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FISHING ACTIVITY IN THE ATLANTIC MOROCCAN OCEAN: MATHEMATICAL MODELING AND OPTIMAL CONTROL

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Abstract. As it spans the Mediterranean sea and the Atlantic Ocean on the north and west respectively, Morocco is characterized by its marine sources' diversity. Fishing Activity is considered to be the backbone of the country's economy. the process of structuring the exploited population, and the marine environment are two major fields that the kingdom must understand to implement management measures to sustain the equilibrium of national fisheries and marine sources. Here, we have built a discrete-time model to describe the dynamic of fishing of two small pelagic species on the Atlantic Coast. The equilibrium between marine population and fishing effort, and maximizing profit are what the control parameter proposed have to achieve. We use a discrete version of Pontryagin's principal maximum to calculate the optimal system. The numerical simulation is carried out using Matlab.

Keywords: Sardina pilchardus; Scomber japonicus; optimal control; existence of optimal control; discrete Pontryagin's maximum principale; maximizing the profit.

2020 AMS Subject Classification: 92C42.

1. Introduction

According to [1], in Morocco "The most important stocks remains too small pelagics whose production exceeds 80% of the weight landed at the national level. 2017 was characterized by

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generally favorable hydrological conditions for small pelagics resources on the Atlantic coast, resulting in an increase in Biomass in this region with 7.59 million tonnes assessed in the fall, while catches in this region biomass are of an order 1 458 155 tonnes and are at the same level as the previous year. There ported catch volumes for these species show an increase of 38% in the northern zone (Tangier-Safi) and 2% in the southern zone(Cape Boujdour-Cap Blanc), whereas these catches decreased by 5% in the central area(Safi-Cap Boujdour). At, catches of small pelagic fish have decreased by 29% compared to 2016";

The pelagics ressources are composed of Sardine (Sardina Pilchardus), Chub Mackerel (Scomber colias), Anchois (Engraulis Encrasicolus), Chinchard (Trachurus Trachurus, and Trachurus Trecas) and Sardinelle (Sardinelle Aurita and Sardinelle amaderensis). These resources are exploited at the level of four major fishing areas:

Mediteranean Zone : Saadia- Cap Spartel

Zone NorthAtlantic: Cap Spartel- Cap Cantin

Zone Atlantic Centre(Zone A+B): Cap Cantin- Cap Boujdour

Zone Atlantic Sud (Zone C): Cap Boujdour- Cap Blanc

It shows that Sardine and Chub Mackerel compete for food and space.

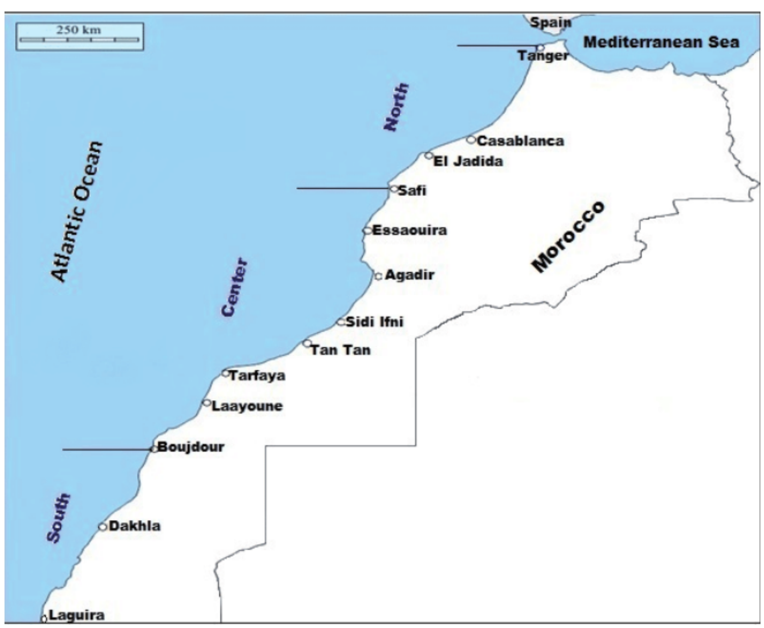

FIGURE 1. The three Zone in the Atlantic Moroccan sea.

And by three types of fleets: Coastal purse seiners, Moroccan RWS, and pelagic freezer trawler. According to ([2],[3]), the sardines (Sardina pilchardus) are the dominant species among small pelagics. In 2006, it accounted for about 66 percent of the total catch of these species. Between 2001 and 2004, we notice a gradual decrease in landings from 770,000 to 640,000 tonnes, then an increase to reach 700,000 tonnes in 2005 followed by a slight decrease in 2006 with 620,000 tonnes. The average catch of sardines over the last five years (2002-2006) is approximately 657,000 tonnes. The second most important species that landed in Morocco in 2006 was mackerel (Scomber Colias) with a total catch of around 160,000 tonnes representing roughly 17 percent of total landings of small pelagics. The European horse mackerel (Trachurus trachurus) is the third most important species in Morocco in 2006 and accounts for about 7 percent of total landings of small pelagics in this country. The round sardinella (Sardinella aurita) comes then with 4 percent of landings. Despite the overall downward trend observed for this species since the late 1990s, a significant increase in catches of S. aurita appears, rising from 1,600 tonnes in 2004 to 33,000 tonnes in 2006.

In 2006, the catch of anchovies (Engraulis encrasicolus) amounted to around 10,000 tonnes, which accounts for about 1 percent of the total small pelagic catch.

According to [1], 1.46 million tonnes of small pelagics, composed of Sardine 73%, followed by Chub Mackerel 17%, Chinchard 7% and Sardinelle 2% and Anchors 1%, the important capture are realized in the Sud and Centre Zone.

In the North, Centre, and Sud Zones, Sardine Pilchardus and Chub Mackerel distribution continue all along the coast of this area, they grow in abundance along the Atlantic Ocean.

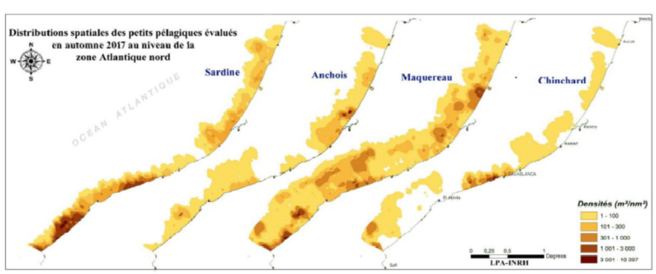

FIGURE 2. Distribution of pelagic species in the North Zone.

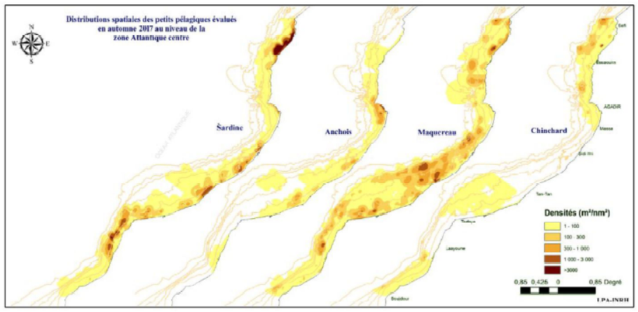

FIGURE 3. Distribution of pelagic species in the Centre Zone.

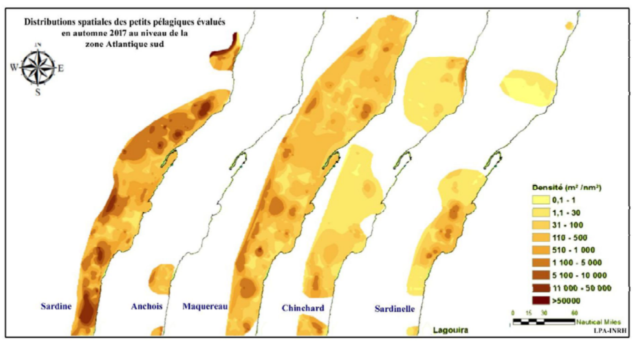

FIGURE 4. Distribution of pelagic species in the Sud Zone.

According to [4], Colin Clark published a book entitled Mathematical Bioeconomics: The optimal management of renewable resources in 1976, this book influenced many researchers to work on problems of management of natural resources. This book shows the importance of biological and economic aspects when studying natural resources. Among these works we can refer to [5] this paper deals with the problem of combined harvesting of two competing fish species, it analyses the dynamic behavior of the exploited system and studies some aspects of the optimal harvest policy. In [6] Chaudhuri studies the problem of dynamic optimization of the exploitation policy connected with the combined harvesting of two competing fish species, and discusses the possibilities of the existence of bioeconomic equilibrium in a system of combined harvesting of the prey-predator system in [7]. Also we can refer to ([8] [9] [10] [11] [12]). All works take problem of management of nature maritime resources in the point of view of stability.

In this work, we propose to study a specific prey-predator model, witch the prey is two marine species (Sarcina Pilchardus and Scomber colias) the predator are fishing efforts of both (Sardina P and Scomber C), plus two equations describing the dynamics of the market price of Sardine and Chub Mackerel. We assume that Sardine and Chub Mackerel compete for food and space. The model in question is a discrete multi-region system since the distribution of Sardine and Chub Mackerel continues along the Atlantic sea. Using a discrete version of Pontryagin's Maximum Principle, we built the optimality system using two controls u_1 u_2 (Conserving an equilibrium state between Sardine(Sardina Pilcardus) and its harvesting and Conserving an equilibrium state between Chub Mackerel (Scomber Colias) and its harvesting, also maximizing the profit). The optimal result is tested using an iterative function compiled on Matlab.

2. Presentation of Model

The model that we will study is a discrete multi-region model, which describes the fishing of two competing pelagic species x (Sardina Pilchardus) and y (Scomber Colias); Essentially these two species (Sardine and Chub Mackerel are found all along the Atlantic (North Zone, Center Zone and finally South Zone). We focus the study on the Atlantic since statistics in the Mediterranean are rare.

Our model is a discrete multi region version of Auger model ([11],[12]), we take the general Cobb-Douglas harvestiong function of fishery as well ([13],[14],[15]).

The model describing the dynamic of fishing of Sardine and Chub Mackerel is:

for $j \in \{1,2,3\}$,

$$(1) \qquad \begin{cases} x_{i+1}^j = x_i^j + r_1^j x_i^j \left(1 - \frac{x_i^j}{K_1^j}\right) - q_1^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j} - \theta^j x_i^j y_i^j \\ y_{i+1}^j = y_i^j + r_2^j y_i^j \left(1 - \frac{y_i^j}{K_2^j}\right) - q_2^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j} - \theta^j x_i^j y_i^j \\ E_{1,i+1}^j = E_{1,i}^j + \gamma_1^j (q_1^j P_{1,i} (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j} - c_1^j E_{1,i}^j) \\ E_{2,i+1}^j = E_{2,i}^j + \gamma_2^j (q_2^j P_{2,i} (y_i^j)^{\alpha_2^j} (E_{i,2}^j)^{\beta_2^j} - c_2^j E_{2,i}^j) \\ P_{1,i+1}^j = P_{1,i}^j + \mu_1^j \left(\frac{A_1^j}{P_{1,i}^j} - q_1^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j}\right) \\ P_{2,i+1}^j = P_{2,i}^j + \mu_2^j \left(\frac{A_2^j}{P_{2,i}^j} - q_2^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j}\right) \end{cases}$$

with initial conditions $x_0^j > 0, \quad y_0^j > 0 \quad E_{1,0}^j > 0, \quad E_{2,0}^j > 0, \quad P_{1,0}^j > 0, \quad P_{2,0}^j > 0.$

Our model contains six equations that we can divide into three parts. The first part contains the first and second equations describing the evolution of the biomass of sardine (Sardina pilchardus) and Chub Mackerel (Scomber Colias) in presence of harvesting and competitions between these two species. The second part contains the third and fourth equations describing the evolution of the fishing efforts for both Sardine and Chub Mackerel, whereas the third part describes the evolution of the market price of these species.

We assume in this model that the evolution of Sardine x and Chub Mackerel y obey to a discrete version of the law of logistic growth. $r_1^j x_i^j (1 - \frac{x_i^j}{K_1^j})$, and $r_2^j y_i^j (1 - \frac{y_i^j}{K_2^j})$. The biomass of these species is decreasing by harvesting. We choose a general harvesting function ([13]) for Sardine x $q_1^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j}$, and the same general harvesting function is used for decreasing the biomass of Chub mackerel y $q_2^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j}$. The biomass x and y are also decreasing by competition for food and spaces between this two species $\theta x_i^j y_i^j$. $r_i^j i \in \{1,2\} j \in \{1,2,3\}$ is the intrinsic growth rate of stock x and y respictively. $K_i^j and i \in \{1,2\} j \in \{1,2,3\}$ are the carying capacities, r_i^j, K_i^j are different from sardine and Chub Mackerel and from different regions in the Atlantic. $q_i^j i \in \{1,2\} j \in \{1,2,3\}$ is the catch-ability coefficients, when θ^j are the competition coefficients. α_i^j, and β_i^j are the biomass and effort output elasticity respectively. The effort fishing for Sardine is E_1^j when the effort fishing for Chub Mackerel is E_2^j $j \in \{1,2,3\}$, the predator in our system is the effort of fishing. $c_i^j i \in \{1,2\} j \in \{1,2,3\}$ describes the cost per unit effort. $P_1^j$and $P_2^j j \in \{1,2,3\}$ are the price market of sardine and Chub Mackerel respectively. The variation of the price market for x and y is the difference between the demand $\frac{A_1^j}{P_1} \frac{A_2^j}{P_2}$, and the supply $q_1^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j}$ and $q_2^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j}$ $j \in \{1,2,3\}$ respectively.

3. An Optimal Control Problem

On the first hand, if fishing is carried on at specific periods in the year when maritime reserves are ecologically low in density, there is a risk of the extinction of some marine species. On other hand, if vessels' activity lowered fishing activity will disappear.

u_1, u_2 will be restricting vessels to use certain fishing techniques, to limit the catch especially when the number of biomass decrease for some natural raisins. The tables below reflect the evolution of the biomass of Sardine and Chub mackerel in Morocco in 2015 and 2007.

Therefore, the balance must be struck between maritime resources density and the frequency of fishing activity.

According to the report of 2015

stock	Zone	Assesement
Sardine(S. pilchardus)	Zone A+B	Non-fully exploited(2013)
Sardine (S. pilchardus)	Zone C	Non fully exploited (2013)
Chub mackerel(Scomber colias)	Whole subregion	Fully exploited

Or according to the rapport of 2007, we have

stock	Zone	Assesement
Sardine(S. pilchardus)	Zone A+B	Stock is overexploited.
Sardine (S. pilchardus)	Zone C	Stock is not fully exploited
Chub mackerel(Scomber colias)	Whole subregion	Stock is not fully exploited

In this work, we introduce two controls parameter. The first one is called u_1 which is the parameter control that leads to conserving state equilibrium between Sardine and its harvesting. The second parameter control is u_2 which leads to conserving stable equilibrium between Chub Mackerel and its harvesting. u_1, u_2 are real constant choosing also to bring much profit.
Our goal is to maximize the objective functional presented as follow, for $j \in \{1,2,3\}$

$$(2) \quad J(u_1^j, u_2^j) = \sum_{i=0}^{t_f-1} \rho_1^j (u_{1,i}^j P_{1,i}^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j})^2 - \xi_1^j (E_{1,i}^j)^2 + \rho_2^j (u_{2,i}^j P_{2,i}^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j})^2 - \xi_2^j (E_{2,i}^j)^2$$

Were the parametre $\rho_1 > 0$, $\rho_2 > 0$, $\xi_1 > 0$ and ξ_2 are the cost coefficients, they are selected to weigh the relative importance $u_{1,i}$, $u_{2,i}$ at time i. t_f is the final time.

$$(3) \qquad J\left({u_{1,i}^j}^*, {u_{2,i}^j}^*\right) = \max_{(u_{1,i}^j, u_{2,i}^j) \in U_{ad}} J(u_{1,i}^j, u_{2,i}^j)$$

for $j \in \{1,2,3\}$ subject to

$$
\begin{cases}
x_{i+1}^j = x_i^j + r_1^j x_i^j (1 - \dfrac{x_i^j}{K_1^j}) - q_1^j u_{1,i}^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j} - \theta^j x_i^j y_i^j \\
y_{i+1}^j = y_i^j + r_2^j y_i^j (1 - \dfrac{y_i^j}{K_2^j}) - q_2^j u_{2,i}^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j} - \theta^j x_i^j y_i^j \\
E_{1,i+1}^j = E_{1,i}^j + \gamma_1^j (q_1^j u_{1,i}^j P_{1,i} (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j} - c_1^j E_{1,i}^j) \\
E_{2,i+1}^j = E_{2,i}^j + \gamma_2^j (q_2^j u_{2,i}^j P_{2,i} (y_i^j)^{\alpha_2^j} (E_{i,2}^j)^{\beta_2^j} - c_2^j E_{2,i}^j) \\
P_{1,i+1}^j = P_{1,i}^j + \mu_1^j (\frac{A_1^j}{P_{1,i}^j} - q_1^j u_{1,i}^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j}) \\
P_{2,i+1}^j = P_{2,i}^j + \mu_2^j (\frac{A_2^j}{P_{2,i}^j} - q_2^j u_{2,i}^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j})
\end{cases} \tag{4}
$$

with initial conditions $x_0^j > 0$, $y_0^j > 0$, $E_{1,0}^j > 0$, $E_{2,0}^j > 0$, $P_{1,0}^j > 0$, $P_{2,0}^j > 0$, where U_{ad} is the set of admissible controls defined by

$$
U_{ad} = \Big\{ \Big(u_{1,i}^j, u_{2,i}^j, \Big) : 0 < u_{1,min}^j \leq u_{i,1}^j \leq u_{1,max}^j < 1, 0 < u_{2,min}^j \leq u_{i,2}^j \leq u_{2,max}^j < 1; \quad i = 0,1,2\ldots t_f - 1 \Big\}. \tag{5}
$$

3.1. Existence theorem. To demonstrate the existence of the optimal pair for J, we use a result of Dabbs and present in ([16], [17], [18], [19]).

Theorem 1. *There exists an optimal control $({u_{1,i}^j}^*, {u_{2,i}^j}^*) \in U_{ad}$ such that*

$$
J({u_{1,i}^j}^*, {u_{2,i}^j}^*) = \max_{(u_{1,i}^j, u_{2,i}^j) \in U_{ad}} J(u_{1,i}^j, u_{2,i}^j).
$$

Proof. For all $j \in \{1,2,3\}$, since the coefficients of the state equations are bounded and there are a finite number of time steps, x^j y^j E_1^j E_2^j, P_1^j, P_2^j are uniformly bounded for all $(u_{1,i}^j, u_{2,i}^j)$ in the control set U_{ad}. Thus $J(u_{1,i}^j, u_{2,i}^j)$ is bounded for all $(u_{1,i}^j, u_{2,i}^j) \in U_{ad}$.
Since $J(u_{1,i}^j, u_{2,i}^j)$ is bounded, $\max_{(u_{1,i}^j, u_{2,i}^j) \in U_{ad}} J(u_{1,i}^j, u_{2,i}^j)$ is finite, and there exists a sequence $({u_{1,i}^j}^k, {u_{2,i}^j}^k) \in U_{ad}$ such that

$$
\lim_{k \to +\infty} ({u_{1,i}^j}^k, {u_{2,i}^j}^k) = \inf_{(u_{1,i}^j, u_{2,i}^j) \in U_{ad}} J(u_{1,i}^j, u_{2,i}^j)
$$

and corresponding sequences of states

$$
x^{j^k} \to x^j
$$

$$y^{jk} \to y^j$$

$$E_1^{jk} \to E_1^j$$

$$E_2^{jk} \to E_2^j$$

$$P_1^{jk} \to P_1^j$$

$$P_2^{jk} \to P_2^j$$

Since there is a finite number of uniformly bounded sequences, there exist $({u_{1,i}^j}^*, {u_{2,i}^j}^*) \in U_{ad}$ and $(x^{j^*}, y^{j^*}, E_1^{j^*}, E_2^{j^*}, P_1^{j^*}, P_2^{j^*}) \in \mathbb{R}^{t_f+1}$ such that, on a subsequence,

$$({u_{1,i}^j}^k, {u_{2,i}^j}^k) \to ({u_{1,i}^j}^*, {u_{2,i}^j}^*).$$

Finally, due to the finite dimensional structure of system (4) and the objective function $J(u_{1,i}^j, u_{2,i}^j)$, $({u_{1,i}^j}^*, {u_{2,i}^j}^*)$ is an optimal control with corresponding states $(x^{j^*}, y^{j^*}, E_1^{j^*}, E_2^{j^*}, P_1^{j^*}, P_2^{j^*})$. Therefore $\max_{(u_{1,i}^j, u_{2,i}^j) \in U_{ad}} J(u_{1,i}^j, u_{2,i}^j)$. □

3.2. Characterization of the optimal control. A necessary condition for an optimal control is obtained by applying the discrete version of the Pontryagin's Maximum Principle . This principle converts (3)-(4) into a problem of maximizing of Hamiltonian $\mathbf{H_i}$ at time step i defined by, for all $j \in \{1,2,3,4\}$,

$$\begin{aligned}(6)\quad \mathbf{H_i^j} \quad &= \rho_1^j (u_{1,i}^j P_{1,i}^j (x_i^j)^{\alpha_1^j} (E_{1,i}^j)^{\beta_1^j})^2 - \xi_1^j (E_{1,i}^j)^2 + \rho_2^j (u_{2,i}^j P_{2,i}^j (y_i^j)^{\alpha_2^j} (E_{2,i}^j)^{\beta_2^j})^2 - \xi_2^j (E_{2,i}^j)^2 \\ &+ \lambda_{1,i+1}^j x_{i+1}^j + \lambda_{2,i+1}^j y_{i+1}^j + \lambda_{3,i+1}^j E_{1,i+1}^j + \lambda_{4,i+1}^j E_{2,i+1}^j + \lambda_{5,i+1}^j P_{1,i+1}^j + \lambda_{6,i+1}^j P_{2,i+1}^j.\end{aligned}$$

In ordre to calculate the adjoints functions, we apply the Pontryagin's principle ([20], [[21], [22]).

Theorem 2. *Let* $({u_{1,i}^j}^*, {u_{2,i}^j}^*)$ *be an optimal control solution of (3), and* $(x_i^{j^*}, y_i^{j^*}, {E_{1,i}^j}^*, {E_{2,i}^j}^*, {P_{1,i}^j}^*, {P_{2,i}^j}^*)$ *the solution of (4) according to* $({u_{1,i}^j}^*, {u_{2,i}^j}^*)$. *There exists adjoints functions* $\lambda_{1,i}^j \lambda_{2,i}^j$, $\lambda_{3,i}^j$, $\lambda_{4,i}^j$, $\lambda_{5,i}^j$ *and* $\lambda_{6,i}^j$ *satisfying*

$$\begin{cases}
\lambda^j_{1,i} = 2\rho^j_1\alpha^j_1(u^j_{1,i}P^j_{1,i}(E^j_{1,i})^{\beta^j_1})^2(x^j_i)^{2\alpha^j_1-1} + \lambda^j_{1,i+1}(1+r^j_1(1-2\frac{x^j_i}{K^j_1}) - \alpha^j_1 q^j_1 u^j_{i,1}(x^j_i)^{\alpha^j_1-1}(E^j_{1,i})^{\beta^j_1} \\
-\theta^j y_i) - \lambda^j_{2,i+1}\theta^j y^j_i + \alpha^j_1 q^j_1 u^j_{1,i}(x^j_i)^{\alpha^j_1-1}E^j_{1,i})^{\beta^j_1}(\lambda^j_{3,i+1}\gamma^j_1 P^j_1 - \lambda^j_{5,i+1}\mu^j_1) \\
\lambda^j_{2,i} = 2\rho^j_2\alpha^j_2(u^j_{2,i}P^j_{2,i}E^j_{2,i}{}^{\beta^j_2})^2 y_i^{j\,2\alpha^j_2-1} + \lambda^j_{2,i+1}(1+r^j_2(1-2\frac{y^j_i}{K^j_2}) - \alpha^j_2 q^j_2 u^j_{2,i} y_i^{j\,\alpha^j_2-1}E^j_{2,i}{}^{\beta^j_2} \\
-\theta^j x_i) - \lambda^j_{1,i+1}\theta^j x^j_i + u^j_{2,i}\alpha^j_2 y_i^{j\,\alpha^j_2-1}E^j_{2,i}{}^{\beta^j_2}(\lambda^j_{4,i+1}\gamma^j_2 P^j_{2,i} - \lambda^j_{6,i+1}\mu^j_2) \\
\lambda^j_{3,i} = 2\beta^j_1\rho^j_1(u^j_{1,i}P_{1,i}(x^j_i)^{\alpha^j_1})^2(E^j_{1,i})^{2\beta^j_1-1} - 2\xi^j_1 E_{1,i} + \beta^j_1 q^j_1 u^j_1(x^j_i)^{\alpha^j_1}(E^j_{1,i})^{\beta^j_1-1} \\
(-\lambda^j_{1,i+1} + \gamma^j_1 P^j_{1,i}\lambda_{3,i+1} - \mu^j_1\lambda_{5,i+1}) + (1-c_1)\lambda_{3,i+1} \\
\lambda^j_{4,i} = 2\beta^j_2\rho^j_2(u^j_{2,i}P_{2,i}(y^j_i)^{\alpha^j_2})^2(E^j_{2,i})^{2\beta^j_2-1} - 2\xi^j_2 E^j_{2,i} + \beta^j_2 q^j_2 u^j_{2,i}(y^j_i)^{\alpha^j_2}(E^j_{2,i})^{\beta^j_2-1} \\
(-\lambda^j_{2,i+1} + \gamma^j_2 P^j_{2,i}\lambda_{4,i+1} - \mu^j_2\lambda_{6,i+1}) + (1-c_2)\lambda_{4,i+1} \\
\lambda^j_{5,i} = 2\rho^j_1 P^j_{1,i}(u^j_{1,i}(x^j_i)^{\alpha^j_1}(E^j_{1,i})^{\beta^j_1})^2 + \lambda^j_{3,i+1}(\gamma^j_1 u^j_{1,i} q^j_1(x^j_i)^{\alpha^j_1}(E^j_{1,i})^{\beta^j_1} + \lambda^j_{5,i+1}(1-\mu^j_1\frac{A^j_1}{(P^j_{1,i})^2}). \\
\lambda^j_{6,i} = 2\rho^j_2 P^j_{2,i}(u^j_{2,i}(y^j_i)^{\alpha^j_2}(E^j_{2,i})^{\beta^j_2})^2 + \lambda^j_{4,i+1}(\gamma^j_2 u^j_{2,i} q^j_2(y^j_i)^{\alpha^j_2}(E^j_{2,i})^{\beta^j_2} + \lambda^j_{6,i+1}(1-\mu^j_2\frac{A^j_2}{(P^j_{2,i})^2}).
\end{cases}$$

with transversality conditions

$$\lambda_1(t_f) = 0,\ \lambda_2(t_f) = 0,\ \lambda_3(t_f) = 0,\ \lambda_4(t_f) = 0,\ \lambda_5(t_f) = 0,\ \lambda_6(t_f) = 0$$

furthermore

$$u^j_{1,i} = \max\left\{\min\left\{\frac{q^j_1(\lambda_{1,i+1} - \gamma^j_1\lambda_{3,i+1} + \mu^j_1\lambda_{5,i+1}}{2\rho^j_1 P_{1,i}(x^j_i)^{\alpha^j_1}(E^j_{1,i})^{\beta^j_1}}, 1\right\}, 0\right\},$$

$$u^j_2 = \max\left\{\min\left\{\frac{q^j_2(\lambda_{2,i+1} - \gamma^j_2\lambda_{4,i+1} + \mu^j_2\lambda_{6,i+1}}{2\rho^j_2 P_{2,i}(y^j_i)^{\alpha^j_2}(E^j_{2,i})^{\beta^j_2}}, 1\right\}, 0\right\}.$$

for all $i \in \{0, 1, 2, \cdot, t_f - 1\}$ *and all* $j \in \{1, 2, 3\}$.

Proof. Using the discrete version of Pontryagin's Maximum Principal

$$\lambda^j_{1,i} = \frac{\partial \mathbf{H^j_i}}{\partial x^j_i}$$

$$\lambda^j_{2,i} = \frac{\partial \mathbf{H^j_i}}{\partial y^j_i}$$

$$\lambda^j_{3,i} = \frac{\partial \mathbf{H^j_i}}{\partial E^j_{1,i}}$$

$$\lambda^j_{4,i} = \frac{\partial \mathbf{H^j_i}}{\partial E^j_{2,i}}$$
$$\lambda^j_{5,i} = \frac{\partial \mathbf{H^j_i}}{\partial P^j_{1,i}}$$
$$\lambda^j_{6,i} = \frac{\partial \mathbf{H^j_i}}{\partial P^j_{2,i}}$$

where

$$\begin{aligned} \mathbf{H^j_i} &= \rho^j_1 (u^j_{1,i} P^j_{1,i} (x^j_i)^{\alpha^j_1} (E^j_{1,i})^{\beta^j_1})^2 - \xi^j_1 (E^j_{1,i})^2 + \rho^j_2 (u^j_{2,i} P^j_{2,i} (y^j_i)^{\alpha^j_2} (E^j_{2,i})^{\beta^j_2})^2 - \xi^j_2 (E^j_{2,i})^2 \\ &+ \lambda^j_{1,i+1} (x^j_i + r^j_1 x^j_i (1 - \frac{x^j_i}{K^j_1}) - u^j_{1,i} q^j_1 (x^j_i)^{\alpha^j_1} (E^j_{i,1})^{\beta^j_1} - \theta^j x^j_i y^j_i) + \lambda^j_{2,i+1} (y^j_i + r^j_2 y^j_i (1 - \frac{y^j_i}{K^j_2}) \\ &- u^j_{2,i} q^j_2 (y^j_i)^{\alpha^j_2} (E^j_{i,2})^{\beta^j_2} - \theta^j x^j_i y^j_i) + \lambda^j_{3,i+1} (E^j_{1,i} + \gamma^j_1 (u^j_{1,i} q^j_1 P^j_{1,i} (x^j_i)^{\alpha^j_1} (E^j_{1,i})^{\beta^j_1} - c^j_1 E^j_{1,i})) \\ &+ \lambda^j_{4,i+1} (E^j_{2,i} + \gamma^j_2 (u^j_{2,i} q^j_2 P^j_{2,i} (y^j_i)^{\alpha^j_2} (E^j_{2,i})^{\beta^j_2} - c^j_2 E^j_{2,i})) + \lambda^j_{5,i+1} (P^j_{1,i} + \mu^j_1 (\frac{A^j_1}{P^j_{1,i}} \\ &- u^j_{1,i} q^j_1 (x^j_i)^{\alpha^j_1} (E^j_{1,i})^{\beta^j_1})) + \lambda^j_{6,i+1} (P^j_{2,i} + \mu^j_2 (\frac{A^j_2}{P^j_{2,i}} - u^j_{2,i} q^j_2 (y^j_i)^{\alpha^j_2} (E^j_{2,i})^{\beta^j_2})) \end{aligned} \tag{7}$$

so

$$\begin{aligned} \lambda^j_{1,i} &= 2\rho^j_1 \alpha^j_1 (u^j_{1,i} P^j_{1,i} {E^j_{1,i}}^{\beta^j_1})^2 {x^j_i}^{2\alpha^j_1 - 1} + \lambda^j_{1,i+1} (1 + r^j_1 (1 - 2\frac{x^j_i}{K^j_1}) - \alpha^j_1 q^j_1 u^j_{i,1} {x^j_i}^{\alpha^j_1 - 1} {E^j_{1,i}}^{\beta^j_1} \\ &- \theta^j y_i) - \lambda^j_{2,i+1} \theta^j y^j_i + \lambda^j_{3,i+1} \gamma^j_1 \alpha^j_1 q^j_1 u^j_{1,i} {x^j_i}^{\alpha^j_1 - 1} {E^j_{1,i}}^{\beta^j_1} - \lambda^j_{5,i+1} \mu^j_1 u_{1,i} \alpha^j_1 {x^j_i}^{\alpha^j_1 - 1} {E^j_{1,i}}^{\beta^j_1} \end{aligned} \tag{8}$$

$$\begin{aligned} \lambda^j_{2,i} &= 2\rho^j_2 \alpha^j_2 (u^j_{2,i} P^j_{2,i} ({E^j_{2,i}}^{\beta^j_2})^2 {y^j_i}^{2\alpha^j_2 - 1} + \lambda^j_{2,i+1} (1 + r^j_2 (1 - 2\frac{y^j_i}{K^j_2}) - \alpha^j_2 q^j_2 u^j_{2,i} {y^j_i}^{\alpha^j_2 - 1} {E^j_{2,i}}^{\beta^j_2} \\ &- \theta^j x_i) - \lambda^j_{1,i+1} \theta^j x^j_i + \lambda^j_{4,i+1} \gamma^j_2 \alpha^j_2 q^j_2 u^j_{2,i} {y^j_i}^{\alpha^j_2 - 1} {E^j_{2,i}}^{\beta^j_2} - \lambda^j_{6,i+1} \mu^j_2 u^j_{2,i} \alpha^j_2 {y^j_i}^{\alpha^j_2 - 1} {E^j_{2,i}}^{\beta^j_2} \end{aligned} \tag{9}$$

$$\begin{aligned} \lambda^j_{3,i} &= 2\beta^j_1 \rho^j_1 (u^j_{1,i} P_{1,i} {x^j_i}^{\alpha^j_1})^2 {E^j_{1,i}}^{2\beta^j_1 - 1} - 2\xi^j_1 E_{1,i} - \lambda^j_{1,i+1} \beta^j_1 q^j_1 u^j_1 {x^j_i}^{\alpha^j_1} {E^j_{1,i}}^{\beta^j_1 - 1} \\ &+ \lambda^j_{3,i+1} (1 + \gamma^j_1 (\beta^j_1 u^j_{1,i} q^j_1 P^j_{1,i} {x^j_i}^{\alpha^j_1} {E^j_{1,i}}^{\beta^j_1 - 1} - c^j_1) - \lambda^j_{5,i+1} \mu^j_1 u^j_{1,i} \beta^j_1 q^j_1 {x^j_i}^{\alpha^j_1} {E^j_{1,i}}^{\beta^j_1 - 1}. \end{aligned} \tag{10}$$

$$\begin{aligned} \lambda^j_{4,i} &= 2\beta^j_2 \rho^j_2 (u^j_{2,i} P_{2,i} (y^j_i)^{\alpha^j_2})^2 (E^j_{2,i})^{2\beta^j_2 - 1} - 2\xi^j_2 E^j_{2,i} - \lambda^j_{2,i+1} \beta^j_2 q^j_2 u^j_{2,i} (y^j_i)^{\alpha^j_2} (E^j_{2,i})^{\beta^j_2 - 1} \\ &+ \lambda^j_{4,i+1} (1 + \gamma^j_2 (\beta^j_2 u^j_{2,i} q^j_2 P^j_{2,i} (y^j_i)^{\alpha^j_2} (E^j_{2,i})^{\beta^j_2 - 1} - c^j_2) - \lambda^j_{6,i+1} \mu^j_2 u^j_{2,i} \beta^j_2 q^j_2 (y^j_i)^{\alpha^j_2} (E^j_{2,i})^{\beta^j_2 - 1}. \end{aligned} \tag{11}$$

(12) $$\lambda^j_{5,i} = 2\rho^j_1 P^j_{1,i}(u^j_{1,i}(x^j_i)^{\alpha^j_1}(E^j_{1,i})^{\beta^j_1})^2 + \lambda^j_{3,i+1}(\gamma^j_1 u^j_{1,i} q^j_1 (x^j_i)^{\alpha^j_1}(E^j_{1,i})^{\beta^j_1} + \lambda^j_{5,i+1}(1-\mu^j_1 \frac{A^j_1}{(P^j_{1,i})^2}).$$

(13) $$\lambda^j_{6,i} = 2\rho^j_2 P^j_{2,i}(u^j_{2,i}(y^j_i)^{\alpha^j_2}(E^j_{2,i})^{\beta^j_2})^2 + \lambda^j_{4,i+1}(\gamma^j_2 u^j_{2,i} q^j_2 (y^j_i)^{\alpha^j_2}(E^j_{2,i})^{\beta^j_2} + \lambda^j_{6,i+1}(1-\mu^j_2 \frac{A^j_2}{(P^j_{2,i})^2}).$$

Additionally, the adjoints functions at $n = t_f$ are

(14) $$\lambda_1(t_f) = 0,\ \lambda_2(t_f) = 0,\ \lambda_3(t_f) = 0,\ \lambda_4(t_f) = 0,\ \lambda_5(t_f) = 0,\ \lambda_6(t_f) = 0.$$

Using

$$\frac{\partial \mathbf{H^j_i}}{\partial u^j_{1,i}} = 0 \quad \frac{\partial \mathbf{H^j_i}}{\partial u^j_{2,i}} = 0$$

at $u^j_{1,i}$ $u^j_{1,i}$ respectively on the admissible set, we obtain the control

(15) $$u^j_{1,i} = \max\left\{\min\left\{\frac{q^j_1(\lambda^j_{1,i+1} - \gamma^j_1\lambda^j_{3,i+1} + \mu^j_1\lambda^j_{5,i+1})}{2\rho^j_1 P^j_{1,i}(x^j_i)^{\alpha^j_1}(E^j_{1,i})^{\beta^j_1}}, 1\right\}, 0\right\},$$

and

(16) $$u^j_{2,i} = \max\left\{\min\left\{\frac{q^j_2(\lambda^j_{2,i+1} - \gamma^j_2\lambda^j_{4,i+1} + \mu^j_2\lambda^j_{6,i+1})}{2\rho^j_2 P^j_{2,i}(y^j_i)^{\alpha^j_2}(E^j_{2,i})^{\beta^j_2}}, 1\right\}, 0\right\},$$

for all $i \in \{0, 1, 2, \cdot, T_f - 1\}$ and all $j \in \{1, 2, 3\}$. □

4. NUMERICAL SIMULATION AND DISCUSSIONS

4.1. Numerical Simulation.

Our aim goal in this section is to test how successful the controls u_1, u_2 are. So, we solve the optimality system numerically using an iterative method. Note that we calculate the state system with an initial guess using a progressive schema in time, while the adjoint functions with transversality conditions are obtained using a regressive schema in time. For the general harvestion function Cobb-Douglas, we take in first case that $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ which are schaefer conditions.

In the second case we take $\alpha_1 = \alpha_2 = 0.65$ $\beta_1 = \beta_2 = 0.60$ according to [23].

In the third case, we take that $\alpha_1 = \alpha_2 = 0.44$ $\beta_1 = \beta_2 = 0.48$ according to [24].

We concentrate the calculus to the Centre and Sud Zones in the Atlantic ocean since we have statistics in just these two zones.

The statistics for Sardine (Sardina P) [3].

Parametre	Zone A+B	Zone C
Intrinsic growth rate r	2.61	0.93
Carring Capacity K (tonnes)	904796	3333333

The statistics for Chub Mackerel (Scomber C)

Intrinsic growth rate r	1.55
Carring Capacity K (thousand of tonnes)	602

Centre Zone: For the centre Zone, the data used in centre zone are,

$r1 = 2.61$; $r2 = 1.55$; $q_1 = 6 * 10^{-7}$; $q_2 = 6 * 10^{-7}$; $\alpha_1 = 1(or = 0.65\ or = 0.44)$; $\alpha_2 = 1(or = 0.65\ or = 0.44)$; $\beta_1 = 1(or = 0.60\ or = 0.48)$; $\beta_2 = 1\ (or = 0.60\ or = 0.48)$; $\theta = 1.34 * 10^{-9}$; $\gamma_1 = 0.01$; $\gamma_2 = 0.01$; $c_1 = 0.006$; $c_2 = 0.006$; $K_1 = 904 * 10^9$; $K_2 = 602 * 10^9$; $A_1 = 40$; $A_2 = 10$; $\mu_1 = 0.002$; $\mu_2 = 0.002$; $\rho_1 = 1$; $\xi_1 = 1$; $\rho_2 = 1$; $\xi_2 = 1$; $u_{1,min} = u_{2,min} 0.1$; $u_{1,max} = u_{2,max} = 0.9$;

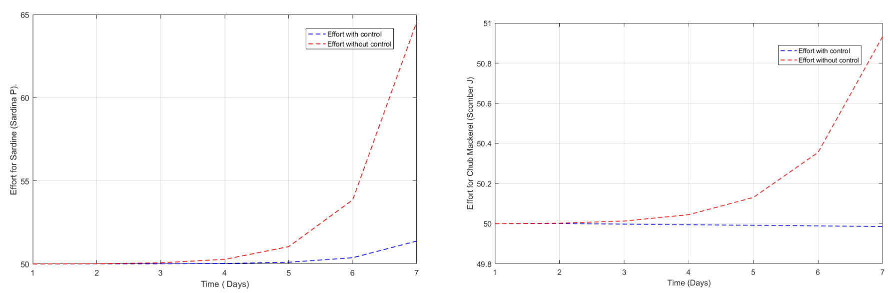

FIGURE 5. The evolution of E_1, E_2 in centre zone in 7 days, for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$

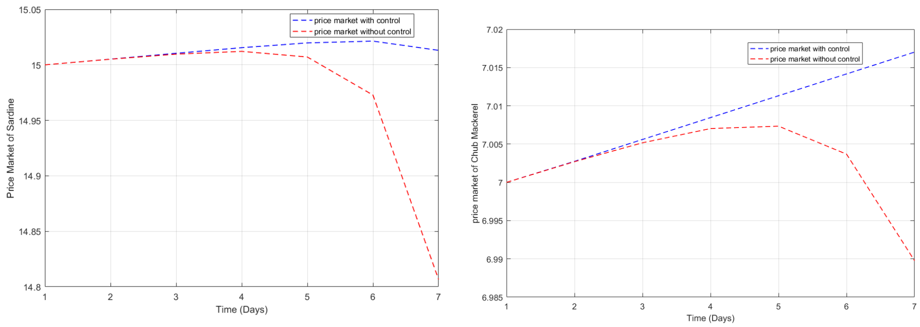

FIGURE 6. The evolution of P_1, P_2 in centre zone in 7 days, for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$

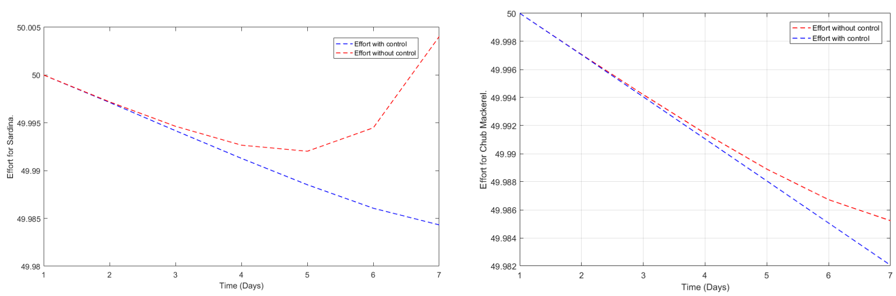

FIGURE 7. The evolution of E_1, E_2 in centre zone in 7 days, for $\alpha_1 = \alpha_2 = 0.66$ $\beta_1 = \beta_2 = 0.60$

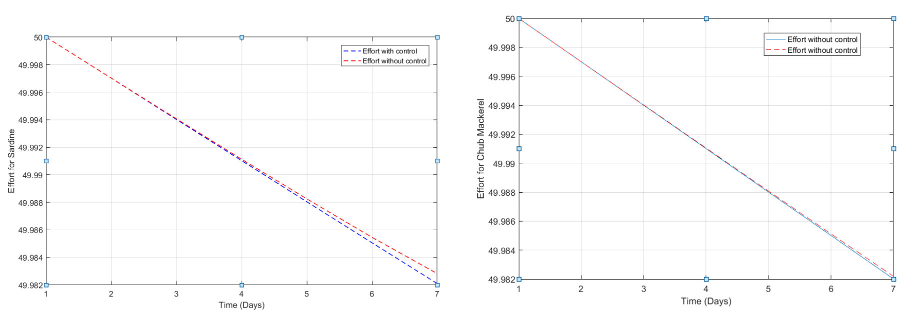

FIGURE 8. The evolution of E_1, E_2 in centre zone in 7 days, for $\alpha_1 = \alpha_2 = 0.44$ $\beta_1 = \beta_2 = 0.48$

In the center zone, we observe that for $\alpha_1 = \alpha_2 = 0.44\ \beta_1 = \beta_2 = 0.48$ the effort decrease in general since the biomass x and y decrease, but the effort provided to fish with control is less than that provided to fish without control in both biomass Sardine and Chub Mackerel. Furthermore, the price market of Sardine and Chub Mackerel with control is greater than the price market of Sardine and Chub Mackerel than the price with no control, which will reflect positively on the profit and we see the fishing activity is not stopped at the end of the simulation. For the second case $\alpha_1 = \alpha_2 = 0.66\ \beta_1 = \beta_2 = 0.60$ we observe that the effort decrease in general but the effort with control provided to fish with control is less than that one provided to fish with control for both Sardine et Chub Mackerel. Likewise for the case $\alpha_1 = \alpha_2 = 0.44\ \beta_1 = \beta_2 = 0.48$. For these two last cases, we observe that the control allowed us to fish with lower expenses.

Sud Zone: For the Sud Zone, the data used in the sud zone are,

$r1 = 0.93$; $r2 = 1.55$; $q_1 = 6 * 10^{-7}$; $q_2 = 6 * 10^{-7}$; $\alpha_1 = 1(or = 0.65\ or = 0.44)$; $\alpha_2 = 1(or = 0.65\ or = 0.44)$; $\beta_1 = 1(or = 0.60\ or = 0.48)$; $\beta_2 = 1\ (or = 0.60\ or = 0.48)$; $\theta = 1.34 * 10^{-9}$; $\gamma_1 = 0.01$; $\gamma_2 = 0.01$; $c_1 = 0.006$; $c_2 = 0.006$; $K_1 = 3333333$; $K_2 = 602 * 10^9$; $A_1 = 40$; $A_2 = 10$; $\mu_1 = 0.002$; $\mu_2 = 0.002$; $\rho_1 = 1$; $\xi_1 = 1$; $\rho_2 = 1$; $\xi_2 = 1$; $u_{1,min} = u_{2,min} 0.1$; $u_{1,max} = u_{2,max} = 0.9$;

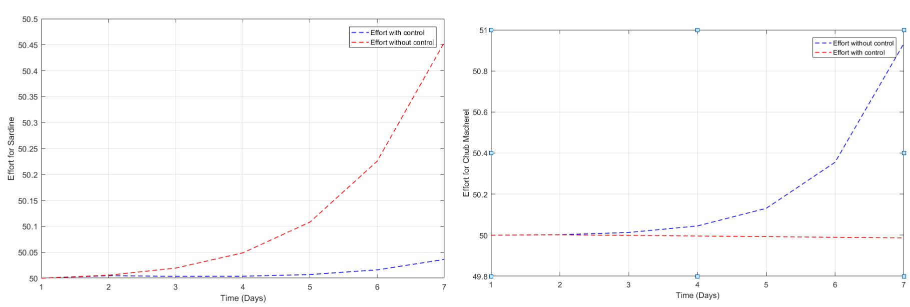

FIGURE 9. The evolution of E_1, E_2 in sud zone in 7 days, for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$

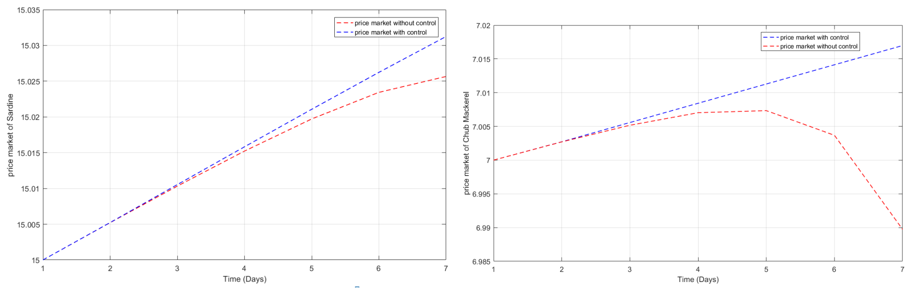

FIGURE 10. The evolution of P_1, P_2 in sud zone in 7 days, for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$

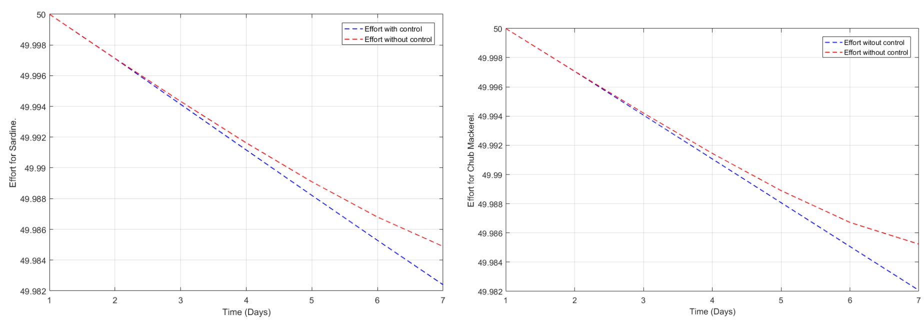

FIGURE 11. The evolution of E_1, E_2 in sud zone in 7 days, for $\alpha_1 = \alpha_2 = 0.65$ $\beta_1 = \beta_2 = 0.60$

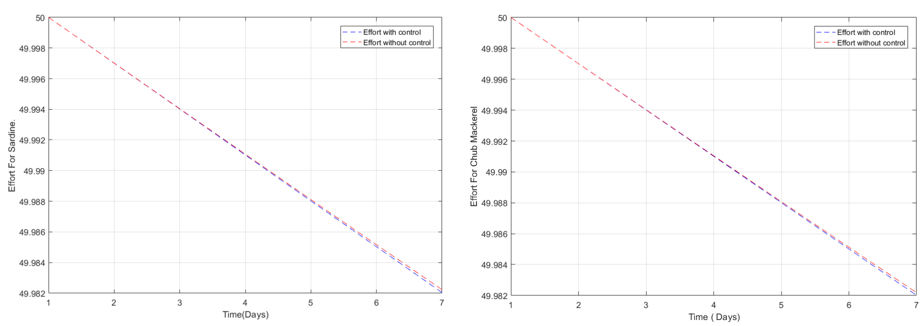

FIGURE 12. The evolution of E_1, E_2 in sud zone in 7 days, for $\alpha_1 = \alpha_2 = 0.44$ $\beta_1 = \beta_2 = 0.48$

For the Sud zone, we observe that the results are similar to what we got in the center zone. For $\alpha_1 = \alpha_2 = 0.44$ $\beta_1 = \beta_2 = 0.48$ the effort decrease in general, but the effort provided to fish with control is less than that provided to fish without control for both Sardine and Chub

Mackerel. Furthermore, the price market of Sardine and Chub Mackerel with control is greater than the price market of Sardine and Chub Mackerel than the price with no control. For the second and third cases $\alpha_1 = \alpha_2 = 0.66\ \beta_1 = \beta_2 = 0.60\ \alpha_1 = \alpha_2 = 0.44\ \beta_1 = \beta_2 = 0.48$, we observe that the effort decrease in general but the effort provided to fish with control is less than that one provided to fish without control for both Sardine et Chub Mackerel.

4.2. Discussions. Figure 5, show the development of $E_1, E_2, P_1, and P_2$ in the presence, absence of controls. We notice on day 4 and day 3 respectively, that $E_1 and E_2$ without control respectively start to rise, as we notice that values reach by E_1 without control are greater than values reaches by E_2 without control. While, values reach by E_1 with control remaining constant, and on the fifth day, these values increase very slightly. On the contrary, E_2 with control remains constant until the last day of hunting, when a slight decrease in its value is noted.

Regarding P_1 without control, the value keeps increasing until the penultimate day when this value decreases. As for, P_2 without control increases until the seventh day of hunting. On the other hand, we note that the value of each P_1, P_2 with control respectively rise but on the fourth, and fifth days respectively these values begin to decrease.

And from it, we conclude, in the case of development of the system without control with high values of $E_1 and E_2$ and low values of $P_1\, and P_2$ this will make losses higher than profits or that will make benefits very low. The opposite occurs when the system develops with control with a low value of $E_1\, E_2$ and a high value of P_1, P_2, which will make the profits bigger or losses will be lower than profits.

Figure 6, with ($\alpha_1 = \alpha_2 = 0.65\, and\, \beta_1 = \beta_2 = 0.60$). This simulation depicts that values of E_1 without control decrease from the first day but on the fourth-day values of E_1 without control increase until the end of hunting. As for, E_2 without control decreases linearly from the first day until the end. E_1 with control decrease also until the end of hunting, we can deduce that the difference between E_1 with and without control become big day per day. The same for E_2 with-without control, at the beginning of hunting E_2 with-without control decrease but the difference between these two becomes clear, since the value of E_2 with control is lower than E_2 without control.

In general, values of E_1 with control decrease daily while, values of E_1 without control increase from the expiry of the hunting half term. Besides that E_2 decrease with the presence and absence of the values of control, but E_2 without control are lower from values of E_2 with control.

Simulation 7, the case of $(\alpha_1 = \alpha_2 = 0.44 \,\text{and}\, \beta_1 = \beta_2 = 0.48)$, values of E_1 without-with decrease from the first day, the difference between them begins to become clear from the fourth day, but remains very small. Same thing for E_2 with the presence and absence of controls.

For the center Zone, with the Schaefer parameter the development of the system with control gives good results since efforts (E_1, E_2) decrease and the price (P_1, P_2) increase with these parameter. We conclude that there is a difference with without control in the system but it remains very small.

Simulation 8, case of $(\alpha_1 = \alpha_2 = 1 \,\text{and}\, \beta_1 = \beta_2 = 1)$, we note that the value of E_1 without control increase, same for E_2 without control. On the other, we note that the value of P_1 without control decreases same for P_2 without control. And on the contrary E_1, E_2 with control decrease while, P_1, P_2 with control increase. As here as before the system with control makes benefits big than losses.

Simulation 9, case of $(\alpha_1 = \alpha_2 = 0.65 \,\text{and}\, \beta_1 = \beta_2 = 0.60)$ in general both E_1, E_2 in the presence-absence of control decrease but, values of E_1, E_2 with control respectively are lower than value of E_1, E_2 without control. Simulation 9, the difference between E_1, E_2 without-with control is very small.

5. Conclusion

Fishing is the backbone of Morocco's economy. As a result of this, several studies have been devoted to researching the biological properties of marine resources and so on, to find ways to catch these resources without compromising the biodiversity that characterizes Morocco. In this paper, we built a discrete multi-region model to describe the fishing activity for both Sardine and Chub Mackerel. We introduce lately two controls to weigh the fishing activity and the biomass of that two species. To calculate the optimality system, we use a discrete version of Pontryagin's maximum principle to calculate the adjoint function and deduce the optimality system. At the end of the simulation, we observe that the fishing activity is not stopped and also the effort provides to fish Sardine and Chub Mackerel with control is less than that one provided to fish with control, and for some cases, we observe that the price market for both Sardine and Chub mackerel is higher than the price with no controls which reflect positively to the profit.

DECLARATIONS

No funds, grants, or other support were received.
This work is original and not have been published elsewhere in any form or language.
The datasets generated during and/or analyzed during the current study are available in Laboratoires centraux-Casablanca,Département pêche, Etat des stocks et des pêcheries Marocaines 2017.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] INRH, Etat des stocks et des pêcheries Marocaines, (2017).

[2] FAO, Report of the FAO working group on the assessment of small pelagic fish off Northwest Africa, FAO Fisheries Report No. 849, (2007).

[3] FAO, Report of the FAO working group on the assessment of small pelagic fish off Northwest Africa, FAO Fisheries Report No. 849, (2015).

[4] P.J. Rago, G.M. Van Dyne, Mathematical bioeconomics: The optimal management of renewable resources (Colin W. Clark), SIAM Rev. 20 (1978), 865–867. https://doi.org/10.1137/1020117.

[5] K. Chaudhuri, A bioeconomic model of harvesting a multispecies fishery, Ecol. Model. 32 (1986), 267–279. https://doi.org/10.1016/0304-3800(86)90091-8.

[6] K. Chaudhuri, Dynamic optimization of combined harvesting of a two-species fishery, Ecol. Model. 41 (1988), 17–25. https://doi.org/10.1016/0304-3800(88)90041-5.

[7] K.S. Chaudhuri, S.S. Ray, On the combined harvesting of a prey-predator system, J. Biol. Syst. 04 (1996), 373–389. https://doi.org/10.1142/s0218339096000259.

[8] R. Mchich, P.M. Auger, R. Bravo de la Parra, et al. Dynamics of a fishery on two fishing zones with fish stock dependent migrations: aggregation and control, Ecol. Model. 158 (2002), 51–62. https://doi.org/10.1016/s0304-3800(02)00237-5.

[9] R. Mchich, N. Charouki, P. Auger, et al. Optimal spatial distribution of the fishing effort in a multi fishing zone model, Ecol. Model. 197 (2006), 274–280. https://doi.org/10.1016/j.ecolmodel.2006.03.026.

[10] R. Mchich, P. Auger, J.-C. Poggiale, Effect of predator density dependent dispersal of prey on stability of a predator–prey system, Math. Biosci. 206 (2007), 343–356. https://doi.org/10.1016/j.mbs.2005.11.005.

[11] P. Auger, R. Mchich, N. Raïssi, et al. Effects of market price on the dynamics of a spatial fishery model: Over-exploited fishery/traditional fishery, Ecol. Complex. 7 (2010), 13–20. https://doi.org/10.1016/j.ecocom.2009.03.005.

[12] S. Ly, P. Auger, M. Balde, A bioeconomic model of a multi-site fishery with nonlinear demand function: Number of sites optimizing the total catch, Acta Biotheor. 62 (2014), 371–384. https://doi.org/10.1007/s10441-014-9222-z.

[13] E. Tsoa, W.E. Schrank, N. Roy, Generalizing Fisheries Models: An Extension of the Schaefer Analysis, Can. J. Fish. Aquat. Sci. 42 (1985), 44–50. https://doi.org/10.1139/f85-006.

[14] T.T. Agnew, Optimal exploitation of a fishery employing a non-linear harvesting function, Ecol. Model. 6 (1979), 47–57. https://doi.org/10.1016/0304-3800(79)90057-7.

[15] A. Eide, F. Skjold, F. Olsen, et al. Harvest functions: The Norwegian bottom trawl cod fisheries, Marine Resource Econ. 18 (2003), 81–93. https://doi.org/10.1086/mre.18.1.42629384.

[16] K. Dabbs, Optimal control in discrete pest control models, Chancellor's Honors Program Projects, (2010). https://trace.tennessee.edu/utk_chanhonoproj/1375.

[17] A. Labzai, O. Balatif, M. Rachik, Optimal control strategy for a discrete time smoking model with specific saturated incidence rate, Discrete Dyn. Nat. Soc. 2018 (2018), 5949303. https://doi.org/10.1155/2018/5949303.

[18] L. El Youssoufi, H. Moutamanni, A. Labzai, et al. Optimal control for a discrete model of hepatitis C with latent, acute and chronic stages in the presence of treatment, Commun. Math. Biol. Neurosci. 2020 (2020), 82. https://doi.org/10.28919/cmbn/4996.

[19] Z. Rachik, A. Labzai, O. Balatif, et al. A multi-region discrete-time mathematical modeling and optimal control of an electoral behavior, J. Math. Comput. Sci. 10 (2020), 2579-2598. https://doi.org/10.28919/jmcs/4948.

[20] C.L. Hwang, L.T. Fan, A discrete version of Pontryagin's maximum principle, Oper. Res. 15 (1967), 139–146. https://doi.org/10.1287/opre.15.1.139.

[21] W. Ding, R. Hendon, B. Cathey, E. Lancaster, R. Germick, Discrete time optimal control applied to pest control problems, Involve. 7 (2014), 479–489. https://doi.org/10.2140/involve.2014.7.479.

[22] L.S. Pontryagin, Mathematical theory of optimal processes, 1st ed., Routledge, (2018). https://doi.org/10.1201/9780203749319.

[23] J.M. Tomkins, J.A. Butlin, A theoretical and empirical approach to fisheries economics, J. Agric. Econ. 26 (1975), 105–125. https://doi.org/10.1111/j.1477-9552.1975.tb02109.x.

[24] J.V. Henderson, M. Tugwell, Exploitation of the lobster fishery: Some empirical results, J. Environ. Econ. Manage. 6 (1979), 287–296. https://doi.org/10.1016/0095-0696(79)90009-3.