals with the mathematical formulation of a real-world food web system. It is assumed that the food web consists of prey, a predator, and a scavenger that feeds on the prey carcasses of the predator.

According to the Holling type II functional response, the food is assumed to be transferred between the food web levels. However, the growth rate of the prey is affected due to the prey’s fear of the predators. The scavenger consumes the carcasses of the predator whenever they existed according to the linear type of functional response. Finally, there is proportional quadratic harvesting on the predator and scavenger. As a result, the dynamics of a food web system that follows the above assumptions can be represented mathematically using the following set of first-order nonlinear differential equations.

$$\begin{aligned}\frac{dX}{dT} &= \frac{rX}{1+f(Y+Z)} - bX^2 - \frac{a_1XY}{b_1+X} - \frac{a_2XZ}{b_2+X},\\ \frac{dY}{dT} &= \frac{a_3XY}{b_1+X} - d_1Y - q_1E_1Y^2,\\ \frac{dZ}{dT} &= \frac{a_4XZ}{b_2+X} + a_5YZ - d_2Z - q_2E_2Z^2,\end{aligned} \tag{1}$$

where $X(0), Y(0)$, and $Z(0)$ are all positive and denote the population density of prey, the population density of predator, and the population density of scavengers at time T respectively. The description of model parameters can be described in the following table (1).

All the above parameters are positive with $f \geq 0$. The next non-dimensional food web model, which is easier to study and analyze, is created by utilizing the following non-dimensional variables and parameters.

Table 1: parameters description

Parameter	Description
r	The net growth rate
f	The fear rate
b	The intraspecific competition
a_1, a_2	The attack rates of predator and scavenger respectively
a_3, a_4	The conversion rates of prey's biomass to predator and scavenger respectively
b_1, b_2	The half-saturation constant of predator and scavenger respectively
a_5	The conversion rate of the predator's carcasses biomass to scavenger
d_1, d_2	The death rates of predators and scavengers respectively
q_1, q_2	The catchability constants of predator and scavenger respectively
E_1, E_2	The harvesting efforts of predator and scavenger respectively

$$t = rT, x = \frac{b}{r}X, y = \frac{a_1 b}{r^2}Y, z = \frac{a_2 b}{r^2}Z$$
$$w_0 = \frac{r^2 f}{a_2 b}, w_1 = \frac{a_2}{a_1}, w_2 = \frac{b_1 b}{r}, w_3 = \frac{b_2 b}{r}, w_4 = \frac{a_3}{r}, w_5 = \frac{d_1}{r},$$
$$w_6 = \frac{q_1 r E_1}{a_1 b}, w_7 = \frac{a_4}{r}, w_8 = \frac{a_5 r}{a_1 b}, w_9 = \frac{d_2}{r}, w_{10} = \frac{q_2 r E_2}{a_2 b}.$$

However, the non-dimensional food web model that corresponds system (1) takes the form:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[\frac{1}{1+w_0(w_1 y+z)} - x - \frac{y}{w_2+x} - \frac{z}{w_3+x}\right] = x f_1(x,y,z), \\
\frac{dy}{dt} &= y\left[\frac{w_4 x}{w_2+x} - w_5 - w_6 y\right] = y f_2(x,y,z), \\
\frac{dz}{dt} &= z\left[\frac{w_7 x}{w_3+x} + w_8 y - w_9 - w_{10} z\right] = z f_3(x,y,z).
\end{aligned} \tag{2}$$

Obviously, system (2) is defined on the following domain $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$. Moreover, the functions of the right-hand side of the system (2) are continuous and have continuous partial derivatives, hence they are Lipschitz functions. Therefore, the solution of system (2) exists and is unique. Furthermore, the solution is uniformly bounded as discussed in the following theorem.

Theorem (1): The solutions of the system (2) are uniformly bounded.

Proof. Assume that $(x(t), y(t), z(t))$ is any solution of system (2), then from the first equation, it is observed that

$$\frac{dx}{dt} \leq x\left[\frac{1}{1+w_0(w_1 y+z)} - x\right] \leq x(1-x).$$

Hence, $x(t) \leq 1$ as $t \to \infty$. Now, define the function: $N_1(t) = x(t) + \frac{y(t)}{w_4}$, then after some algebraic calculation we have

$$\frac{dN_1}{dt} \leq x - \frac{w_5}{w_4} y \leq (1+w_5)x - w_5\left(x + \frac{y}{w_4}\right).$$

Hence it is getting that

$$\frac{dN_1}{dt} + w_5 N_1 \leq (1+w_5).$$

By using Gronwall inequality, we obtain that $y \leq \frac{w_4(1+w_5)}{w_5} = \beta_1$ as $t \to \infty$.

Now, we define $N_2(t) = x + \frac{y(t)}{w_4} + \frac{z(t)}{w_7}$, then

$$\frac{dN_2}{dt} \leq 2x - x - \frac{w_5}{w_4} y - \frac{(w_9 - w_8\beta_1)}{w_7} z \leq 2 - \mu\left(x + \frac{y}{w_4} + \frac{z}{w_7}\right),$$

where $\mu = min\,\{1, w_5, w_9 - w_8\beta_1\}$. Thus, it is getting that

$$\frac{dN_2}{dt} + \mu N_2 \leq 2 \text{ gives } N_2 \leq \frac{2}{\mu} \text{ as } t \to \infty.$$

Hence all solutions are uniformly bounded and that guarantees their validity.

3. Local Stability Analysis

There are at most five non-negative equilibrium points of the system (2), existing conditions and stability analyses of them are described below:

The evanescence equilibrium point $P_0 = (0,0,0)$ always exists.

The predation-free equilibrium point $P_x = (1,0,0)$ always exists too.

The scavenger-free equilibrium point that denoted by $P_{xy} = (\hat{x}, \hat{y}, 0)$, where

$$\hat{y} = \frac{w_4\hat{x} - w_5(w_2 + \hat{x})}{(w_2 + \hat{x})w_6}, \tag{3}$$

with $\hat{x}$ is a positive root of the fourth-order equation:

$$\gamma_1 x^4 + \gamma_2 x^3 + \gamma_3 x^2 + \gamma_4 x + \gamma_5 = 0,$$

where:

$$\gamma_1 = -w_0 w_1 w_6 (w_4 + w_5) - w_6^2 < 0,$$

$$\gamma_2 = -2w_0 w_1 w_2 w_6 (w_4 - \tfrac{3}{2} w_5) + w_6^2 (1 - 3w_2),$$

$$\gamma_3 = (-w_0 w_1 w_4^2 + 2w_0 w_1 w_4 w_5 - w_0 w_1 w_5^2 - w_4 w_6 - w_0 w_1 w_2^2 w_4 w_6 + w_5 w_6 + 3w_0 w_1 w_2^2 w_5 w_6 + 3w_2 w_6^2 - 3w_2^2 w_6^2),$$

$$\gamma_4 = 2w_0 w_1 w_2 w_5 (w_4 - w_5) - w_2 w_6 (w_4 - 2w_5) + w_0 w_1 w_2^3 w_5 w_6 - w_2^2 w_6^2 (w_2 - 3),$$

$$\gamma_5 = -w_2^2 [w_0 w_1 w_5^2 - w_6 (w_5 + w_2 w_6)]$$

Therefore, P_{xy} exists uniquely in the interior of the positive quadrant of the xy −plane provided that the following sufficient conditions are met.

$$0 < \frac{w_2 w_5}{(w_4 - w_5)} < \hat{x}, \tag{4a}$$

$$w_0 w_1 w_5^2 < w_6 (w_5 + w_2 w_6), \tag{4b}$$

$$\gamma_2 < 0, \tag{4c}$$

$$\gamma_4 > 0. \tag{4d}$$

The predator-free equilibrium point $P_{xz} = (\bar{x}, 0, \bar{z})$, where

$$\bar{z} = \frac{w_7 \bar{x} - w_9 (w_3 + \bar{x})}{(w_3 + \bar{x}) w_{10}}, \tag{5}$$

with $\bar{x}$ is a positive root of the fourth-order equation:

$$\omega_1 z^4 + \omega_2 z^3 + \omega_3 z^2 + \omega_4 z + \omega_5 = 0,$$

where:

$$\omega_1 = -w_0 w_{10} (w_7 - w_9) - w_{10}^2 < 0,$$

$$\omega_2 = (-2w_0 w_3 w_7 w_{10} + 3w_0 w_3 w_9 w_{10} + w_{10}^2 - 3w_3 w_{10}^2),$$

$$\omega_3 = (-w_0 w_7^2 + 2w_0 w_7 w_9 - w_0 w_9^2 - w_7 w_{10} - w_0 w_3^2 w_7 w_{10} + w_9 w_{10} + 3w_0 w_3^2 w_9 w_{10} + 3w_3 w_{10}^2 - 3w_3^2 w_{10}^2),$$

$$\omega_4 = (2w_0w_3w_7w_9 - 2w_0w_3w_9^2 - w_3w_7w_{10} + 2w_3w_9w_{10} + w_0w_3^3w_9w_{10} + 3w_3^2w_{10}^2 - w_3^3w_{10}^2),$$
$$\omega_5 = -w_3^2\,[w_0w_9^2 - w_{10}(w_9 + w_3w_{10})].$$

Accordingly, P_{xz} exists uniquely in the interior of the positive quadrant of the $xz-$plane provided that the following sufficient conditions are met.

$$0 < \frac{w_9w_3}{(w_7-w_9)} < \bar{x}, \tag{6a}$$

$$w_0w_9^2 < w_{10}(w_9 + w_3w_{10}), \tag{6b}$$

$$\omega_2 < 0, \tag{6c}$$

$$\omega_4 > 0. \tag{6d}$$

The coexistence equilibrium point $P_{xyz} = (x^*, y^*, z^*)$, where

$$\left.\begin{aligned} y^* &= \frac{(w_4-w_5)x^* - w_2w_5}{w_6(w_2+x^*)} \\ z^* &= \frac{1}{w_{10}}\left[\frac{w_7x^*}{w_3+x^*} + w_8y^* - w_9\right] \end{aligned}\right\}, \tag{7}$$

with x^* is a positive root of the seven-order equation:

$$\delta_1x^7 + \delta_2x^6 + \delta_3x^5 + \delta_4x^4 + \delta_5x^3 + \delta_6x^2 + \delta_7x + \delta_8 = 0,$$

where: $\delta_1 = w_0w_6w_{10}(w_1w_{10} + w_8)(w_4 - w_5) + {w_{10}}^2{w_6}^2 + w_0w_{10}{w_6}^2(w_7 - w_9) > 0,$

$$\delta_8 = {w_{10}}^2{w_2}^2{w_3}^3w_5(w_0w_1w_5 - w_6) + w_{10}{w_2}^3{w_3}^2(w_0w_1{w_5}^2w_8 - w_{10}{w_6}^2) + {w_2}^3{w_3}^2w_5w_8w_{10}(w_0w_5 - w_6) + w_0{w_2}^3{w_3}^2{w_5}^2{w_8}^2 + w_0w_1w_{10}{w_2}^3{w_3}^2w_5w_6w_9 + w_{10}{w_2}^2{w_3}^2w_6w_9(w_0w_3w_5 - w_{10}w_2w_6) + 2w_0{w_2}^3{w_3}^2w_5w_6w_8w_9 + w_0{w_2}^3{w_3}^2{w_6}^2{w_9}^2,$$

while, $\delta_i; i = 2,3,4,5,6,7$ are complicated functions of systems' parameters.

Accordingly, P_{xyz} exists uniquely in the interior of the positive octant ($int.\,\mathbb{R}_+^3$) provided that the following sufficient conditions are met.

$$0 < \frac{w_2w_5}{(w_4-w_5)} < x^*, \tag{8a}$$

$$w_9 < w_7, \tag{8b}$$

with one set of the following sets of conditions:

$$\left.\begin{aligned} &\delta_2 > 0, \delta_3 > 0, \delta_4 > 0, \delta_5 > 0, \delta_6 < 0, \delta_7 < 0, \delta_8 < 0, \\ &\delta_2 > 0, \delta_3 > 0, \delta_4 > 0, \delta_5 < 0, \delta_6 < 0, \delta_7 < 0, \delta_8 < 0, \\ &\delta_2 > 0, \delta_3 > 0, \delta_4 > 0, \delta_5 > 0, \delta_6 > 0, \delta_7 > 0, \delta_8 < 0, \\ &\delta_2 < 0, \delta_3 < 0, \delta_4 < 0, \delta_5 < 0, \delta_6 < 0, \delta_7 < 0, \delta_8 < 0. \end{aligned}\right\} \tag{8c}$$

For the stability analysis of the above equilibrium points, consider the Jacobian matrix M at the point (x, y, z) that can be written as:

$$M((x, y, z) = \begin{bmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{bmatrix}, \tag{9}$$

where

$$\frac{\partial f_1}{\partial x} = -1 + \frac{y}{(w_2+x)^2} + \frac{z}{(w_3+x)^2}, \ \frac{\partial f_1}{\partial y} = -\frac{w_0 w_1}{(1+w_0(w_1 y+z))^2} - \frac{1}{w_2+x},$$

$$\frac{\partial f_1}{\partial z} = -\frac{w_0}{(1+w_0(w_1 y+z))^2} - \frac{1}{w_3+x}, \ \frac{\partial f_2}{\partial x} = \frac{w_2 w_4}{(w_2+x)^2}, \ \frac{\partial f_2}{\partial y} = -w_6, \ \frac{\partial f_2}{\partial z} = 0,$$

$$\frac{\partial f_3}{\partial x} = \frac{w_3 w_7}{(w_3+x)^2}, \ \frac{\partial f_3}{\partial y} = w_8, \ \frac{\partial f_3}{\partial z} = -w_{10}$$

Now, by substituting the above equilibrium points in Jacobian $M(x, y, z)$ one at a time and then computing their eigenvalues it is observed that:

At the evanescence equilibrium point, the eigenvalues of the matrix $M(P_0)$ are given by (1, $-w_5$, and $-w_9$). As a result, P_0 is a saddle point.

At the predation-free equilibrium point, the eigenvalues of the matrix $M(P_x)$ are determined as:

$$\lambda_{11} = -1, \lambda_{12} = \frac{w_4}{w_2+1} - w_5, \text{ and } \lambda_{13} = \frac{w_7}{w_3+1} - w_9. \tag{10}$$

Consequently, P_x is a locally asymptotically stable if and only if the following conditions are met:

$$w_4 < w_5(w_2 + 1). \tag{11a}$$

$$w_7 < w_9(w_3 + 1). \tag{11b}$$

For the scavenger-free equilibrium point the Jacobian matrix can be written as:

$$M(P_{xy}) = \begin{bmatrix} \hat{x}\left(-1 + \frac{\hat{y}}{(w_2+\hat{x})^2}\right) & -\hat{x}\left(\frac{w_0 w_1}{(1+w_0 w_1 \hat{y})^2} + \frac{1}{w_2+\hat{x}}\right) & -\hat{x}\left(\frac{w_0}{(1+w_0 w_1 \hat{y})^2} + \frac{1}{w_3+\hat{x}}\right) \\ \hat{y}\left(\frac{w_2 w_4}{(w_2+\hat{x})^2}\right) & -w_6\hat{y} & 0 \\ 0 & 0 & \frac{w_7\hat{x}}{w_3+\hat{x}} + w_8\hat{y} - w_9 \end{bmatrix} \tag{12}$$

The characteristic equation of $M(P_{xy})$ can be written as follows:

$$\left[\lambda^2 - T_{xy}\lambda + D_{xy}\right]\left[\frac{w_7\hat{x}}{w_3+\hat{x}} + w_8\hat{y} - w_9 - \lambda\right] = 0, \tag{13}$$

where:

$$T_{xy} = \hat{x}\left(-1 + \frac{\hat{y}}{(w_2+\hat{x})^2}\right) - w_6\hat{y}\ ,$$

$$D_{xy} = -w_6\hat{x}\hat{y}\left(-1 + \frac{\hat{y}}{(w_2+\hat{x})^2}\right) + \hat{x}\hat{y}\left(\frac{w_0w_1}{(1+w_0w_1\hat{y})^2} + \frac{1}{w_2+\hat{x}}\right)\left(\frac{w_2w_4}{(w_2+\hat{x})^2}\right).$$

Therefore, the eigenvalues of $M(P_{xy})$ are determined as $\lambda_{2i} = \frac{T_{xy}}{2} \pm \frac{1}{2}\sqrt{{T_{xy}}^2 - 4D_{xy}}$, for $i = 1,2$ and $\lambda_{23} = \frac{w_7\hat{x}}{w_3+\hat{x}} + w_8\hat{y} - w_9$. Accordingly, all the eigenvalues have negative real parts and then P_{xy} is said to be locally asymptotically stable if and only if the following conditions hold.

$$\hat{x} < \frac{w_3(w_9-w_8\hat{y})}{w_7-(w_9-w_8\hat{y})}. \tag{14a}$$

$$\hat{y} < (w_2+\hat{x})^2 \tag{14b}$$

For the predator-free equilibrium point the Jacobian matrix can be written as:

$$M(P_{xz}) = \begin{bmatrix} \bar{x}\left(-1+\frac{\bar{z}}{(w_3+\bar{x})^2}\right) & -\bar{x}\left(\frac{w_0w_1}{(1+w_0\bar{z})^2}+\frac{1}{w_2+\bar{x}}\right) & -\bar{x}\left(\frac{w_0}{(1+w_0\bar{z})^2}+\frac{1}{w_3+\bar{x}}\right) \\ 0 & \frac{w_4\bar{x}}{w_2+\bar{x}} - w_5 & 0 \\ \bar{z}\left(\frac{w_3w_7}{(w_3+\bar{x})^2}\right) & w_8\bar{z} & -w_{10}\bar{z} \end{bmatrix} \tag{15}$$

The characteristic equation of $M(P_{xz})$ can be written as follows:

$$[\lambda^2 - T_{xz}\lambda + D_{xz}]\left[\frac{w_4\bar{x}}{w_2+\bar{x}} - w_5 - \lambda\right] = 0, \tag{16}$$

where:

$$T_{xz} = \bar{x}\left(-1 + \frac{\bar{z}}{(w_3+\bar{x})^2}\right) - w_{10}\bar{z}\ ,$$

$$D_{xz} = -w_{10}\bar{x}\bar{z}\left(-1 + \frac{\bar{z}}{(w_3+\bar{x})^2}\right) + \bar{x}\bar{z}\left(\frac{w_0}{(1+w_0\bar{z})^2} + \frac{1}{w_3+\bar{x}}\right)\left(\frac{w_3w_7}{(w_3+\bar{x})^2}\right).$$

Therefore, the eigenvalues of $M(P_{xz})$ are determined as $\lambda_{2i} = \frac{T_{xz}}{2} \pm \frac{1}{2}\sqrt{{T_{xz}}^2 - 4D_{xz}}$, for $i = 1,3$ and $\lambda_{22} = \frac{w_4\bar{x}}{w_2+\bar{x}} - w_5$. Accordingly, all the eigenvalues have negative real parts and then P_{xz} is said to be locally asymptotically stable if and only if the following conditions hold.

$$\bar{x} < \frac{w_2w_5}{w_4-w_5}. \tag{17a}$$

$$\bar{z} < (w_3+\bar{x})^2. \tag{17b}$$

Theorem (2): Suppose that the coexistence equilibrium point of the system (2) exists, then it is a locally asymptotically stable provided that the following sufficient conditions hold

$$\frac{y^*}{(x^*+w_2)^2}+\frac{z^*}{(x^*+w_3)^2}<1. \tag{18a}$$

$$a_{11}a_{22}a_{33}-a_{13}a_{21}a_{33}<0. \tag{18b}$$

Proof: The Jacobian matrix of the system (2) at $P_{xyz}=(x^*,y^*,z^*)$, can be written as:

$$M(P_{xyz})=\left[a_{ij}\right]_{3*3}, \tag{19}$$

where:

$$a_{11}=x^*\left(-1+\frac{y^*}{(w_2+x^*)^2}+\frac{z^*}{(w_3+x^*)^2}\right),\ a_{12}=-x^*\left(\frac{w_0w_1}{(1+w_0(w_1y^*+z^*))^2}+\frac{1}{w_2+x^*}\right),$$

$$a_{13}=-x^*\left(\frac{w_0}{(1+w_0(w_1y^*+z^*))^2}+\frac{1}{w_3+x^*}\right),\ a_{21}=\frac{w_2w_4y^*}{(w_2+x^*)^2},\ a_{22}=-w_6y^*, a_{23}=0,$$

$$a_{31}=\frac{w_3w_7z^*}{(w_3+x^*)^2},\ a_{32}=w_8z^*,\ a_{33}=-w_{10}z^*.$$

Therefore the characteristic equation of $M(P_{xyz})$ can be written as:

$$\lambda^3+A\lambda^2+B\lambda+C=0, \tag{20}$$

where

$$A=-(a_{11}+a_{22}+a_{33}),\ B=a_{11}a_{22}-a_{12}a_{21}+a_{11}a_{33}-a_{13}a_{31}+a_{22}a_{33},$$

$$C=-(a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}).$$

with

$$\begin{aligned}\Delta = AB-C = &-(a_{11}+a_{22})[a_{11}a_{22}-a_{12}a_{21}]\\ &-(a_{11}+a_{33})[a_{11}a_{33}-a_{13}a_{31}]-(a_{22}a_{33})[a_{11}+a_{22}+a_{33}]\\ &-a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}.\end{aligned}$$

Note that, an application to the Routh-Hurwitz criterion, which is required that $A>0$, $C>0$, and $\Delta>0$, ensures that all the roots of equation (20) have negative real parts. Direct computation shows that all the requirements of the Routh-Hurwitz criterion are satisfied under sufficient conditions (18a)-(18b). Hence, P_{xyz} is locally asymptotically stable.

4. PERSISTENCE

The survival of all species as time passes without limit is addressed in this section. In the deterministic sense, a species' persistence refers to its continuing existence. Persistence, on the other hand, indicates that $\lim_{t\to\infty}\inf x_i(t)>0$ for each population $x_i(t)$ when $x_i(0)>0$. This means that each of the system's (2) trajectories is eventually restricted away from the coordinate planes. As a result, if each variable $x, y,$ and z survives, system (2) is said to persist. Accordingly, the possibility of the existence of periodic dynamics in the boundary planes is investigated first. There are two subsystems that can be driven from the system (2). These are written as follows:

Subsystem A, which is in the xy −plane, is written as:

$$\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+w_0 w_1 y} - x - \frac{y}{w_2+x}\right] = h_1(x,y), \\ \frac{dy}{dt} &= y\left[\frac{w_4 x}{w_2+x} - w_5 - w_6 y\right] = h_2(x,y). \end{aligned} \tag{21}$$

Subsystem B, which is in the xz −plane, is written as:

$$\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+w_0 z} - x - \frac{z}{w_3+x}\right] = g_1(x,z), \\ \frac{dz}{dt} &= z\left[\frac{w_7 x}{w_3+x} - w_9 - w_{10} z\right] = g_2(x,z). \end{aligned} \tag{22}$$

Consider the Dulac functions as $B_1(x,y) = \frac{1}{xy}$, and $B_2(x,z) = \frac{1}{xz}$, which satisfy $B_i > 0; i = 1,2,$ and C^1 functions in the $Int.\mathbb{R}_+^2$ of the $xy-$ and $xz-$planes respectively. Hence, direct computation shows that

$$\Delta(x,y) = \frac{\partial(B_1\, h_1)}{\partial x} + \frac{\partial(B_1\, h_2)}{\partial y} = -\frac{1}{y} + \frac{1}{(w_2 + x)^2} - \frac{w_6}{x}$$

Then $\Delta(x,y)$ does not identically zero in the $Int.\mathbb{R}_+^2$ of the xy −plane and does not change the sign if and only if the following condition holds:

$$\left.\begin{aligned} &\frac{1}{(w_2+x)^2} > \frac{w_6}{x} + \frac{1}{y} \\ &\qquad OR \\ &\frac{1}{(w_2+x)^2} < \frac{w_6}{x} + \frac{1}{y} \end{aligned}\right\} \tag{23}$$

A similar result is obtained regarding $\Delta(x,z)$ if and only if the following condition holds:

$$\left.\begin{aligned} &\frac{1}{(w_3+x)^2} > \frac{w_{10}}{x} + \frac{1}{z} \\ &\qquad OR \\ &\frac{1}{(w_3+x)^2} < \frac{w_{10}}{x} + \frac{1}{z} \end{aligned}\right\} \tag{24}$$

Thus, under the conditions (23) and (24), there is no closed curve in the $int.\mathbb{R}_+^2$ of the $xy-$ and xz −planes according to the Dulac-Bendixson criterion [20]. As a result, the Poincare-Bendixon theorem [21] states that the unique equilibrium point in the $int.\mathbb{R}_+^2$ of the $xy-$ and xz −planes, as defined by P_{xy} and P_{xz}, is globally asymptotically stable whenever they are locally asymptotically stable.

Theorem (3): If conditions (23)-(24) and the following requirements are met, system (2) is uniformly persistent.

$$w_4 > w_5(w_2 + 1), \tag{25a}$$

$$w_7 > w_9(w_3 + 1), \tag{25b}$$

$$\hat{x} > \frac{w_3(w_9 - w_8\hat{y})}{w_7 - (w_9 - w_8\hat{y})}, \tag{25c}$$

$$\bar{x} > \frac{w_2 w_5}{w_4 - w_5}. \tag{25d}$$

Proof: According to the method of average Lyapunov function [22], define the following function $\varphi(x, y, z) = x^{q_1} y^{q_2} z^{q_3}$, where $q_j, \forall j = 1,2,3$ are positive constants. Hence, $\varphi(x, y, z) > 0$, for all $(x, y, z) \in int. \mathbb{R}_+^3$ and $\varphi(x, y, z) \to 0$ when anyone of their variables approaches zero. Therefore, it is obtained that

$$\Omega(x, y, z) = \frac{\varphi'(x, y, z)}{\varphi(x, y, z)} = q_1 \left[\frac{1}{1 + w_0(w_1 y + z)} - x - \frac{y}{w_2 + x} - \frac{z}{w_3 + x}\right]$$
$$+ q_2 \left[\frac{w_4 x}{w_2 + x} - w_5 - w_6 y\right] + q_3 \left[\frac{w_7 x}{w_3 + x} + w_8 y - w_9 - w_{10} z\right].$$

Now, according to the average Lyapunov function the proof is done if $\Omega(P) > 0$ for any attractor point P in the boundary planes, for a suitable selection of constants $q_i > 0$, $i = 1,2,3$.

Since

$$\Omega(P_0) = q_1 - w_5 q_2 - w_9 q_3,$$

$$\Omega(P_x) = q_2 \left[\frac{w_4}{w_2+1} - w_5\right] + q_3 \left[\frac{w_7}{w_3+1} - w_9\right],$$

$$\Omega(P_{xy}) = q_3 \left[\frac{w_7 \hat{x}}{w_3 + \hat{x}} + w_8 \hat{y} - w_9\right],$$

$$\Omega(P_{xz}) = q_2 \left[\frac{w_4 \bar{x}}{w_2 + \bar{x}} - w_5\right].$$

Then, selecting q_1 to be a sufficiently large value leads to $\Omega(P_0) > 0$. However, $\Omega(P_x) > 0$, $\Omega(P_{xy}) > 0$, and $\Omega(P_{xz}) > 0$ provided that conditions (25a), (25b), (25c), and (25d) hold respectively. Thus the proof is done.

5. GLOBAL STABILITY

The basin of attraction that belongs to $int.\mathbb{R}_+^3$ for each locally asymptotically stable point is determined utilizing the method of the Lyapunov function in this section. The equilibrium point is said to be globally asymptotically stable if its basin of attraction is the $int.\mathbb{R}_+^3$.

Theorem (4): Suppose that P_x is locally asymptotically stable, then it is globally asymptotically stable if the following conditions are met:

$$\frac{w_5}{w_4} > w_0 w_1 + \frac{1}{w_2} \tag{26a}$$

$$\frac{w_9}{w_7} > \frac{w_8}{w_7}\beta_1 + w_0 + \frac{1}{w_3} \tag{26b}$$

Proof: Define the real-valued function, $V_1 = q_1\left(x - \tilde{x} - \tilde{x}\ln\frac{x}{\tilde{x}}\right) + q_2 y + q_3 z$, with $\tilde{x} = 1$. Direct computation shows that $V_1 : U_1 \to \mathbb{R}$, where $U_1 = \{(x,y,z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z \geq 0\}$, so that $V_1(P_x) = 0$, and $V_1(x,y,z) > 0$, for all $(x,y,z) \in U_1 - P_x$. Moreover, straightforward computation gives that:

$$\frac{dV_1}{dt} = q_1 \frac{dx}{dt}\left(\frac{x-\tilde{x}}{x}\right) + q_2\frac{dy}{dt} + q_3\frac{dz}{dt},$$

$$\begin{aligned}\frac{dV_1}{dt} \leq & -q_1(x-\tilde{x})^2 + \frac{q_1 w_0 w_1 \tilde{x} y}{1 + w_0 w_1 y + w_0 z} + \frac{q_1 w_0 \tilde{x} z}{1 + w_0 w_1 y + w_0 z} \\ & -(q_1 - w_4 q_2)\frac{xy}{w_2 + x} - (q_1 - w_7 q_3)\frac{xz}{w_3 + x} + \frac{q_1 \tilde{x} y}{w_2 + x} \\ & + \frac{q_1 \tilde{x} z}{w_3 + x} - q_2 w_5 y + q_3 w_8 y z - q_3 w_9 z.\end{aligned}$$

Selecting the positive constant values as $q_1 = 1, q_2 = \frac{1}{w_4}$, and $q_3 = \frac{1}{w_7}$, then using maximize concept with the upper bound constant β_1 yields that:

$$\frac{dV_1}{dt} \leq -(x-1)^2 - \left(\frac{w_5}{w_4} - w_0 w_1 - \frac{1}{w_2}\right) y - \left(\frac{w_9}{w_7} - \frac{w_8}{w_7}\beta_1 - w_0 - \frac{1}{w_3}\right) z.$$

Hence, conditions (26a), and (26b) give that $\frac{dV_1}{dt} < 0$. Hence, P_x is globally asymptotically stable.

Theorem (5): Suppose that P_{xy} is locally asymptotically stable, then it is globally asymptotically stable if the following conditions are met:

$$\hat{y} < w_2(w_2 + \hat{x}). \tag{27a}$$

$$\alpha_{12}^2 < 4\alpha_{11}\alpha_{22}. \tag{27b}$$

$$\frac{\hat{x}}{w_3} + \frac{w_8}{w_7}\beta_1 < \frac{w_9}{w_7}. \tag{27c}$$

Proof: Define the real-valued function, $V_2 = \alpha_1\left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + \alpha_2\left(y - \hat{y} - \hat{y}\ln\frac{y}{\hat{y}}\right) + \alpha_3 z$. Direct computation shows that $V_2 \colon U_2 \to \mathbb{R}$, where $U_2 = \{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y > 0, z \geq 0\}$, so that $V_2(P_{xy}) = 0$, and $V_2(x, y, z) > 0$, for all $(x, y, z) \in U_2 - P_{xy}$. Moreover, straightforward computation gives that:

$$\frac{dV_2}{dt} = \alpha_1\left(\frac{x-\hat{x}}{x}\right)\frac{dx}{dt} + \alpha_2\left(\frac{y-\hat{y}}{y}\right)\frac{dy}{dt} + \alpha_3\frac{dz}{dt}.$$

Direct calculation and using maximize concept with the upper bound constant β_1 yields that:

$$\begin{aligned}\frac{dV_2}{dt} \leq & -\left[1 - \frac{\hat{y}}{(w_2+x)(w_2+\hat{x})}\right]\alpha_1(x-\hat{x})^2 - w_6\alpha_2(y-\hat{y})^2 \\ & -\left[\frac{\alpha_1 w_0 w_1}{(1+w_0w_1y+w_0z)(1+w_0w_1\hat{y})} + \frac{\alpha_1(w_2+\hat{x}) - w_2w_4\alpha_2}{(w_2+x)(w_2+\hat{x})}\right](x-\hat{x})(y-\hat{y}) \\ & -(\alpha_1 - w_7\alpha_3)\frac{xz}{w_3+x} - \left(w_9\alpha_3 - \frac{\alpha_1\hat{x}}{w_3} - w_8\alpha_3\beta_1\right)z.\end{aligned}$$

Now, selecting the positive constant values as $\alpha_1 = 1, \alpha_2 = \frac{w_2+\hat{x}}{w_2w_4}$, and $\alpha_3 = \frac{1}{w_7}$ gives:

$$\begin{aligned}\frac{dV_2}{dt} \leq & -\left[1 - \frac{\hat{y}}{(w_2+x)(w_2+\hat{x})}\right](x-\hat{x})^2 - w_6\frac{w_2+\hat{x}}{w_2w_4}(y-\hat{y})^2 \\ & -\left[\frac{w_0w_1}{(1+w_0w_1y+w_0z)(1+w_0w_1\hat{y})}\right](x-\hat{x})(y-\hat{y}) \\ & -\left(\frac{w_9}{w_7} - \frac{\hat{x}}{w_3} - \frac{w_8}{w_7}\beta_1\right)z.\end{aligned}$$

Using conditions (27a)-(27b) it is obtained that:

$$\frac{dV_2}{dt} \leq -[\sqrt{\alpha_{11}}(x-\hat{x}) + \sqrt{\alpha_{22}}(y-\hat{y})]^2 - \left[\frac{w_9}{w_7} - \frac{\hat{x}}{w_3} - \frac{w_8}{w_7}\beta_1\right]z,$$

where:

$$\alpha_{11} = 1 - \frac{\hat{y}}{(w_2+x)(w_2+\hat{x})},\ \alpha_{12} = \frac{w_0w_1}{(1+w_0w_1y+w_0z)(1+w_0w_1\hat{y})},\ \alpha_{22} = w_6\frac{w_2+\hat{x}}{w_2w_4}.$$

Hence, condition (27c) gives that $\frac{dV_2}{dt} < 0$. Hence, P_{xy} is globally asymptotically stable.

Theorem (6): Suppose that P_{xz} is locally asymptotically stable, then it is globally asymptotically stable if the following conditions are met:

$$\bar{z} < w_3(w_3 + \bar{x}). \tag{28a}$$

$$\frac{w_0w_1\bar{x}}{1+w_0\bar{z}} + \frac{\bar{x}}{w_2} < \frac{w_5}{w_4}. \tag{28b}$$

$$k_{13}^2 < 2k_{11}k_{33}. \tag{28c}$$

$k_{23}^2 < 2k_{22}k_{33}$. (28d)

Proof: Define the real-valued function, $V_3 = \rho_1\left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + \rho_2 y + \rho_3\left(z - \bar{z} - \bar{z}\ln\frac{z}{\bar{z}}\right)$. Direct computation shows that $V_3: U_3 \to \mathbb{R}$, where $U_3 = \{(x,y,z) \in \mathbb{R}_+^3: x > 0, y \geq 0, z > 0\}$, so that $V_3(P_{xz}) = 0$, and $V_3(x,y,z) > 0$, for all $(x,y,z) \in U_3 - P_{xz}$. Moreover, straightforward computation gives that:

$$\frac{dV_3}{dt} = \rho_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt} + \rho_2\frac{dy}{dt} + \rho_3\left(\frac{z-\bar{z}}{z}\right)\frac{dz}{dt}.$$

Similarly, direct calculation using maximize concept yields that:

$$\begin{aligned}\frac{dv_3}{dt} \leq & -\rho_1\left[1 - \frac{\bar{z}}{(w_3+x)(w_3+\bar{x})}\right](x-\bar{x})^2 - \rho_3 w_{10}(z-\bar{z})^2 - \rho_2 w_6 y^2 \\ & -\left[\frac{\rho_1 w_0}{(1+w_0w_1y+w_0z)(1+w_0\bar{z})} + \frac{\rho_1(w_3+\bar{x}) - \rho_3 w_3 w_7}{(w_3+x)(w_3+\bar{x})}\right](x-\bar{x})(z-\bar{z}) \\ & -(\rho_1 - w_4\rho_2)\frac{xy}{w_2+x} - \left[\rho_2 w_5 - \frac{\rho_1 w_0 w_1 \bar{x}}{(1+w_0\bar{z})} - \frac{\rho_1\bar{x}}{w_2}\right]y + \rho_3 w_3 y(z-\bar{z}).\end{aligned}$$

Now, selecting the positive constant values as $\rho_1 = 1, \rho_2 = \frac{1}{w_4}$, and $\rho_3 = \frac{w_3+\bar{x}}{w_3\, w_7}$ leads to:

$$\begin{aligned}\frac{dv_3}{dt} \leq & -\left[1 - \frac{\bar{z}}{(w_3+x)(w_3+\bar{x})}\right](x-\bar{x})^2 - w_{10}\frac{w_3+\bar{x}}{w_3\, w_7}(z-\bar{z})^2 - \frac{w_6}{w_4}y^2 \\ & -\left[\frac{w_0}{(1+w_0w_1y+w_0z)(1+w_0\bar{z})}\right](x-\bar{x})(z-\bar{z}) \\ & -\left[\frac{w_5}{w_4} - \frac{w_0w_1\bar{x}}{(1+w_0\bar{z})} - \frac{\bar{x}}{w_2}\right]y + w_3\frac{w_3+\bar{x}}{w_3\, w_7}y(z-\bar{z}).\end{aligned}$$

Using conditions (28a), (28c), and (28d) it is obtained that:

$$\begin{aligned}\frac{dv_3}{dt} \leq & -\left[\sqrt{k_{11}}(x-\bar{x}) + \sqrt{\frac{k_{33}}{2}}(z-\bar{z})\right]^2 - \left[\sqrt{k_{22}}y - \sqrt{\frac{k_{33}}{2}}(z-\bar{z})\right]^2 \\ & -\left[\frac{w_5}{w_4} - \frac{w_0w_1\bar{x}}{1+w_0\bar{z}} - \frac{\bar{x}}{w_2}\right]y,\end{aligned}$$

where:

$$k_{11} = 1 - \frac{\bar{z}}{(w_3+x)(w_3+\bar{x})},\ k_{22} = \frac{w_6}{w_4},\ k_{33} = w_{10}\frac{w_3+\bar{x}}{w_3\, w_7},$$

$$k_{13} = \frac{w_0}{(1+w_0w_1y+w_0z)(1+w_0\bar{z})},\ k_{23} = w_3\frac{w_3+\bar{x}}{w_3\, w_7}.$$

Hence, condition (28b) gives that $\frac{dV_3}{dt} < 0$. Hence, P_{xz} is globally asymptotically stable.

Theorem (7): Suppose that P_{xyz} is locally asymptotically stable, then it is globally asymptotically stable if the following conditions are met:

$$\frac{y^*}{\Lambda_2{\Lambda_2}^*}+\frac{z^*}{\Lambda_3{\Lambda_3}^*}<1. \tag{29a}$$

$$(l_{12})^2<l_{11}l_{22}. \tag{29b}$$

$$(l_{13})^2<l_{11}l_{33}. \tag{29c}$$

$$(l_{23})^2<l_{22}l_{33}. \tag{29d}$$

Proof: Define the real-valued function, $\mathrm{V}_4=\left(x-x^*-x^*\ln\frac{\mathrm{x}}{x^*}\right)+\left(y-y^*-y^*\,ln\frac{y}{y^*}\right)+\left(z-z^*-z^*\ln\frac{\mathrm{z}}{z^*}\right)$. Direct computation shows that $\mathrm{V}_4\colon U_4\to\mathbb{R}$, where $U_4=\{(x,y,z)\in\mathbb{R}_+^3\colon x>0,y>0,z>0\}$, so that $\mathrm{V}_4(P_{xyz})=0$, and $\mathrm{V}_4(x,y,z)>0$, for all $(x,y,z)\in U_4-P_{xyz}$. Moreover, straightforward computation gives that:

$$\frac{d\mathrm{V}_4}{dt}=\gamma_1\left(\frac{x-x^*}{x}\right)\frac{dx}{dt}+\gamma_2\left(\frac{y-y^*}{y}\right)\frac{dy}{dt}+\gamma_3\left(\frac{z-z^*}{z}\right)\frac{dz}{dt}.$$

Straightforward calculation yields that

$$\frac{d\mathrm{V}_4}{dt}=-\left[1-\frac{y^*}{\Lambda_2{\Lambda_2}^*}-\frac{z^*}{\Lambda_3{\Lambda_3}^*}\right](x-x^*)^2-w_6(y-y^*)^2-w_{10}(z-z^*)^2$$
$$-\left[\frac{w_0w_1}{\Lambda_1{\Lambda_1}^*}+\frac{{\Lambda_2}^*+w_2w_4}{\Lambda_2{\Lambda_2}^*}\right](x-x^*)(y-y^*)-\left[\frac{w_0}{\Lambda_1{\Lambda_1}^*}+\frac{{\Lambda_3}^*+w_3w_7}{\Lambda_3{\Lambda_3}^*}\right](x-x^*)(z-z^*)$$
$$+w_8(y-y^*)(z-z^*),$$

where:

$$\Lambda_1=(1+w_0w_1y+w_0z),\ {\Lambda_1}^*=(1+w_0w_1y^*+w_0z^*),\ {\Lambda_2}^*=(w_2+x^*),$$
$$\Lambda_2=(w_2+x),\ {\Lambda_2}^*=(w_2+x^*),\ \Lambda_3=(w_3+x),\ {\Lambda_3}^*=(w_3+x^*).$$

Using conditions (29b), (29c), and (29d) it is obtained that:

$$\frac{d\mathrm{V}_4}{dt}\le-\frac{1}{2}\left[\sqrt{l_{11}}(x-x^*)+\sqrt{l_{22}}(y-y^*)\right]^2-\frac{1}{2}\left[\sqrt{l_{11}}(x-x^*)+\sqrt{l_{33}}(z-z^*)\right]^2$$
$$-\frac{1}{2}\left[\sqrt{l_{22}}(y-y^*)-\sqrt{l_{33}}(z-z^*)\right]^2,$$

where:

$$l_{11}=1-\frac{y^*}{\Lambda_2{\Lambda_2}^*}-\frac{z^*}{\Lambda_3{\Lambda_3}^*},\ l_{22}=w_6,\ l_{33}=w_{10},\ l_{12}=\frac{w_0w_1}{\Lambda_1{\Lambda_1}^*}+\frac{{\Lambda_2}^*+w_2w_4}{\Lambda_2{\Lambda_2}^*},$$
$$l_{13}=\frac{w_0}{\Lambda_1{\Lambda_1}^*}+\frac{{\Lambda_3}^*+w_3w_7}{\Lambda_3{\Lambda_3}^*},\ l_{23}=w_8.$$

Hence, condition (29a) gives that $\frac{d\mathrm{V}_4}{dt}<0$. Hence, P_{xyz} is globally asymptotically stable.

6. The Local Bifurcation Analysis

An application of Sotomayor's theorem [21] is employed in this section to analyze the occurrence of local bifurcation around the system's plausible stable equilibrium locations (2). Because the presence of a non-hyperbolic equilibrium point is a required but not sufficient condition for bifurcation, a parameter that causes the Jacobian matrix to have a zero real part eigenvalue will be used as a candidate bifurcation parameter, as proven in the theorems below.

System (2) should be rewritten as $\frac{\mathrm{dX}}{\mathrm{dt}} = F(X)$, where $X = (x, y, z)^T$, and $F(x) = (xf_1, yf_2, zf_3)^T$. The second directional derivative of the general Jacobian matrix can thus be represented as follows for the general vector $V = (v_1, v_2, v_3)^T$:

$$D^2F(V,V) = [c_{i1}]_{3\times 1} \tag{30}$$

where:

$$c_{11} = \left[-2 + \frac{2w_2 y}{(w_2+x)^3} + \frac{2w_3 z}{(w_3+x)^3}\right] v_1^2 - 2\left[\frac{w_0 w_1}{(1+w_0w_1y+w_0z)^2} + \frac{w_2}{(w_2+x)^2}\right] v_1 v_2$$
$$-2\left[\frac{w_0}{(1+w_0w_1y+w_0z)^2} + \frac{w_3}{(w_3+x)^2}\right] v_1 v_3 + 2\left[\frac{w_0^2 w_1^2 x}{(1+w_0w_1y+w_0z)^3}\right] v_2^2$$
$$+4\left[\frac{w_0^2 w_1 x}{(1+w_0w_1y+w_0z)^3}\right] v_2 v_3 + 2\left[\frac{w_0^2 x}{(1+w_0w_1y+w_0z)^3}\right] v_3^2.$$

$$c_{21} = -2\left[\frac{w_2 w_4 y}{(w_2+x)^3}\right] v_1^2 + 2\left[\frac{w_2 w_4}{(w_2+x)^2}\right] v_1 v_2 - 2w_6 v_2^2.$$

$$c_{31} = -2\left[\frac{w_3 w_7 z}{(w_3+x)^3}\right] v_1^2 + 2\left[\frac{w_3 w_7}{(w_3+x)^2}\right] v_1 v_3 + 2w_8 v_2 v_3 - 2w_{10} v_3^2.$$

Theorem (8): If condition (11a) holds, the system (2) will exhibit a Transcritical bifurcation around the predation-free equilibrium point $P_x = (1,0,0)$ when the parameter w_9 is equal to $\frac{w_7}{w_3+1} = \widetilde{w}_9$.

Proof. The Jacobian matrix of the system (2) at P_x with $w_9 = \widetilde{w}_9$ can be written in the form:

$$M(P_x, \widetilde{w}_9) = \begin{pmatrix} -1 & -\frac{1}{w_2+1} - w_0 w_1 & -\frac{1}{w_3+1} - w_0 \\ 0 & \frac{w_4}{w_2+1} - w_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the eigenvalues of $M(P_x, \widetilde{w}_9)$, are given by $\lambda_{11} = -1$, $\lambda_{12} = \frac{w_4}{w_2+1} - w_5 < 0$ under condition (11a), and $\tilde{\lambda}_{13} = 0$.

Let $\tilde{V} = (v_{11}, v_{12}, v_{13})^T$ be the eigenvector of $M(P_x, \widetilde{w}_9)$ associated with $\tilde{\lambda}_{13} = 0$, then it is obtained that $\tilde{V} = (\tau_1 v_{13}, 0, v_{13})^T$, where $v_{13} \neq 0$, $v_{13} \in \mathbb{R}$, and $\tau_1 = -\frac{1}{w_3+1} - w_0$.

Let $\widetilde{\Psi} = (\Psi_{11}, \Psi_{12}, \Psi_{13})^T$, be the eigenvector of $M(P_x, \widetilde{w}_9)^T$ associated with $\tilde{\lambda}_{13} = 0$, then it is obtained that $\widetilde{\Psi} = (0,0,\Psi_{13})$, where $\Psi_{13} \neq 0$, $\Psi_{13} \in \mathbb{R}$.

Since $\frac{\partial F}{\partial w_9} = F_{w_9} = (0,0,-z)^T$, that leads $F_{w_9}(P_x, \widetilde{w}_9) = (0,0,0)^T$.

Therefore, $\widetilde{\Psi}^T[F_{w_9}(P_x, \widetilde{w}_9)] = 0$. Also, direct computation shows that:

$$DF_{w_9}(P_x, \widetilde{w}_9) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow DF_{w_9}(P_x, \widetilde{w}_9)\, \tilde{V} = (0,0,-v_{13})^T,$$

Then $\widetilde{\Psi}^T\left[DF_{w_9}(P_x, \widetilde{w}_9)\, \tilde{V}\right] = -\Psi_{13} v_{13} \neq 0$.

Now, from equation (30) it is observed that

$$D^2\left[F(P_x, \widetilde{w}_9)\left(\tilde{V}, \tilde{V}\right)\right] = \begin{pmatrix} -2\tau_1^2 v_{13}^2 - 2\left[w_0 + \frac{w_3}{(w_3+1)^2}\right]\tau_1 v_{13}^2 + 2w_0^2 v_{13}^2 \\ 0 \\ 2\frac{w_3 w_7}{(w_3+1)^2}\tau_1 v_{13}^2 + 2w_{10} v_{13}^2 \end{pmatrix}.$$

Hence:

$$\widetilde{\Psi}^T\left[F(P_x, \widetilde{w}_9)\left(\tilde{V}, \tilde{V}\right)\right] = 2\frac{w_3 w_7}{(w_3+1)^2}\tau_1 v_{13}^2 \Psi_{13} + 2w_{10} v_{13}^2\, \Psi_{13} \neq 0,$$

which means a Transcritical bifurcation occurs in the sense of Sotomayor theorem and the proof is done.

Theorem (9): If condition (14b) holds, the system (2) will exhibit a Transcritical bifurcation around the scavenger-free equilibrium point $P_{xy} = (\hat{x}, \hat{y}, 0)$ when the parameter w_9 is equal to $\widehat{w}_9 = \frac{w_7 \hat{x}}{(w_3+\hat{x})} + w_8 \hat{y}$.

Proof. The Jacobian matrix of the system (2) at P_{xy} with $w_9 = \widehat{w}_9$ can be written in the form:

$$M\left(P_{xy}, \widehat{w}_9\right) = \begin{bmatrix} -\hat{x} + \frac{\hat{x}\hat{y}}{(w_2+\hat{x})^2} & -\left(\frac{w_0 w_1 \hat{x}}{(1+w_0 w_1 \hat{y})^2} + \frac{\hat{x}}{w_2+\hat{x}}\right) & -\left(\frac{w_0 \hat{x}}{(1+w_0 w_1 \hat{y})^2} + \frac{\hat{x}}{w_3+\hat{x}}\right) \\ \frac{w_2 w_4 \hat{y}}{(w_2+\hat{x})^2} & -w_6 \hat{y} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left(b_{ij}\right).$$

Under the condition (14b), $M(P_{xy}, \widetilde{w}_9)$ has two negative real parts eigenvalues, whereas $\hat{\lambda}_{23} = 0$ gives the third eigenvalue.

Let $\hat{V} = (v_{21}, v_{22}, v_{23})^T$ be the eigenvector of $M\left(P_{xy}, \widehat{w}_9\right)$ associated with $\hat{\lambda}_{23} = 0$, then it is obtained that $\hat{V} = (\tau_2 v_{23}, \tau_3 v_{23}, v_{23})^T$, where $v_{23} \neq 0$, $v_{23} \in \mathbb{R}$, and $\tau_2 = -\frac{b_{13}b_{22}}{b_{11}b_{22}-b_{12}b_{21}} < 0$, $\tau_3 = \frac{b_{13}b_{21}}{b_{11}b_{22}-b_{12}b_{21}} < 0$.

Let $\widehat{\Psi} = (\Psi_{21}, \Psi_{22}, \Psi_{23})^T$, be the eigenvector of $\left(P_{xy}, \widehat{w}_9\right)^T$ associated with $\hat{\lambda}_{23} = 0$, then it is obtained that $\widehat{\Psi} = (0,0,\Psi_{23})$, where $\Psi_{23} \neq 0$, $\Psi_{23} \in \mathbb{R}$.

Since $\frac{\partial F}{\partial w_9} = F_{w_9} = (0,0,-z)^T$, that leads $F_{w_9}(P_{xy}, \widehat{w}_9) = (0,0,0)^T$.

Therefore, $\widehat{\Psi}^T[F_{w_9}(P_{xy}, \widehat{w}_9)] = 0$. Also, direct computation shows that:

$$DF_{w_9}(P_{xy}, \widehat{w}_9) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Longrightarrow DF_{w_9}(P_{xy}, \widehat{w}_9)\, \hat{V} = (0,0,-v_{23})^T,$$

Then $\widehat{\Psi}^T\left[DF_{w_9}(P_{xy}, \widehat{w}_9)\, \hat{V}\right] = -\Psi_{23} v_{23} \neq 0$.

Now, from equation (30) it is observed that

$$D^2F\left(P_{xy}, \widehat{w}_9\right)\left(\hat{V}, \hat{V}\right) = (\hat{c}_{i1})_{3\times 1}$$

where:

$$\begin{aligned}\hat{c}_{11} = {} & \left[-2 + \frac{2w_2\hat{y}}{(w_2+\hat{x})^3}\right](\tau_2 v_{23})^2 - 2\left[\frac{w_0 w_1}{(1+w_0 w_1\hat{y})^2} + \frac{w_2}{(w_2+\hat{x})^2}\right]\tau_2\tau_3(v_{23})^2 \\ & -2\left[\frac{w_0}{(1+w_0 w_1\hat{y})^2} + \frac{w_3}{(w_3+\hat{x})^2}\right]\tau_2(v_{23})^2 + 2\left[\frac{w_0^2 w_1^2\hat{x}}{(1+w_0 w_1\hat{y})^2}\right](\tau_3 v_{23})^2 \\ & +4\left[\frac{w_0^2 w_1\hat{x}}{(1+w_0 w_1\hat{y})^3}\right]\tau_3(v_{23})^2 + 2\left[\frac{w_0^2\hat{x}}{(1+w_0 w_1\hat{y})^3}\right](v_{23})^2.\end{aligned}$$

$$\hat{c}_{21} = -2\left[\frac{w_2 w_4\hat{y}}{(w_2+\hat{x})^3}\right](\tau_2 v_{23})^2 + 2\left[\frac{w_2 w_4}{(w_2+\hat{x})^2}\right]\tau_2\tau_3(v_{23})^2 - 2w_6(\tau_3 v_{23})^2.$$

$$\hat{c}_{31} = 2\left[\frac{w_3 w_7}{(w_3+\hat{x})^2}\right]\tau_2(v_{23})^2 + 2w_8\tau_3(v_{23})^2 - 2w_{10}(v_{23})^2.$$

Therefore, it is obtained that:

$$\widehat{\Psi}^T\left[D^2F\left(P_{xy},\widehat{w}_9\right)\left(\widehat{V},\widehat{V}\right)\right] = \Psi_{23}(v_{23})^2\left[2\left[\frac{w_3w_7}{(w_3+\hat{x})^2}\right]+2w_8\tau_3-2w_{10}\right].$$

This gives that $\Psi^{\mathrm{T}}D^2F(P_2,\widehat{w}_9)(V,V)\neq 0$, due to the fact that $\tau_2<0$, and $\tau_3<0$.

Hence a Transcritical bifurcation occurs near P_{xy} when $w_9=\widehat{w}_9$.

Theorem (10): If condition (17b) holds, the system (2) will exhibit a Transcritical bifurcation around the predator-free equilibrium point $P_{xz}=(\bar{x},0,\bar{z})$ when the parameter w_5 is equal to $\overline{w}_5=\frac{w_4\bar{x}}{(w_2+\bar{x})}$.

Proof. The Jacobian matrix of the system (2) at P_{xz} with $w_5=\overline{w}_5$ can be written in the form:

$$M(P_{xz},\overline{w}_5\,)=\begin{bmatrix} -\bar{x}+\frac{\overline{\bar{x}\,\bar{z}}}{(w_3+\bar{x})^2} & -\left(\frac{w_0w_1\bar{x}}{(1+w_0\bar{z})^2}+\frac{\bar{x}}{w_2+\bar{x}}\right) & -\left(\frac{w_0\bar{x}}{(1+w_0\bar{z})^2}+\frac{\bar{x}}{w_3+\bar{x}}\right) \\ 0 & 0 & 0 \\ \frac{w_3w_7\bar{z}}{(w_3+\bar{x})^2} & w_8\bar{z} & -w_{10}\bar{z} \end{bmatrix}=\left(c_{ij}\right).$$

Clearly, under the condition (17b), $M(P_{xz},\overline{w}_5\,)$ has two negative real parts eigenvalues, whereas $\bar{\lambda}_{32}=0$ gives the third eigenvalue.

Let $\bar{V}=(v_{31},v_{32},v_{33})^T$ be the eigenvector of $M(P_{xz},\overline{w}_5\,)$ associated with $\bar{\lambda}_{32}=0$, then it is obtained that $\bar{V}=(\tau_4v_{32},v_{32},\tau_5v_{32})^T$, where $v_{32}\neq 0$, $v_{32}\in\mathbb{R}$, and $\tau_4=-\frac{c_{13}c_{32}-c_{12}c_{33}}{c_{11}c_{33}-c_{13}c_{31}}<0$, $\tau_5=\frac{c_{12}c_{31}-c_{11}c_{32}}{c_{11}c_{33}-c_{13}c_{31}}$.

Let $\bar{\Psi}=(\Psi_{31},\Psi_{32},\Psi_{33})^T$, be the eigenvector of $M(P_{xz},\overline{w}_5\,)^T$ associated with $\bar{\lambda}_{32}=0$, then it is obtained that $\bar{\Psi}=(0,\Psi_{32},0)$, where $\Psi_{32}\neq 0$, $\Psi_{32}\in\mathbb{R}$.

Now since, $\frac{\partial F}{\partial w_5}=F_{w_5}=(0,-y,0)^T$, then $F_{w_5}(P_{xz},\overline{w}_5)=(0,0,0)^T$. Accordingly, it is obtained that $\bar{\Psi}^T[F_{w_5}(P_{xz},\overline{w}_5)]=0$. Furthermore, since

$$DF_{w_5}(P_{xz},\overline{w}_5)=\begin{pmatrix}0&0&0\\0&-1&0\\0&0&0\end{pmatrix}\Longrightarrow DF_{w_5}(P_{xz},\overline{w}_5)\bar{V}=(0,-v_{32},0)^T.$$

Therefore, $\bar{\Psi}^{\mathrm{T}}\left[DF_{w_5}(P_{xz},\overline{w}_5)\bar{V}\right]=-\Psi_{32}v_{32}\neq 0$.

Finally, from equation (30) it is observed that

$$D^2F(P_{xz},\overline{w}_5)](\bar{V},\bar{V})=(\bar{c}_{i1})_{3\times 1}$$

where

$$\bar{c}_{11} = 2\left[- + \frac{w_3\bar{z}}{(w_3+\bar{x})^3}\right](\tau_4 v_{32})^2 - 2\left[\frac{w_0 w_1}{(1+w_0\bar{z})^2} + \frac{w_2}{(w_2+\bar{x})^2}\right]\tau_4(v_{32})^2$$
$$-2\left[\frac{w_0}{(1+w_0\bar{z})^2} + \frac{w_3}{(w_3+\bar{x})^2}\right]\tau_4\tau_5(v_{32})^2 + 2\left[\frac{w_0^2 w_1^2\bar{x}}{(1+w_0\bar{z})^3}\right](v_{32})^2$$
$$+4\left[\frac{w_0^2 w_1\bar{x}}{(1+w_0\bar{z})^3}\right]\tau_5(v_{32})^2 + 2\left[\frac{w_0^2\bar{x}}{(1+w_0\bar{z})^3}\right](\tau_5 v_{32})^2.$$
$$\bar{c}_{21} = 2\left[\frac{w_2 w_4}{(w_2+\bar{x})^2}\right]\tau_4(v_{32})^2 - 2w_6(v_{32})^2.$$
$$\bar{c}_{31} = -2\left[\frac{w_3 w_7\bar{z}}{(w_3+\bar{x})^3}\right](\tau_4 v_{32})^2 + 2\left[\frac{w_3 w_7}{(w_3+\bar{x})^2}\right]\tau_4\tau_5(v_{32})^2 + 2w_8\tau_5(v_{32})^2 - 2w_{10}(\tau_5 v_{32})^2.$$

Hence, it results in that

$$\bar{\Psi}^{\mathrm{T}}[D^2F(P_{xz},\bar{w}_5)(\bar{V},\bar{V})] = 2\Psi_{32}(v_{32})^2\left[\left[\frac{w_2 w_4}{(w_2+\bar{x})^2}\right]\tau_4 - w_6\right].$$

This gives that $\bar{\Psi}^{\mathrm{T}}[D^2F(P_{xz},\bar{w}_5)(\bar{V},\bar{V})] \neq 0$ due to the fact that $\tau_4 < 0$.

Hence a Transcritical bifurcation occurs near P_{xz} when $w_5 = \bar{w}_5$.

Theorem (11): The system (2) undergoes a saddle-node bifurcation near the coexistence equilibrium point P_{xyz} when the parameter w_7 passes from the positive value $w_7{}^*$ provided that the following condition is satisfied.

$$1 < \frac{y^*}{(w_2+x^*)^2} + \frac{z^*}{(w_3+x^*)^2}, \tag{31a}$$

$$c_{11}{}^*\tau_8 + c_{21}{}^*\tau_9 + c_{31}{}^* \neq 0, \tag{31b}$$

where $w_7{}^* = \frac{(w_3+x^*)^2}{w_3 z^*}\left[\frac{a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}-a_{12}a_{21}a_{33}}{a_{13}a_{22}}\right]$, and all other symbols are given in the proof.

Proof: Consider the Jacobian matrix of system (2) at $w_7 = w_7{}^*$:

$$M(P_{xyz}, w_7{}^*) = [a_{ij}]_{3*3},$$

where the Jacobian elements are given in equation (19) with $a_{31} = \frac{w_3 w_7{}^* z^*}{(w_3+x^*)^2}$. Then, by using condition (31a), direct computation shows that the determinant of the $M(P_{xyz}, w_7{}^*)$ is equaled to zero. Therefore, $M(P_{xyz}, w_7{}^*)$ has a zero eigenvalue $\lambda^* = 0$ that makes P_{xyz} a nonhyperbolic point.

Let $V^* = (v_{41}, v_{42}, v_{43})^T$ be the eigenvector of $M(P_{xyz}, w_7{}^*)$ corresponding to $\lambda^* = 0$. Then it is obtained that $V^* = (\tau_6 v_{43}, \tau_7 v_{43}, v_{43})^T$, where $v_{43} \neq 0$, $v_{43} \in \mathbb{R}$ and $\tau_6 = -\frac{a_{13}a_{22}}{a_{11}a_{22}-a_{12}a_{21}} < 0$, $\tau_7 = \frac{a_{13}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} < 0$.

Let $\Psi^* = (\Psi_{41}, \Psi_{42}, \Psi_{43})^T$, be the eigenvector of $\left[M\left(P_{xyz}, {w_7}^*\right)\right]^T$ corresponding to $\lambda^* = 0$. Then it is obtained that, $\Psi^* = (\tau_8 \Psi_{43}, \tau_9 \Psi_{43}, \Psi_{43})^T$, where $\Psi_{43} \neq 0$, $\Psi_{43} \in \mathbb{R}$, and $\tau_8 = \frac{a_{21}a_{32}-a_{22}a_{31}}{a_{11}a_{22}-a_{12}a_{21}} > 0$, $\tau_9 = \frac{a_{12}a_{31}-a_{11}a_{32}}{a_{11}a_{22}-a_{12}a_{21}} < 0$.

Since, $\frac{\partial F}{\partial w_7} = F_{w_7} = \left(0,0,\frac{xz}{w_3+x}\right)^T$, then $F_{w_5}\left(P_{xyz}, {w_7}^*\right) = \left(0,0,\frac{x^* z^*}{w_3+x^*}\right)^T$.

Therefore, ${\Psi^*}^T\left[F_{w_7}\left(P_{xyz}, {w_7}^*\right)\right] = \frac{x^* z^*}{w_3+x^*}\Psi_{43} \neq 0$. Hence the first requirements of the saddle-node bifurcation in the sense of Sotomayor is satisfied. Now, according to equation (30) it is obtained that:

$$D^2F\left(P_{xyz}, {w_7}^*\right)(V^*, V^*) = ({c_{i1}}^*)_{3\times 1},$$

Where:

$$\begin{aligned}
{c_{11}}^* = {} & 2\left[-1 + \frac{w_2 y^*}{(w_2+x^*)^3} + \frac{w_3 z^*}{(w_3+x^*)^3}\right](\tau_6 v_{43})^2 \\
& -2\left[\frac{w_0 w_1}{(1+w_0 w_1 y^* + w_0 z^*)^2} + \frac{w_2}{(w_2+x^*)^2}\right]\tau_6 \tau_7 (v_{43})^2 \\
& -2\left[\frac{w_0}{(1+w_0 w_1 y^* + w_0 z^*)^2} + \frac{w_3}{(w_3+x^*)^2}\right]\tau_6 (v_{43})^2 + 2\left[\frac{w_0^2 w_1^2 x^*}{(1+w_0 w_1 y^* + w_0 z^*)^3}\right](\tau_7 v_{43})^2 \\
& +4\left[\frac{w_0^2 w_1 x^*}{(1+w_0 w_1 y^* + w_0 z^*)^3}\right]\tau_7 (v_{43})^2 + 2\left[\frac{w_0^2 x^*}{(1+w_0 w_1 y^* + w_0 z^*)^3}\right](v_{43})^2.
\end{aligned}$$

$${c_{21}}^* = -2\left[\frac{w_2 w_4 y^*}{(w_2+x^*)^3}\right](\tau_6 v_{43})^2 + 2\left[\frac{w_2 w_4}{(w_2+x^*)^2}\right]\tau_6 \tau_7 (v_{43})^2 - 2w_6(\tau_7 v_{43})^2.$$

$${c_{31}}^* = -2\left[\frac{w_3 w_7 z^*}{(w_3+x^*)^3}\right](\tau_6 v_{43})^2 + 2\left[\frac{w_3 w_7}{(w_3+x^*)^2}\right]\tau_6 (v_{43})^2 + 2w_8 \tau_7 (v_{43})^2 - 2w_{10}(v_{43})^2.$$

Therefore, using condition (31b) the following is obtained.

$${\Psi^*}^T\left[D^2F\left(P_{xyz}, {w_7}^*\right)(V^*, V^*)\right] = [{c_{11}}^* \tau_8 + {c_{21}}^* \tau_9 + {c_{31}}^*]\Psi_{43} \neq 0.$$

Hence, system (2) undergoes a saddle-node bifurcation near the P_{xyz} when $w_7 = {w_7}^*$.

7. Hopf- Bifurcation Analysis

In section, the possibility of occurrence of Hopf bifurcation is investigated as shown in the following theorem.

Theorem (12): If the condition (18a) holds, then the system (2) has a Hop-bifurcation around P_{xyz} when $w_4 = {w_4}^*$ if and only if the following conditions hold

$$[A({w_4}^*)B({w_4}^*)]' < C'({w_4}^*), \tag{32}$$

where A, B and C are the characteristic polynomial coefficients that given in equation (20), with

$${w_4}^* = \frac{{B_1^*}^2}{w_2 y^*}\left[\frac{a_{11}a_{22}(a_{11}+a_{22})+(a_{11}+a_{33})[a_{11}a_{33}-a_{13}a_{31}]+(a_{22}a_{33})[a_{11}+a_{22}+a_{33}]+a_{11}a_{22}a_{33}}{a_{13}a_{32}+(a_{11}+a_{22})a_{12}}\right].$$

Proof. It is simple to determine from the Jacobian matrix $M(P_{xyz})$ and their characteristic equation, which are given by equations (18) and (19), respectively, that $AB = C$ when $w_4 = {w_4}^*$. Hence, the characteristic equation becomes

$$(\lambda^2 + B)(\lambda + A) = 0. \tag{33}$$

Consequently, we obtain that $\lambda_1 = -A$ and $\lambda_{2,3} = \pm i\sqrt{B} = \pm i\delta_2(w_4)$. Clearly, we have $A > 0$ and $B > 0$ due to condition (18a). Therefore, the Jacobian matrix have three eigenvalues:

One negative real eigenvalue and two pure imaginary complex conjugate when $w_4 = {w_4}^*$. As a result, the first criteria for having a Hopf bifurcation is met.

Moreover, where w_4 belongs to neighborhood of ${w_4}^*$, the complex conjugate eigenvalues become $\lambda_{2,3} = \delta_1(w_4) \pm i\delta_2(w_4)$.

Now, substituting $\delta_1(w_4) \pm i\delta_2(w_4)$ in characteristic equation (19) and then take the derivative of the resulting equation with respect to w_4. After that comparing the two sides with equating their real and imaginary parts, we obtain that

$$\begin{aligned} H_1(w_4)\delta_1'(w_4) - H_2(w_4)\delta_2'(w_4) &= -H_3(w_4), \\ H_2(w_4)\delta_1'(w_4) + H_1(w_4)\delta_2'(w_4) &= -H_4(w_4), \end{aligned} \tag{34}$$

where:

$$H_1(w_4) = 3[\delta_1(w_4)]^2 + 2A(w_4)\delta_1(w_4) - 3[\delta_2(w_4)]^2 + B(w_4).$$

$$H_2(w_4) = 6\delta_1(w_4)\delta_2(w_4) + 2A(w_4)\delta_2(w_4).$$

$$H_3(w_4) = A'(w_4)[\delta_1(w_4)]^2 - A'(w_4)[\delta_2(w_4)]^2 + B'(w_4)\delta_1(w_4) + C'(w_4).$$

$$H_4(w_4) = 2A'(w_4)\delta_1(w_4)\delta_2(w_4) + B'(w_4)\delta_2(w_4).$$

Solving that linear system (34) for $\delta_1'(w_4)$ gives

$$\delta_1'(w_4) = -\frac{H_1(w_4)H_3(w_4)+H_2(w_4)H_4(w_4)}{[H_2(w_4)]^2+(w_4)[H_2(w_4)]^2}.$$

Hence the proof is done only if and only if

$$\delta_1'(w_4) \neq 0 \text{ or equivalently } H_1(w_4)H_3(w_4) + H_2(w_4)H_4(w_4) \neq 0.$$

Now, since $\delta_1({w_4}^*) = 0$ and $\delta_2({w_4}^*) = \sqrt{B({w_4}^*)}$. Hence, it is obtained that

$$H_1({w_4}^*) = -2B({w_4}^*).$$

$$H_2({w_4}^*) = -2A({w_4}^*)\sqrt{B({w_4}^*)}.$$

$$H_3({w_4}^*) = -A'({w_4}^*)[B({w_4}^*)] + C'(w_4).$$

$$H_4({w_4}^*) = B'({w_4}^*)\sqrt{B({w_4}^*)}.$$

Consequently, it is obtained that:

$$\begin{aligned} &H_1({w_4}^*)H_3({w_4}^*) + H_2({w_4}^*)H_4({w_4}^*) \\ &\quad = 2A'({w_4}^*)B^2({w_4}^*) - 2B({w_4}^*)C'({w_4}^*) + 2A({w_4}^*)B'({w_4}^*)B({w_4}^*) \\ &\quad = -2B({w_4}^*)[C'({w_4}^*) - (A'({w_4}^*)B({w_4}^*) + A({w_4}^*)B'({w_4}^*). \end{aligned}$$

Clearly, condition (32) ensures that $H_1({w_4}^*)H_3({w_4}^*) + H_2({w_4}^*)H_4({w_4}^*) \neq 0$.

Hence the system has a Hopf bifurcation because $\delta_1'(w_4) > 0$ under the condition (32), which complete the proof.

8. NUMERICAL SIMULATION

The global dynamics of the system (2) are quantitatively investigated in this section. The goals of the study are to verify our analytical findings and better understand the consequences of changing the system's parameters on the system's dynamics (2). As a result, system (2) can be numerically solved for various sets of parameters and initial conditions.

Different sets of parameter values can be adopted for the following biologically possible set of hypothetical parameter values, and system (2) has a globally asymptotically stable coexistence equilibrium point, as illustrated in figure (1) below.

$$\begin{gathered} w_0 = 0.25, w_1 = 1, w_2 = 0.25, w_3 = 0.25, w_4 = 0.15, w_5 = 0.1 \\ w_6 = 0.1, w_7 = 0.15, w_8 = 0.1, w_9 = 0.1, w_{10} = 0.1 \end{gathered} \tag{35}$$

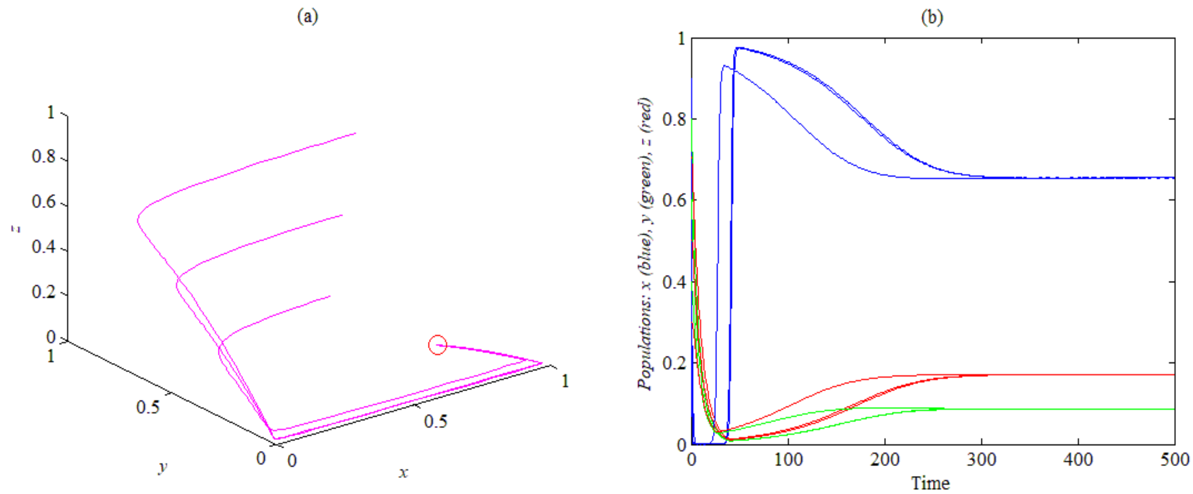

Fig.1: The trajectories of system (2) for the data (35) starting from different initial points. (a) 3D phase portrait for a globally asymptotically coexistence equilibrium point $(0.65, 0.08, 0.17)$. (b) Time series for the attractor in (a).

Obviously, figure (1) ensures the obtained theoretical finding regarding the existence of globally asymptotically stable coexistence equilibrium points under certain conditions. Now, by modifying one parameter at a time, the effect of changing the parameter values on the dynamics of the system (2) is explored, and the resulting trajectory is shown.

It has been shown that as the fear rate w_0 rises, the overall number of individuals in a species declines, and the predators face extinction see figure (2). Similar effect has been detected regarding varying of w_1.

It is observed that for the ranges $w_2 \in (0,0.06)$, $w_2 \in (0.06,0.11)$, $w_2 \in (0.11,0.4)$, and $w_2 \in (0.4,1)$ the trajectory of the system (2) approaches to periodic in the xy −plane, periodic in the $int.\mathbb{R}^3_+$, globally asymptotically stable coexistence equilibrium point, and predator-free equilibrium point, see figure (3) for selected values.

Further investigation for the effect of fear rate w_0 on the periodic dynamics of the system (2) has been done, and then the obtained results are drawn in figure (4).

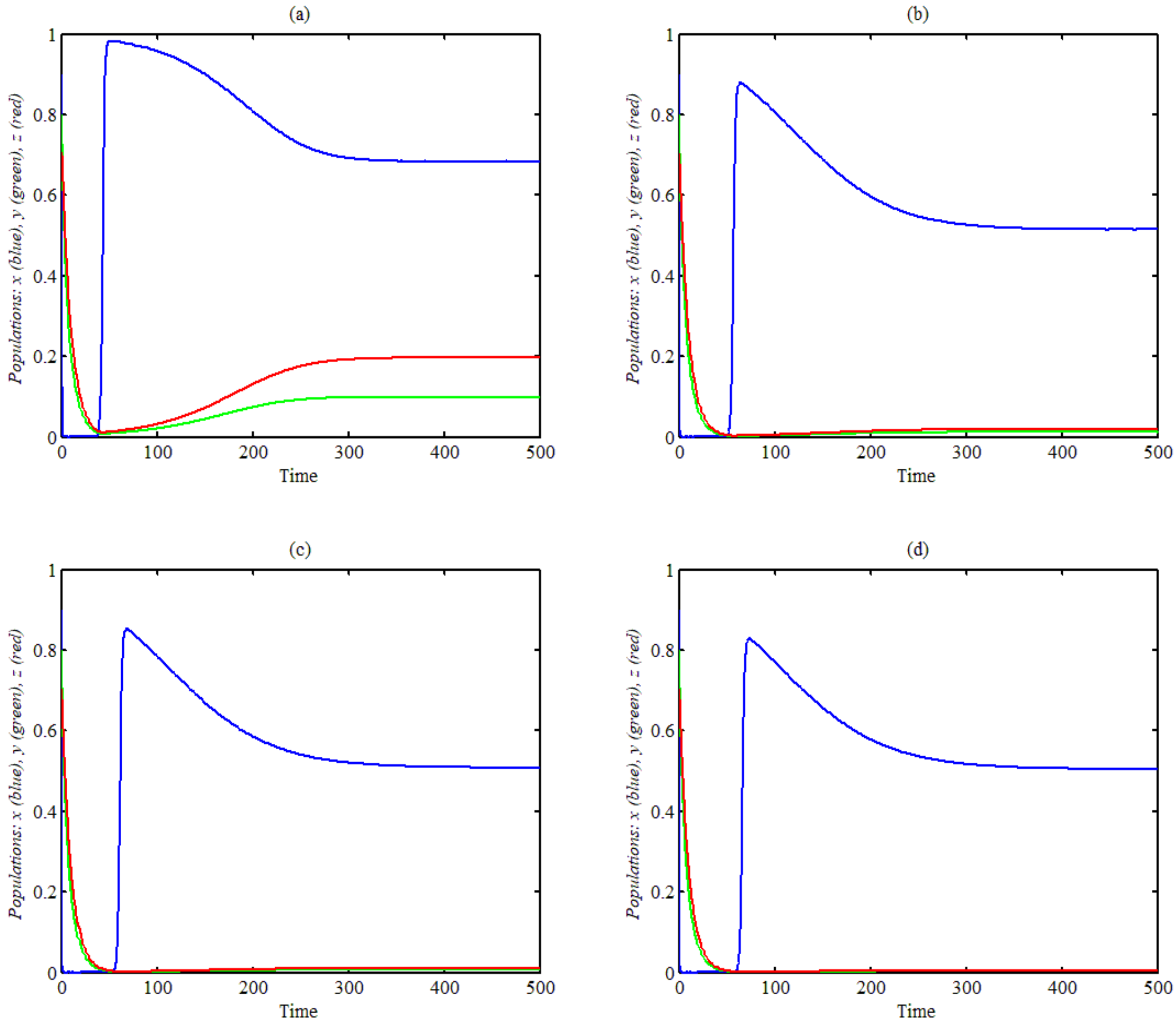

Fig.2: The trajectories of system (2) for the data (35) as a function of time: (a) Approach to coexistence equilibrium point $(0.68, 0.1, 0.19)$, when $w_0 = 0$ (b) Approaches to coexistence equilibrium point $(0.51, 0.01, 0.02)$, when $w_0 = 25$. (c) Approaches to point $(0.5, 0.006, 0.01)$, when $w_0 = 50$. (d) Approaches to point $(0.5, 0.004, 0.006)$, when $w_0 = 100$.

Although the scavenger population shrinks as w_0 rises, the periodic dynamics show a decline and transfer to a stable point.

The effect of varying in the parameter w_3 is investigated and presented in figure (5) at typical values. It is observed that, or $w_3 \geq 0.65$ the scavenger faces extinction, and the solution attracting to P_{xy}, for $w_3 \leq 0.09$ the predator faces extinction, and the solution attracting to P_{xz}. While, for $w_3 \leq 0.07$, P_{xz} becomes unstable and the solution goes to periodic dynamic in the xz −plane. Otherwise, the system (2) has a globally asymptotically stable coexistence point.

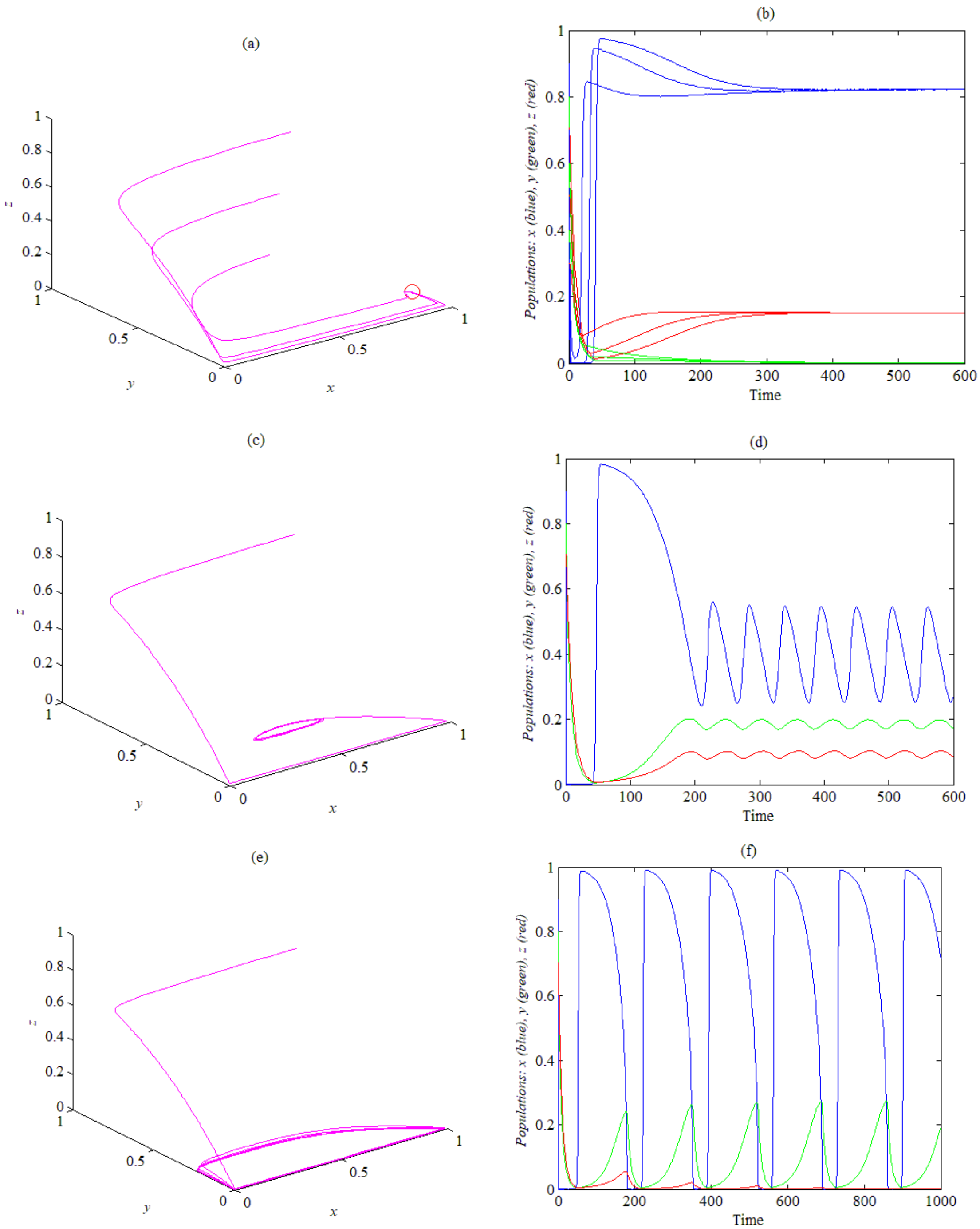

Fig. 3: The trajectory of the system (2) for data (35): (a) Approaches to a predator-free equilibrium point $(0.82,0,0.15)$, when $w_2 = 0.5$. (b) Time series for the trajectories in (a). (c) Approaches to a 3D periodic dynamics, when $w_2 = 0.1$. (d) Time series for the trajectory in (c). (e) Approaches to a 2D periodic attractor in the xy −plane, when $w_2 = 0.05$. (f) Time series for the trajectory in (e).

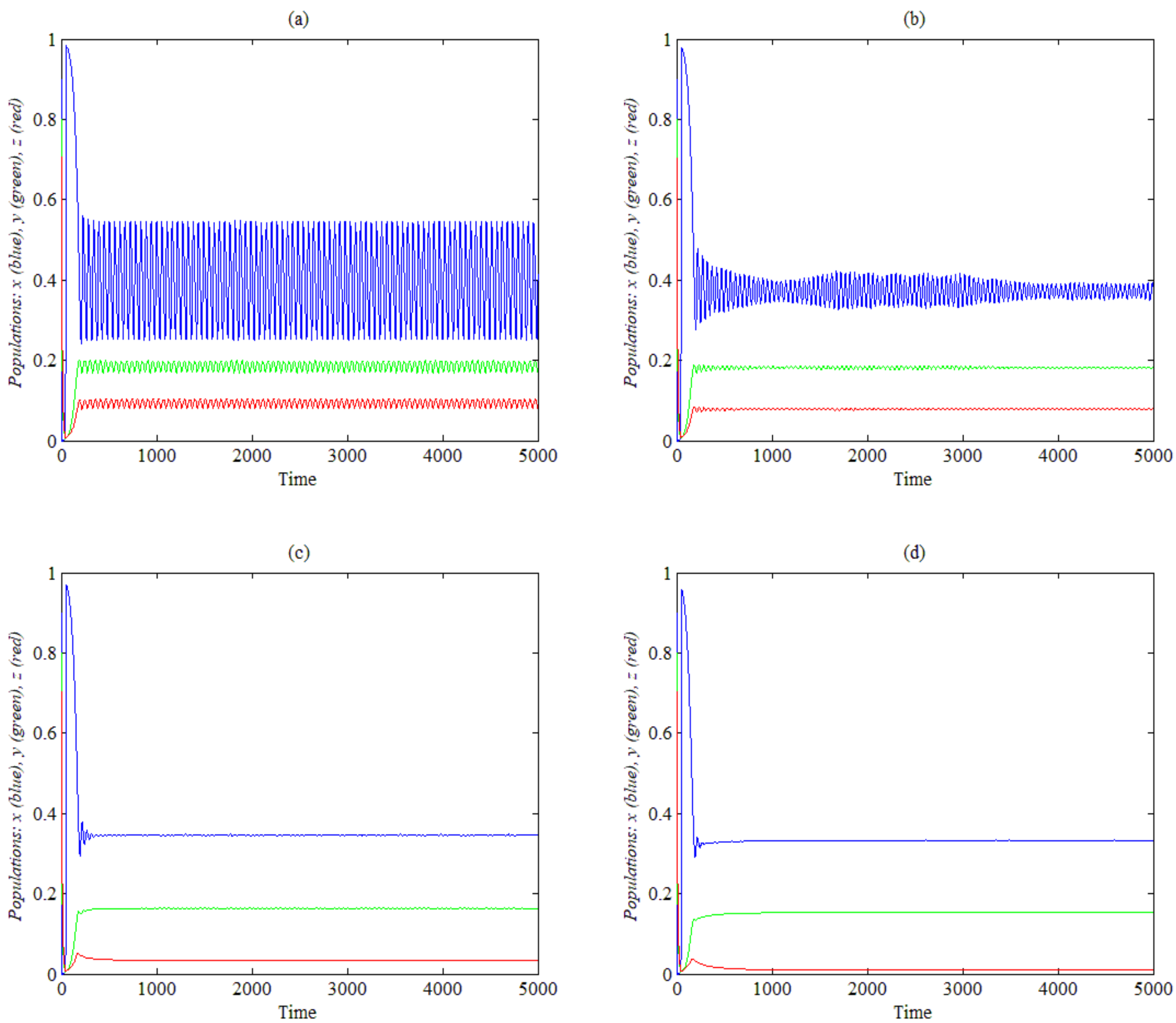

Fig. 4: The trajectories of system (2) for the data (35) with $w_2 = 0.1$, and different values of w_0 as a function of time: (a) 3D periodic attractor, when $w_0 = 0.25$ (b) 3D periodic attractor with small period size when $w_0 = 0.5$. (c) 3D periodic attractor with a very small period, when $w_0 = 1.5$. (d) Asymptotic stable coexistence equilibrium point, when $w_0 = 2.6$.

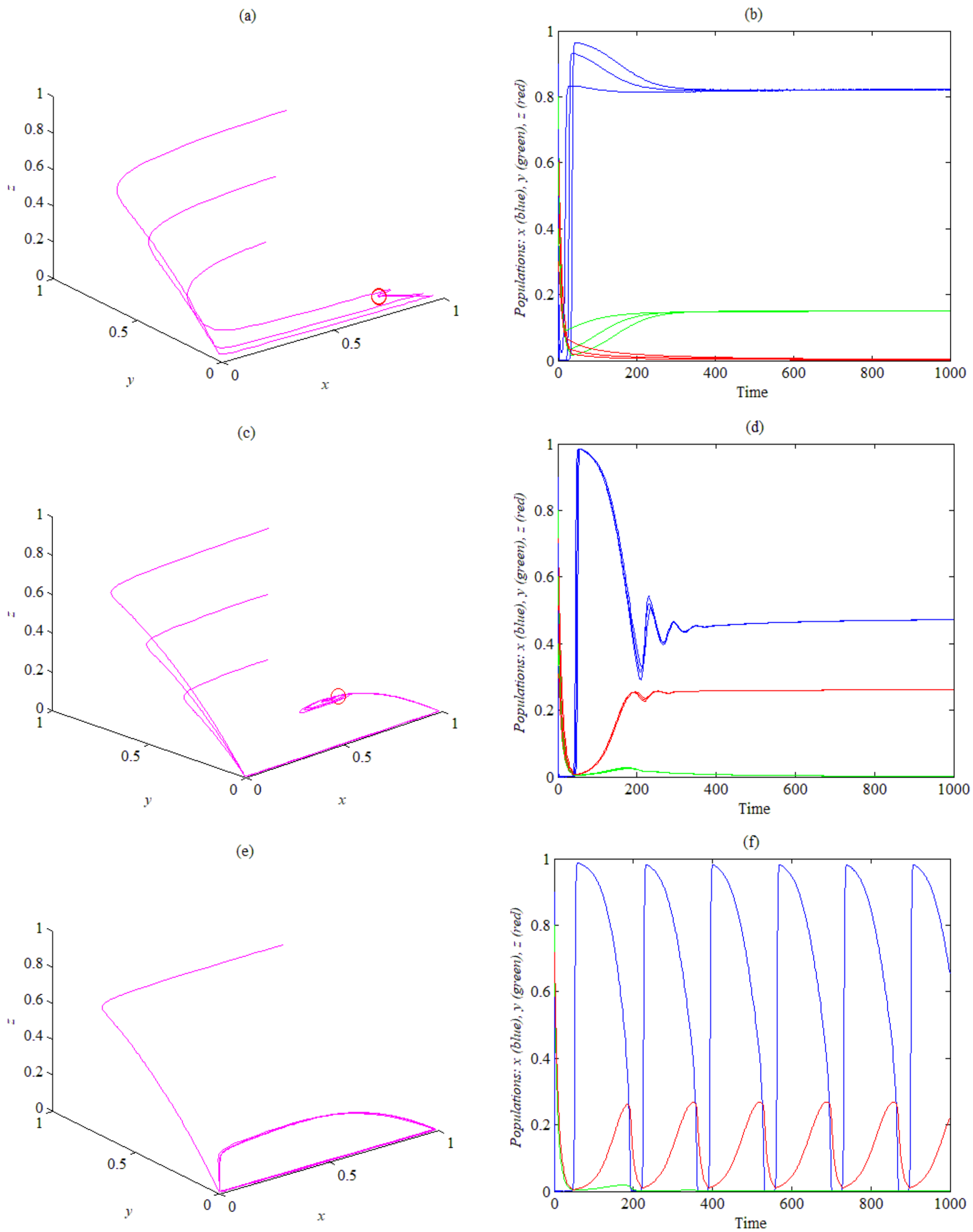

Fig. 5: The trajectory of the system (2) for data (35): (a) Approaches to a scavenger-free equilibrium point $(0.82,0.15,0)$, when $w_3 = 0.65$. (c) Approaches to a predator-free equilibrium point $(0.47,0,0.26)$, when $w_3 = 0.09$. (e) Approaches to a 2D periodic attractor in the xz −plane, when $w_3 = 0.07$. (b), (d), and (f) Represent the time series for the trajectories in (a), (c), and (e) respectively.

In figure (6), the influence of varying the parameter w_4 is shown at selected values. It is noted that for $0.21 \leq w_4 < 0.28$ the system (2) has a periodic dynamics in $int.\mathbb{R}^3_+$, for $0.29 \leq w_4$ the scavenger disappears and the system (2) approaches periodic dynamics in the xy −plane. While for $w_4 < 0.13$ the predator species disappears and the system approaches asymptotically to P_{xz}. The system (2) has a globally asymptotic coexistence point otherwise.

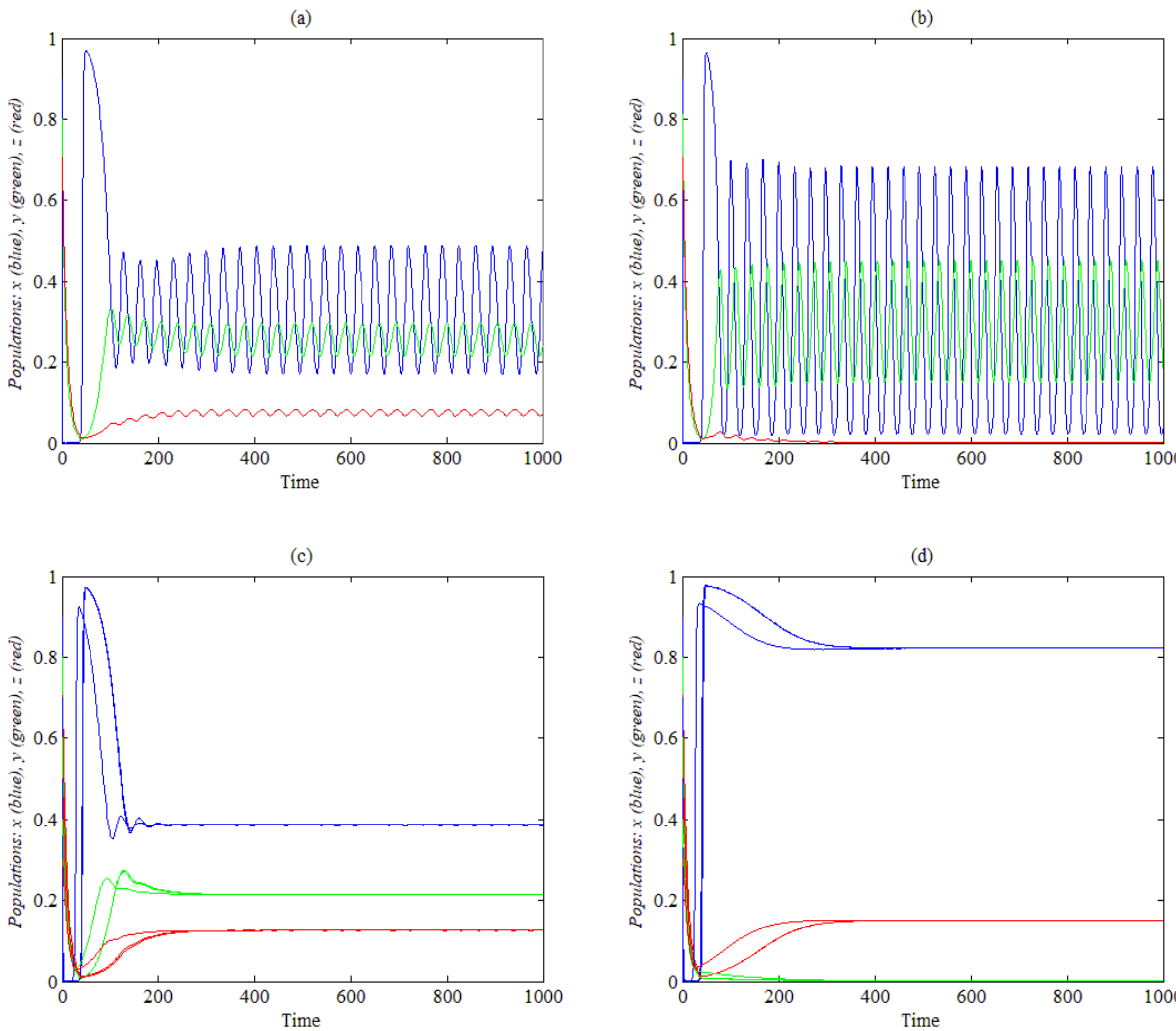

Fig. 6: The trajectories of system (2) for the data (35) with different values of w_4 as a function of time: (a) 3D periodic attractor, when $w_4 = 0.23$ (b) 2D periodic attractor in xy −plane when $w_4 = 0.3$. (c) Asymptotic stable coexistence equilibrium point $(0.38,0.21,0.12)$ when $w_4 = 0.2$. (d) Asymptotic stable predator-free point $(0.82,0,0.15)$ when $w_4 = 0.12$.

The influence of changing w_5 is considered in figure (7) at selected values. It is noted that $w_5 \geq 0.12$ the system (2) approaches P_{xz}, however for $w_5 < 0.07$ there is a 3D periodic dynamics, and for $w_5 < 0.04$ there is 2D periodic dynamics in xy −plane. Finally, it has a globally coexistence point otherwise.

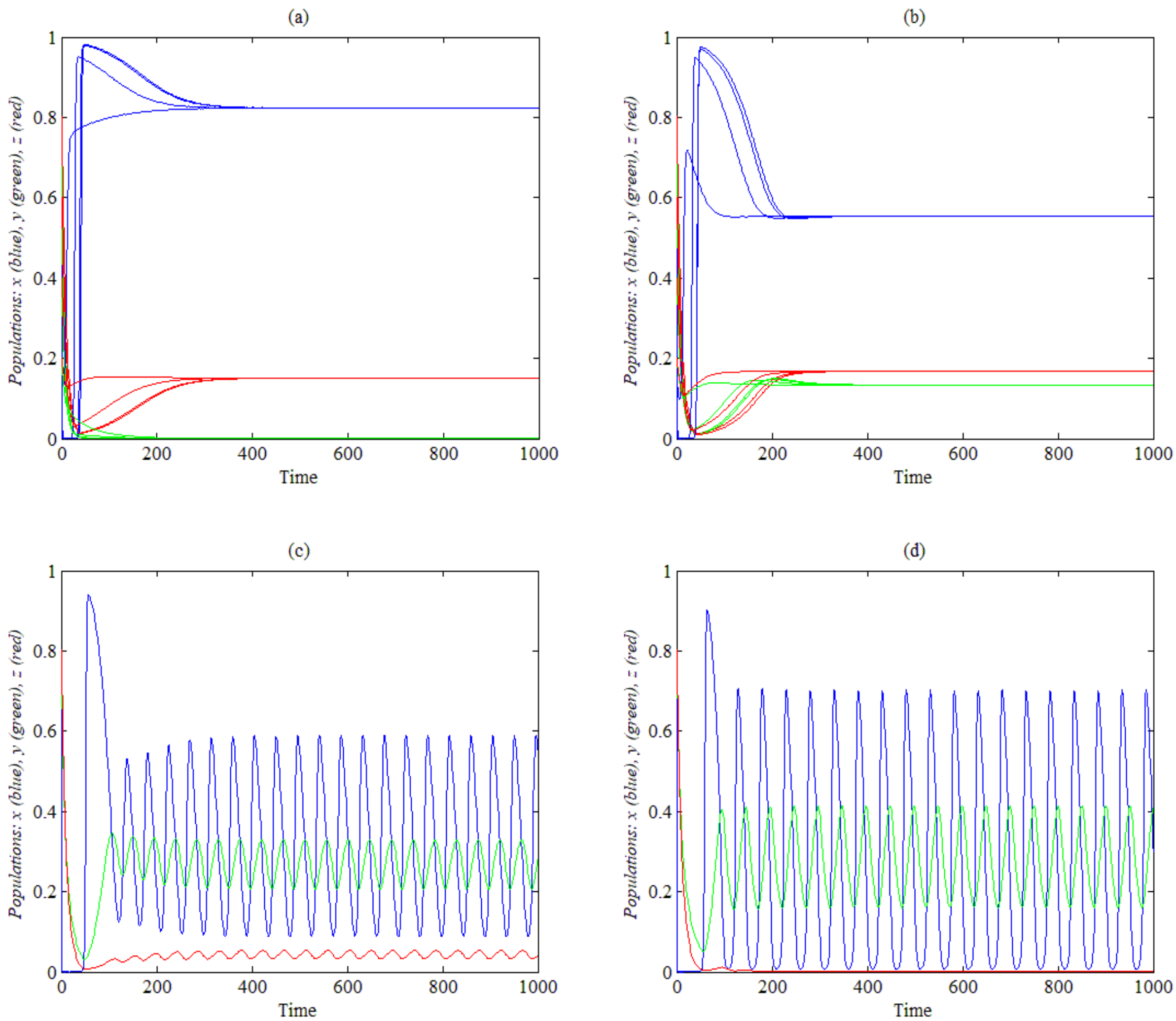

Fig. 7: The trajectories of system (2) for the data (35) with different values of w_5 as a function of time: (a) Asymptotic stable predator-free point $(0.82,0,0.15)$ when $w_5 = 0.13$. (b) Asymptotic stable coexistence equilibrium point $(0.55,0.13,0.16)$ when $w_5 = 0.09$. (c) 3D periodic attractor, when $w_5 = 0.05$. (d) 2D periodic attractor in xy −plane when $w_5 = 0.03$.

It is observed that increasing w_6 (w_{10}) leads to decreasing in predators (scavengers), see figure (8a) below. Moreover, it transfers the 3D periodic dynamics to P_{xyz}, as seen in figure (8b)-(8c).

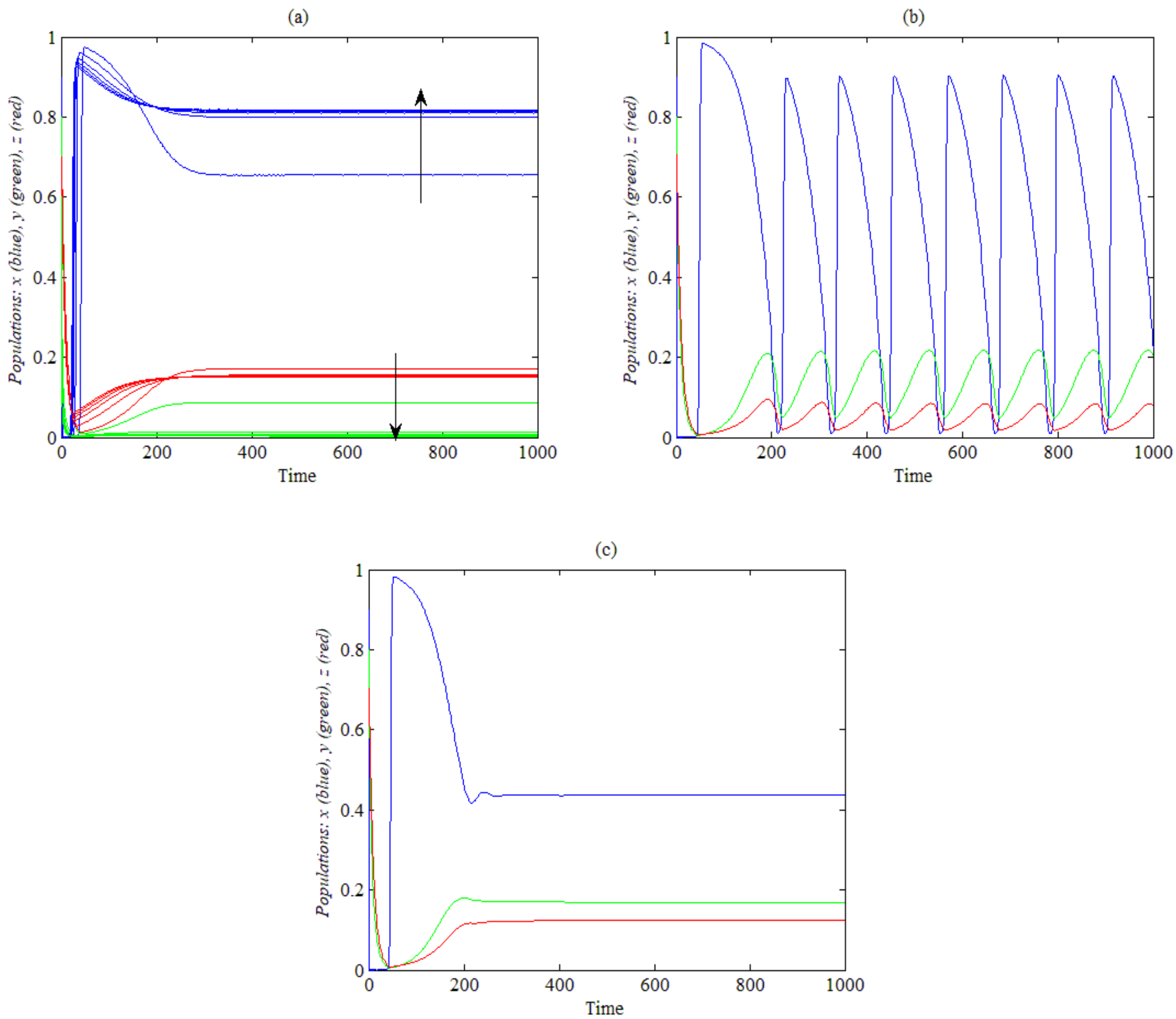

Fig. 8: The trajectories of system (2) for the data (35) as a function of time: (a) The predator population decreases as $w_6 = 0.1, 1.1, 2.1, 3.1, 4.1, 5.1$, ultimately approaching to stable predator-free point. (b) 3D periodic attractor, when $w_2 = 0.1$, and $w_6 = 0.09$. (c) Asymptotic stable coexistence equilibrium point $(0.43, 0.17, 0.12)$ when $w_2 = 0.1$, and $w_6 = 0.13$.

The influence of changes in w_7 is discussed in figure (9) at selected values. It is noted that for $w_7 \geq 0.23$ there is a 2D periodic dynamics in xz −plane, however, for $0.2 \leq w_7 < 0.23$ the solution approaches P_{xz}. While, for $w_7 \leq 0.11$ the solution approaches P_{xy}. Otherwise, the solution approaches P_{xyz}.

Further, it is noted that varying in w_8 has a quantitative effect on the population size of predator species.

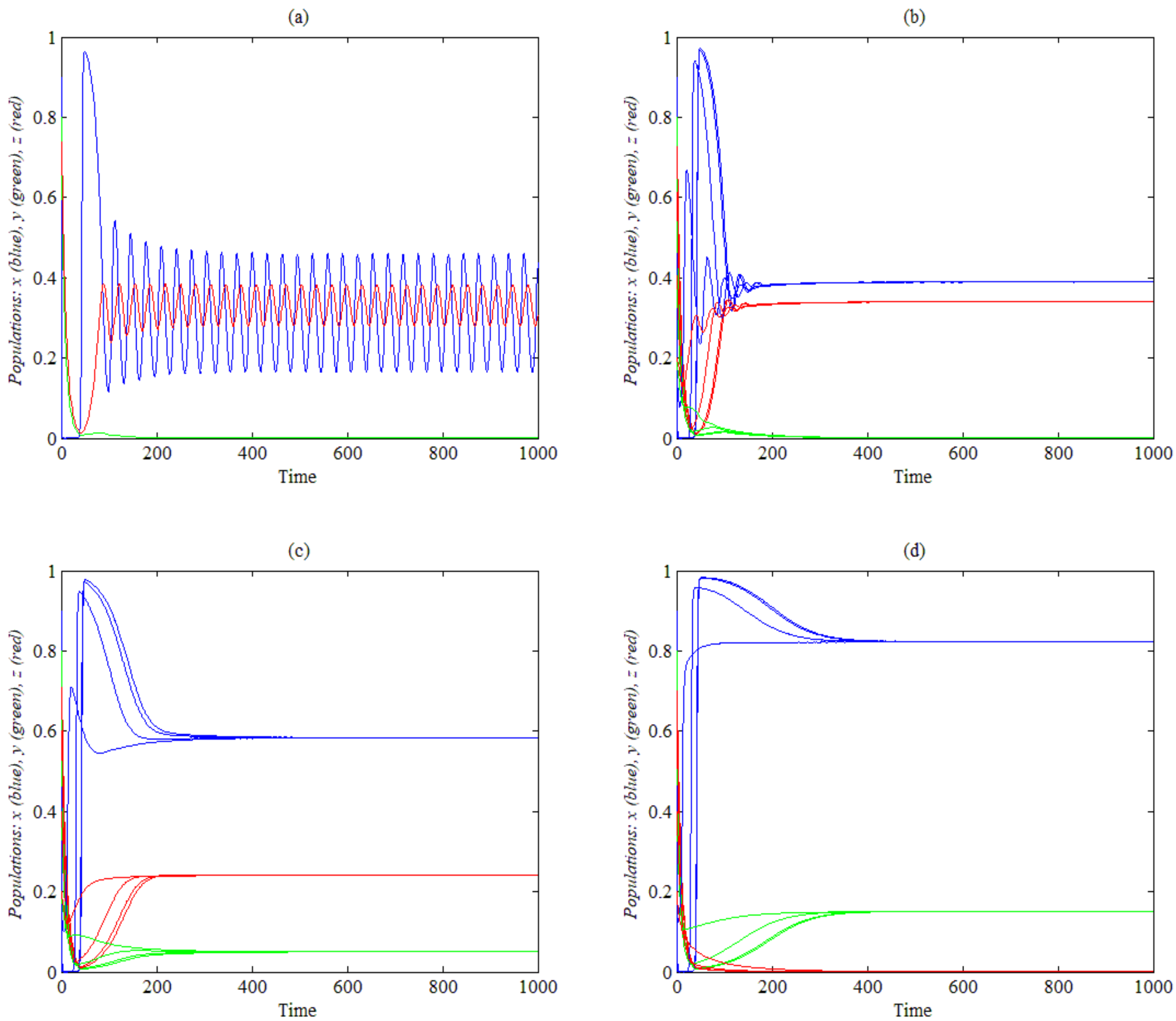

Fig. 9: The trajectories of system (2) for the data (35) as a function of time: (a) 2D periodic attractor in xz −plane when $w_7 = 0.25$. (b) Approaches predator-free point $(0.38,0,0.34)$ when $w_7 = 0.2$. (c) Asymptotic stable coexistence equilibrium point $(0.43,0.17,0.12)$ when $w_7 = 0.17$. (d) Approaches scavenger-free point $(0.82,0.15,0)$ when $w_7 = 0.1$.

Finally, figure (10) explain the influence of changing w_9 at selected values. It is noted that $w_9 > 0.13$ the system approaches P_{xy}, While, for $w_9 \leq 0.06$ it approaches P_{xz}, and for $w_9 \leq 0.04$ the system has a periodic in xz −plane. Otherwise, the system approaches P_{xyz}.

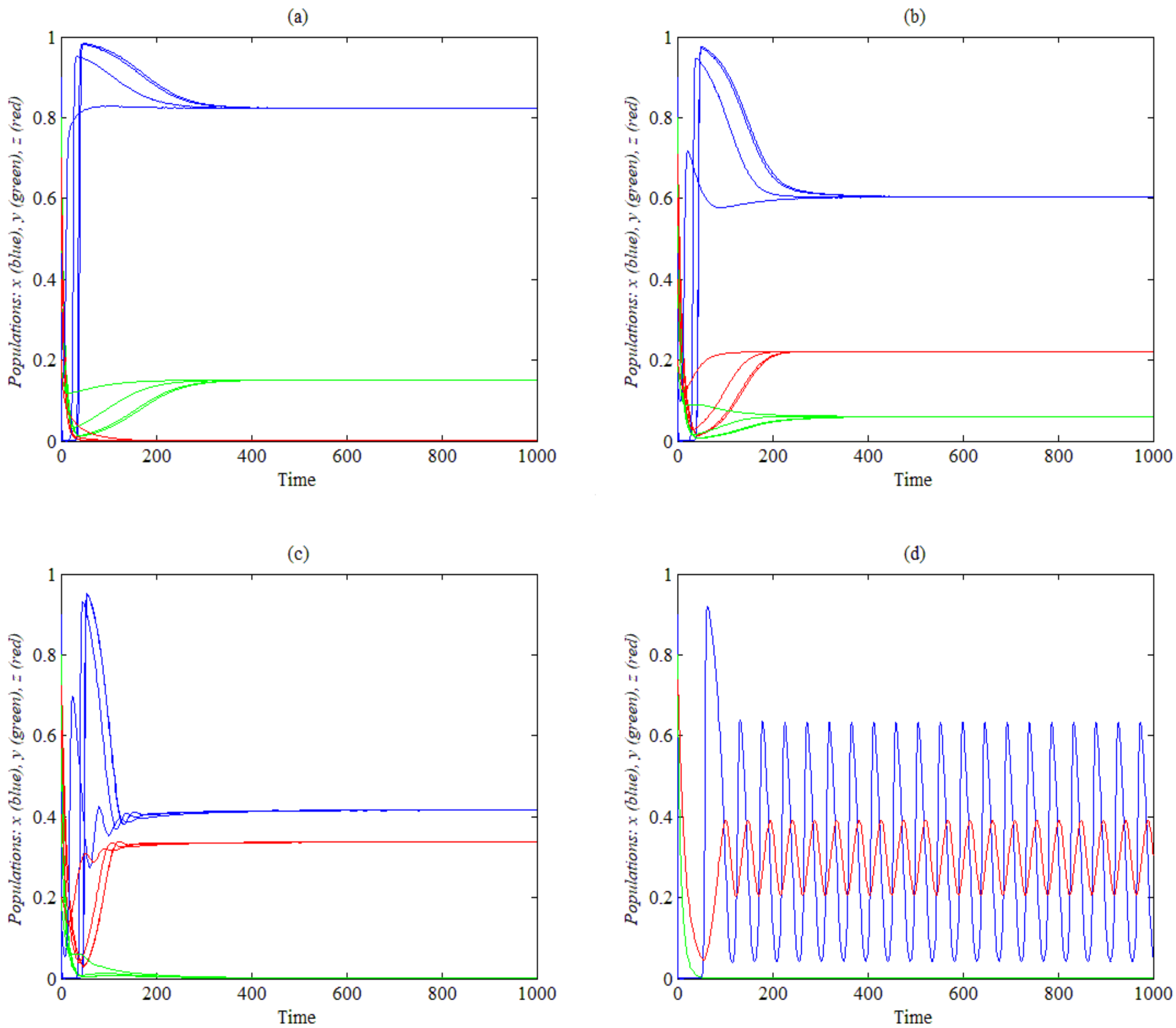

Fig. 10: The trajectories of system (2) for the data (35) with different values of w_9 as a function of time: (a) Asymptotic stable scavenger-free point $(0.82,0.15,0)$ when $w_9 = 0.15$. (b) Asymptotic stable coexistence equilibrium point $(0.6,0.06,0.22)$ when $w_9 = 0.09$. (c) Asymptotic stable predator-free point $(0.41,0,0.33)$ when $w_9 = 0.06$. (d) 2D periodic attractor in xz −plane when $w_9 = 0.04$.

9. CONCLUSIONS AND DISCUSSION

A food web system with a prey-predator-scavenger is mathematically formulated in this study, with food being transmitted according to the Holling type II functional response. Fear plays a role, as does proportional quadratic harvesting. All of the solution's properties are studied. There are only five nonnegative equilibrium points in the system. The topics of stability, persistence, local bifurcations, and Hopf-bifurcation are all thoroughly explored. The numerical simulation was used to examine global dynamics and determine the impact of changing parameters, particularly fear and harvesting, on the system's behavior. Using the set of hypothetical data, the following observations were made.

1. Although the system has many bifurcation points resulting from varying in their parameters, the system (2) has a globally asymptotically stable coexistence equilibrium point different ranges of parameters.
2. The presence of fear has a stabilizing effect on the system's dynamics. However, if the fear level is higher than a certain threshold, predator and scavenger populations may become extinct.
3. Increasing the half-saturation constant of the predator species leads to extinction in predator. However lowering their value below a certain threshold leads to destabilizing the system and Hopf-bifurcation occurs in the interior of first octant, and then the solution approaches to periodic dynamics in the xy −plane due to extinction in scavenger.
4. Increasing the half-saturation constant of the scavenger species leads to extinction in scavenger. However lowering their value below a certain threshold leads to extinction in predator species, and then the solution approaches to predator-free point first and then to periodic dynamics in the xz −plane.
5. The system (2) undergoes a Hopf bifurcation in the first octant as the conversion rate of prey biomass to the predator population increases and passes a threshold value. Further increasing this parameter leads to extinction in scavenger and periodic dynamics in the

xy −plane take place. Lowering this parameter, otherwise, makes the solution approaches a predator-free equilibrium point.

6. Increasing the death rate of predator species above a threshold value leads to extinction in predators and the system approaches a predator-free equilibrium point. On the other hand, lowering their value below a threshold value destabilizes the system and a Hopf bifurcation take place in the first octant. Lowering the value of this parameter further causes extinction in scavengers and the solution approaches to periodic in the xy −plane.
7. Increasing the harvest rate of predator (scavenger) causes decreasing in the predator (scavenger) population and vice versa. It is also observed that these parameters have a stabilizing role in the system behavior.
8. Rising the value of conversion rate of prey biomass to the scavenger above a threshold value leads to extinction in predators and the solution approaches predator-free equilibrium point first and then to periodic in the xz −plane. On the other hand, lowering this value below a threshold value leads to extinction in scavenger and the solution approaches to scavenger-free point.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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