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STABILITY ANALYSIS AND NEIMARK-SACKER BIFURCATION OF A NONSTANDARD FINITE DIFFERENCE SCHEME FOR LOTKA-VOLTERRA PREY-PREDATOR MODEL

R. AHMED[1,*], A. AHMAD[2], N. ALI[2]

[1]Department of Mathematics, Air University Multan Campus, Multan, Pakistan

[2]Department of Mathematics, National College of Business Administration & Economics, Rahim Yar Khan, Pakistan

Abstract. To discretize the Lotka-Volterra prey-predator model, we adopted a dynamically consistent nonstandard finite difference scheme. The existence of fixed points and their topological categorization are examined. It is proved using bifurcation theory that the system experiences Neimark-Sacker bifurcation. Moreover, the hybrid control method is used to control the Neimark-Sacker bifurcation. Additionally, numerical simulations are performed to show the system's complexity and consistency with analytical results.

Keywords: nonstandard finite difference scheme; Lotka-Volterra; stability; bifurcation; chaos control.

2010 AMS Subject Classification: 39A28, 39A30, 92D25.

1. Introduction

Predator-prey models offer a wide variety of ecological and biological applications. Numerous studies on the dynamics of prey-predator models have been conducted. Although many essential elements of the nonlinear dynamics of prey-predator models are linked to continuous dynamical systems, the properties of discrete dynamical systems are still largely unknown. In

*Corresponding author

E-mail address: rizwanahmed488@gmail.com

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contrast to a continuous system, a discrete dynamical structure has a solitary dynamical nature. A discrete dynamical system may be used to define various crucial and practical challenges in everyday life. The discretization of a continuous system may be done in several different ways. Then one can discuss the numerical solution to think about the analytical features of a solution that is difficult to compute. As a result, rigorous critical inspections of discrete dynamical systems have made significant contributions to domains like engineering, physics, and biology [1, 2].

The classical Lotka-Volterra prey-predator model, which assumes that the functional response of prey is linear, is governed by the nonlinear differential system given by [3]:

$$
\begin{cases} \frac{dN(\tau)}{d\tau} = N(\tau)(\alpha - \beta N(\tau) - \gamma P(\tau)), \\ \frac{dP(\tau)}{d\tau} = P(\tau)(\gamma\delta N(\tau) - D), \end{cases} \tag{1.1}
$$

where $N(\tau)$ and $P(\tau)$ are the population densities of prey and predator, respectively, at time τ, α represents the maximum per capita growth rate for prey species, β denotes the strength of the intra-specific competition of the prey population, γ represents the strength of intraspecific competition between prey and its predator, δ is the conversion rate of prey into the predator, and D represents per capita death rate of predator species. The following transformations can be used to produce the non-dimensional form of (1.1):

$$
x(t) = \frac{\alpha\delta N(\tau)}{D}, \; y(t) = \frac{\gamma P(\tau)}{D}, \; t = D\tau.
$$

In addition, by incorporating the new parameters, we are able to get the non-dimensional version of (1.1) that is as follows:

$$
\begin{cases} \frac{dx}{dt} = x(t)(a - bx(t) - y(t)), \\ \frac{dy}{dt} = y(t)(cx(t) - 1), \end{cases} \tag{1.2}
$$

where a, b and c are positive constants.

Discrete models are preferred over continuous models for a variety of reasons. Discretization is required to construct discrete models that approach the exact analytical solutions to continuous models. In addition, digital computers are needed for model simulations, which necessitate the usage of discrete models. In [3], Elsadany and Matouk investigated the following fractional-order discretized analog of (1.2).

$$\begin{cases} x_{n+1} = x_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(x_n(a - bx_n - y_n)), \\ y_{n+1} = y_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(y_n(-1 + cx_n)), \end{cases} \tag{1.3}$$

where $0 < \alpha \leq 1$. They addressed local equilibrium dynamics, and Neimark-Sacker bifurcation was proved only by numerical simulations.

In [4], Liu and Xiao studied the following discrete counterpart of (1.2) by using the Euler method to obtain a discrete predator-prey model.

$$\begin{cases} x_{n+1} = x_n + \delta(rx_n(1 - x_n) - bx_n y_n), \\ y_{n+1} = y_n + \delta(-d + bx_n)y_n, \end{cases} \tag{1.4}$$

where r, b, d and δ are positive. They investigated the local stability and bifurcation of the fixed points.

In [5], the authors studied the following discrete counterpart of (1.2) by using the piecewise constant argument to obtain a discrete predator-prey model.

$$\begin{cases} x_{n+1} = x_n Exp(a - bx_n - y_n), \\ y_{n+1} = y_n Exp(cx_n - 1), \end{cases} \tag{1.5}$$

They investigated stability analysis of equilibria, flip bifurcation, and Neimark-Sacker bifurcation analytically and numerically. Moreover, chaos control methods are employed to control the chaotic behaviors of the model.

Discrete and continuous models are dynamically consistent if they possess similar dynamical behavior, for example, boundedness and permanence of solution, local stability of fixed points, bifurcation, and chaos. We employ a non-standard finite difference scheme of Mickens-type [6, 7, 8, 9, 10] to discretize the model (1.2). This scheme is dynamically consistent.

$$\begin{cases} \frac{x_{n+1}-x_n}{h} = ax_n - bx_nx_{n+1} - x_ny_n, \\ \frac{y_{n+1}-y_n}{h} = cx_ny_n - y_n, \end{cases} \tag{1.6}$$

where x_n and y_n are approximations of $x(t_n)$ and $y(t_n)$, respectively, and h is the time step size. The system (1.6) can be written as:

$$\begin{cases} x_{n+1} = \frac{(1+h(a-y_n))x_n}{1+bhx_n}, \\ y_{n+1} = (1+h(cx_n-1))y_n, \end{cases} \tag{1.7}$$

The following is how the paper is organized: The existence and topological classification of fixed points of the model (1.7) are examined in section 2. We explore local bifurcation analysis at unique positive fixed point of the model (1.7) in section 3 applying the center manifold theory and bifurcation theory. To support our theoretical results, we provide several numerical examples in section 5. The section 6 has some concluding remarks.

2. Existence and Stability of Fixed Points

In this section, we investigated the conditions for the existence and stability of fixed points in the system (1.7). The solutions of the following system are the fixed points $(\bar{x},\bar{y})$ of the system (1.7):

$$\begin{cases} \bar{x} = \frac{(1+h(a-\bar{y}))\bar{x}}{1+bh\bar{x}}, \\ \bar{y} = (1+h(c\bar{x}-1))\bar{y}, \end{cases}$$

We can check out that the system (1.7) has three fixed points by doing some simple algebraic calculations.

$$E_0(0,0),\ E_1(\frac{a}{b},0),\ E_2(\frac{1}{c},\frac{ac-b}{c}).$$

The point E_2 is the unique fixed point of system (1.7) if $ac > b$.

The eigenvalues of the variational matrix computed at the fixed points of the model (1.7) determine the local stability of the model's fixed points. The variational matrix of the model (1.7) at a point $(\bar{x}, \bar{y})$ is defined as follows:

$$J(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{1+ah-h\bar{y}}{(1+bh\bar{x})^2} & -\frac{h\bar{x}}{1+bh\bar{x}} \\ ch\bar{y} & 1+h(-1+c\bar{x}). \end{bmatrix} \tag{2.1}$$

The following results are used to classify the fixed points of the system (1.7).

Lemma 2.1. [11] *Let $P(s) = s^2 + As + B$ be the characteristic polynomial of the variational matrix determined at a fixed point $(\bar{x}, \bar{y})$ and s_1, s_2 are two roots of the equation $P(s) = 0$, then $(\bar{x}, \bar{y})$ is*

(i) sink and therefore locally asymptotically stable if $|s_{1,2}| < 1$,

(ii) source and therefore unstable if $|s_{1,2}| > 1$,

(iii) saddle if $|s_1| < 1$ and $|s_2| > 1$ (or $|s_1| > 1$ and $|s_2| < 1$),

(iv) non-hyperbolic point if either $|s_1| = 1$ or $|s_2| = 1$.

Lemma 2.2. [11] *Let $P(s) = s^2 + As + B$. Assume that $P(1) > 0$. If s_1, s_2 are two roots of $P(s) = 0$, then*

(i) $|s_{1,2}| < 1$ iff $P(-1) > 0$ and $B < 1$,

(ii) $|s_1| < 1$ and $|s_2| > 1$ (or $|s_1| > 1$ and $|s_2| < 1$) iff $P(-1) < 0$,

(iii) $|s_1| > 1$ and $|s_2| > 1$ iff $P(-1) > 0$ and $B > 1$,

(iv) $s_1 = -1$ *and* $|s_2| \neq 1$ *iff* $P(-1) = 0$ *and* $A \neq 0, 2$,

(v) s_1 *and* s_2 *are complex and* $|s_{1,2}| = 1$ *iff* $A^2 - 4B < 0$ *and* $B = 1$.

Lemma 2.3. *The fixed point* $E_0(0,0)$ *is saddle point if* $0 < h < 2$, *source if* $h > 2$, *and non-hyperbolic if* $h = 2$.

Proof. The variational matrix evaluated at fixed point $E_0(0,0)$ is

$$J(E_0) = \begin{bmatrix} 1+ah & 0 \\ 0 & 1-h \end{bmatrix}.$$

The eigenvalues of $J(E_0)$ are $s_1 = 1 + ah$ and $s_2 = 1 - h$. Clearly $s_1 > 1$ because $a > 0$ and $h > 0$.

□

Lemma 2.4. *The fixed point* $E_1(\frac{a}{b}, 0)$ *is:*

(i) sink and therefore locally asymptotically stable if $b > ac$ *and* $h < \frac{2b}{b-ac}$,

(ii) saddle point if

(a) $b < ac$

(b) $b > ac,\ h > \frac{2b}{b-ac}$

(iii) non-hyperbolic point if

(a) $b = ac$

(b) $b > ac,\ h = \frac{2b}{b-ac}$

Proof. The variational matrix evaluated at fixed point $E_1(\frac{a}{b}, 0)$ is

$$J(E_1) = \begin{bmatrix} \frac{1}{1+ah} & -\frac{ah}{b+abh} \\ 0 & 1+(-1+\frac{ac}{b})h \end{bmatrix}.$$

The eigenvalues of $J(E_1)$ are $s_1 = \frac{1}{1+ah}$ and $s_2 = 1 + (-1 + \frac{ac}{b})h$. Clearly $s_1 > 1$ because $a > 0$ and $h > 0$.

□

The variational matrix evaluated at $E_2(\frac{1}{c}, \frac{ac-b}{c})$ is

$$J(E_2) = \begin{bmatrix} \frac{c}{c+bh} & -\frac{h}{c+bh} \\ -bh+ach & 1 \end{bmatrix}. \tag{2.2}$$

The characteristic polynomial of $J(E_2)$ is

$$P(s) = s^2 + As + B,$$

where

$$A = -\frac{2c+bh}{c+bh},\ B = \frac{c-bh^2+ach^2}{c+bh}$$

By simple computations, we obtain

$$\begin{aligned} P(0) &= \frac{c-bh^2+ach^2}{c+bh}, \\ P(1) &= \frac{(ac-b)h^2}{c+bh}, \\ P(-1) &= \frac{-bh(-2+h)+c(4+ah^2)}{c+bh}. \end{aligned}$$

It is clear that $P(1) > 0$ because $ac > b$. Moreover, by simple computations one can show that $P(-1) > 0$ for $ac > b$.

$$\begin{aligned} P(-1) =& \frac{-bh(-2+h)+c(4+ah^2)}{c+bh} \\ &= \frac{2bh-bh^2+4c+ach^2}{c+bh} \\ &= \frac{2bh+4c+(ac-b)h^2}{c+bh} \end{aligned}$$

We get the local dynamics of the fixed point $E2$ by using the lemma (2.2).

Proposition 2.5. *Assume that $ac > b$. The fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ of the system* (1.7) *is*

(i) sink and therefore locally asymptotically stable if

$$\frac{b}{c} < a < \frac{b+bh}{ch},$$

(ii) source and therefore unstable if

$$a > \frac{b+bh}{ch},$$

(iv) non-hyperbolic point if

$$a = \frac{b+bh}{ch}.$$

If $ac > b$ and $a = \frac{b+bh}{ch}$, it is evident that the eigenvalues $s_{1,2}$ of $J(E_2)$ are complex satisfying the property $|s_{1,2}| = 1$. Therefore, the system (1.7) undergoes Neimark-Sacker bifurcation at fixed point E_2 if the parameters differ in a small neighbourhood of Σ.

$$\Sigma = \left\{ a,b,c,h \in \mathbb{R}^+ \middle| ac > b,\ a = \frac{b+bh}{ch} \right\}.$$

3. Neimark-Sacker Bifurcation at $E_2(\frac{1}{c}, \frac{ac-b}{c})$

This section focuses on the Neimark-Sacker bifurcation at the system's (1.7) unique positive fixed point. We refer interested readers to [12, 13] for an in-depth analysis of bifurcation theory. Bifurcation analysis has been studied extensively in recent years by several researchers [14, 15, 16, 17, 18].

Consider the domain

$$\Sigma_1 = \left\{ a_1,b,c,h \in \mathbb{R}^+ \middle| a_1 c > b,\ a_1 = \frac{b+bh}{ch} \right\}.$$

Assuming that $(a_1,b,c,h) \in \Sigma_1$, and δ be small perturbation in a_1, we consider the following perturbation of the system (1.7)

$$\begin{cases} x_{n+1} = \frac{(1+h(a+\delta-y_n))x_n}{1+bhx_n}, \\ y_{n+1} = (1+h(cx_n-1))y_n, \end{cases} \tag{3.1}$$

We define $u_n = x_n - \frac{1}{c}$, $v_n = y_n - \frac{(a+\delta)c-b}{c}$, to translate the unique positive fixed point $E_2(\frac{1}{c}, \frac{(a+\delta)c-b}{c})$ of (3.1) to $(0,0)$. The translation map reduces the system (3.1) to

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{c}{c+bh} & -\frac{h}{c+bh} \\ b+ch\delta & 1 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} F(u_n, v_n) \\ chu_n v_n \end{bmatrix}, \tag{3.2}$$

where

$$F(a_n, b_n) = -\frac{c^2 h}{(c+bh)^2} u_n v_n - \frac{bc^2 h}{(c+bh)^2} u_n^2 + \frac{bc^3 h^2}{(c+bh)^3} u_n^2 v_n + \frac{b^2 c^3 h^2}{(c+bh)^3} u_n^3 + O((|u_n| + |v_n|)^4)$$

The characteristic equation for the linear part of the system (3.2) at $(0,0)$ is

$$s^2 - p(\delta)s + q(\delta) = 0, \tag{3.3}$$

with

$$p(\delta) = \frac{2c+bh}{c+bh},\ q(\delta) = \frac{c+bh+ch^2\delta}{c+bh}.$$

The solutions of the equation (3.3) are complex satisfying the property $|s_{1,2}| = 1$, which are given by

$$s_{1,2} = \frac{p(\delta) \pm i\sqrt{4q(\delta) - p^2(\delta)}}{2}.$$

By computations we obtain $|s_1| = |s_2| = \sqrt{q(\delta)}$, and

$$\left(\frac{d|s_1|}{d\delta}\right)_{\delta=0} = \left(\frac{d|s_2|}{d\delta}\right)_{\delta=0} = \frac{ch^2}{2(c+bh)} > 0.$$

Moreover, it is required that $s_1^k, s_2^k \neq 1$ for $k \in \{1,2,3,4\}$ at $\delta = 0$ which is analogous to $p(0) \neq \pm 2, 0, 1$. Since $b, c, h \in \mathbb{R}^+$, therefore $p(0) > 0$. Moreover, we can write $p(0) = 1 + \frac{c}{c+bh} > 1$. Setting $p(0) = 2$ implies that $bh = 0$ which is not possible. Therefore $p(0) \neq \pm 2, 0, 1$.

The following transformation is used to obtain the canonical form of the linear part of (3.2) at $\delta = 0$.

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} -\frac{h}{c+bh} & 0 \\ \frac{bh}{2(c+bh)} & -\frac{\sqrt{bh(4c+3bh)}}{2(c+bh)} \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix}. \tag{3.4}$$

Under the transformation (3.4), the system (3.2) becomes

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} F(e_n, f_n) \\ G(e_n, f_n) \end{bmatrix}, \tag{3.5}$$

where

$$\mu = \frac{2c+bh}{2(c+bh)},\ \nu = \frac{\sqrt{bh(4c+3bh)}}{2(c+bh)},$$

$$F(e_n, f_n) = A_1 e_n^2 + A_2 e_n^3 + A_3 e_n f_n + A_4 e_n^2 f_n + O((|e_n| + |f_n|)^4),$$

$$G(e_n, f_n) = B_1 e_n^2 + B_2 e_n^3 + B_3 e_n f_n + B_4 e_n^2 f_n,$$

where

$$A_1 = \frac{bc^2h^2}{2(c+bh)^3},\ A_2 = \frac{b^2c^3h^4}{2(c+bh)^5},\ A_3 = \frac{c^2\sqrt{bh^3(4c+3bh)}}{2(c+bh)^3},\ A_4 = \frac{c^3\sqrt{b^3h^7(4c+3bh)}}{2(c+bh)^5},$$
$$B_1 = \frac{bch^3(2c^2+2b^2h^2+b(c+4ch))}{2(c+bh)^3\sqrt{bh(4c+3bh)}},\ B_2 = \frac{b^3c^3h^5}{2(c+bh)^5\sqrt{bh(4c+3bh)}},$$
$$B_3 = -\frac{ch^2(2c^2+2b^2h^2+bc(-1+4h))}{2(c+bh)^3},\ B_4 = \frac{b^2c^3h^4}{2(c+bh)^5}.$$

The following quantity L determines the direction in which the closed invariant curve arises in a system with Neimark-Sacker bifurcation.

$$L = \left(\left[-Re\left(\frac{(1-2s_1)s_2^2}{1-s_1}\eta_{20}\eta_{11} \right) - \frac{1}{2}|\eta_{11}|^2 - |\eta_{02}|^2 + Re(s_2\eta_{21}) \right] \right)_{\delta=0},$$

where

$$\eta_{20} = \frac{1}{8}\left[F_{e_ne_n} - F_{f_nf_n} + 2G_{e_nf_n} + i(G_{e_ne_n} - G_{f_nf_n} - 2F_{e_nf_n})\right],$$
$$\eta_{11} = \frac{1}{4}\left[F_{e_ne_n} + F_{f_nf_n} + i(G_{e_ne_n} + G_{f_nf_n})\right],$$
$$\eta_{02} = \frac{1}{8}\left[F_{e_ne_n} - F_{f_nf_n} - 2G_{e_nf_n} + i(G_{e_ne_n} - G_{f_nf_n} + 2F_{e_nf_n})\right],$$
$$\eta_{21} = \frac{1}{16}\left[F_{e_ne_ne_n} + F_{e_nf_nf_n} + G_{e_ne_nf_n} + G_{f_nf_nf_n} + i(G_{e_ne_ne_n} + G_{e_nf_nf_n} - F_{e_ne_nf_n} - F_{f_nf_nf_n})\right].$$

From the above calculations, we obtain the following theorem for the existence and direction of Neimark-Sacker bifurcation.

Theorem 3.1. *Suppose that $a, b, c, h \in \mathbb{R}^+$ and $ac > b$. If $L \neq 0$, then the system* (1.7) *undergoes Neimark-Sacker bifurcation at the fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ when the parameter a varies within a neighbourhood of $a_1 = \frac{b+bh}{ch}$. Additionally, an attracting invariant closed curve will bifurcate from the fixed point if the value of L is less than zero, and a repelling invariant closed curve will bifurcate from the fixed point if the value of L is greater than zero.*

4. Hybrid Control Method to Control Bifurcation and Chaos

Controlling chaos in discrete systems has recently become a popular research area. It is ideal to optimize the system in terms of some success criteria in dynamical systems while avoiding chaos. In discrete models, chaos control may be done using several ways, including the state feedback control method (OGY method) [19, 20], the pole-placement technique [21], and the hybrid control method [22]. This section just covers the hybrid control approach. This technique was designed to manage the period-doubling bifurcation, but it can also control the Neimark-Sacker bifurcation.

We consider the following controlled system associated to (1.7).

$$\begin{cases} x_{n+1} = \frac{\alpha(1+h(a-y_n))x_n}{1+bhx_n} + (1-\alpha)x_n, \\ y_{n+1} = \alpha(1+h(cx_n-1))y_n + (1-\alpha)y_n, \end{cases} \tag{4.1}$$

where $0 < \alpha < 1$. The controlled system (4.1) and the original system (1.7) both have same fixed points. The variational matrix of (4.1) evaluated at E_2 is

$$J(E_2) = \begin{bmatrix} \frac{c+bh-\alpha bh}{c+bh} & -\frac{\alpha h}{c+bh} \\ \alpha h(-b+ac) & 1 \end{bmatrix}.$$

We obtain the characteristic polynomial for $J(E_2)$ as:

$$P(s) = s^2 + \frac{(-2c+bh(-2+\alpha))}{c+bh}s + \frac{c+ach^2\alpha^2 - bh(-1+\alpha+h\alpha^2)}{c+bh}.$$

Keeping in view the Jury condition, system (4.1) is controllable if $\frac{b}{c} < a < \frac{b+\alpha bh}{\alpha ch}$.

5. Numerical Simulations

In this section, we present some numerical examples to support our theoretical findings of the model's multiple qualitative properties, which are discussed in the previous sections.

5.1. Neimark-Sacker bifurcation of the system (1.7) at E_2 by using a as bifurcation parameter: We use the following values for the parameters and initial condition: $b = 4.1,\ c = 2.5,\ h = 1.5, x_0 = 0.35,\ y_0 = 1.1$. The point $E_2(0.4, 1.09333)$ represents the positive fixed point of the system (1.7) when it is applied to these parametric values. The eigenvalues of $J(E_2)$ for $a = 2.73333$ are $s_1 = 0.644509 - 0.764597i, s_2 = 0.644509 + 0.764597i$, which indicates that the system (1.7) is undergoing Neimark-Sacker bifurcation at $E_2(0.4, 1.09333)$ as the bifurcation parameter a crosses $a = a_1 = 2.73333$. Bifurcation diagrams for the prey and the predator , respectively, are shown in the figures (1a, 1b), which are for the range $a \in [2.6, 3]$. For these parametric values the fixed point E_2 is locally asymptotically stable iff $0 < a < 2.73333$. Phase portraits of the model (1.7) are shown in the figures (1c-1f) for various values of the parameter a. According to these figures, the fixed point $E_2(0.4, 1.09333)$ is locally asymptotically stable for $0 < a < 2.73333$, but it becomes unstable at $a = 2.73333$, which is the point at which the model (1.7) experiences Neimark-Sacker bifurcation. At the point where $a = 2.73333$, an invariant closed curve emerges, and as a rises, the radius of this curve grows. In addition, when these numbers are used, the value of L is -0.0344048.

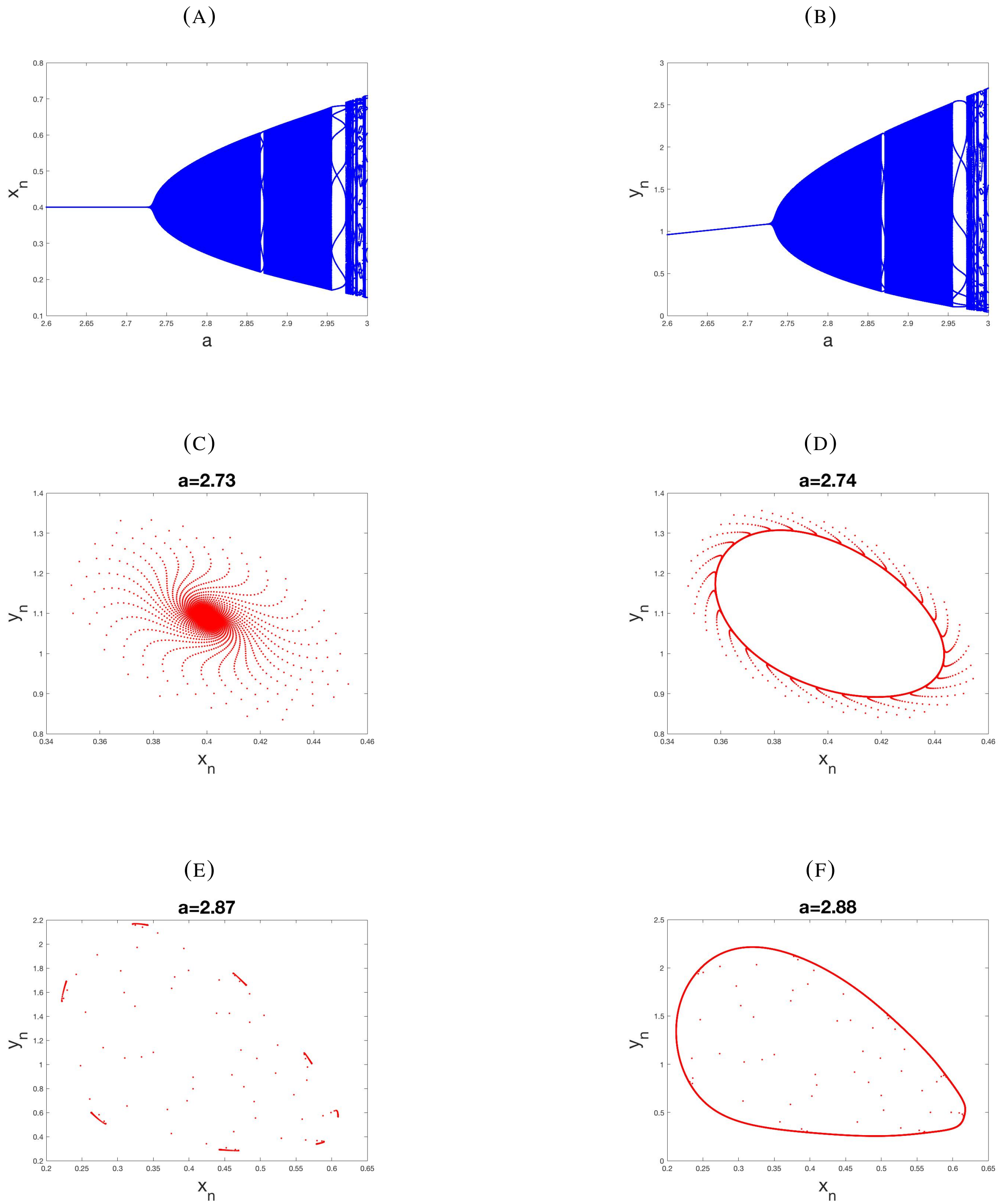

FIGURE 1. Bifurcation diagrams of (1.7) for $b = 4.1,\ c = 2.5,\ h = 1.5, x_0 = 0.35,\ y_0 = 1.1, a \in [2.6, 3]$, and phase portraits for some values of a.

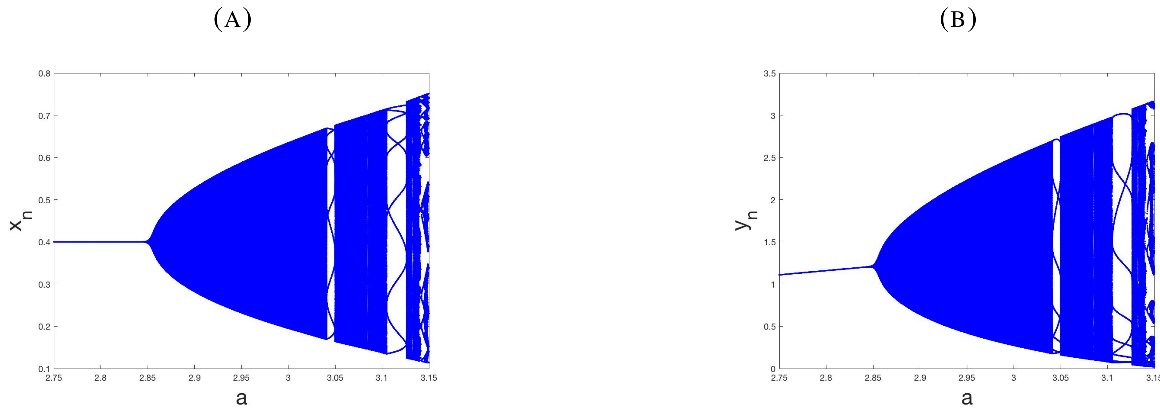

FIGURE 2. Bifurcation diagrams of (1.7) for $b = 4.1,\ c = 2.5,\ h = 1.5,\ \alpha = 0.9,\ x_0 = 0.35,\ y_0 = 1.1$ and $a \in [2.75, 3.15]$.

5.2. Hybrid Control Method: We take the parameter values and initial condition as $b = 4.1,\ c = 2.5,\ h = 1.5,\ \alpha = 0.9,\ x_0 = 0.35,\ y_0 = 1.1$. For these values, the system (4.1) experiences Neimark-Sacker bifurcation as bifurcation parameter a passes through the critical value $a = 2.85481$. Figures (2a, 2b) depict bifurcation diagrams for both prey and predator populations for $a \in [2.75, 3.15]$. Bifurcation diagrams show that the fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ of the system (1.7) is stable for smaller values of a and unstable for larger values of a. Moreover, comparing the bifurcation diagrams ((1a),(1b)) and ((2a),(2b)) it is concluded that the bifurcation is delayed from $a = 2.73333$ to $a = 2.85481$ by using the hybrid control method.

5.3. Neimark-Sacker bifurcation of the system (1.7) at E_2 by using h as bifurcation parameter: We take the parameter values and initial condition as $a = 2.75,\ b = 4.1,\ c = 2.5, x_0 = 0.35,\ y_0 = 1.1$. For these values, the system (1.7) experiences Neimark-Sacker bifurcation as bifurcation parameter h passes through the critical value $h = 1.47748$. Figures (3a, 3b) depict bifurcation diagrams for both prey and predator populations for $h \in [1.3, 1.8]$. Bifurcation diagrams show that the fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ of the system (1.7) is stable for smaller values of h and unstable for larger values of h.

(A) (B)

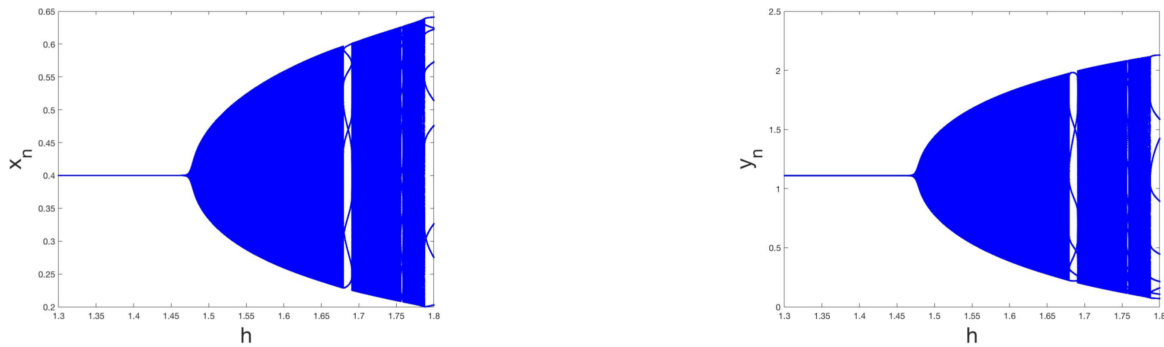

FIGURE 3. Bifurcation diagrams of (1.7) for $a = 2.75,\ b = 4.1,\ c = 2.5, x_0 = 0.35,\ y_0 = 1.1$ and $h \in [1.3, 1.8]$.

5.4. Neimark-Sacker bifurcation of the system (1.7) **at** E_2 **by using** b **as bifurcation parameter:** We take the parameter values and initial condition as $a = 2.75,\ c = 2.5,\ h = 1.5,\ x_0 = 0.35,\ y_0 = 1.1$. For these values, the system (1.7) experiences Neimark-Sacker bifurcation as bifurcation parameter b passes through the critical value $b = 4.125$. Figures (4a, 4b) depict bifurcation diagrams for both prey and predator populations for $b \in [3.75, 4.25]$. Bifurcation diagrams show that the fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ of the system (1.7) is stable for larger values of b and unstable for smaller values of b.

5.5. Neimark-Sacker bifurcation of the system (1.7) **at** E_2 **by using** c **as bifurcation parameter:** We take the parameter values and initial condition as $a = 2.75,\ b = 4.1,\ h = 1.5,\ x_0 = 0.35,\ y_0 = 1.1$. For these values, the system (1.7) experiences Neimark-Sacker bifurcation as bifurcation parameter c passes through the critical value $c = 2.48485$. Figures (5a, 5b) depict bifurcation diagrams for both prey and predator populations for $c \in [2.35, 2.75]$. Bifurcation diagrams show that the fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ of the system (1.7) is stable for smaller values of c and unstable for larger values of c.

(A) (B)

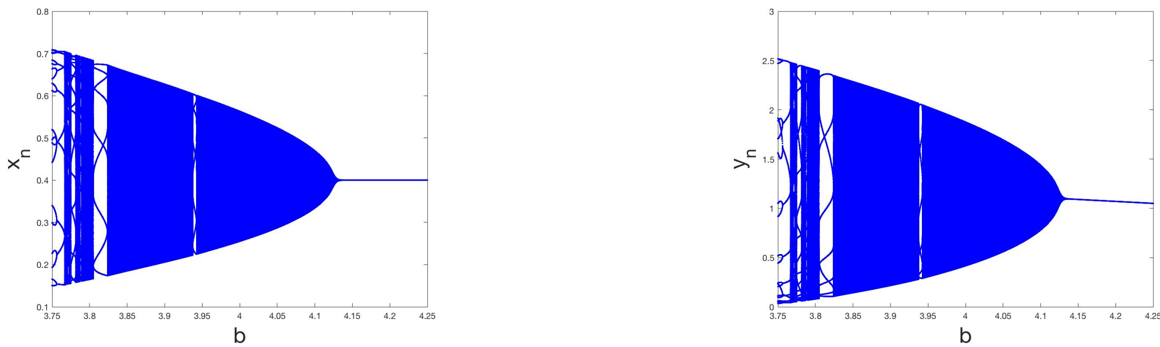

FIGURE 4. Bifurcation diagrams of (1.7) for $a = 2.75,\ c = 2.5,\ h = 1.5, x_0 = 0.35,\ y_0 = 1.1$ and $b \in [3.75, 4.25]$.

(A) (B)

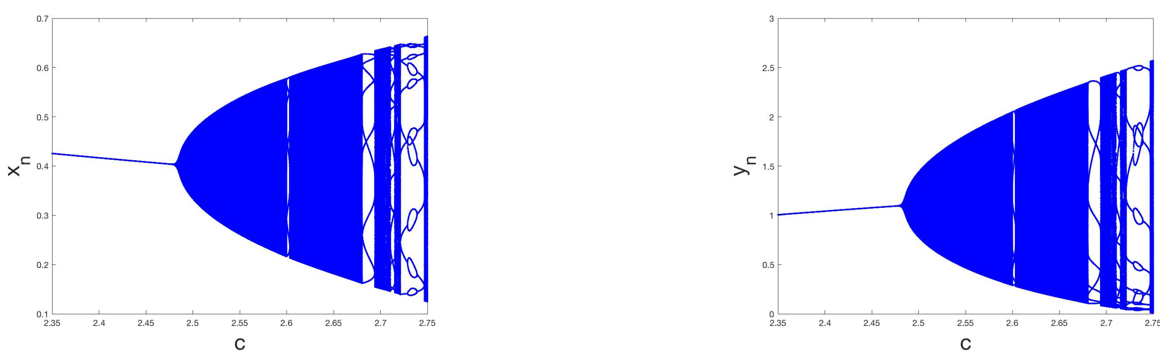

FIGURE 5. Bifurcation diagrams of system (1.7) for $a = 2.75,\ b = 4.1,\ h = 1.5, x_0 = 0.35,\ y_0 = 1.1$ and $c \in [2.35, 2.75]$.

6. CONCLUSION

This study used a dynamically consistent nonstandard finite difference scheme to discretize the classical Lotka-Volterra prey-predator model with a linear functional response. The existence and topological classification of fixed points of the discrete model (1.7) are discussed. It is proved that the system experiences Neimark-Sacker bifurcation at the unique positive fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ when the bifurcation parameter a varies in a small neighbourhood of $a_1 = \frac{b+bh}{ch}$. The hybrid control method is used to control the Neimark-Sacker bifurcation. It is observed that the discrete counterparts of (1.2) obtained by using the Euler method [4] and the piecewise constant argument method [5] experience period-doubling bifurcation and Neimark-Sacker bifurcation. The discrete counterpart of (1.2) obtained by using a nonstandard finite difference scheme experiences only Neimark-Sacker bifurcation. Moreover, in numerical simulations it is observed that for small values of parameters a, c and h the fixed point $E_2(\frac{1}{c}, \frac{ac-b}{c})$ of the system (1.7) is stable, and E_2 is unstable for large values of parameters a, c and h. The fixed point E_2 is unstable for small values of b and it is stable for large values of b.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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