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REGIONAL OPTIMAL CONTROL APPROACH FOR A SPATIOTEMPORAL PREY-PREDATOR MODEL

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Abstract. In this paper, we consider a spatiotemporal Prey-Predator system with a diffusion term. Our main objective is to characterize the two optimal controls that minimize the density of prey population also to minimize the density of the predator population to reach the ecological balance in two imposed different regions ω_1 and ω_2 (it is not excluded that $\omega_1 = \omega_2$). For that reason we prove the existence of a pair of control and provide a characterization of optimal control in term of state and adjoint function. Finally, we present numerical simulation to justify our main results and to support the theoretical conclusions.

Keywords: spatial-temporal transmission; prey-predator system; optimal control; numerical method.

2010 AMS Subject Classification: 92D40.

1. Introduction

Mathematical methods are frequently used in biology, specifically for the study of complicated dynamic systems, such as the interaction of different animal species in the natural environment. Alfred Lotka of the United States and Vito Volterra of Italy created a model to

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explain how predator populations and their prey interact under various conditions. The Lotka-Volterra mathematical model (often called ”predator-prey”) is applicable to the description of different processes in biology, ecology, medicine, social research, history, radiophysics, and other sciences [1].

No animal species lives in complete isolation. Since all animals must eat to live, they must interact, if not with other animals, at least with plants.

The proposed model we will build and then study considers the situation where one animal population serves as food for another. The important point is that one species feeds on the other and that both grow in accordance with a reasonable set of biological laws. [2]. Predator-prey relationships are defined as interactions between two species where one is the hunted food source for the other.

There are litteraly hundreds of examples of predator-prey relation, a few of them are the lion-zebra, bear-salmon and fox-rabbit.

Predators and prey exist at even the simplest shapes on earth, they respond proactively to each other.

The predator-prey relationship changes over time as many differents generations of each species are interacting. At the meantime, they affect the future success and survival of the other.

The process of working out will select the adaptations of the population, when scientists first began to study population dynamics or changes in a population infinitely, they observed that the relationships between predators and prey are very tight and any flaw or disorder can cause a lot of disturbance in the ecosystem [3, 4], and that because of the relationships between predators and prey, these population fluctuations are linked. In this sense, since the traditional and simplest model of the prey-predator interaction by Lotka and Volterra, a large number of models have been developed to study prey-predator relationships and to model their interrelation and competition [5, 6, 7, 8].

The structure of this work is the following; Section 2 is staunched to the basic mathematical model and the associated optimal control problem. Section 3, we demonstrate the existence of a global strong solution for our system. In Section 4, we prove the existence of an optimal solution . Neccesary optimality conditions are comfirmed in Section 5. As application, the numerical results related to our control problem are given in Section 6. In the end, we conclude the article in Section 7.

2. The Basic Mathematical Model

2.1. The model without controls. In this work, we propose an optimal control problem which is based on a spatio-temporal prey-predator model. We provide an extension of these models by adding the spatial behavior of the population. We write $X(t,\theta)$, $Y(t,\theta)$ to indicate that the populations have spatial and temporal behavior for the prey population density and predator population density, respectively. The time t belongs to a finite interval $[0,T]$, while θ varies in a bounded domain $\Omega \subseteq \mathbb{R}^2$. The population dynamics is given by the following system

$$(1) \qquad \begin{cases} \dfrac{\partial X}{\partial t} = b + \alpha\Delta X - \beta XY - \mu X \\ \\ \dfrac{\partial Y}{\partial t} = \Lambda + \gamma\Delta Y + \beta XY - \varphi Y \end{cases} \qquad (t,\theta) \in Q = [0,T] \times \Omega$$

with the homogenous Neumann boundary conditions

$$(2) \qquad \frac{\partial X}{\partial \eta} = \frac{\partial Y}{\partial \eta} = 0, \quad (t,\theta) \in \Sigma = [0,T] \times \partial\Omega$$

where $\dfrac{\partial}{\partial \eta}$ is the outward normal derivative, b is the birth rate of the prey population, Λ is the birth rate of the predator population, β is the coefficient of interaction between the prey and the predators population, μ, φ are the natural death rate of the prey and the predator comumunity respectively. α and γ are the self-diffusion coefficients for the prey and predator community. The inital distribution of the two popoulations is supposed to be

$$(3) \qquad X(0,\theta) = X_0 \in L^2(\Omega) \textit{ and } Y(0,\theta) = Y_0 \in L^2(\Omega)$$

2.2. The model with controls. As a strategy of control, we adopt a regional treatment program, so into the model (1), we include two controls $u(.)\chi(.)$ and $v(.)\chi(.)$ where $u(.)$ and $v(.) \in L^2(0,T;\mathbb{R})$ and χ_ω is the characteristic function of ω. The control $u\chi_{\omega_1}$ represents the control measure to hold the ecological equilibruim per unit of time and space while the controle $v\chi_{\omega_2}$ represents also the control measure to keep the ecological equilibruim per time and space. The two controls represent the effect of the measurement and the treatment applied to the prey and predator population who act in the subdomains $\omega_1, \omega_2 \subset \Omega$. The dynamic of the regional controlled system is given by

$$\begin{cases} \dfrac{\partial X}{\partial t} = b + \alpha\Delta X - \beta XY - \mu X - u\chi_{\omega_1} X \\ \\ \dfrac{\partial Y}{\partial t} = \Lambda + \gamma\Delta Y + \beta XY - \varphi Y - v\chi_{\omega_2} Y \end{cases} \qquad (t,\theta) \in Q = [0,T] \times \Omega \tag{4}$$

$$\frac{\partial X}{\partial \eta} = \frac{\partial Y}{\partial \eta} = 0, \quad (t,\theta) \in \Sigma = [0,T] \times \partial\Omega \tag{5}$$

$$X(0,\theta) = X_0 \in L^2(\Omega) \; and \; Y(0,\theta) = Y_0 \in L^2(\Omega) \tag{6}$$

Our goal is to minimize the density of the prey population in region ω_1, and to minimize also the density of the predators in region ω_2. Mathematically, the problem is equivalent to minimize the objective functional

(7)

$$J(u,v) = \|X - R\|^2_{L^2([0,T]\times\omega_1)} + \|Y - S\|^2_{L^2([0,T]\times\omega_2)} + (\frac{p}{2}\|u\|^2_{L^2([0,T]\times\omega_1)} + \frac{q}{2}\|v\|^2_{L^2([0,T]\times\omega_2)})$$

where S and R are two constant functions that represent the ecological average of preys and predators respectively in regions ω_1 and ω_2, besides p and q are the positive weights associated with the two controls. Our objective is to find minimal control variables u and v in order to minimize the objective functional defined in (7) by reaching the equilibruim balance in both regions. U_{ad} is the control set defined by

$$U_{ad} = \left\{(u\chi_{\omega_1}, v\chi_{\omega_2}) \in (L^2\left([0,T] \times L^2(\Omega)\right))^2;\; 0 \leq u \leq 1 \, and \; 0 \leq v \leq 1\right\} \tag{8}$$

- $W^{1,2}([0,T];H(\Omega))$ the space of all absolutely continuous functions $f : [0,T] \longmapsto H(\Omega)$ having the property that $\dfrac{\partial f}{\partial t} \in W^{1,2}([0,T];H(\Omega))$ where $H(\Omega) = \left(L^2(\Omega)\right)^2$
- $L(T,\Omega) = L^2\left(0,T;H^2(\Omega)\right) \cap L^\infty\left(0,T;H^1(\Omega)\right)$

3. EXISTENCE OF GLOBAL SOLUTION

We study in this section the existence of a (global) strong solution, of system (4-6). As this model describes the population for biological reasons, the populations X and Y should remain nonnegative and bounded.

Let $c=(c_1,c_2)=(X,Y)$ the solution of the system (4-6) with $c^0=(X_0,Y_0)=\left(c_1^0,c_2^0\right)$. Denote by A the linear operator defined as follow

$$\begin{aligned} & A: D(A)\subset H(\Omega)\longrightarrow H(\Omega) \\ & Ac=(\alpha\Delta c_1,\gamma\Delta c_2)\in D(A),\forall c\in D(A) \end{aligned} \tag{9}$$

$$D(A)=\left\{c\in\left(H^2(\Omega)\right)^2,\frac{\partial c_1}{\partial\eta}=\frac{\partial c_2}{\partial\eta}=0,a.e\,\partial\Omega\right\} \tag{10}$$

Theorem 1. *Let Ω be a bounded domain from $\mathbb{R}^2$, with the boundary of class $C^{2+\alpha}$,$\alpha>0$. As the rates $b,\Lambda,\mu,\varphi,\beta>0$, $(u,v)\in U_{ad}$, $c^0\in D(A)$ and $c_i^0\geq 0$ on Ω (for $i=1,2$), the system (4–6) has a unique (global) strong solution $c\in W^{1,2}([0,T];H(\Omega))$ such that*

$$c_1,c_2\in L(T,\Omega)\cap L^\infty(Q)$$

in addition, there exists $\Gamma>0$ independent of (u,v) (and of the corresponding solution c) such that for a $t\in[0,T]$

$$\left\|\frac{\partial c_i}{\partial t}\right\|_{L^2(Q)}+\|c_i\|_{L^2\left(0,T,H^2(\Omega)\right)}+\|c_i\|_{H^1(\Omega)}+\|c_i\|_{L^\infty(Q)}\leq\Gamma, for\ i=1,2 \tag{11}$$

Proof. For the proof of the existence of a (global) strong solution for system (4-6), let

$$\begin{cases} f_1(c(t))=b+-\beta c_1c_2-\mu c_1-u\chi_{\omega_1}c_1 \\ f_2(c(t))=\Lambda+\beta c_1c_2-\varphi c_2-v\chi_{\omega_2}c_2 \end{cases} \tag{12}$$

The nonlinear term in (12) and we consider the function $f(c(t))=(f_1(c(t)),f_2(c(t))$, then we can be rewritten the system (4-6) in the space $H(\Omega)$ under the form

$$\begin{cases} \dfrac{\partial c}{\partial t}=Ac+f(c(t)), \qquad t\in[0,T] \\ c(0)=c^0 \end{cases}$$

As the operator A defined in (9-10) is dissipating and self-adjoint and generates a C_0-semigroup of contractions on $H(\Omega)$ see ([9, 10]), since $|c_i|\leq N\,for\,i=1,2$ where N is a constant, that represents the total population , thus function $f=(f_1,f_2)$ becomes Lipschitz continuous in $c=(c_1,c_2)$ uniformly with respect to $t\in[0,T]$, problem (4-6) admits a unique strong solution $c=(c_1,c_2)\in W^{1,2}([0,T];H(\Omega))$ See [11, 12], with $c_1,c_2\in L^2\left(0,T;H^2(\Omega)\right)$.

In order to prove that $c\in L^{\infty}(Q)$, we put $M=\max\left\{\|f_i\|_{L^{\infty}(Q)},\left\|c_i^0\right\|_{L^{\infty}(\Omega)}\, for\, i=1,2\right\}$, it is obvious to see that the function $V_1(t,\theta)=c_1-Mt-\left\|c_1^0\right\|_{L^{\infty}(\Omega)}$ satisfies the system

$$\begin{aligned}\frac{\partial V_1}{\partial t}(t,\theta) &= \alpha\triangle V_1+f_1(t,c(t))-M \quad t\in[0,T]\\ V_1(0,\theta) &= c_1^0-\left\|c_1^0\right\|_{L^{\infty}(\Omega)}\end{aligned} \tag{13}$$

the solution of this system can be written as

$$V_1(t)=S(t)(c_1^0-\|c_1\|_{L^{\infty}(\Omega)})+\int_0^t S(t-s)(f_1(c(s))-M)\,ds,$$

with $\{S(t),t\geq 0\}$ is the C_0-semi-group generated by the operator $\bar{A}:D(B)\subset L^2(\Omega)\longrightarrow L^2(\Omega)$ where $\bar{A}u=\lambda\Delta c_1$ and $D\left(\bar{A}\right)=\left\{c_1\in H^2(\Omega),\frac{\partial c_1}{\partial\eta}=0,a.e\,\partial\Omega\right\}$. Since $c_1^0-\left\|c_1^0\right\|_{L^{\infty}(\Omega)}\leq 0$ and $f_1(c(s))-M\leq 0$, it follows that $V_1(t,\theta)\leq 0,\forall(t,\theta)\in Q$.

According to the same manner we can prove that the function $V_2(t,\theta)=c_1+Mt+\left\|c_1^0\right\|_{L^{\infty}(\Omega)}$ is nonnegative. Then $|c_1(t,\theta)|\leq Mt+\left\|c_1^0\right\|_{L^{\infty}(\Omega)}\forall(t,\theta)\in Q$ and analogously

$$|c_2(t,\theta)|\leq Mt+\left\|c_2^0\right\|_{L^{\infty}(\Omega)}\forall(t,\theta)\in Q \tag{14}$$

Thus, we have proved that $c_i\in L^{\infty}(\Omega)\ \forall(t,\theta)\in Q$ for $i=1,2$.

By the first equation of (4) one obtains

$$\begin{aligned}&\int_0^t\int_\Omega\left|\frac{\partial c_1}{\partial t}\right|^2 dsd\theta+\alpha^2\int_0^t\int_\Omega|\triangle c_1|^2 dsd\theta-2\alpha\int_0^t\int_\Omega\frac{\partial c_1}{\partial t}\triangle c_1 dsd\theta\\ =&\int_0^t\int_\Omega(b-\beta c_1c_2-\mu c_1-u\chi_{\omega_1}(\theta)c_1)^2 dsd\theta\end{aligned}$$

Using the regularity of c_1 and the Greens formula, we can write

$$2\alpha\int_0^t\int_\Omega\frac{\partial c_1}{\partial t}\triangle c_1 dsd\theta = \alpha\int_\Omega|\nabla c_1|^2 d\theta-\alpha\int_\Omega\left|\nabla c_1^0\right|^2 d\theta$$

Then

$$\begin{aligned}&\int_0^t\int_\Omega\left|\frac{\partial c_1}{\partial t}\right|^2 dsd\theta+\alpha^2\int_0^t\int_\Omega|\triangle c_1|^2 dsd\theta+\alpha\int_\Omega|\nabla c_1|^2 d\theta-\alpha\int_\Omega\left|\nabla c_1^0\right|^2 d\theta\\ =&\int_0^t\int_\Omega(b-\beta c_1c_2-\mu c_1-u\chi_{\omega_1}(\theta)c_1)^2 dsd\theta\end{aligned}$$

Since $c_i^0 \in H^2(\Omega)$ and $\|c_i\|_{L^\infty(Q)}$for $i=1,2$ are bounded independently of v and u, we submit that $c_1 \in L^\infty\left(0,T,H^1\left(\Omega\right)\right)$ and the first inequality in (14) holds for $i=1$. The remaining cases can be treated similarly.

Let show the positiveness of c_1 and c_2, first we show the positiveness of c_2, we set $c_2 = c_2^+ - c_2^-$ with

$$c_2^+\left(t,\theta\right) \;=\; sup\left\{c_2\left(t,\theta\right),0\right\} \; and \, c_2^-\left(t,\theta\right) = inf\left\{c_2\left(t,\theta\right),0\right\}$$

One multiplies the second equation of the system (4) by c_2^- integrates over Ω, we have

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \left(c_2^-\right)^2(t,\theta)\,d\theta\right) = \; & \int_\Omega \left|\gamma\nabla c_2^-\left(t,\theta\right)\right|^2 d\theta + \varphi\int_\Omega \left(c_2^-\right)^2(t,\theta)\,d\theta + \int_\Omega \Lambda\left(c_2^-\right)(t,\theta)\,d\theta \\ & -\beta\int_\Omega c_1\left(c_2^-\right)^2(t,\theta)\,d\theta + \int_\Omega \chi_{\omega_2} v\left(c_2^-\right)^2(t,\theta)\,d\theta\end{aligned}$$

As $c_1 \le N$ then $-\beta c_1 \ge -\beta N$, we have $-\dfrac{1}{2}\dfrac{d}{dt}\left(\int_\Omega \left(c_2^-\right)^2(t,\theta)\,d\theta\right) \ge -\beta\int_\Omega N\left(c_2^-\right)^2(t,\theta)\,d\theta$. Gronwall's inequality leads to

$$\int_\Omega \left(c_2^-\right)^2(t,\theta)\,d\theta \le e^{t\beta N}\int_\Omega \left(c_2^-\right)^2(0,\theta)\,d\theta$$

Then

$$c_2^- = 0$$

One deduces that $c_2\left(t,\theta\right) \ge 0$, $\forall\left(t,\theta\right) \in Q$. In addition, we consider the system

$$\frac{\partial c_1}{\partial t} = \alpha\Delta c_1 + b - \beta c_1 c_2 - \mu c_1 - u\chi_{\omega_1}(\theta)c_1 \tag{15}$$

where

$$F\left(c_1,c_2\right) = b - \beta c_1 c_2 - \mu c_1 - u\chi_{\omega_1}(\theta)c_1$$

It is obvious to see that the function F is continuously differentiable satisfying $F\left(0,c_2\right) = b$ for all $c_1, c_2 \ge 0$. Since initial data of system (4-6) are nonnegative, we deduce the positivity of c_1 and c_2 [13]. One deduces that $c_1\left(t,\theta\right) \ge 0$ and $c_2\left(t,\theta\right) \ge 0 \,\forall\left(t,\theta\right) \in Q$. □

4. THE EXISTENCE OF THE OPTIMAL SOLUTION

In this section, we will prove the existence of an optimal control for problem (4-6) subject to reaction diffusion system and $(u,v) \in U_{ad}$. The main result of this section is the following.

Theorem 2. *Under the conditions of theorem (1) the optimal control problem (4-6) admits an optimal solution* (c^*, u^*, v^*).

Proof. Let

$$J^* = \inf\{J(c,u,v)\} \tag{16}$$

where $(u,v) \in U_{ad}$ and c is the solution of (4-6).

Obviously $J^*$is finite. Therefore there exists a sequence(c^n, u^n, v^n) with $(u^n, v^n) \in U_{ad}$, $c^n = \left(c_1^n, c_2^n\right) \in W^{1,2}([0,T]; H(\Omega))$, such that

$$\begin{cases} \dfrac{\partial c_1^n}{\partial t} = b + \alpha\Delta c_1^n - \beta c_1^n c_2^n - \mu c_1^n - u^n \chi_{\omega_1} c_1^n \\ \dfrac{\partial c_2^n}{\partial t} = \Lambda + \gamma\Delta c_2^n + \beta c_1^n c_2^n - \varphi c_2^n - v^n \chi_{\omega_2} c_2^n \end{cases} \quad (t,\theta) \in Q = [0,T] \times \Omega \tag{17}$$

$$\frac{\partial c_1^n}{\partial \eta} = \frac{\partial c_2^n}{\partial \eta} = 0 \qquad (t,\theta) \in \Sigma \tag{18}$$

$$c_i^n(0,\theta) = c_i^0, for. i = 1,2 \qquad \theta \in \Omega \tag{19}$$

and

$$J^* \leq J(c^n, (u^n, v^n)) \leq J^* + \frac{1}{n} \qquad (\forall n \geq 1) \tag{20}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we infer that $c_1^n(t)$ is compact in $L^2(\Omega)$. Show that $\left\{c_1^n(t), n \geq 1\right\}$ is equicontinuous in $C\left([0,T] : L^2(\Omega)\right)$. As $\dfrac{\partial c_1^n}{\partial t}$ is bounded in $L^2(Q)$, $i = 1,2$, this implies that for all $s,t \in [0,T]$

$$\left| \int_\Omega (c_1^n)^2(t,\theta)\, d\theta - \int_\Omega (c_1^n)^2(s,\theta)\, d\theta \right| \leq K|t-s| \tag{21}$$

for any s, $t \in [0,T]$. The Ascoli-Arzela Theorem (See [14]) implies that c_1^n is compact in $C\left([0,T] : L^2(\Omega)\right)$. We conclude that there exist a subsequence denoted again c_1^n such that

$c_1^n \longrightarrow c_1^*$ in $L^2(\Omega)$, uniformly with respect to t.

Analogously $c_i^n \longrightarrow c_i^*$in $L^2(\Omega)$, $i = 2$ uniformly with respect to t.

The boundedness of Δc_i^n in $L^2(Q)$ implies its weak convergence in $L^2(Q)$ on a subsequence denoted again $\triangle c_i^n$ then for all distribution ψ

$$\int_Q \psi \Delta c_i^n = \int_Q c_i^n \triangle \psi \to \int_Q c_i^* \triangle \psi = \int_Q \psi \Delta c_i^*$$

Which implies that $\Delta c_i^n \rightharpoonup \Delta c_i^*$in $L^2(Q)$ for $i = 1,2$. Here and everywhere below the sign $\rightharpoonup$ denotes the weak convergence in the specified space. Estimates lead to

$$\frac{\partial c_i^n}{\partial t} \rightharpoonup \frac{\partial c_i^*}{\partial t} \, in \, L^2(Q), i = 1,2$$

$$c_i^n \rightharpoonup c_i^* \, in \, L^2\left(0,T : H^2(\Omega)\right), i = 1,2$$

$$c_i^n \rightharpoonup c_i^* \, in \, L^\infty\left(0,T : H^1(\Omega)\right), i = 1,2$$

Writing $c_1^n c_2^n - c_1^* c_2^* = \left(c_1^n - c_1^*\right) c_2^n + c_1^* \left(c_2^n - c_2^*\right)$ and making use of the convergences $c_i^n \longrightarrow c_i^*$in $L^2(Q)$, $i = 1,2$ and of the boundedness of c_1^*, c_2^n in $L^\infty(Q)$, one arrives at $c_1^n c_2^n \mapsto c_1^* c_2^*$in $L^2(Q)$. We also have $v^n \rightharpoonup v^*$ and $u^n \rightharpoonup u^*$in $L^2(Q)$ on a subsequence denoted again v^n and u^n. Since U_{ad} is a closed and convex set in $L^2(Q)$, it is weakly closed, so $(u^*, v^*) \in U_{ad}$ and as above $u^n \chi_{\omega_1}(\theta) c_1^n \to \chi_{\omega_1}(\theta) u^* c_1^*$ in $L^2([0,T] \times \omega_1)$ also $v^n \chi_{\omega_2}(\theta) c_2^n \to v^* \chi_{\omega_2}(\theta) c_2^*$in $L^2([0,T] \times \omega_2)$. Now, we may pass to the limit in $L^2(Q)$ as $n \to \infty$ in (17-19) to deduce that $(c^*, (u^*, v^*))$ is an optimal solution. □

5. Necessary Optimality Conditions

In this section, we establish the optimality condition corresponding to problem (1) and we investigate a caracterization of optimal control [15, 16].

Theorem 3. *The mapping* $c : U_{ad} \to W^{1,2}([0,T], H(\Omega))$ *with* $c_i \in L(T,\Omega)$ *is Gateaux differentiable with respect to* $w^* = \begin{pmatrix} u^* \\ v^* \end{pmatrix}$. *For* $w = \begin{pmatrix} u \\ v \end{pmatrix} \in U_{ad}$, $c'(w) w^* = C$ *is the unique solution in* $W^{1,2}([0,T], H(\Omega))$ *with* $C_i \in L(T,\Omega)$ *of the problem*

$$\left\{ \begin{array}{l} \dfrac{\partial C}{\partial t} = AC + JC + Gw \; sur \; Q \\ C(0,\theta) = 0 \end{array} \right. \tag{22}$$

with

$$J=\begin{pmatrix} -\beta c_2^* - \mu - u^*\chi_{\omega_1}(\theta) & -\beta c_1^* \\ \beta c_2^* & -\beta c_1^* - \varphi - v^*\chi_{\omega_2}(\theta) \end{pmatrix}, G=\begin{pmatrix} -c_1^*\chi_{\omega_1}(\theta) & 0 \\ 0 & -c_2^*\chi_{\omega_2}(\theta) \end{pmatrix}$$

Proof. In order to establish the result of this theorem, let (c^*, w^*) be an optimal pair and $w^\varepsilon = w^* + \varepsilon w\ (\varepsilon > 0) \in L^2(Q)$. Denote by $c^\varepsilon = \left(c_1^\varepsilon, c_2^\varepsilon\right)$ and $c^* = (c_1^*, c_2^*)$ the solution of (4-6) corresponding to w^ε and w^*, respectively. Put $c_i^\varepsilon = c_i^* + \varepsilon C^\varepsilon$ for $i = 1,2$

Subtracting system (4-6) corresponding c^* from the system corresponding to c^ε we get

(23)

$$\begin{cases} \dfrac{\partial C_1^n}{\partial t} = \alpha\Delta C_1^n + (-\beta c_2^* - \mu - u^*\chi_{\omega_1}(\theta))C_1^n + (-\beta c_1^*)C_2^n - u^*\chi_{\omega_1}(\theta)c_1^* \\ \dfrac{\partial C_2^n}{\partial t} = \gamma\Delta C_2^n + (\beta c_2^*)C_1^n - (-\beta c_1^* - \varphi - v^*\chi_{\omega_2}(\theta))C_2^n - v^*\chi_{\omega_2}(\theta)\,c_2^* \end{cases} \quad (t,\theta) \in Q = [0,T]\times\Omega$$

(24)
$$\frac{\partial C_1^\varepsilon}{\partial \eta} = \frac{\partial C_2^\varepsilon}{\partial \eta} = 0\,(t,\theta) \in \Sigma = [0,T]\times\partial\Omega$$

(25)
$$C_i^\varepsilon(0,\theta) = 0\ \ \theta \in \Omega,\ for\, i = 1,2$$

Now, we show that C_i^ε are bounded in $L^2(Q)$ uniformly with respect to ε and that c_i^ε in $L^2(Q)$. To this end, denote

$$\begin{aligned} J^\varepsilon \quad &= \quad \begin{pmatrix} -\beta c_2^\varepsilon - \mu - u^*\chi_{\omega_1}(\theta) & -\beta c_1^\varepsilon \\ \beta c_2^\varepsilon & -\beta c_1^\varepsilon - \varphi - v^*\chi_{\omega_2}(\theta) \end{pmatrix} \\ \text{and} \quad G = & \quad \begin{pmatrix} -c_1^*\chi_{\omega_1}(\theta) & 0 \\ 0 & -c_2^*\chi_{\omega_2}(\theta) \end{pmatrix} \end{aligned}$$

Then the system (23-25) can be written in the form

(26)
$$\begin{cases} \frac{\partial C^\varepsilon}{\partial t} = AC^\varepsilon + J^\varepsilon C^\varepsilon + Gw, \text{ on } [0,T] \\ C^\varepsilon(0) = 0 \end{cases}$$

We consider $(S(t), t \geq 0)$ the semi-group generated by A, then the solution of system (26) is given by

(27)
$$C^\varepsilon(t) = \int_0^t S(t-s)J^\varepsilon(s)C^\varepsilon(s)\,ds + \int_0^t S(t-s)(Gw)(s)\,ds,$$

since the elements of the matrix J^{ε} are bounded uniformly with respect to ε, the Gronwall's inequality we guide to

$$\|C_i^{\varepsilon}\|_{L^2(Q)} \leq \Gamma \tag{28}$$

for some constant $\Gamma > 0$ $(i = 1,2)$. Then

$$\|c_i^{\varepsilon} - c_i^*\|_{L^2(Q)} = \varepsilon \|C_i^{\varepsilon}\|_{L^2(Q)} \tag{29}$$

Thus, $c_i^{\varepsilon} \to c_i^*$in $L^2(Q)$, $i = 1,2$. Let

$$J = \begin{pmatrix} -\beta c_2^* - \mu - u^* \chi_{\omega_1}(\theta) & -\beta c_1^* \\ \beta c_2^* & -\beta c_1^* - \varphi - v^* \chi_{\omega_2}(\theta) \end{pmatrix}$$

$$\text{and} \quad G = \begin{pmatrix} -c_1^* \chi_{\omega_1}(\theta) & 0 \\ 0 & -c_2^* \chi_{\omega_2}(\theta) \end{pmatrix}$$

Then the system (23-25) can be written as

$$\begin{cases} \frac{\partial C}{\partial t} = AC + JC + Gw \, \text{on } [0,T] \\ C(0) = 0 \end{cases} \tag{30}$$

and its solution is given by

$$C(t) = \int_0^t S(t-s) J(s) C(s) \, ds + \int_0^t S(t-s)(Gw)(s) \, ds, \tag{31}$$

By (27) and (31) one deduces that

$$C^{\varepsilon}(t) - C(t) = \int_0^t S(t-s) J^{\varepsilon}(s)(C^{\varepsilon} - C) + C(s)(J^{\varepsilon}(s) - J(s)) \, ds. \tag{32}$$

Since all the elements of the matrix J^{ε}tend to the corresponding elements of the matrix J in $L^2(Q)$, by using the Gronwall's inequality, we derive that Thus $C_i^{\varepsilon} \to C_i^*$in $L^2(Q)$ as $\varepsilon \to 0$, for $i = 1,2$. □

Let $p = (p_1, p_2)$ the adjoint variable, we can write the dual system associated to our problem

$$\begin{cases} -\dfrac{\partial p}{\partial t} - Ap - J^* p = D^* D c^* \qquad t \in [0,T] \\ p(T,\theta) = 0 \\ \dfrac{\partial p}{\partial \eta} = 0 \end{cases} \tag{33}$$

wehre w^* is the optimal control, $c^* = (c_1^*, c_2^*)$ is the optimal state and D is the matrix defined by $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\rho = (p, q)$.

Lemma 4. *Under hypotheses of (26-32), if $(c^*, (u^*, v^*))$ is an optimal pair, then the dual system (33) admits a unique strong solution $p \in W^{1,2}([0,T], H(\Omega))$ with $p_i = (p_1, p_2) \in L(T, \Omega)$ for $i = 1, 2$.*

Proof. The lemma can be proved by making the change of variable $s = T - t$ and the change of functions $q_i(s, \theta) = p_i(T - s, \theta) = p_i(t, \theta), (t, \theta) \in Q$, $i = 1, 2$. and applying the same method like in the proof above. □

Now, we can find the first order necessary condition

Theorem 5. *Let w^*be an optimal control of (23-25) and let $c^* \in W^{1,2}(0, T; H(\Omega))$ with $c_i^* \in L(T, \Omega)$ for $i = 1, 2$ be the optimal state, that is c^* is the solution to (1.2) with the control w^*. Then, there exists a unique solution $p \in W^{1,2}(0, T, H(\Omega))$ of the linear problem*

$$\begin{cases} -\frac{\partial p}{\partial t} - Ap - J^* p = D^* D c^* \qquad t \in [0, T] \\ p(T, \theta) = 0 \\ \dfrac{\partial p}{\partial \eta} = 0 \end{cases} \tag{34}$$

and

$$u^* = min\left(1, max\left(0, \frac{c_1^* \chi_{\omega_1}(\theta)}{p} p_1\right)\right) \text{ and } v^* = min\left(1, max\left(0, \frac{c_2^* \chi_{\omega_2}(\theta)}{q} p_2\right)\right). \tag{35}$$

Proof. Suppose w^* is an optimal control and $c^* = (c_1^*, c_2^*) = (c_1, c_2)(w^*)$ are the corresponding state variables. Putting $w^\varepsilon = w^* + \varepsilon h \in U_{ad}, h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in (L^2(0, T; L^2(\Omega)))^2$ and corresponding state solution $c^\varepsilon = \left(c_1^\varepsilon, c_2^\varepsilon\right) = (c_1, c_2)(w^\varepsilon)$.

Since the minimum of the objective functional is attained at w^*, we have

$$J(w^*)(h) = lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(J(w^{\varepsilon}) - J(w^*)\right)$$
$$= lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\int_0^T\int_{\omega_1}(c_1^{\varepsilon})^2-(c_1^*)^2d\theta dt+\int_0^T\int_{\omega_2}(c_2^{\varepsilon})^2-(c_2^*)^2d\theta dt$$
$$+p\int_0^T\int_{\omega_1}(u^{\varepsilon})^2-(u^*)^2d\theta dt+q\int_0^T\int_{\omega_2}(v^{\varepsilon})^2-(v^*)^2d\theta dt)$$
$$= lim_{\varepsilon\to 0}(\int_0^T\int_{\omega_1}(\frac{c_1^{\varepsilon}-c_1^*}{\varepsilon})(c_1^{\varepsilon}+c_1^*)d\theta dt+\int_0^T\int_{\omega_2}(\frac{c_2^{\varepsilon}-c_2^{\varepsilon}}{\varepsilon})(c_2^{\varepsilon}+c_2^*)d\theta dt$$
$$+p\int_0^T\int_{\omega_1}(\varepsilon h_1)^2+2h_1u^*d\theta dt+q\int_0^T\int_{\omega_2}(\varepsilon h_2)^2+2h_2v^*d\theta dt)$$

as $lim_{\varepsilon\to 0}\frac{c_2^{\varepsilon}-c_2^*}{\varepsilon} = lim_{\varepsilon\to 0}\frac{c(w^*+\varepsilon h)-c_2^*}{\varepsilon} = c'(w^*)h,\ c_2^{\varepsilon}\longrightarrow c_2^*\, in\, L^2(Q)\, and\, c_2^{\varepsilon}, c_2^*\in L^{\infty}(Q).$

Then we obtain

$$J(w^*)(h) \quad = 2\int_0^T\int_{\Omega}(c_1^*)c'(w^*)h\,d\theta dt+2\int_0^T\int_{\Omega}(c_2^*)c'(w^*)h\,d\theta dt+2p\int_0^T\int_{\Omega}h_1u^*d\theta dt$$
$$+2q\int_0^T\int_{\Omega}h_2v^*d\theta dt = 2\int_0^T\langle Dc^*,DC\rangle_{H(\Omega)}+2\rho\int_0^T\langle w^*,h\rangle_{(L^2(\Omega))^2}dt$$

Since J is Gateaux differentiable at w^* and U_{ad} is convex, it is seen that $J'(w^*)(z-w^*)\geq 0$ for all $z\in U_{ad}$

$$J'(w^*)(z-w^*) = 2\int_0^T\langle Dc^*,DC\rangle_{H(\Omega)}+2\rho\int_0^T\langle w^*,z-w^*\rangle_{L^2(\Omega)}dt$$

We have

$$\begin{aligned}
\int_0^T\langle Dc^*,DC\rangle_{H(\Omega)} &= \int_0^T\langle D^*Dc^*,C\rangle_{H(\Omega)}dt\\
&= \int_0^T\left\langle -\frac{\partial p}{\partial t}-Ap-Jp,C\right\rangle_{H(\Omega)}dt\\
&= \int_0^T\left\langle P,\frac{\partial C}{\partial t}-AC-JC\right\rangle_{H(\Omega)}dt\\
&= \int_0^T\langle P,G(z-w^*)\rangle_{H(\Omega)}dt\\
&= \int_0^T\langle G^*P,(z-w^*)\rangle_{(L^2(\Omega))^2}dt
\end{aligned}$$

We deduce that $J^{'}(w^*)(z-w^*) \geq 0$ for all $z \in U_{ad}$ equivalent to $\int_0^T \langle G^*P,(z-w^*)\rangle_{(L^2(\Omega))^2} dt \geq 0$ for all $z \in U_{ad}$. By standard arguments varying z, we get

$$w^* = -\frac{1}{\rho} G^* P$$

Afterwards

$$u^* = \frac{c_1^* \chi_{\omega_1}(\theta)}{\rho} p_1 \; and \; v^* = \frac{c_2^* \chi_{\omega_2}(\theta)}{\rho} p_2$$

As $(u^*,v^*) \in U_{ad}$, we have

$$u^* = min\left(1, max\left(0, \frac{c_1^* \chi_{\omega_1}(\theta)}{p} p_1\right)\right) \; and \, v^* = min\left(1, max\left(0, \frac{c_2^* \chi_{\omega_2}(\theta)}{q} p_2\right)\right)$$

□

6. NUMERICAL SIMULATIONS

In this section, we present numerical results that show and reinforce the effectiveness of our control strategy. This strategy is based on applying two control terms representing the control measure to keep the ecological balance in time and space. We developed a code in MATLAB^{TM}, and simulated our results using different data. Regarding the numerical method, we give numerical simulations to our optimality system which is formulated by state equations with initial conditions and boundary conditions (4-6). We apply the forward-backward sweep method (FBSM) [17, 18] to solve our optimality system in an iterative process. The state equations are solved using a direct method in time by employing Euler explicit method, in order to discretize the second order derivatives $\triangle X_T$ and $\triangle Y_T$ we use the second order Euler explicit method, initial control variables are guessed in the beginning of the iterative method, next, the adjoint equations are solved backward in time. Finally, the control variables are updated with the existing state and the adjoint solutions. The iterative process is repeated until a tolerable criterion is reached.

The numerical results are presented in the spatial space with two dimensions. The parameter values using in this part of numerical simulations are cited in table **1**. Without loss of generality, we consider a $40km \times 30km$ rectangular grid denoted .

Table 1. Parameter values of marines species

	X_0	b	α	β	μ
ω_1	800	0.02	0.6	0.024	0.001
	Y_0	Λ	γ	β	φ
ω_2	100	0.03	0.6	0.024	0.0015

For all the figures shown below, the red part of the colored bars holds a very large number whereas the blue part holds the smaller numbers. As can be seen in the following figures, this fish population ranges from 0 to 1000. To demonstrate the impact of our controls on the sustainability of the prey and predator population, we deal with two cases: without and with controls. We assume the same initial state in both cases to maintain the validity of the results.

The model we suggest is a spatio-temporal prey-predator model. In other words, to show the efficiency of the suggested model and the influence on the prey and the predator in a certain area, we will present a numerical simulation over a time period of t=250 days. From figure 1, the case without control, we notice that the density of prey strongly reduces, which may lead to their absence. Nevertheless, we observe that there is a large increase of predators during these 250 days, as can be seen in figure 2. The remarks noted during these simulations lead us to think about the definition of suitable control strategies which take these remarks into account.

To prove and clarify the usefulness of the control strategy, we present a regional control to conserve a particular region to attain ecological balance. We consider two treatment areas as a rectangle $\omega_1 = [20,30] \times [0,15]$ at the border, and $\omega_2 = [10,20] \times [15,25]$ in the center. This strategy involves the introduction of two controls reflecting the effect of the applied measurement program on the prey and predator population.

Figures 3-5 show the situation with control. In this situation, we have controlled the operation of prey-predators to sustain their existence. So, in this case, the only compulsion we have is through predation on prey populations.

In this second case, we maximize the optimal controls related to the development of the genus of the two marine species. Based on Figure 3, the prey density level lowers with a minimum value, which can guarantee its abundance for a long period of time in the ω_i, $i = 1,2$. This minimal decrease is explained by the presence of predation. From figure 4, we illustrate that the optimal control value employed in this case is effective in guaranteeing the density over time. Indeed,The density of predators is fairly high in Figures 4-5, as these populations are still feeding on prey and their fishing is well controlled in the ω_i, $i = 1,2$. In this research, we have found that the optimal control values that ensure the durability of marine populations are equal to 0.52 and 0.73.

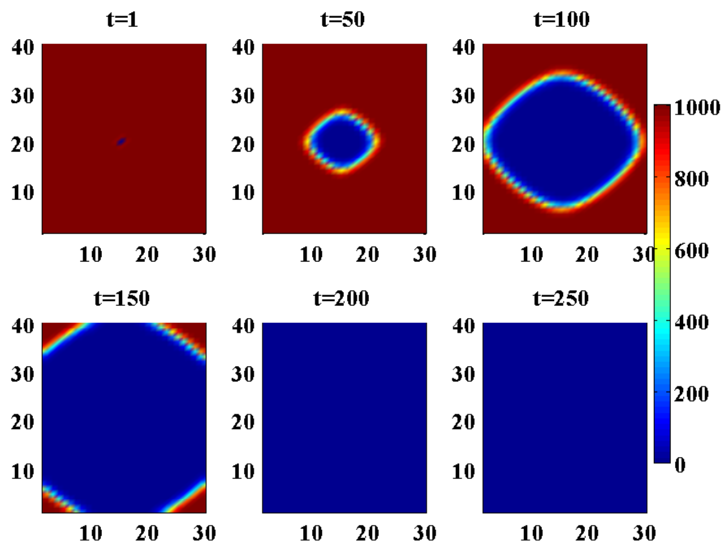

FIGURE 1. Evolution of X without controls along time

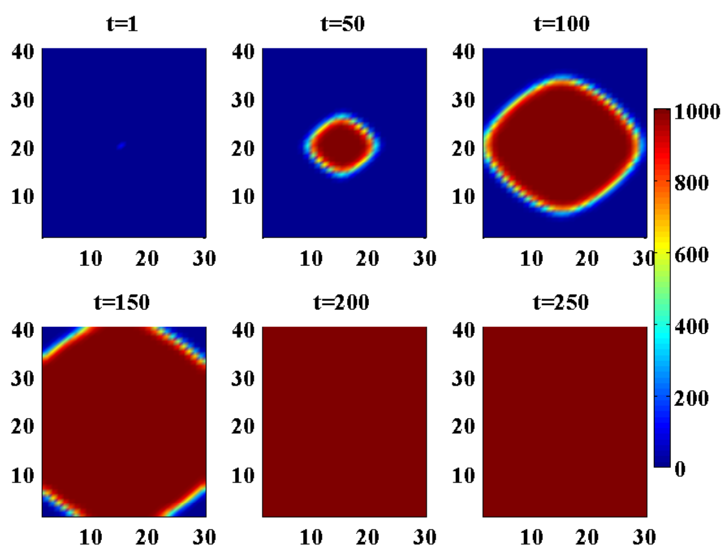

FIGURE 2. Evolution of Y without controls along time.

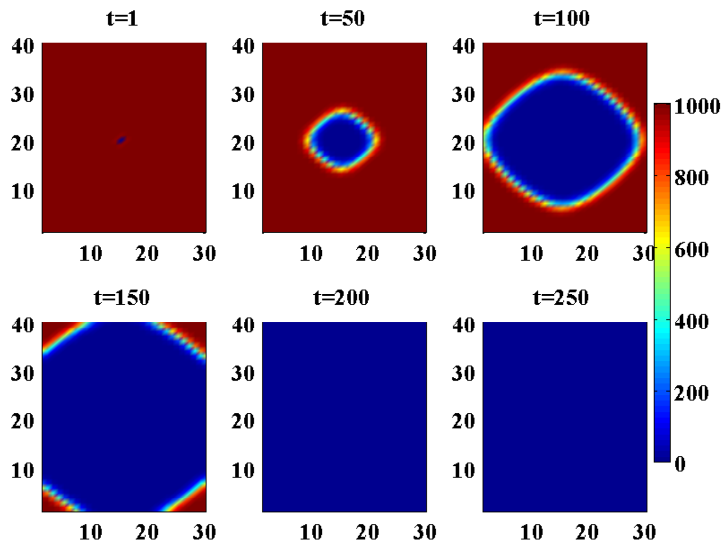

FIGURE 3. Evolution of X with control in region $\omega_1$and ω_2.

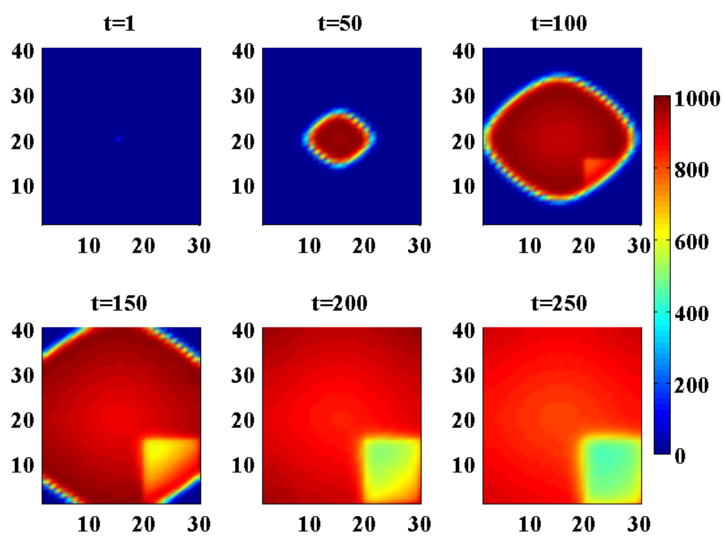

FIGURE 4. Evolution of X with control in region ω_1

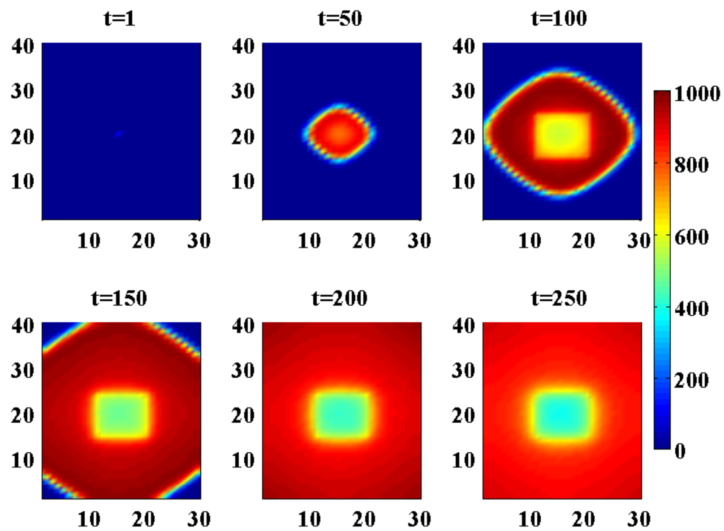

FIGURE 5. Evolution of Y with control in region ω_2.

7. CONCLUSION

In this work, we are concerned with the regional side of optimal control, we studied a distributed optimal control pair in the form of minimization of the population density of prey and predators in two diverse regions. We show the existence of solutions to our state system as well as the existence of an optimal control. For a certain functional objective, an optimal control is described in terms of the related state and the associated function. A numerical simulation is performed, showing that the two optimal controls are very effective in reducing the total number of the prey population as well as the number of predators in order to reach an ecological balance.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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