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THE IMPACT OF FEAR AND HARVESTING ON PLANKTON-FISH SYSTEM DYNAMICS INCORPORATING HARMFUL PHYTOPLANKTON IN THE CONTAMINATED ENVIRONMENT

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Abstract: In this study, an aquatic system with fish, zooplankton, and harmful phytoplankton interacting with one another through the food chain in a polluted environment was proposed. It was thought about how the dynamics of the food chain were affected by fear and harvesting. An alternate food source was included for fish to eat to make the model more realistic. The characteristics of every solution were investigated. Both a local and global analysis of the system's stability was conducted. The system's persistence was investigated. The effects of changing the system's parameters were investigated using the local bifurcation theorem. To validate the discovered theoretical result and comprehend the effect of the system's parameters, a numerical example was then provided. It has been found that the system responds very quickly to changing the parameter values and exhibits a variety of attractors, including stable limit cycles and bi-stable behavior.

Keywords: aquatic system; stability; bifurcation; persistence.

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1. INTRODUCTION

It is widely acknowledged that plankton is essential to the functioning of the marine ecosystem. The two basic forms of plankton are phytoplankton and zooplankton. To simulate phytoplankton-zooplankton interactions, numerous models have already been created [1-2]. The impact of fish and plankton biomass has received a lot of attention in the marine ecosystem. Research on the dynamics of populations exploited in fisheries should be prioritized due to the economic significance of fishing, fishermen's desire to maximize returns from natural stands, and the requirement for responsible authorities to conserve stocks through measures. In aquatic food webs, phytoplankton makes up the majority of the main energy sources and contributes significantly to fixed global production. Zooplankton, which feeds fish and other aquatic animals, consumes phytoplankton. In reality, phytoplankton can provide other species with a significant amount of oxygen after absorbing carbon dioxide from the surrounding environment. Plankton is therefore the foundation of all aquatic food chains and plays a crucial part in the study of marine ecology [3-7].

Top-down and bottom-up impacts on plankton-fish dynamics were examined [1, 4]. The authors have talked about how zooplankton is regulated by fish predation at high fish densities while algal biomass is low or nutrient-limited. In contrast, zooplankton is food-limited and phytoplankton abundance is regulated by zooplankton feeding at low fish numbers. To demonstrate how an increase in fish predation may alter plankton dynamics in the model and to clarify which type of bifurcations may occur, isocline analysis and simulations were discussed in [2]. Anthropogenic pollution of freshwater and marine systems has gained attention in recent years. There is a lot of research on bloom dynamics, with a focus on toxic algal blooms in particular. In aquatic environments, toxic plankton blooms have significantly increased over the last 20 years [8-9]. At least eight distinct modes and processes exist for hazardous phytoplankton species to induce mortality, physiological impairment, or other adverse in situ impacts, according to studies [10-11]. Studies of marine plankton are common and relevant since it is well known that toxin-producing phytoplankton has a substantial impact on fish growth. Through the creation of toxins, plankton species have defensive mechanisms against predation. The dynamics of phytoplankton-zooplankton are significantly impacted by such defensive behavior [5, 12–15]. Such algal blooms and fish-on-zooplankton predation have a significant deleterious impact on zooplankton and the marine ecology.

Numerous scholars have considered the study of aquatic food chains in a marine environment that has been contaminated by external toxicity. It has been found that external toxicity is crucial to the aquatic ecology; several dynamical behaviors in this food chain have been found, for instance [13, 16-17]. Later on, consideration was given to a three-species plankton-fish system that includes nonlinear harvesting and external toxicity [16]. An external toxic substance can have a direct or indirect impact on a species' growth, and Holling type II functional responses are when a predator feeds on an affected prey. Through sensitivity analysis and the numerical computation of the Lyapunov exponents, the existence of limit cycles has been observed in relation to the distinct coexisting equilibrium point. Talib et al. [17] recently designed and evaluated an aquatic food chain model that included fish, phytoplankton, and zooplankton and lived in a contaminated environment. When describing the growth of fish and the movement of food up the food chain, they employed modified Leslie-Gower models with Holling type IV functional responses, respectively. The food chain has complex dynamics, including chaos, they discovered. Additionally, the presence of harmful materials serves as a stabilizing element in the model.

On the other hand, understanding how a predator affects its prey in predator-prey interactions can be done from two different perspectives: direct killing and indirect method. The indirect strategy is founded on the predation fear brought on by the prey's anti-predator activities. The majority of mathematical ecology research focused on direct predation by predators on prey. However, several experimental investigations show that in addition to direct predation, prey species' fear of predators substantially alters their physiology and behavior [18]. Predation risk from the predator population can sometimes compel prey species to modify their preferred habitat, grazing areas, and reproductive zone, which has an impact on their long-term survival and fecundity rate. Recently, a 3D plankton-fish dynamical system with species of phytoplankton, zooplankton, and fish was suggested and examined to determine the effects of anti-predator behavior brought on by the fear effect and zooplankton refuge [19]. They came to the conclusion that the plankton-fish ecosystem's sustainability and coexistence are maintained by the current ecological model's distinct intervals of zooplankton refuge and fear impact.

In this paper, however, an aquatic food chain system in a contaminated environment that consists of harmful phytoplankton-zooplankton-fish is proposed and studied. The impact of predation fear, external toxic substances, and harvesting are combined in the proposed food chain model.

2. CONSTRUCTION OF A MATHEMATICAL MODEL

Consider the following simple aquatic food chain model, which includes harmful phytoplankton, zooplankton, and fish, as well as a Holling type-II functional response that depicts food movement across the chain:

$$\begin{aligned}
\frac{dP}{dT} &= rP\left[1-\frac{P}{L}\right]-\frac{a_1PZ}{a_2+P}-b_1P^3,\\
\frac{dZ}{dT} &= \frac{ea_1PZ}{(a_2+P)(1+kF)}-d_1PZ-\frac{a_3ZF}{a_4+Z}-b_2Z^2-d_2Z,\\
\frac{dF}{dT} &= c_1F\left[1-\frac{F}{c_2+c_3Z}\right]-\frac{qEF}{l_1E+l_2F}-b_3F^2,
\end{aligned} \tag{1}$$

where $P(T)$, $Z(T)$, and $F(T)$ represent the density at time T for the phytoplankton, zooplankton, and fish respectively, and $P(0)\geq 0$, $Z(0)\geq 0$, and $F(0)\geq 0$. The food chain model (1) is constructed according to the following hypotheses:

1. The environment in which the food chain exists is contaminated, and the pollution directly damages phytoplankton while indirectly affecting other species through their consumption of phytoplankton.
2. In the absence of zooplankton, phytoplankton grows logistically and creates a toxic substance as a defense against zooplankton predation.
3. In the absence of other species, zooplankton decays exponentially; yet, it grows by feeding on phytoplankton, according to the Holling type II functional response. Fear of fish is thought to have an impact on their growth. However, zooplankton decline occurs as a result of phytoplankton's antipredator properties and a contaminated environment that has an

indirect impact. Finally, according to the Holling type II functional response, it is attacked by fish.

4. It is thought that the fish grows logistically with a carrying capacity based on zooplankton, implying that fish have alternative food sources. While a fish population declines as a result of nonlinear harvesting and the indirect influence of a polluted environment.

The following fear function is utilized in the growth term of zooplankton $G(k,F) = \frac{1}{(1+kF)}$, which satisfies the following properties.

$G(0,F) = 1$: This suggests that in the absence of the fear component, zooplankton species' reproduction rates remain unchanged.

$G(k,0) = 1$: This suggests that zooplankton reproduction is unaffected by the absence of fish species.

$\lim_{k\to\infty} G(k,F) = 0$: As a result of the large increase in antipredator behavior, the growth rate of zooplankton species becomes zero.

$\lim_{F\to\infty} G(k,F) = 0$: When the fish population is sufficiently large, the growth rate of zooplankton species becomes zero.

$\frac{\partial G(k,F)}{\partial K} = \frac{-F}{(1+kF)^2} < 0$: This suggests that as anti-predator behavior increases, the growth of zooplankton species declines.

$\frac{\partial G(k,F)}{\partial F} = \frac{-k}{(1+kF)^2} < 0$: This indicates that as the fish population grows, the growth of zooplankton species diminishes.

Keeping the above in mind, the description of model parameters is given in Table (1).

Table 1: The descriptions of parameters

Parameter	**Descriptions**
r	The intrinsic growth rate of prey
L	The carrying capacity of prey
a_1	The attack rate of zooplankton to the phytoplankton
a_2	The half-saturation constant of the zooplankton
b_1	The coefficient of external toxic substances that affects the phytoplankton population
b_2	The coefficient of external toxic substances that affects the zooplankton population
b_3	The coefficient of external toxic substances that affects the fish population
e	The conversion rate of the food to zooplankton
k	Level of Fear in zooplankton
d_1	The liberation rate of toxic substances by the harmful phytoplankton
a_3	The attack rate of fish on the zooplankton
a_4	The half-saturation constant of the fish
d_2	The natural death rate of the zooplankton
c_1	The intrinsic growth rate of fish
c_2	The half-saturation constant of the fish in the absence of zooplankton
c_3	The fish's preference rate of zooplankton
E	The harvest effort.
q	The catchability rate
l_1 and l_2	The appropriate constants.

It's worth noting that system (1) includes 20 parameters, which makes the analysis more difficult. As a result, the following dimensionless variables and parameters are used in the system (1) to decrease a large number of parameters and then simplify our equations, and the following dimensionless system is obtained.

$$\begin{aligned}\frac{dx}{dt} &= x\left[(1-x) - \frac{y}{u_1+x} - u_2x^2\right] = xf_1(x,y,z),\\ \frac{dy}{dt} &= y\left[\frac{u_3x}{(u_1+x)(1+u_4z)} - u_5x - \frac{z}{u_6+y} - u_7y - u_8\right] = yf_2(x,y,z),\\ \frac{dz}{dt} &= z\left[u_9\left(1-\frac{u_{10}z}{u_{11}+y}\right) - \frac{u_{12}}{u_{13}+z} - u_{14}z\right] = zf_3(x,y,z),\end{aligned} \tag{2}$$

where the dimensionless variables and parameters are given as:

$$T = \frac{t}{r},\ x = \frac{P}{L},\ y = \frac{a_1}{rL}Z,\ z = \frac{a_1a_3}{r^2L}F,$$

$$u_1 = \frac{a_2}{L},\ u_2 = \frac{b_1L^2}{r},\ u_3 = \frac{ea_1}{r},\ u_4 = \frac{r^2Lk}{a_1a_3},\ u_5 = \frac{d_1L}{r},\ u_6 = \frac{a_1a_4}{rL},\ u_7 = \frac{b_2L}{a_1},$$

$$u_8 = \frac{d_2}{r},\ u_9 = \frac{c_1}{r},\ u_{10} = \frac{r}{c_3a_3},\ u_{11} = \frac{c_2a_1}{c_3rL},\ u_{12} = \frac{qEa_1a_3}{r^3LL_2},\ u_{13} = \frac{L_1Ea_1a_3}{L_2r^2L},\ u_{14} = \frac{rb_3L}{a_1a_3}.$$

System (2) clearly has 14 parameters, which simplifies the system analysis. Furthermore, the domain of system (2) is defined by $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$.

System (2) has a unique solution because the right-hand side functions are continuous and have continuous partial derivatives. In addition, the following theorem establishes the solution's positivity and boundedness.

Theorem 1: The nonnegative solutions of the aquatic system (2) are uniformly bounded and reside in the positive octant $D = \{(x,y,z) \in \mathbb{R}^3_+ : x(0) > 0, y(0) > 0, z(0) > 0\}$.

Proof. From the equations of system (2), it is obtained that:

$$x(t) = x(0)\exp\int_0^t\left[(1-x(s)) - \frac{y(s)}{u_1+x(s)} - u_2x(s)^2\right]ds,$$

$$y(t) = y(0)\exp\int_0^t\left[\frac{u_3x(s)}{(u_1+x(s))(1+u_4z(s))} - u_5x(s) - \frac{z(s)}{u_6+y(s)} - u_7y(s) - u_8\right]ds,$$

$$z(t) = z(0)\exp\int_0^t\left[u_9\left(1-\frac{u_{10}z(s)}{u_{11}+y(s)}\right) - \frac{u_{12}}{u_{13}+z(s)} - u_{14}z(s)\right]ds.$$

Therefore, all solutions belong to D all the time provided that $x(0) > 0, y(0) > 0, z(0) > 0$.

Now, from the first equation, it is gained that $\frac{dx}{dt} \leq x(1-x)$, which gives for $t \to \infty$ that $x(t) < 1$. Let, $W_1 = x(t) + \frac{y}{u_3}$, then

$$\frac{dW_1}{dt} \leq (1+u_8)x - u_8x - \frac{u_8}{u_3}y \Rightarrow \frac{dW_1}{dt} + u_8W_1 \leq (1+u_8).$$

Therefore, using Gronwall inequality, It is gained that $W_1 \leq \frac{(1+u_8)}{u_8}$ as $t \to \infty$. As a result, it is obtained that $y \leq u_3 \frac{(1+u_8)}{u_8} = \zeta_1$ as $t \to \infty$. Using the resulting upper bound of the variable y in the third equation gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = u_9 z\left(1 - \frac{u_{10}z}{u_{11}+y}\right) - \frac{u_{12}z}{u_{13}+z} - u_{14}\mathrm{z}^2 \leq u_9 z\left(1 - \frac{u_{10}z}{u_{11}+\zeta_1}\right)$$

Straightforward computation gives that $z \leq \frac{u_{11}+\zeta_1}{u_{10}} = \zeta_2$ as $t \to \infty$. Thus, the proof is done.

3. Stability Analysis

This section looks at the local stability of the dynamical system (2) around various biologically feasible steady-state points (SSPs). There are at most seven nonnegative SSPs in the system (2):

The disappear steady-state point (DSSP) that is given by $E_0 = (0,0,0)$ exists at all times.

The phytoplankton steady-state point (PHSSP), which is represented by $E_{\mathrm{x}} = (\bar{x}, 0,0)$, exists at all times, were

$$\bar{x} = \frac{-1+\sqrt{1+4u_2}}{2u_2}. \tag{3}$$

The fish steady-state point (FSSP) can be written as $E_z = (0,0,\bar{z})$, were

$$\bar{z} = \frac{-B_2+\sqrt{B_2^2-4B_1B_3}}{2B_1}, \tag{4}$$

where $B_1 = u_9u_{10} + u_{11}u_{14} > 0$, $B_2 = u_9u_{10}u_{13} + u_{11}u_{13}u_{14} - u_9u_{11}$, and $B_3 = u_{11}(u_{12} - u_9u_{13})$. Obviously, FSSP exists if and only if the following requirement is met

$$u_{12} < u_9u_{13}. \tag{5}$$

The fish-free steady-state point (FFSSP) denoted as $E_{xy} = (\hat{x}, \hat{y}, 0)$, were

$$\hat{y} = (1 - \hat{x} - u_2\hat{x}^2)(u_1 + \hat{x}). \tag{6}$$

While $\hat{x}$ represents a positive root of the following equation:

$$\begin{gathered} -u_2u_7x^4 - (u_7 + 2u_1u_2u_3)x^3 + (u_5 + u_7 - 2u_1u_7 - {u_1}^2u_2u_7)x^2 \\ +(-u_3 + u_1u_5 + 2u_1u_7 - {u_1}^2u_7 + u_8)x + {u_1}^2u_7 + u_1u_8 = 0 \end{gathered} \tag{7}$$

As a result, FFSSP exists uniquely in the interior of a positive quadrant of xy −plane provided that the following requirements are met

$$u_2\hat{x}^2 + \hat{x} < 1, \tag{8a}$$

with applying one of the conditions listed below to verify that Eq. (7) has a single positive root

$$\left.\begin{array}{c} u_5+u_7<2u_1u_7+{u_1}^2u_2u_7 \\ u_3+{u_1}^2u_7<u_1u_5+2u_1u_7+u_8 \\ u_5+u_7<2u_1u_7+{u_1}^2u_2u_7 \text{ and } u_1u_5+2u_1u_7+u_8<u_3+{u_1}^2u_7 \\ 2u_1u_7+{u_1}^2u_2u_7<u_5+u_7 \text{ and } u_1u_5+2u_1u_7+u_8<u_3+{u_1}^2u_7 \end{array}\right\} \quad (8b)$$

The zooplankton-free steady-state point (ZFSSP) that denoted as $E_{xz}=(\bar{x},0,\bar{z})$, where $\bar{x}$ and $\bar{z}$ are given by Eqs. (3) and (4) respectively. As a result, ZFSSP exists uniquely in the interior of a positive quadrant of $xz-$plane provided that condition (5) is met.

The survival steady-state point (SSSP), which is denoted by $E_{xyz}=(x^*,y^*,z^*)$, is obtained by solving the system:

$$\begin{array}{l} f_1(x,y,z)=0, \\ f_2(x,y,z)=0, \\ f_3(x,y,z)=0. \end{array} \quad (9a)$$

The third equation gives that:

$$y=\frac{(u_{13}+z)[u_9u_{11}-(u_9u_{10}+u_{11}u_{14})\mathrm{z}]-u_{11}u_{12}}{[u_{12}+(u_{14}\mathrm{z}-u_9)(u_{13}+z)]}. \quad (9b)$$

Substituting the value of y in the first and second equations of (9a) gives the following two isoclines:

$$g_1(x,z)=(1-x)-\frac{\frac{(u_{13}+z)[u_9u_{11}-(u_9u_{10}+u_{11}u_{14})\mathrm{z}]-u_{11}u_{12}}{[u_{12}+(u_{14}\mathrm{z}-u_9)(u_{13}+z)]}}{u_1+x}-u_2x^2=0 \quad (9c)$$

$$\begin{array}{c} g_2(\mathrm{x,z})=\frac{u_3x}{(u_1+x)(1+u_4z)}-u_5x-\frac{z}{u_6+\frac{(u_{13}+z)[u_9u_{11}-(u_9u_{10}+u_{11}u_{14})\mathrm{z}]-u_{11}u_{12}}{[u_{12}+(u_{14}\mathrm{z}-u_9)(u_{13}+z)]}} \\ -u_7\frac{(u_{13}+z)[u_9u_{11}-(u_9u_{10}+u_{11}u_{14})\mathrm{z}]-u_{11}u_{12}}{[u_{12}+(u_{14}\mathrm{z}-u_9)(u_{13}+z)]}-u_8=0 \end{array}. \quad (9d)$$

Clearly, for $z\rightarrow 0$, it is obtained that

$$g_1(x,0)=-u_2x^3-(1+u_1u_2)x^2+(1-u_1)x+u_1+u_{11}=0$$

$$g_2(x,0)=-u_5x^2+(u_3-u_1u_5+u_7u_{11}-u_8)x+u_1(u_7u_{11}-u_8)=0$$

As a result, using the Discard Rule of sign, $g_1(x,0)$ intersect the $x-$axis at a unique positive point say x_1, while $g_2(x,0)$ intersects the $x-$axis at a unique positive point x_2 provided that the following condition holds.

$$u_8<u_7u_{11}. \quad (10a)$$

Consequently, the two isoclines (9c) and (9d) have a unique intersection point in the interior of a positive quadrant of the $xz-$plane that is denoted by (x^*,z^*) provided that in addition to condition (9e) the following sufficient conditions are met.

$$\left.\begin{array}{c} x_1 < x_2 \\ \frac{dz}{dx} = -\frac{(\partial g_1/\partial x)}{(\partial g_1/\partial z)} > 0 \\ \frac{dz}{dx} = -\frac{(\partial g_2/\partial x)}{(\partial g_2/\partial z)} < 0 \end{array}\right\}, \tag{10b}$$

Substituting the value of z^* in Eq. (9b) gives a unique value $y(z^*) = y^*$ that is positive provided that the following condition holds.

$$u_9(u_{13} + z^*) - \frac{u_9 u_{10}}{u_{11}} z^*(u_{13} + z^*) < u_{14} z^*(u_{13} + z^*) + u_{12} < u_9(u_{13} + z^*). \tag{10c}$$

Hence the SSSP exists uniquely under the conditions (10a), (10b), and (10c).

The Jacobian matrix (JM) of the system (2) at the point (x, y, z) can be written as:

$$J = \begin{bmatrix} x\frac{df_1}{dx} + f_1 & x\frac{df_1}{dy} & x\frac{df_1}{dz} \\ y\frac{df_2}{dx} & y\frac{df_2}{dy} + f_2 & y\frac{df_2}{dz} \\ z\frac{df_3}{dx} & z\frac{df_3}{dy} & z\frac{df_3}{dz} + f_3 \end{bmatrix}, \tag{11}$$

where

$$\frac{df_1}{dx} = -1 + \frac{y}{(u_1+x)^2} - 2u_2 x,$$

$$\frac{df_1}{dy} = -\frac{1}{u_1+x},$$

$$\frac{df_1}{dz} = \text{zero},$$

$$\frac{df_2}{dx} = \frac{u_1 u_3}{(u_1+x)^2(1+u_4 z)} - u_5,$$

$$\frac{df_2}{dy} = \frac{z}{(u_6+y)^2} - u_7,$$

$$\frac{df_2}{dz} = \frac{-u_3 u_4 x}{(u_1+x)(1+u_4 z)^2} - \frac{1}{u_6+y},$$

$$\frac{df_3}{dx} = zero,$$

$$\frac{df_3}{dy} = \frac{u_9 u_{10} z}{(u_{11}+y)^2},$$

$$\frac{df_3}{dz} = \frac{-u_9 u_{10}}{(u_{11}+y)} + \frac{u_{12}}{(u_{13}+z)^2} - u_{14}.$$

Therefore, direct computation shows that the eigenvalues of $J(E_0)$ are determined as:

$$\lambda_{01} = 1 > 0,\ \lambda_{02} = -u_8 < 0,\ \lambda_{03} = u_9 > 0. \tag{12}$$

Hence, DSSP is a saddle point.

The eigenvalues of $J(E_x)$ can be written as

$$\lambda_{11} = -\bar{x}(1 + 2u_2\bar{x}) < 0,\ \lambda_{12} = \frac{u_3\bar{x}}{u_1+\bar{x}} - u_5\bar{x} - u_8,\ \lambda_{13} = u_9 - \frac{u_{12}}{u_{13}}. \quad (13)$$

Thus, PHSSP is locally asymptotically stable if and only if the following conditions are met:

$$\frac{u_3\bar{x}}{u_1+\bar{x}} < u_5\bar{x} + u_8, \quad (14a)$$

$$u_9 < \frac{u_{12}}{u_{13}}. \quad (14b)$$

The eigenvalues of $J(E_z)$ can be determined as

$$\lambda_{21} = 1 > 0,\ \lambda_{22} = \frac{-\bar{z}}{u_6} - u_8,\ \lambda_{23} = -\frac{u_9u_{10}\bar{z}}{u_{11}} + \frac{u_{12}\bar{z}}{(u_{13}+\bar{z})^2} - u_{14}\bar{z}. \quad (15)$$

As a result, FSSP is an unstable point.

The JM at $E_{\mathrm{xy}} = (\hat{x}, \hat{y}, 0)$ can be written as

$$\mathrm{J}(E_{xy}) = \begin{bmatrix} \hat{x}\left(-1 + \frac{\hat{y}}{(u_1+\hat{x})^2} - 2u_2\hat{x}\right) & -\frac{\hat{x}}{u_1+\hat{x}} & 0 \\ \frac{u_1u_3\hat{y}}{(u_1+\hat{x})^2} - u_5\hat{y} & -u_7\hat{y} & -\frac{u_3u_4\hat{x}\hat{y}}{u_1+\hat{x}} - \frac{\hat{y}}{u_6+\hat{y}} \\ 0 & 0 & u_9 - \frac{u_{12}}{u_{13}} \end{bmatrix}. \quad (16)$$

Accordingly, the characteristic equation can be written as:

$$\left[\lambda^2 - T_{xy}\lambda + D_{xy}\right]\left(u_9 - \frac{u_{12}}{u_{13}} - \lambda\right) = 0, \quad (17)$$

where

$$T_{xy} = \hat{x}\left(-1 + \frac{\hat{y}}{(u_1+\hat{x})^2} - 2u_2\hat{x}\right) - u_7\hat{y},$$

$$D_{xy} = \hat{x}\left(-1 + \frac{\hat{y}}{(u_1+\hat{x})^2} - 2u_2\hat{x}\right)(-u_7\hat{y}) - \left(-\frac{\hat{x}}{u_1+\hat{x}}\right)\left(\frac{u_1u_3\hat{y}}{(u_1+\hat{x})^2} - u_5\hat{y}\right).$$

Straightforward computation shows that the eigenvalues $\mathrm{J}(E_{xy})$ can be represented as

$$\lambda_{31,32} = \frac{T_{xy} \pm \sqrt{{T_{xy}}^2 - 4D_{xy}}}{2},\ \text{and}\ \lambda_{33} = u_9 - \frac{u_{12}}{u_{13}}. \quad (18)$$

Hence all the eigenvalues have negative real parts and then FFSSP is locally asymptotically stable if and only if the following conditions are satisfied.

$$\frac{\hat{y}}{(u_1+\hat{x})^2} < 1 + 2u_2\hat{x}, \quad (19a)$$

$$u_5\hat{y} < \frac{u_1 u_3 \hat{y}}{(u_1+\hat{x})^2}, \tag{19b}$$

$$u_9 < \frac{u_{12}}{u_{13}}. \tag{19c}$$

Similarly, the JM at $E_{xz} = (\bar{x}, 0, \bar{z})$ can be represented as

$$\mathrm{J}(E_{xz}) = \left[b_{ij}\right]_{3\times 3}. \tag{20}$$

where

$$b_{11} = -\bar{x} - 2u_2\bar{x}^2,\ b_{12} = -\frac{\bar{x}}{u_1+\bar{x}},\ b_{13} = 0,$$

$$b_{21} = 0,\ b_{22} = \frac{u_3\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})} - u_5\bar{x} - \frac{\bar{z}}{u_6} - u_8,\ b_{23} = 0,$$

$$b_{31} = 0,\ b_{32} = \frac{u_9 u_{10}\bar{z}^2}{u_{11}^2},\ b_{33} = -\frac{u_9 u_{10}\bar{z}}{u_{11}} + \frac{u_{12}\bar{z}}{(u_{13}+\bar{z})^2} - u_{14}\bar{z}.$$

Therefore, the eigenvalues of $\mathrm{J}(E_{xz})$ are determined as

$$\left.\begin{array}{c} \lambda_{41} = -\bar{x} - 2u_2\bar{x}^2 < 0 \\ \lambda_{42} = \frac{u_3\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})} - u_5\bar{x} - \frac{\bar{z}}{u_6} - u_8 \\ \lambda_{43} = -\frac{u_9 u_{10}\bar{z}}{u_{11}} + \frac{u_{12}\bar{z}}{(u_{13}+\bar{z})^2} - u_{14}\bar{z} \end{array}\right\} \tag{21}$$

Accordingly, ZFSSP is locally asymptotically stable if and only if the following conditions are met.

$$\frac{u_3\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})} < u_5\bar{x} + \frac{\bar{z}}{u_6} + u_8, \tag{22a}$$

$$\frac{u_{12}\bar{z}}{(u_{13}+\bar{z})^2} < \frac{u_9 u_{10}\bar{z}}{u_{11}} + u_{14}\bar{z}. \tag{22b}$$

The JM at E_{xyz} can be computed as

$$\mathrm{J}(E_{xyz}) = \left[a_{ij}\right]_{3\times 3}, \tag{23}$$

where

$$a_{11} = -x^* + \frac{x^* y^*}{u_1+x^*} - 2u_2 {x^*}^2,\ a_{12} = -\frac{x^*}{u_1+x^*},\ a_{13} = 0,$$

$$a_{21} = \frac{u_1 u_3 y^*}{(u_1+x^*)^2(1+u_4 z^*)} - u_5 y^*,\ a_{22} = \frac{y^* z^*}{(u_6+y^*)^2} - u_7 y^*,$$

$$a_{23} = -\frac{u_3 u_4 x^* y^*}{(u_1+x^*)(1+u_4 z^*)^2} - \frac{1}{u_6+y^*},\ a_{31} = 0,$$

$$a_{32} = \frac{u_1 u_{10} {z^*}^2}{(u_1+y^*)^2},\ a_{33} = -\frac{u_9 u_{10} z^*}{u_{11}+y^*} + \frac{u_{12} z^*}{(u_{13}+z^*)^2} - u_{14} z^*.$$

As a result, the local stability criteria for SSSP are established by the following theorem.

Theorem 2. The SSSP is locally asymptotically stable if and only if the following requirements are met.

$$\frac{y^*}{u_1+x^*} < 2u_2x^* + 1, \tag{24a}$$

$$u_5 < \frac{u_1u_3}{(u_1+x^*)^2(1+u_4z^*)}, \tag{24b}$$

$$\frac{z^*}{(u_6+y^*)^2} < u_7, \tag{24c}$$

$$\frac{u_{12}}{(u_{13}+z^*)^2} < u_{14} + \frac{u_9u_{10}}{u_{11}+y^*}. \tag{24d}$$

Proof. If $J(E_{xyz})$ has three eigenvalues with negative real parts, the proof is complete. Since the eigenvalues of $J(E_{xyz})$ are the roots of the equation

$$\lambda^3 + H_1\lambda^2 + H_2\lambda + H_3 = 0, \tag{25}$$

where

$$H_1 = -(a_{11} + a_{22} + a_{33}),$$

$$H_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32}$$

$$H_3 = -(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33})$$

$$\begin{aligned} H_1H_2 - H_3 = &-(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21}) \\ &-(a_{22} + a_{33})(a_{22}a_{33} - a_{23}a_{32}) \\ &-a_{11}a_{33}(a_{11} + a_{33}) - 2a_{11}a_{22}a_{33}. \end{aligned}$$

According to the Routh-Hurwitz criterion all the roots of Eq. (25) have negative real parts if and only if $H_1 > 0$, $H_3 > 0$, and $H_1H_2 - H_3 > 0$. Therefore, direct computation shows that, the supplied requirements are assured to satisfy the Routh-Hurwitz requirements. As a result, SSSP is asymptotically stable locally.

4. PERSISTENCE

This section deals with the permanence of all species when time goes on indefinitely. Persistence refers to a species' continued existence in the deterministic sense. When $x_i(0) > 0$, however, persistence shows that $\lim_{t\to\infty} \inf x_i(t) > 0$ for each individual species of $x_i(t)$. This means that the trajectories of the system (2) are eventually confined away from the border planes. As a result, system (2) is said to persist if each variable $x, y,$ and z is permanence. Hence, the initial step is to look at the potential of periodic dynamics in border planes.

The system (2) has two subsystems that can be controlled by it. The first subsystem that fall in xy −plane and denoted by I, while the second one denoted by II and fall into the xz −plane.

Subsystem I is written as:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x\left[(1-x) - \frac{y}{u_1+x} - u_2x^2\right] = h_1(x,y), \\ \frac{\mathrm{d}y}{dt} &= y\left[\frac{u_3x}{(u_1+x)} - u_5x - u_7\mathrm{y} - u_8\right] = h_2(x,y). \end{aligned} \tag{26}$$

Subsystem II is written as:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x[(1-x) - u_2x^2] = g_1(x,z), \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= z\left[u_9\left(1 - \frac{u_{10}}{u_{11}}z\right) - \frac{u_{12}}{u_{13}+z} - u_{14}\mathrm{z}\right] = g_2(x,z). \end{aligned} \tag{27}$$

Consider $M_1(x,y) = \frac{1}{xy}$, and $M_2(x,z) = \frac{1}{xz}$, which satisfy $M_i > 0; i = 1,2,$ and C^1 functions in the $int.\mathbb{R}_+^2$ of the $xy-$ and $xz-$ planes respectively. As a result, straightforward computation reveals that:

$$D(x,y) = \frac{\partial(M_1\, h_1)}{\partial x} + \frac{\partial(M_1\, h_2)}{\partial y} = -\frac{1}{y} + \frac{1}{(u_1+x)^2} - \frac{2u_2x}{y} - \frac{u_7}{x}$$

Accordingly, $D(x,y)$ does not equal zero in the $int.\mathbb{R}_+^2$ of the xy −plane and does not change the sign provided the next condition is met:

$$\left.\begin{aligned} \frac{1}{y} + \frac{2u_2x}{y} + \frac{u_7}{x} < \frac{1}{(u_1+x)^2} \\ OR \\ \frac{1}{y} + \frac{2u_2x}{y} + \frac{u_7}{x} > \frac{1}{(u_1+x)^2} \end{aligned}\right\} \tag{28}$$

Regarding $D(x,z)$ similar finding is got provided that the following condition is met:

$$\left.\begin{aligned} \frac{1}{z} + \frac{2u_2x}{z} + \frac{u_9u_{10}}{u_{11}x} + \frac{u_{14}}{x} < \frac{u_{12}}{x(u_{13}+z)^2} \\ OR \\ \frac{1}{z} + \frac{2u_2x}{z} + \frac{u_9u_{10}}{u_{11}x} + \frac{u_{14}}{x} > \frac{u_{12}}{x(u_{13}+z)^2} \end{aligned}\right\} \tag{29}$$

According to the Dulac-Bendixson criterion [20], there is no closed curve in the $int.\mathbb{R}_+^2$ of the $xy-$ and xz −planes under conditions (28) and (29). As a result, the Poincare-Bendixon theorem [20] asserts that whenever the border plane’s steady-states, as defined by E_{xy} and E_{xz}, are locally asymptotically stable, the unique equilibrium point in $int.\mathbb{R}_+^2$ is globally asymptotically stable.

Theorem 3. If the following requirements with the conditions (28)-(29) are met, system (2) is uniformly persistent.

$$\frac{u_3\bar{x}}{u_1+\bar{x}} > u_5\bar{x} + u_8. \tag{30a}$$

$$u_9 > \frac{u_{12}}{u_{13}}. \tag{30b}$$

$$\frac{u_3\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})} > u_5\bar{x} + \frac{\bar{z}}{u_6} + u_8. \tag{30c}$$

Proof: Define the following function using the average Lyapunov function method [21]. $\rho(x,y,z) = x^{n_1}\, y^{n_2}\, z^{n_3}$, where $n_j, \forall j = 1,2,3$ represent the positive constants. Thus, $\rho(x,y,z) > 0$, for all $(x,y,z) \in int.\mathbb{R}^3_+$ and $\rho(x,y,z) \to 0$ when any of their variables gets close to zero. Therefore, it is gained that

$$\begin{aligned}\sigma(x,y,z) = \frac{\rho'(x,y,z)}{\rho(x,y,z)} = n_1\left[(1-x) - \frac{y}{u_1+x} - u_2x^2\right] \\ +n_2\left[\frac{u_3x}{(u_1+x)(1+u_4z)} - u_5x - \frac{z}{u_6+y} - u_7\mathrm{y} - u_8\right] \\ +n_3\left[u_9\left(1-\frac{u_{10}z}{u_{11}+y}\right) - \frac{u_{12}}{u_{13}+z} - u_{14}\mathrm{z}\right].\end{aligned}$$

Now, if $\sigma(E) > 0$ for every attractor point E on the border planes, given a sufficient selection of constants $n_i > 0$, $i = 1,2,3$, the proof is done using the average Lyapunov function.

Because:

$$\sigma(E_0) = n_1 - u_8n_2 + \left(u_9 - \frac{u_{12}}{u_{13}}\right)n_3,$$

$$\sigma(E_x) = n_2\left(\frac{u_3\bar{x}}{(u_1+\bar{x})} - u_5\bar{x} - u_8\right) + n_3\left(u_9 - \frac{u_{12}}{u_{13}}\right),$$

$$\sigma(E_z) = n_1 - \left(\frac{\bar{z}}{u_6} + u_8\right)n_2$$

$$\sigma(E_{xy}) = n_3\left(u_9 - \frac{u_{12}}{u_{13}}\right),$$

$$\sigma(E_{xz}) = n_2\left(\frac{u_3\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})} - u_5\bar{x} - \frac{\bar{z}}{u_6} - u_8\right).$$

Thus, choosing n_1 to be a sufficiently large value leads to $\sigma(E_0) > 0$, and $\sigma(E_z) > 0$. However, $\sigma(E_x) > 0$, $\sigma(E_{xy}) > 0$, and $\sigma(E_{xz}) > 0$ under the conditions (30a), (30b), and (30c) respectively. Thus, the proof is done.

5. GLOBAL STABILITY ANALYSIS

The study of the global stability analysis for the steady-state points of system (2), which previously examined their local stability, is investigated theoretically in this section with the use of a suitable Lyapunov method, as proven in the following theorems:

Theorem 4. Assume that PHSSP is locally asymptotically stable, then it is a globally asymptotically stable provided that the following conditions hold

$$\bar{x} < \frac{u_1 u_8}{u_3}. \tag{31a}$$

$$u_9 < \frac{u_{12}}{u_{13}+\zeta_2}. \tag{31b}$$

where ζ_2 represents the upper bound of z that is given in theorem (1).

Proof. Consider the function $Q_1 = C_1\left(x - \bar{x} - \bar{x}\, ln\, \frac{x}{\bar{x}}\right) + C_2 y + C_3 z$ that is positive definite on $\mathcal{B}_1 = \{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z \geq 0\}$. Straightforward computation gives:

$$\frac{dQ_1}{dt} = C_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt} + C_2\frac{dy}{dt} + C_3\frac{dz}{dt}$$

That gives

$$\frac{dQ_1}{dt} \leq -C_1[1 + u_2(x+\bar{x})](x-\bar{x})^2 - \frac{xy}{u_1+x}[C_1 - C_2 u_3]$$
$$-y\left[C_2 u_8 - C_1\frac{\bar{x}}{u_1}\right] - C_3 z\left[\frac{u_{12}}{u_{13}+z} - u_9\right].$$

Selecting the values of positive constants as $C_1 = u_3$, and $C_2 = C_3 = 1$ yields:

$$\frac{dQ_1}{dt} \leq -u_3[1 + u_2(x+\bar{x})](x-\bar{x})^2 - y\left[u_8 - u_3\frac{\bar{x}}{u_1}\right] - z\left[\frac{u_{12}}{u_{13}+z} - u_9\right].$$

Therefore, the function $\frac{dQ_1}{dt}$ is negative definite under the conditions (31a)-(31b). Hence PHSSP is globally asymptotically stable.

Theorem 5. Suppose that the FFSSP is locally asymptotically stable then it is globally asymptotically stable provided that in addition to condition (31b) the following conditions hold.

$$P_{12}^2 < 2P_{11}P_{22}. \tag{32a}$$

$$P_{23}^2 < 2P_{22}P_{33}. \tag{32b}$$

Proof. Consider the function $Q_2 = \left[x - \hat{x} - \hat{x}ln\,\frac{x}{\hat{x}}\right] + \left[y - \hat{y} - \hat{y}ln\,\frac{y}{\hat{y}}\right] + z$ that is positive definite on $\mathcal{B}_2 = \{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y > 0, z \geq 0\}$. Straightforward computation gives:

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} = \left(\frac{x-\hat{x}}{x}\right)\frac{dx}{dt} + \left(\frac{y-\hat{y}}{y}\right)\frac{dy}{dt} + \frac{dz}{dt}$$

Hence, the following is obtained:

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} = -P_{11}(x-\hat{x})^2 + P_{12}(x-\hat{x})(y-\hat{y}) - P_{22}(y-\hat{y})^2$$
$$-P_{23}(y-\hat{y})z - P_{33}z^2 - \left[\frac{u_{12}}{u_{13}+z} - u_3\right]z$$

where $P_{11} = [1 + \hat{y} + u_2(x + \hat{x})]$, $P_{12} = \left[(u_1 + \hat{x}) + \frac{u_1}{(u_1+x)(1+u_4 z)(u_1+\hat{x})} - u_5\right]$, $P_{22} = u_7$, $P_{23} = \left[\frac{u_4\hat{x}}{(1+u_4 z)(u_1+\hat{x})} + \frac{1}{(u_8+y)}\right]$, and $P_{33} = \left[\frac{u_9 u_{10}}{u_{11}+y} + u_{14}\right]$. Therefore, using the conditions (32a)-(32b) yields:

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} \leqslant -\left[\sqrt{P_{11}}(x-\hat{x}) - \sqrt{\frac{P_{22}}{2}}(y-\hat{y})\right]^2$$
$$-\left[\sqrt{\frac{P_{22}}{2}}(y-\hat{y}) + \sqrt{P_{33}}\ z\right]^2 - \left[\frac{u_{12}}{u_{13}+z} - u_9\right]z.$$

Therefore, using condition (31b) the function $\frac{\mathrm{d}Q_2}{\mathrm{d}t}$ is negative definite. Therefore, FFSSP is globally asymptotically stable.

Theorem 6. Suppose that the ZFSSP is locally asymptotically stable then it is globally asymptotically stable provided that the following conditions are met.

$$\frac{u_{12}}{u_{13}(u_{13}+\bar{z})} < \frac{u_9 u_{10}}{u_{14}+\zeta_1} + u_{14}. \tag{33a}$$

$$\frac{u_3}{u_1} < u_5, \tag{33b}$$

$$q_{12}^2 < 2q_{11}q_{22}, \tag{33c}$$

$$q_{23}^2 < 2q_{22}q_{33}. \tag{33d}$$

where ζ_1 stands for the upper bound of the variable y, while the other symbols are given in the proof.

Proof. Consider the function $Q_3 = \left[x - \bar{x} - \bar{x}ln\,\frac{x}{\bar{x}}\right] + y + \left[z - \bar{z} - \bar{z}ln\,\frac{z}{\bar{z}}\right]$ that is positive definite on $\mathcal{B}_3 = \{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z > 0\}$. Straightforward computation gives:

$$\frac{\mathrm{d}Q_3}{\mathrm{d}t} = \left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt} + \frac{dy}{dt} + \left(\frac{z-\bar{z}}{z}\right)\frac{dz}{dt}$$

Hence, the following is obtained:

$$\frac{dQ_3}{dt} = -q_{11}(x-\bar{x})^2 - q_{12}(x-\bar{x})y - q_{22}y^2 - \left[u_5 - \frac{u_3}{(u_1+x)(1+u_4 z)}\right]xy$$
$$-\frac{yz}{u_6+y} + q_{23}(z-\bar{z})y - q_{33}z^2 - u_8 y$$

where $q_{11} = 1 + u_2(x+\bar{x})$, $q_{12} = \frac{1}{u_1+x}$, $q_{22} = u_7$, $q_{23} = \frac{u_9 u_{10}\bar{z}}{u_{11}(u_{11}+y)}$, $q_{33} = \frac{u_9 u_{10}}{u_{11}+y} + u_{14} - \frac{u_{12}}{(u_{13}+z)(u_{13}+\bar{z})}$. Therefore, using the conditions (33a)-(33d) yields:

$$\frac{dQ_3}{dt} \le -\left[\sqrt{q_{11}}(x-\bar{x}) + \sqrt{\frac{q_{22}}{2}}y\right]^2 - u_8 y - \left[\sqrt{\frac{q_{22}}{2}}y - \sqrt{q_{33}}(z-\bar{z})\right]^2.$$

Therefore, the function $\frac{dQ_3}{dt}$ is negative definite. Therefore, ZFSSP is globally asymptotically stable.

Theorem 7. Suppose that the SSSP is locally asymptotically stable then it is globally asymptotically stable provided that the following conditions are met.

$$\frac{z^*}{u_6(u_6+y^*)} < u_7. \tag{34a}$$

$$\frac{u_{12}}{u_{13}(u_{13}+z^*)} < \frac{u_9 u_{10}}{(u_{11}+\zeta_1)} + u_{14}. \tag{34b}$$

$$r_{12}^2 < 2r_{11}r_{22}. \tag{34c}$$

$$r_{23}^3 < 2r_{22}r_{33}. \tag{34d}$$

where ζ_1 stands for the upper bound of the variable y, while the other symbols are given in the proof.

Proof. Consider the function:

$$Q_4 = \left[x - x^* - x^* ln\,\frac{x}{x^*}\right] + \left[y - y^* - y^* ln\,\frac{y}{y^*}\right] + \left[z - z^* - z^* ln\,\frac{z}{z^*}\right],$$

which is positive definite on $\mathcal{B}_4 = \{(x,y,z) \in \mathbb{R}_+^3 : x > 0, y > 0, z > 0\}$. Straightforward computation gives:

$$\frac{dQ_4}{dt} = \left(\frac{x-x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y-y^*}{y}\right)\frac{dy}{dt} + \left(\frac{z-z^*}{z}\right)\frac{dz}{dt}$$

Hence, the following is obtained:

$$\frac{dQ_4}{dt} = -r_{11}(x-x^*)^2 + r_{12}(x-x^*)(y-y^*) - r_{22}y^2$$
$$-r_{23}(y-y^*)(z-z^*) - r_{33}(z-z^*)^2,$$

where

$$r_{11} = 1 + \frac{y^*}{(u_1+x^*)(u_1+x)} + u_2(x + x^*).$$

$$r_{12} = \frac{1}{(u_1+x)} + \frac{u_1}{(u_1+x)(u_1+x^*)(1+u_4 z)} - u_5.$$

$$r_{23} = \frac{u_4 x^*}{(1+u_4 z)(u_1+x^*)(1+u_4 z^*)} + \frac{1}{(u_6+y)} - \frac{u_9 u_{10} z^*}{(u_{11}+y)(u_{11}+y^*)}.$$

$$r_{22} = u_7 - \frac{z^*}{(u_6+y)(u_6+y^*)}.$$

$$r_{33} = \frac{u_9 u_{10}}{u_{11}+y} + u_{14} - \frac{u_{12}}{(u_{13}+z)(u_{13}+z^*)}.$$

Therefore, using the conditions (34a) -(34d) yields:

$$\frac{\mathrm{d}Q_4}{\mathrm{d}t} \leq -\left[\sqrt{r_{11}}(x - x^*) - \sqrt{\frac{r_{22}}{2}}(y - y^*)\right]^2 - \left[\sqrt{\frac{r_{22}}{2}}(y - y^*) + \sqrt{r_{33}}(z - z^*)\right]^2$$

Therefore, the function $\frac{\mathrm{d}Q_4}{\mathrm{d}t}$ is negative definite. Therefore, SSSP is globally asymptotically stable.

6. BIFURCATION ANALYSIS

This section investigates the local bifurcation at the likely SSPs of the system (2) using Sotomayor's theorem [20]. It is commonly known that a non-hyperbolic SSP is a necessary but insufficient condition for bifurcation to occur. As a result, the candidate bifurcation parameter is chosen in order to attain the SSP. It will be non-hyperbolic at a particular value of that parameter. Rewrite system (2) in the following format:

$$\frac{\boldsymbol{dx}}{\boldsymbol{dt}} = \boldsymbol{F}(\boldsymbol{X}, \alpha)$$

where $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\boldsymbol{F}(\boldsymbol{X}, \alpha) = \begin{pmatrix} x f_1(x, y, z, \alpha) \\ y f_2(x, y, z, \alpha) \\ z f_3(x, y, z, \alpha) \end{pmatrix}$, and $\alpha \in \mathbb{R}$ be any parameter.

As a result, the second directional derivative of $\boldsymbol{F}(\boldsymbol{X}, \alpha)$ in the system (2) can be expressed as:

$$D^2\boldsymbol{F}(\boldsymbol{X}, \alpha)(\mho, \mho) = \left(c_{i1}{}^{[2]}\right)_{3\times 1}, \tag{35}$$

where:

$$c_{11}{}^{[2]} = 2\left[-1 + \frac{u_1 y}{(u_1+x)^3} - 3u_2 x\right] v_1^2 - \frac{2u_1}{(u_1+x)^2} v_1 v_2.$$

$$c_{21}{}^{[2]} = -\frac{2u_1u_3y}{(u_1+x)^3(1+u_4z)}v_1^2 + 2\left[\frac{u_1u_3}{(u_1+x)^2(1+u_4z)} - u_5\right]v_1v_2 - \left[\frac{2u_1u_3u_4y}{(u_1+x)^2(1+u_4z)^2}\right]v_1v_3$$
$$+2\left[\frac{u_6z}{(u_6+y)^3} - u_7\right]v_2^2 - 2\left[\frac{u_3u_4x}{(u_1+x)(1+u_4z)^2} + \frac{u_6}{(u_6+y)^2}\right]v_2v_3 + \frac{2u_3u_4^2xy}{(u_1+x)(1+u_4z)^3}v_3^2.$$

$$c_{31}{}^{[2]} = -\frac{2u_9u_{10}z^2}{(u_{11}+y)^3}v_2^2 + \frac{4u_9u_{10}}{(u_{11}+y)^2}v_2v_3 + 2\left[\frac{u_{12}u_{13}}{(u_{13}+z)^3} - \frac{u_9u_{10}}{(u_{11}+y)} - u_{14}\right]v_3^2.$$

with $\mho = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be a non-zero real vector. While the third directional derivative of $\boldsymbol{F}(\boldsymbol{X}, \alpha)$ in the system (2) can be expressed as:

$$D^3\boldsymbol{F}(\boldsymbol{X}, \alpha)(\mho, \mho, \mho) = \left(c_{i1}{}^{[3]}\right)_{3\times1}, \tag{36}$$

where:

$$c_{11}{}^{[3]} = -6\left[\frac{u_1y}{(u_1+x)^4} + u_2\right]v_1^3 + \frac{6u_1}{(u_1+x)^3}v_1^2v_2.$$

$$c_{21}{}^{[3]} = \frac{6u_1u_3y}{(u_1+x)^4(1+u_4z)}v_1^3 - \frac{6u_1u_3}{(u_1+x)^3(1+u_4z)}v_1^2v_2 + \frac{6u_1u_3u_4y}{(u_1+x)^3(1+u_4z)^2}v_1^2v_3$$
$$-\frac{6u_1u_3u_4}{(u_1+x)^2(1+u_4z)^2}v_1v_2v_3 + \frac{6u_1u_3u_4^2y}{(u_1+x)^2(1+u_4z)^3}v_1v_3^2 - \frac{6u_6z}{(u_6+y)^4}v_2^3 + \frac{6u_6}{(u_6+y)^3}v_2^2v_3 .$$
$$+\frac{6u_3u_4^2x}{(u_1+x)(1+u_4z)^3}v_2v_3^2 - \frac{6u_3u_4^3xy}{(u_1+x)(1+u_4z)^4}v_3^3$$

$$c_{31}{}^{[3]} = \frac{6u_9u_{10}z^2}{(u_{11}+y)^4}v_2^3 - \frac{12u_9u_{10}}{(u_{11}+y)^3}v_2^2v_3 + \frac{2u_9u_{10}}{(u_{11}+y)^2}v_2v_3^2 - \frac{6u_{12}u_{13}}{(u_{13}+z)^4}v_3^3.$$

Theorem 8. Assume that condition (14a) is satisfied. Then system (2) possesses a transcritical bifurcation (TB) at PHSSP when u_9 passes through the value $u_9^* = \frac{u_{12}}{u_{13}}$, provided that the following condition is met.

$$\frac{u_{12}}{u_{13}{}^2} - \frac{u_9^*u_{10}}{u_{11}} - u_{14} \neq 0. \tag{37}$$

Otherwise, a pitchfork bifurcation (PB) takes place.

Proof. It is easy to verify that the JM of the system (2) at PHSSP with $u_9 = u_9^*$ can be written as:

$$J(E_x, u_9^*) = J_x = \begin{bmatrix} -\bar{x}(1+2u_2\bar{x}) & \frac{-\bar{x}}{u_1+\bar{x}} & 0 \\ 0 & \frac{u_3\bar{x}}{u_1+\bar{x}} - u_5\bar{x} - u_8 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, the following is obtained

The eigenvalues of J_x are $\lambda_{11}(u_9^*) = -\bar{x}(1+2u_2\bar{x}) < 0$, $\lambda_{12}(u_9^*) = \frac{u_3\bar{x}}{u_1+\bar{x}} - u_5\bar{x} - u_8 < 0$, and $\lambda_{13}(u_9^*) = 0$.

The eigenvector $V_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix}$ that corresponding $\lambda_{13}(u_9^*)$ is determined by $V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The eigenvector $\psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{13} \end{pmatrix}$ that corresponding $\lambda_{13}(u_9^*)$ of ${J_x}^T$ is determined by $\psi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Moreover,

$$\frac{d\boldsymbol{F}(\boldsymbol{X},u_9)}{du_9} = \boldsymbol{F}_{u_9}(\boldsymbol{X},u_9) = \left(0,0,z - \frac{u_{10}z^2}{u_{11}+y}\right)^T \Rightarrow {\psi_1}^T F_{u_9}(E_x,u_9^*) = 0.$$

$$D\boldsymbol{F}_{u_9}(E_x,u_9^*)V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow {\psi_1}^T\left[D\boldsymbol{F}_{u_9}(E_x,u_9^*)V_1\right] = 1 \neq 0.$$

$$D^2\boldsymbol{F}(E_x,u_9^*)(V_1,V_1) = \begin{bmatrix} 0 \\ 0 \\ 2\left(\frac{u_{12}}{{u_{13}}^2} - \frac{u_9^*u_{10}}{u_{11}} - u_{14}\right) \end{bmatrix}.$$

Accordingly, it is produced that

$${\psi_1}^T[D^2\boldsymbol{F}(E_x,u_9^*)(V_1,V_1)] = 2\left(\frac{u_{12}}{{u_{13}}^2} - \frac{u_9^*u_{10}}{u_{11}} - u_{14}\right).$$

As a result, TB occurs in the sense of Sotomayor's theorem under condition (37), and the proof is complete. However, if condition (37) is broken, ${\psi_1}^T[D^2\boldsymbol{F}(E_x,u_9^*)(V_1,V_1)] = 0$ is returned. Moreover, since:

$$D^3\boldsymbol{F}(E_x,u_9^*)(V_1,V_1,V_1) = \begin{bmatrix} 0 \\ 0 \\ -\frac{6u_{12}}{{u_{13}}^3} \end{bmatrix}.$$

Accordingly, it is produced that

$${\psi_1}^T[D^3\boldsymbol{F}(E_x,u_9^*)(V_1,V_1,V_1)] = -\frac{6u_{12}}{{u_{13}}^3} \neq 0.$$

Hence, PB occurs and the proof is done.

Theorem 9. Assume that conditions (19a) and (19b) are satisfied. Then system (2) possesses a TB at FFSSP when u_9 passes through the value $u_9^* = \frac{u_{12}}{u_{13}}$ provided that the following condition is met.

$$4\frac{u_9^* u_{10}}{(u_{11}+\hat{y})^2}\gamma_2 + 2\left[\frac{u_{12}}{u_{13}{}^2} - \frac{u_9^* u_{10}}{(u_{11}+\hat{y})} - u_{14}\right] \neq 0. \tag{38}$$

Otherwise, a pitchfork bifurcation (PB) takes place.

Proof. It is easy to verify that the JM of the system (2) at FFSSP with $u_9 = u_9^*$ can be written as:

$$J(E_{xy}, u_9^*) = J_{xy} = \begin{bmatrix} \hat{x}\left(-1 + \frac{\hat{y}}{(u_1+\hat{x})^2} - 2u_2\right) & -\frac{\hat{x}}{u_1+\hat{x}} & 0 \\ \hat{y}\left(\frac{u_1 u_3}{u_1+\hat{x}} - u_5\right) & -u_7\hat{y} & -\hat{y}\left(\frac{u_3 u_4 \hat{x}}{u_1+\hat{x}} + \frac{1}{u_6+\hat{y}}\right) \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, the following is obtained

The eigenvalues of J_{xy} are $\lambda_{31,32}(u_9^*) = \frac{T_{xy} \pm \sqrt{T_{xy}{}^2 - 4D_{xy}}}{2}$ with T_{xy} and D_{xy} are given in Eq. (17), and $\lambda_{33}(u_9^*) = 0$. Clearly, $\lambda_{31}(u_9^*)$ and $\lambda_{32}(u_9^*)$ have negative real parts under conditions (19a) and (19b).

The eigenvector $V_2 = \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix}$ that corresponding $\lambda_{33}(u_9^*)$ is determined by $V_2 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 1 \end{pmatrix}$, where $\gamma_1 = \frac{\beta_{12}\beta_{23}}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} > 0$, and $\gamma_2 = -\frac{\beta_{11}\beta_{23}}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} < 0$, where β_{ij} are the J_{xy} elements.

The eigenvector $\psi_2 = \begin{pmatrix} \psi_{21} \\ \psi_{22} \\ \psi_{23} \end{pmatrix}$ that corresponding $\lambda_{33}(u_9^*)$ of $J_{xy}{}^T$ is determined by $\psi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Moreover,

$$\frac{d\boldsymbol{F}(\boldsymbol{X}, u_9)}{du_9} = \boldsymbol{F}_{u_9}(\boldsymbol{X}, u_9) = \left(0, 0, z - \frac{u_{10} z^2}{u_{11}+y}\right)^T \Rightarrow \psi_2{}^T F_{u_9}(E_{xy}, u_9^*) = 0.$$

$$D\boldsymbol{F}_{u_9}(E_{xy}, u_9^*)V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow \psi_2{}^T\left[D\boldsymbol{F}_{u_9}(E_{xy}, u_9^*)V_2\right] = 1 \neq 0.$$

$$D^2\boldsymbol{F}(E_{xy}, u_9^*)(V_2, V_2) = \begin{bmatrix} c_{11}{}^{[2]}(E_{xy}, u_9^*) \\ c_{21}{}^{[2]}(E_{xy}, u_9^*) \\ c_{31}{}^{[2]}(E_{xy}, u_9^*) \end{bmatrix},$$

where

$$c_{11}{}^{[2]}(E_{xy}, u_9^*) = 2\left[-1 + \frac{u_1\hat{y}}{(u_1+\hat{x})^3} - 3u_2\hat{x}\right]\gamma_1{}^2 - \frac{2u_1}{(u_1+\hat{x})^2}\gamma_1\gamma_2.$$

$$c_{21}{}^{[2]}(E_{xy},u_9^*) = -2\frac{u_1u_3\hat{y}}{(u_1+\hat{x})^3}\gamma_1{}^2 + 2\left[\frac{u_1u_3}{(u_1+\hat{x})^2} - u_5\right]\gamma_1\gamma_2 - \left[\frac{2u_1u_3u_4\hat{y}}{(u_1+\hat{x})^2}\right]\gamma_1$$
$$-2u_7\gamma_2{}^2 - 2\left[\frac{u_3u_4\hat{x}}{(u_1+\hat{x})} + \frac{u_6}{(u_6+\hat{y})^2}\right]\gamma_2 + \frac{2u_3u_4^2\hat{x}\hat{y}}{(u_1+\hat{x})}.$$

$$c_{31}{}^{[2]}(E_{xy},u_9^*) = 4\frac{u_9^*u_{10}}{(u_{11}+\hat{y})^2}\gamma_2 + 2\left[\frac{u_{12}}{u_{13}{}^2} - \frac{u_9^*u_{10}}{(u_{11}+\hat{y})} - u_{14}\right].$$

Therefore, it is obtained that

$$\psi_2{}^T\left[D^2\boldsymbol{F}\left(E_{xy},u_9^*\right)(V_2,V_2)\right] = 4\frac{u_9^*u_{10}}{(u_{11}+\hat{y})^2}\gamma_2 + 2\left[\frac{u_{12}}{u_{13}{}^2} - \frac{u_9^*u_{10}}{(u_{11}+\hat{y})} - u_{14}\right].$$

As a result, TB occurs under condition (38), and the proof is complete. However, if condition (38) is broken, $\psi_2{}^T\left[D^2\boldsymbol{F}\left(E_{xy},u_9^*\right)(V_2,V_2)\right] = 0$ is returned. Moreover, since:

$$D^3\boldsymbol{F}\left(E_{xy},u_9^*\right)(V_2,V_2,V_2) = \begin{bmatrix} c_{11}{}^{[3]}(E_{xy},u_9^*) \\ c_{21}{}^{[3]}(E_{xy},u_9^*) \\ c_{31}{}^{[3]}(E_{xy},u_9^*) \end{bmatrix},$$

where:

$$c_{11}{}^{[3]}\left(E_{xy},u_9^*\right) = -6\left[\frac{u_1\hat{y}}{(u_1+\hat{x})^4} + u_2\right]\gamma_1^3 + \frac{6u_1}{(u_1+\hat{x})^3}\gamma_1^2\gamma_2.$$

$$c_{21}{}^{[3]}\left(E_{xy},u_9^*\right) = \frac{6u_1u_3\hat{y}}{(u_1+\hat{x})^4}\gamma_1^3 - \frac{6u_1u_3}{(u_1+\hat{x})^3}\gamma_1^2\gamma_2 + \frac{6u_1u_3u_4\hat{y}}{(u_1+\hat{x})^3}\gamma_1^2 - \frac{6u_1u_3u_4}{(u_1+\hat{x})^2}\gamma_1\gamma_2$$
$$+\frac{6u_1u_3u_4^2\hat{y}}{(u_1+\hat{x})^2}\gamma_1 + \frac{6u_6}{(u_6+\hat{y})^3}\gamma_2^2 + \frac{6u_3u_4^2\hat{x}}{(u_1+\hat{x})}\gamma_2 - \frac{6u_3u_4^3\hat{x}\hat{y}}{(u_1+\hat{x})}.$$

$$c_{31}{}^{[3]}\left(E_{xy},u_9^*\right) = -\frac{12u_9^*u_{10}}{(u_{11}+\hat{y})^3}\gamma_2^2 + \frac{2u_9^*u_{10}}{(u_{11}+\hat{y})^2}\gamma_2 - \frac{6u_{12}}{u_{13}{}^3}.$$

Therefore, it is obtained that

$$\psi_2{}^TD^3\boldsymbol{F}\left(E_{xy},u_9^*\right)(V_2,V_2,V_2) = -\frac{12u_9^*u_{10}}{(u_{11}+\hat{y})^3}\gamma_2^2 + \frac{2u_9^*u_{10}}{(u_{11}+\hat{y})^2}\gamma_2 - \frac{6u_{12}}{u_{13}{}^3} < 0.$$

Hence, PB occurs and the proof is done.

Theorem 10. Assume that condition (22b) is satisfied. Then system (2) possesses a TB at ZFSSP when u_8 passes through the value $u_8^* = \frac{u_3\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})} - u_5\bar{x} - \frac{\bar{z}}{u_6}$ provided that the following condition is met.

$$2\left[\frac{u_1u_3}{(u_1+\bar{x})^2(1+u_4\bar{z})} - u_5\right]\gamma_3 + 2\left[\frac{\bar{z}}{u_6{}^2} - u_7\right] - 2\left[\frac{u_3u_4\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})^2} + \frac{1}{u_6}\right]\gamma_4 \neq 0. \quad (39)$$

Otherwise, a pitchfork bifurcation (PB) takes place if and only if the following condition holds:

$$c_{21}{}^{[3]}(E_{xz},u_8^*) \neq 0. \quad (40)$$

Proof. It is easy to verify that the JM of the system (2) at ZFSSP with $u_8 = u_8^*$ can be written as:

$$J(E_{xz}, u_8^*) = J_{xz} = \begin{bmatrix} -\bar{x} - 2u_2\bar{x}^2 & -\frac{\bar{x}}{u_1+\bar{x}} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{u_9 u_{10}\bar{z}^2}{u_{11}^2} & -\frac{u_9 u_{10}\bar{z}}{u_{11}} + \frac{u_{12}\bar{z}}{(u_{13}+\bar{z})^2} - u_{14}\bar{z} \end{bmatrix}.$$

Obviously, the following is obtained

The eigenvalues of J_{xz} are

$$\lambda_{41}(u_8^*) = -\bar{x} - 2u_2\bar{x}^2 < 0,\ \lambda_{42}(u_8^*) = 0,\ \lambda_{43}(u_8^*) = -\frac{u_9 u_{10}\bar{z}}{u_{11}} + \frac{u_{12}\bar{z}}{(u_{13}+\bar{z})^2} - u_{14}\bar{z}.$$

Obviously, $\lambda_{43}(u_8^*)$ is negative under the condition (22b).

The eigenvector $V_3 = \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix}$ that corresponding $\lambda_{42}(u_8^*)$ is determined by $V_3 = \begin{pmatrix} \gamma_3 \\ 1 \\ \gamma_4 \end{pmatrix}$, where

$\gamma_3 = -\frac{\rho_{12}}{\rho_{11}} < 0$, and $\gamma_4 = -\frac{\rho_{32}}{\rho_{33}} > 0$ under the condition (22b), where ρ_{ij} are the J_{xz} elements.

The eigenvector $\psi_3 = \begin{pmatrix} \psi_{31} \\ \psi_{32} \\ \psi_{33} \end{pmatrix}$ that corresponding $\lambda_{42}(u_8^*)$ of $J_{xz}{}^T$ is determined by $\psi_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Moreover,

$$\frac{d\boldsymbol{F}(\boldsymbol{X}, u_8)}{du_8} = \boldsymbol{F}_{u_8}(\boldsymbol{X}, u_8) = (0, -y, 0)^T \ \Rightarrow\ \psi_3{}^T F_{u_8}(E_{xz}, u_8^*) = 0.$$

$$D\boldsymbol{F}_{u_8}(E_{xz}, u_8^*)V_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \ \Longrightarrow\ \psi_3{}^T\left[D\boldsymbol{F}_{u_8}(E_{xz}, u_8^*)V_3\right] = -1 \neq 0.$$

$$D^2\boldsymbol{F}(E_{xz}, u_8^*)(V_3, V_3) = \begin{bmatrix} c_{11}{}^{[2]}(E_{xz}, u_8^*) \\ c_{21}{}^{[2]}(E_{xz}, u_8^*) \\ c_{31}{}^{[2]}(E_{xz}, u_8^*) \end{bmatrix},$$

where:

$$c_{11}{}^{[2]}(E_{xz}, u_8^*) = 2[-1 - 3u_2\bar{x}]\gamma_3^2 - \frac{2u_1}{(u_1+\bar{x})^2}\gamma_3.$$

$$\begin{aligned} c_{21}{}^{[2]}(E_{xz}, u_8^*) = {} & 2\left[\frac{u_1 u_3}{(u_1+\bar{x})^2(1+u_4\bar{z})} - u_5\right]\gamma_3 + 2\left[\frac{\bar{z}}{u_6{}^2} - u_7\right] \\ & -2\left[\frac{u_3 u_4\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})^2} + \frac{1}{u_6}\right]\gamma_4 \end{aligned}.$$

$$c_{31}{}^{[2]}(E_{xz}, u_8^*) = -\frac{2u_9 u_{10}\bar{z}^2}{u_{11}{}^3} + \frac{4u_9 u_{10}}{u_{11}{}^2}\gamma_4 + 2\left[\frac{u_{12}u_{13}}{(u_{13}+\bar{z})^3} - \frac{u_9 u_{10}}{u_{11}} - u_{14}\right]\gamma_4^2.$$

Therefore, it is obtained that

$$\psi_3{}^T[D^2\boldsymbol{F}(E_{xz},u_8^*)(V_3,V_3)] = 2\left[\frac{u_1u_3}{(u_1+\bar{x})^2(1+u_4\bar{z})} - u_5\right]\gamma_3 + 2\left[\frac{\bar{z}}{u_6{}^2} - u_7\right]$$
$$-2\left[\frac{u_3u_4\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})^2} + \frac{1}{u_6}\right]\gamma_4.$$

Hence, TB occurs under condition (39), and the proof is complete. However, if condition (39) is broken, $\psi_3{}^T[D^2\boldsymbol{F}(E_{xz},u_8^*)(V_3,V_3)] = 0$ is returned. Moreover, since

$$D^3\boldsymbol{F}(E_{xz},u_8^*)(V_3,V_3,V_3) = \begin{bmatrix} c_{11}{}^{[3]}(E_{xz},u_8^*) \\ c_{21}{}^{[3]}(E_{xz},u_8^*) \\ c_{31}{}^{[3]}(E_{xz},u_8^*) \end{bmatrix},$$

where:

$$c_{11}{}^{[3]}(E_{xz},u_8^*) = -6u_2\gamma_3^3 + \frac{6u_1}{(u_1+\bar{x})^3}\gamma_3^2.$$

$$c_{21}{}^{[3]}(E_{xz},u_8^*) = -\frac{6u_1u_3}{(u_1+\bar{x})^3(1+u_4\bar{z})}\gamma_3^2 - \frac{6u_1u_3u_4}{(u_1+\bar{x})^2(1+u_4\bar{z})^2}\gamma_3\gamma_4$$
$$-\frac{6\bar{z}}{u_6{}^3} + \frac{6}{u_6{}^2}\gamma_4 + \frac{6u_3u_4^2\bar{x}}{(u_1+\bar{x})(1+u_4\bar{z})^3}\gamma_4^2.$$

$$c_{31}{}^{[3]}(E_{xz},u_8^*) = \frac{6u_9u_{10}\bar{z}^2}{u_{11}{}^4} - \frac{12u_9u_{10}}{u_{11}{}^3}\gamma_4 + \frac{2u_9u_{10}}{u_{11}{}^2}\gamma_4^2 - \frac{6u_{12}u_{13}}{(u_{13}+\bar{z})^4}\gamma_4^3.$$

Therefore, it is obtained that

$$\psi_3{}^T D^3\boldsymbol{F}(E_{xz},u_8^*)(V_3,V_3,V_3) = c_{21}{}^{[3]}(E_{xz},u_8^*).$$

Hence, PB occurs if and only if condition (40) is met and the proof is done.

Theorem 11. Assume that the conditions (24a)-(24c) are met. Then system (2) undergoes a saddle-node bifurcation (SNB) near the SSSP as the parameter u_{12} passes through the value u_{12}^*, where:

$$u_{12}^* = \frac{(u_{13}+z^*)^2}{z^*}\left[\frac{a_{11}a_{23}a_{32}}{a_{11}a_{22}-a_{12}a_{21}} + \left(\frac{u_9u_{10}+u_{11}u_{14}}{u_{11}}\right)z^*\right]$$

If and only if the following requirements are satisfied.

$$\frac{u_{12}}{(u_{13}+z^*)^2} > u_{14} + \frac{u_9u_{10}}{u_{11}+y^*}. \tag{41}$$

$$\gamma_7 c_{11}{}^{[2]}(E_{xyz},u_{12}^*) + \gamma_8 c_{21}{}^{[2]}(E_{xyz},u_{12}^*) + c_{31}{}^{[2]}(E_{xyz},u_{12}^*) \neq 0. \tag{42}$$

Proof. Direct competition shows that when $u_{12} = u_{12}^*$, then the determinant of JM of the system (2) at the SSSP, which is given in equation (25), is determined as $H_3 = 0$ due to condition (41). Hence the JM at (E_{xyz},u_{12}^*) becomes:

$$J(E_{xyz},u_{12}^*) = J_{xyz} = (a_{ij})_{3\times3},$$

where a_{ij} for $i, j = 1,2,3$ are given in equation (20) with $a_{33} = a_{33}(u_{12}^{*})$, and it has a zero eigenvalue, say $\lambda_{*} = 0$. That is means SSSP is a nonhyperbolic point. Consequently, the following is produced.

The eigenvector $V_4 = \begin{pmatrix} v_{41} \\ v_{42} \\ v_{43} \end{pmatrix}$ that is the corresponding $\lambda_{*} = 0$ is determined by $V_4 = \begin{pmatrix} \gamma_5 \\ \gamma_6 \\ 1 \end{pmatrix}$, where $\gamma_5 = \frac{a_{12}a_{23}}{a_{11}a_{22}-a_{12}a_{21}}$, and $\gamma_6 = -\frac{a_{11}a_{23}}{a_{11}a_{22}-a_{12}a_{21}}$, here $\gamma_5 > 0$, and $\gamma_6 < 0$ under the conditions (24a)-(24c).

The eigenvector $\psi_4 = \begin{pmatrix} \psi_{31} \\ \psi_{32} \\ \psi_{33} \end{pmatrix}$ that is the corresponding $\lambda_{*} = 0$ of $J_{xyz}{}^{T}$ is determined by $\psi_4 = \begin{pmatrix} \gamma_7 \\ \gamma_8 \\ 1 \end{pmatrix}$, where $\gamma_7 = \frac{a_{21}a_{32}}{a_{11}a_{22}-a_{12}a_{21}}$, and $\gamma_8 = -\frac{a_{11}a_{32}}{a_{11}a_{22}-a_{12}a_{21}}$, with $\gamma_7 < 0$, and $\gamma_8 < 0$ under the conditions (24a)-(24c).

Moreover,

$$\frac{d\boldsymbol{F}(\boldsymbol{X},u_{12})}{du_{12}} = \boldsymbol{F}_{u_{12}}(\boldsymbol{X},u_{12}) = \left(0,0,-\frac{z}{u_{13}+z}\right)^{T} \Rightarrow \psi_4{}^{T}F_{u_{12}}\left(E_{xyz},u_{12}^{*}\right) = -\frac{z^{*}}{u_{13}+z^{*}} \neq 0.$$

$$D^2\boldsymbol{F}\left(E_{xyz},u_{12}^{*}\right)(V_4,V_4) = \begin{bmatrix} c_{11}{}^{[2]}(E_{xyz},u_{12}^{*}) \\ c_{21}{}^{[2]}(E_{xyz},u_{12}^{*}) \\ c_{31}{}^{[2]}(E_{xyz},u_{12}^{*}) \end{bmatrix},$$

where:

$$c_{11}{}^{[2]}\left(E_{xyz},u_{12}^{*}\right) = 2\left[-1+\frac{u_1y^{*}}{(u_1+x^{*})^3}-3u_2x^{*}\right]\gamma_5^2-\frac{2u_1}{(u_1+x^{*})^2}\gamma_5\gamma_6.$$

$$\begin{aligned} c_{21}{}^{[2]}\left(E_{xyz},u_{12}^{*}\right) = &-\frac{2u_1u_3y^{*}}{(u_1+x^{*})^3(1+u_4z^{*})}\gamma_5^2+2\left[\frac{u_1u_3}{(u_1+x^{*})^2(1+u_4z^{*})}-u_5\right]\gamma_5\gamma_6 \\ &-\left[\frac{2u_1u_3u_4y^{*}}{(u_1+x^{*})^2(1+u_4z^{*})^2}\right]\gamma_5+2\left[\frac{u_6z^{*}}{(u_6+y^{*})^3}-u_7\right]\gamma_6^2 \\ &-2\left[\frac{u_3u_4x^{*}}{(u_1+x^{*})(1+u_4z^{*})^2}+\frac{u_6}{(u_6+y^{*})^2}\right]\gamma_6+\frac{2u_3u_4^2x^{*}y^{*}}{(u_1+x^{*})(1+u_4z^{*})^3}. \end{aligned}$$

$$c_{31}{}^{[2]}\left(E_{xyz},u_{12}^{*}\right) = -\frac{2u_9u_{10}z^{*2}}{(u_{11}+y^{*})^3}\gamma_6^2+\frac{4u_9u_{10}}{(u_{11}+y^{*})^2}\gamma_6+2\left[\frac{u_{12}^{*}u_{13}}{(u_{13}+z^{*})^3}-\frac{u_9u_{10}}{(u_{11}+y^{*})}-u_{14}\right].$$

Therefore, it is obtained that

$$\begin{aligned} \psi_4{}^{T}\left[D^2\boldsymbol{F}\left(E_{xyz},u_{12}^{*}\right)(V_4,V_4)\right] = &\gamma_7c_{11}{}^{[2]}\left(E_{xyz},u_{12}^{*}\right)+\gamma_8c_{21}{}^{[2]}\left(E_{xyz},u_{12}^{*}\right) \\ &+c_{31}{}^{[2]}\left(E_{xyz},u_{12}^{*}\right). \end{aligned}$$

Hence, SNB occurs under condition (42), and the proof is complete.

7. NUMERICAL SIMULATION

In order to clarify the control parameters that have been specified that affect the system's dynamic in the previous parts, a numerical example for system (2) was provided in this section. The system (2) will be numerically solved using MATLAB version R2013a. To get the numerical results, the following hypothetical values for the parameters were used.

$$\begin{gathered} u_1 = 0.2, u_2 = 0.4, u_3 = 2, u_4 = 0.01, u_5 = 0.01, u_6 = 1, u_7 = 0.4 \\ u_8 = 0.02, u_9 = 1, u_{10} = 0.4, u_{11} = 2, u_{12} = 0.2, u_{13} = 0.2, u_{14} = 0.4. \end{gathered} \tag{43}$$

Now, solving the system (2) using the set of data (43) gives asymptotically stable SSSP as shown in Fig. (1).

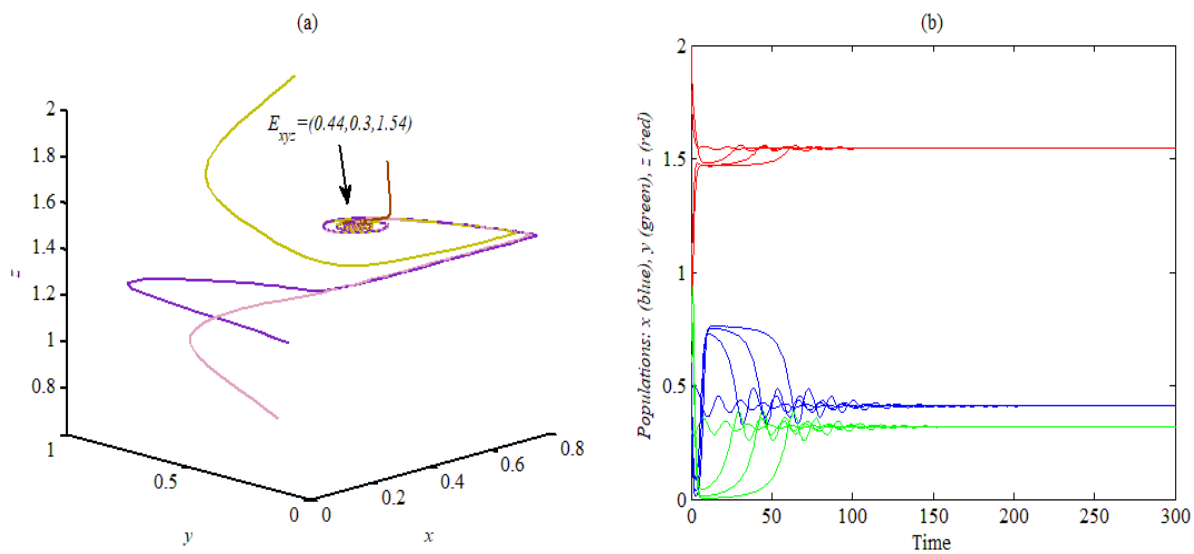


Figure 1: The system's (2) trajectories utilizing data set (43) and various starting positions. (a) The SSSP is approached by a 3D phase portrait. (b) Time-dependent population trajectories.

Now, the influence of varying the parameter u_1 on the system's dynamic is investigated in Fig. (2).

According to Fig. (2), the parameter u_1 plays a vital role in the qualitative behavior of the system (2) due to the existence of a number of bifurcation points in their range. The role of altering u_2 on the behavior of the system (2) is clarified in Fig. (3).

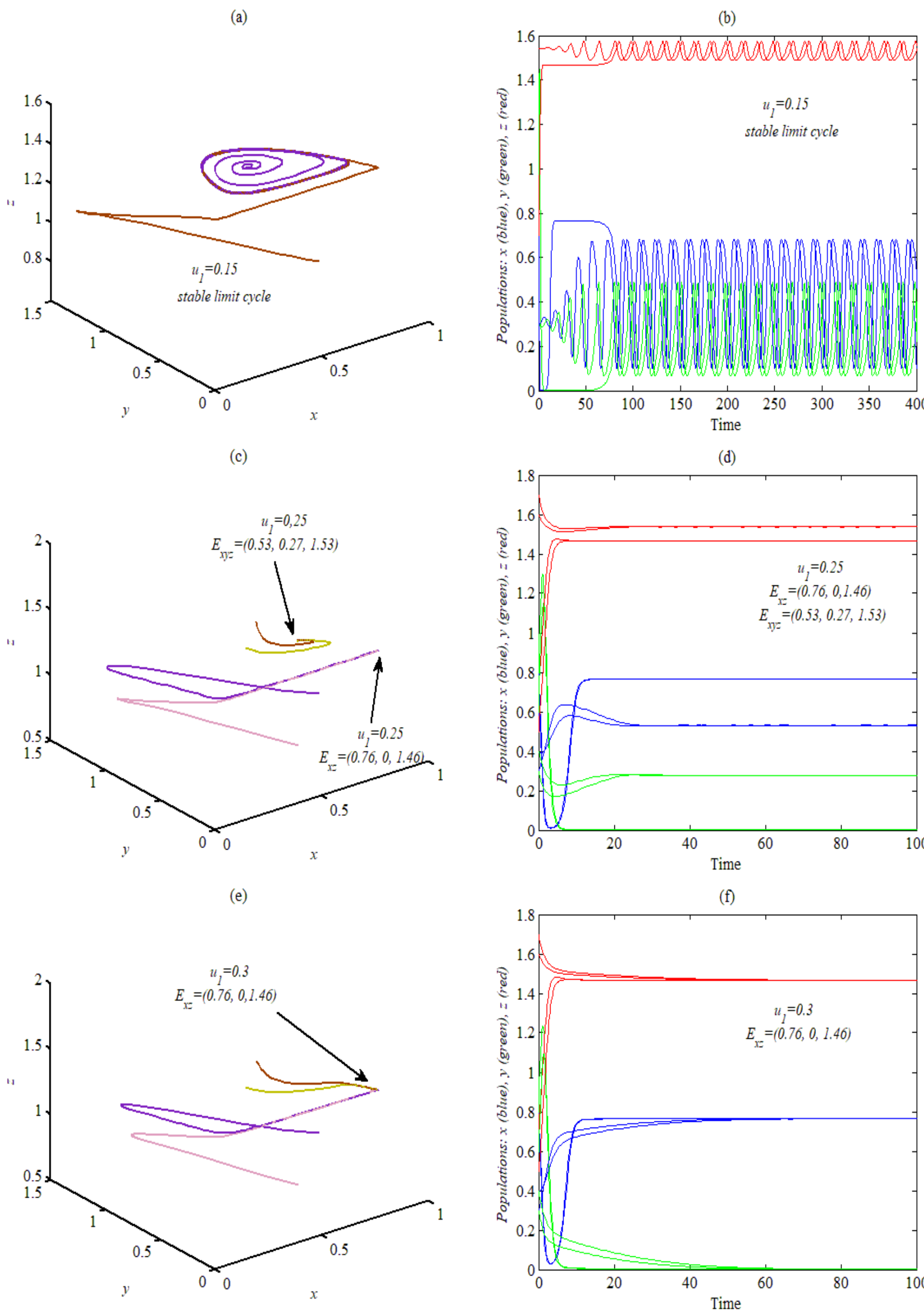


Figure 2: The system's (2) trajectories utilizing data set (43) with various starting positions and various values of u_1. (a) The 3D limit cycle is approached by a 3D phase portrait when $u_1 = 0.15$. (b) Time-dependent population trajectories when $u_1 = 0.15$. (c) Bi-stable case between ZFSSP and SSSP when $u_1 = 0.25$. (d) Time-dependent population trajectories when $u_1 = 0.25$. (e) The ZFSSP is approached by a 3D phase portrait when $u_1 = 0.3$. (f) Time-dependent population trajectories when $u_1 = 0.3$.

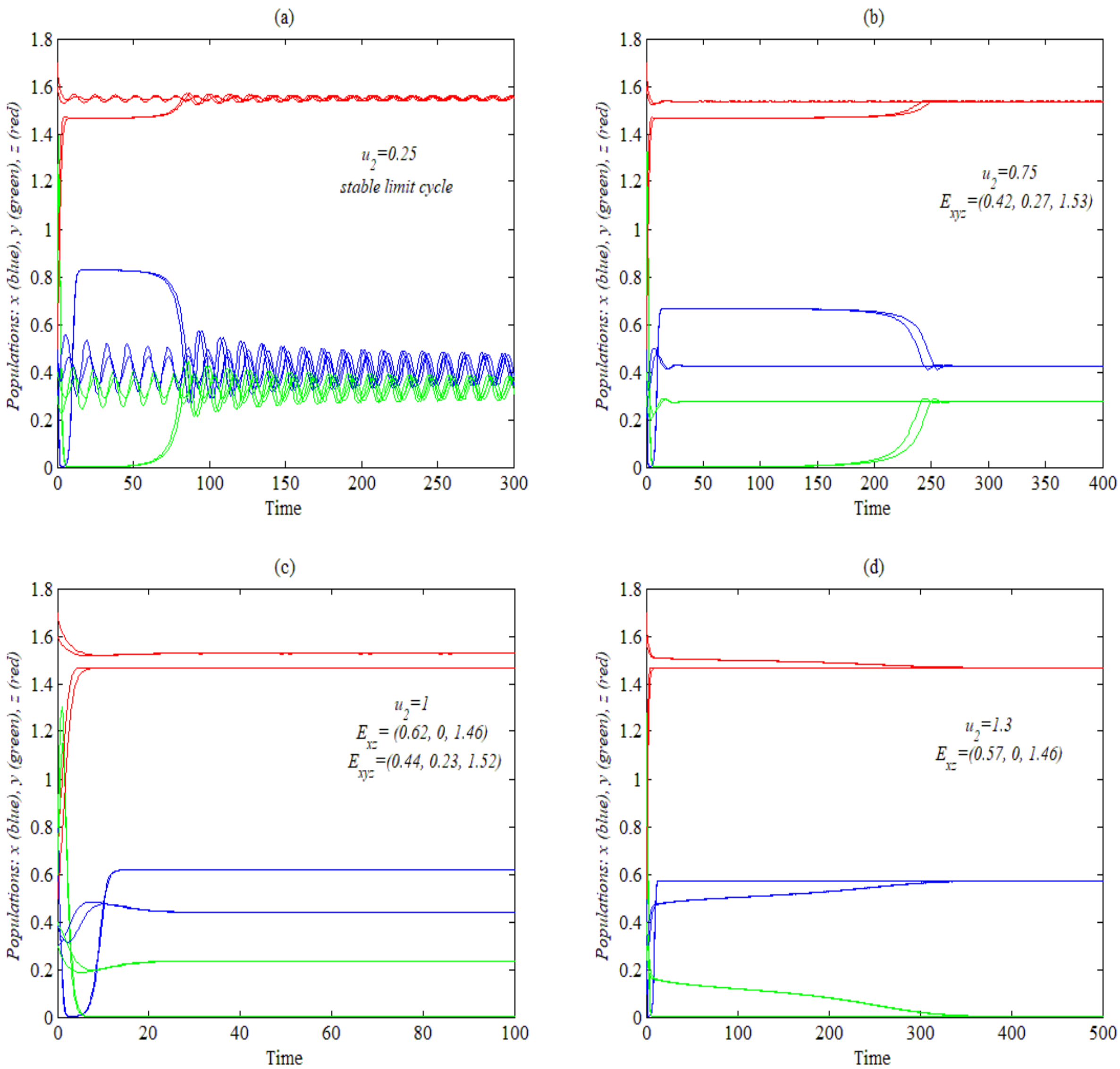


Figure 3: Time-dependent population trajectories of the system (2) utilizing data set (43) with various starting positions and various values of u_2, shows that: (a) Stable 3D limit cycle when $u_2 = 0.25$. (b) Asymptotic stable SSSP when $u_2 = 0.75$. (c) Bi-stable case between ZFSSP and SSSP when $u_2 = 1$. (d) Asymptotic stable ZFSSP $u_2 = 1.3$.

According to Fig. (3), the parameters u_1 and u_2 have a similar influence on the dynamic of the system (2). Now, the altering of u_3 values is investigated in Fig. (4).

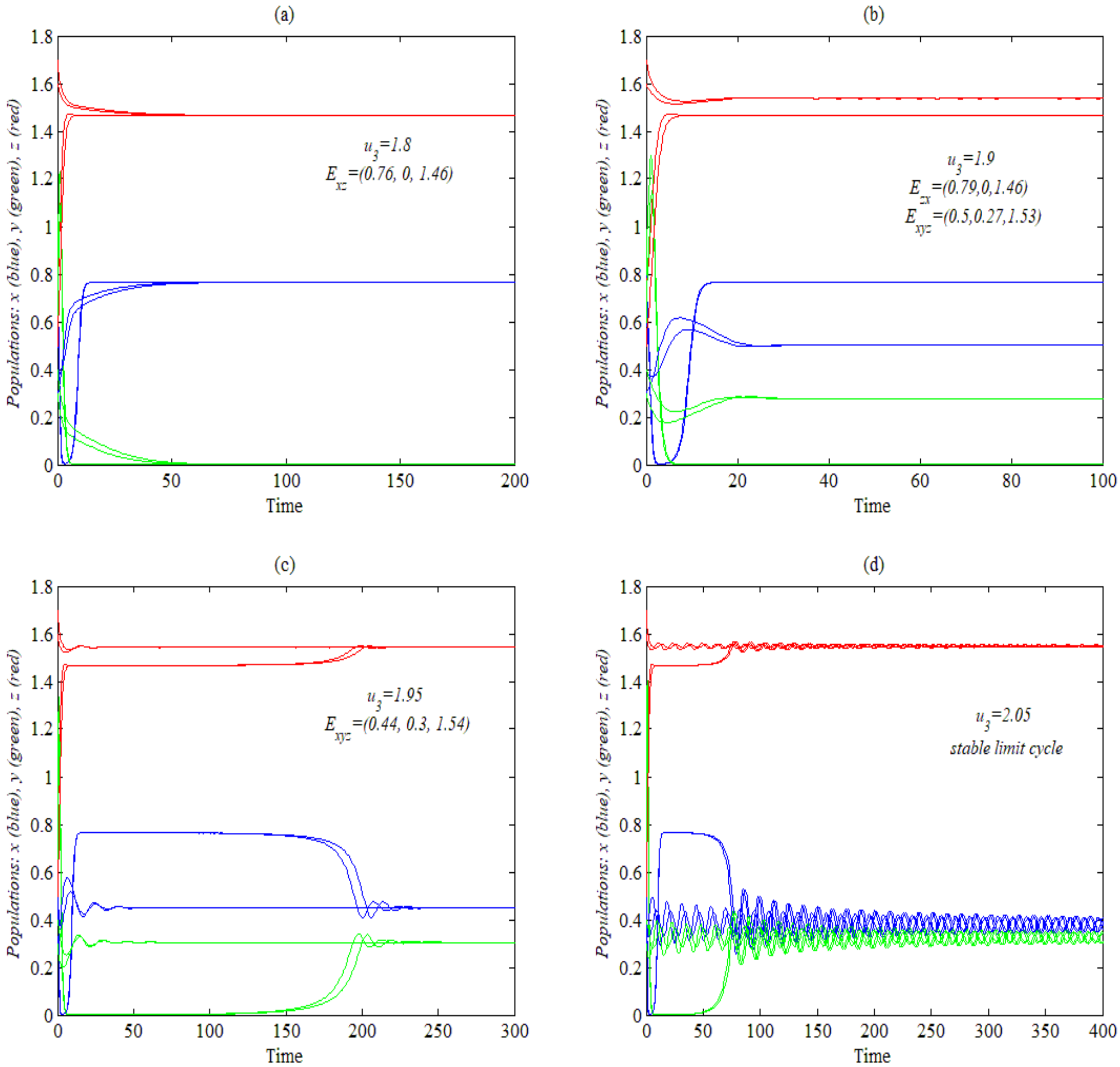


Figure 4: Time-dependent population trajectories of the system (2) utilizing data set (43) with various starting positions and various values of u_3, shows that: (a) Asymptotic stable ZFSSP $u_3 = 1.8$. (b) Bi-stable case between ZFSSP and SSSP when $u_3 = 1.9$. (c) Asymptotic stable SSSP when $u_3 = 1.95$. (d) Stable 3D limit cycle when $u_3 = 2.05$.

Clearly, as shown in Fig. (3) the system (2) displays different types of attractors when the parameter u_3 varies. Although the system (2) dynamic is settled at SSSP for the range $u_4 \in (0,1.85]$, the impact of $u_4 > 1.85$ on the system's (2) dynamics is presented in Fig. (5).

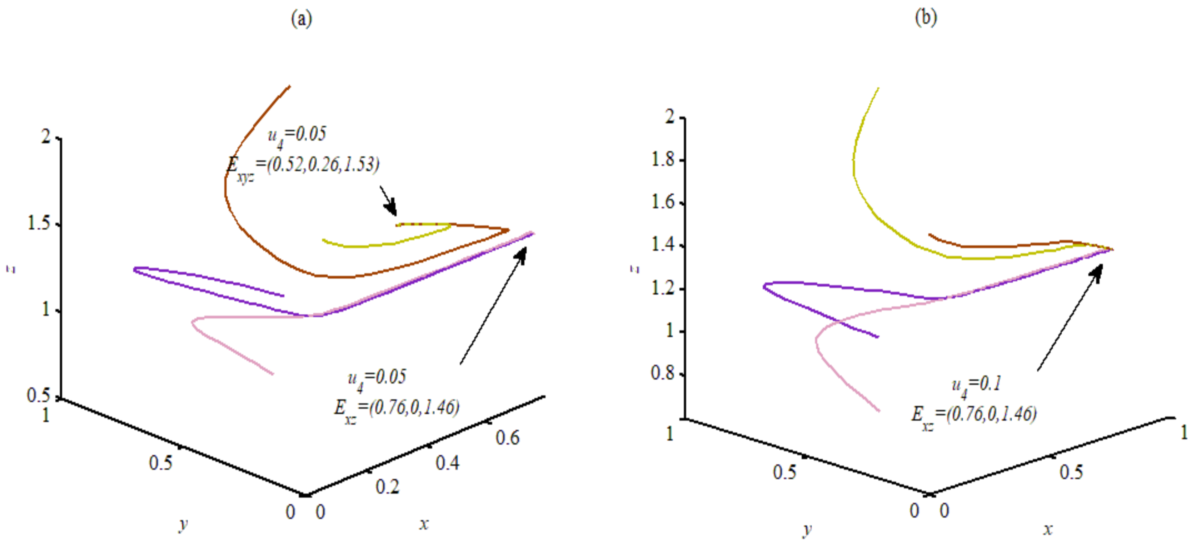
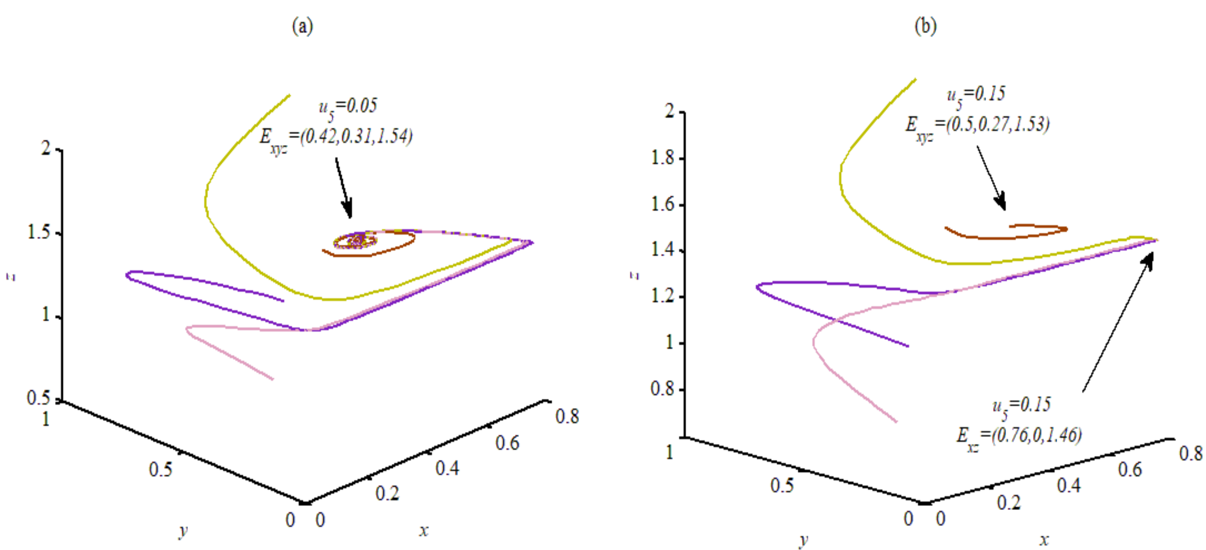


Figure 5: The system's (2) trajectories utilizing data set (43) with various starting positions and various values of u_4. (a) Bi-stable case between ZFSSP and SSSP when $u_4 = 0.05$. (b) The ZFSSP is approached by a 3D phase portrait when $u_4 = 0.1$.

Again Fig. (5) clarifies the vital influence of altering the parameter u_4 on the behavior of the system (2). However, the influence of altering the parameter u_5 on the dynamic of the system (2) is shown in Fig. (6).



(c)

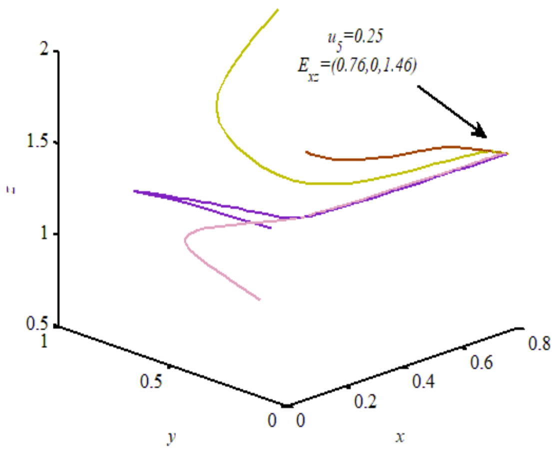


Figure 6: The system's (2) trajectories utilizing data set (43) with various starting positions and various values of u_5. (a) The SSSP is approached by a 3D phase portrait when $u_5 = 0.05$. (b) Bi-stable case between ZFSSP and SSSP when $u_5 = 0.15$. (c) The ZFSSP is approached by a 3D phase portrait when $u_5 = 0.25$.

Due to the Figs. (5) and (6), it is concluded that the parameters u_4 and u_5 have a similar influence on the dynamic of the system (2). Although the system (2) dynamic is settled at ZFSSP for the range $u_6 \in (0,0.91]$, the influence of altering the value of $u_6 > 0.95$ on the dynamic of the system (2) is examined in Fig. (7).

(a) (b)

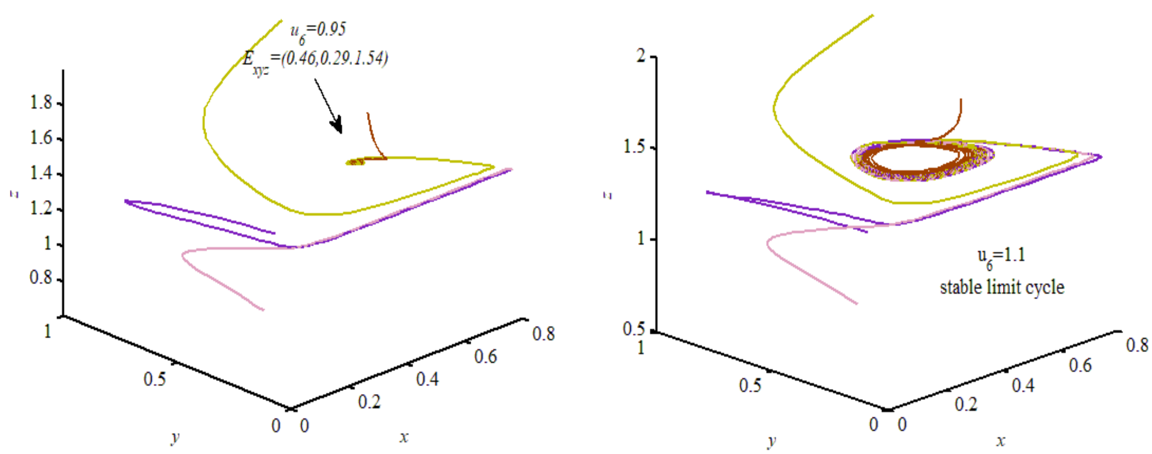
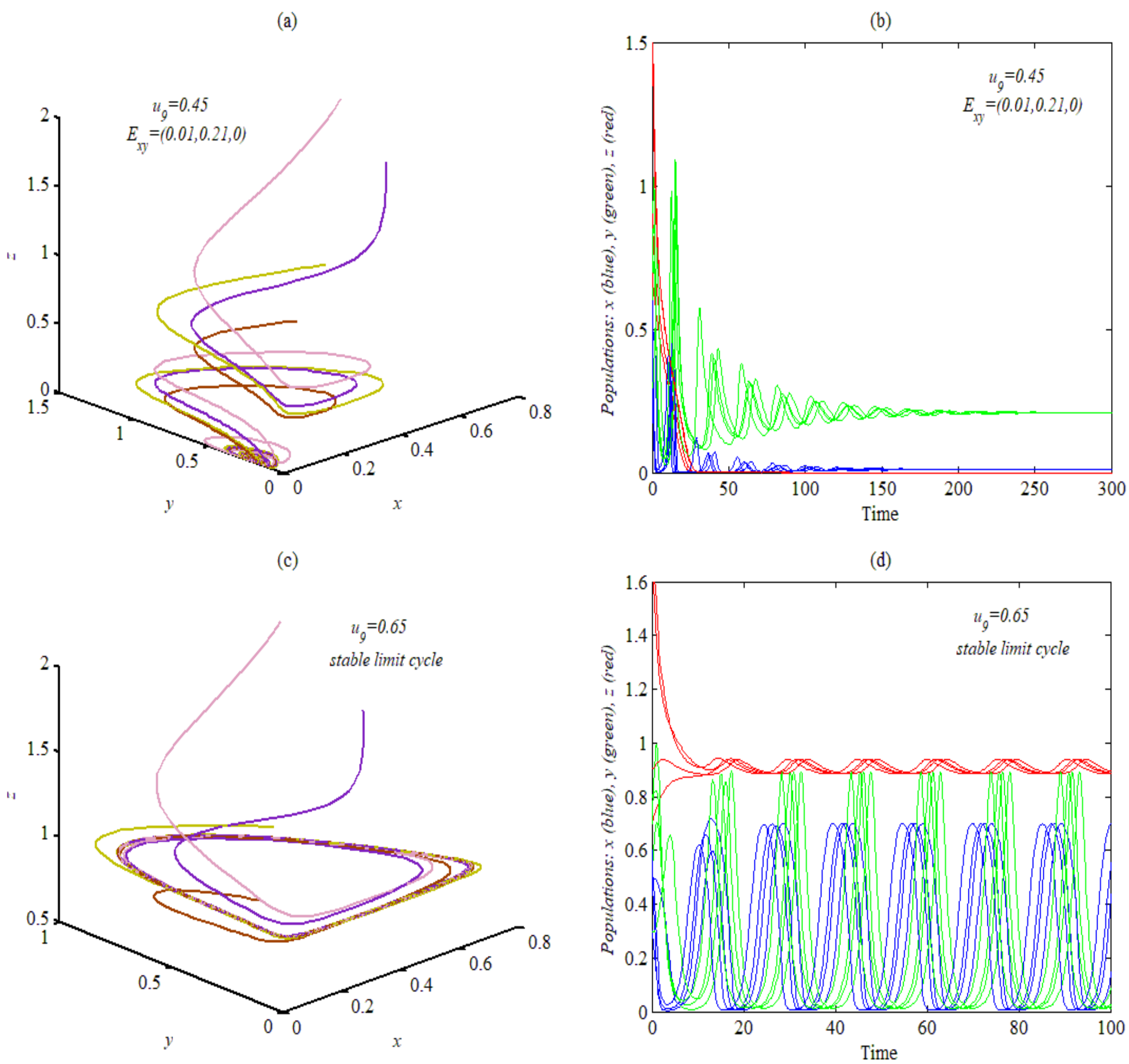


Figure 7: The system's (2) trajectories utilizing data set (43) with various starting positions and various values of u_6. (a) The SSSP is approached by a 3D phase portrait when $u_6 = 0.95$ (b) The 3D limit cycle is approached by a 3D phase portrait when $u_6 = 1.1$.

Hence, as shown in Fig. (7), the parameter u_6 range includes a number of bifurcation points that affect the system's (2) dynamic. Moreover, it was observed that the system (2) approached a stable limit cycle for the range $u_7 \in (0,0.35]$, while it had asymptotic stable SSSP otherwise. On the other hand, system (2) behaves similarly when the parameters u_4, u_5, and u_8 change their values in ascending order.

Now, Figs. (8) and (9) show the influence of altering the value of u_9 only, and together with u_3 respectively, on the dynamic of the system (2).

Not that, according the Fig. (8), the system (2) transfers from the FFSSP in the xy −plane to the ZFSSP in the xz −plane passing through periodic dynamics in range $u_9 \in [0.54,0.98]$, SSSP in the range $u_9 \in [0.99,1.05]$, and then bi-stable case in the range $u_9 \in [1.06,1.09]$.



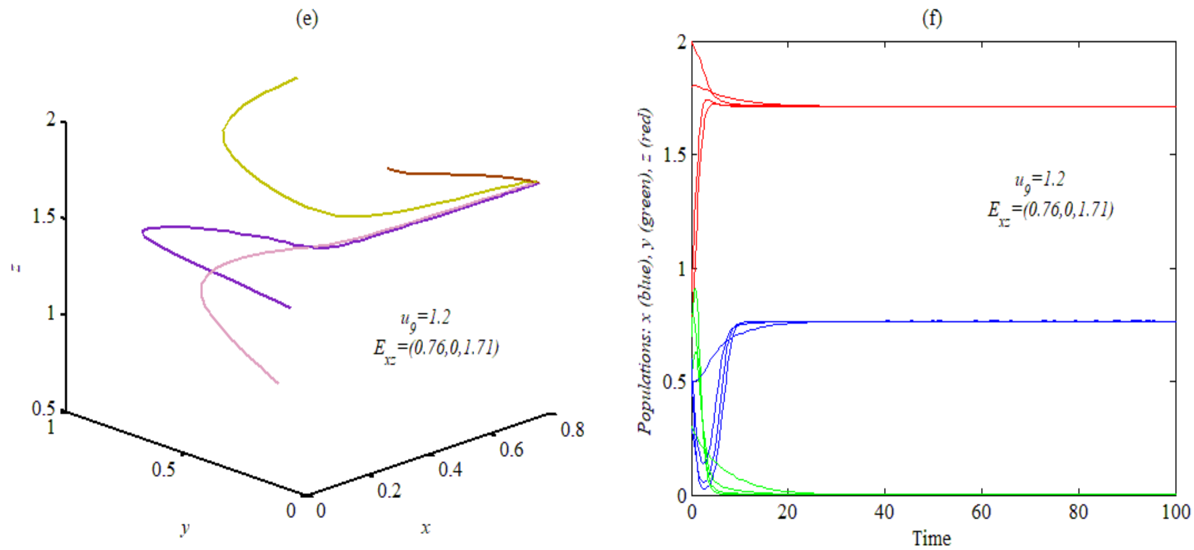
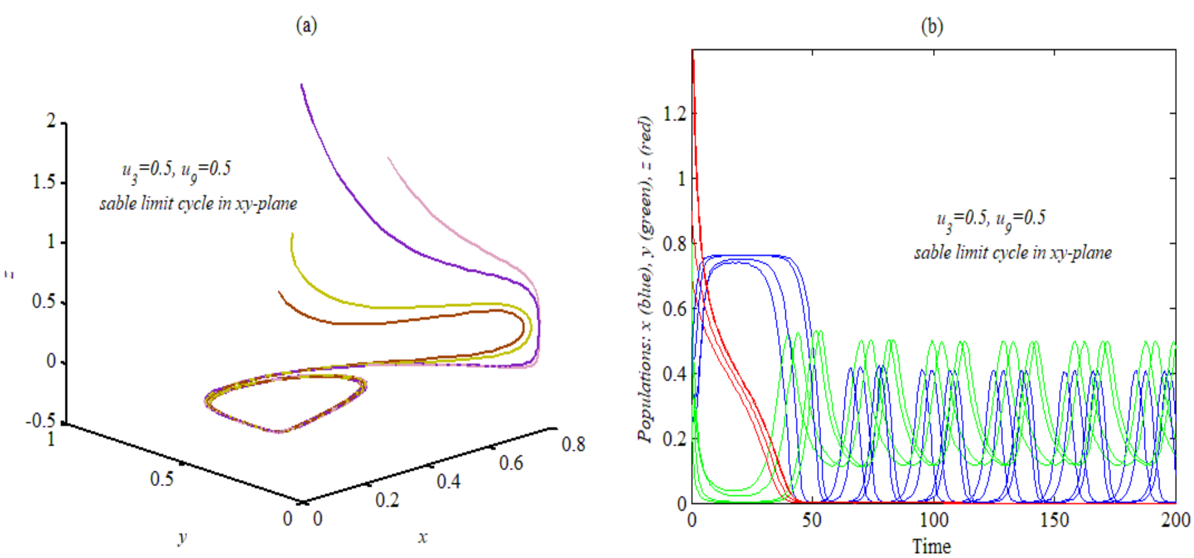


Figure 8: The system's (2) trajectories utilizing data set (43) with various starting positions and various values of u_9. (a) The FFSSP is approached by a 3D phase portrait when $u_9 = 0.45$. (b) Time-dependent population trajectories when $u_9 = 0.45$. (c) The 3D limit cycle is approached by a 3D phase portrait when $u_9 = 0.65$. (d) Time-dependent population trajectories when $u_9 = 0.65$. (e) The ZFSSP is approached by a 3D phase portrait when $u_9 = 1.2$. (f) Time-dependent population trajectories when $u_9 = 1.2$.



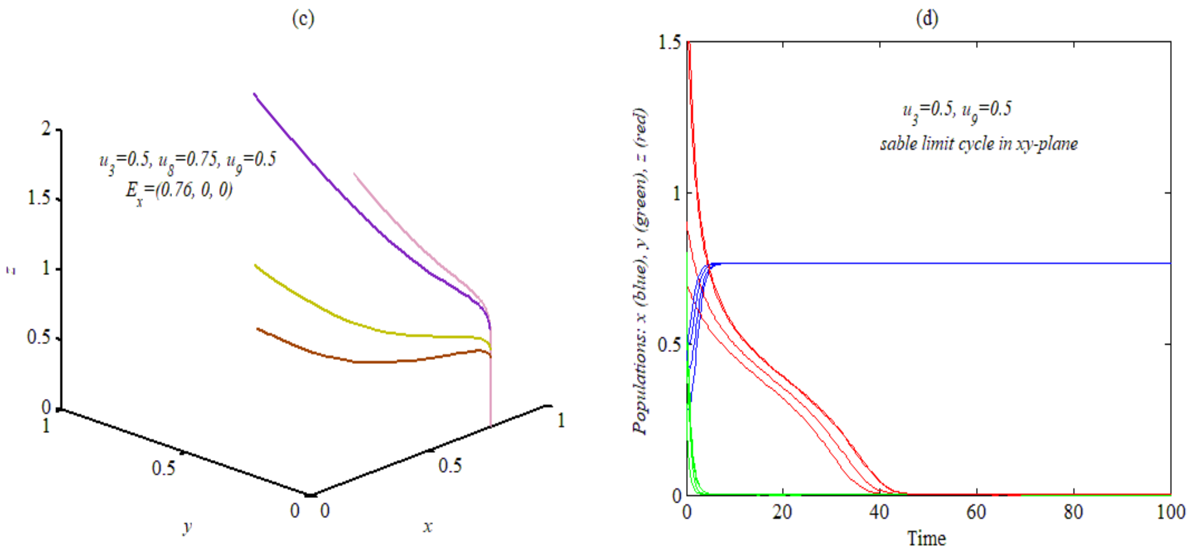
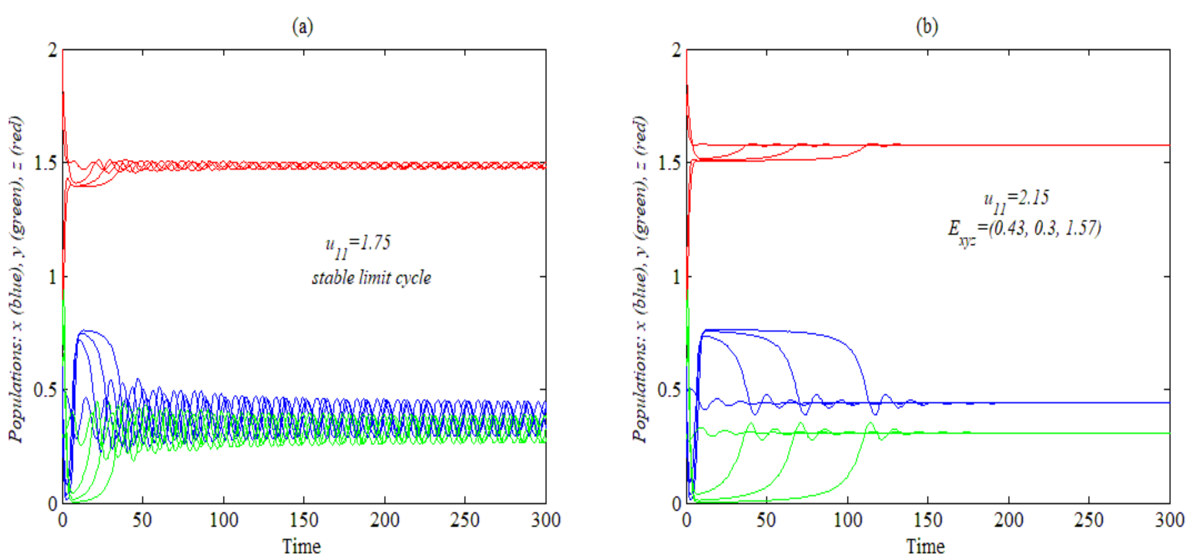


Figure 9: The system's (2) trajectories utilizing data set (43) with various starting positions and various values of u_3, u_8, and u_9. (a) The 2D limit cycle in the xy −plane is approached by the trajectories of the system (2) when $u_3 = 0.5$ and $u_9 = 0.5$. (b) Time-dependent population trajectories when $u_3 = 0.5$ and $u_9 = 0.5$. (c) The PHSSP is approached by the trajectories of the system (2) when $u_3 = 0.5$, $u_8 = 0.75$, and $u_9 = 0.5$. (f) Time-dependent population trajectories when $u_3 = 0.5$, $u_8 = 0.75$, and $u_9 = 0.5$.

Clearly, Fig. (9) shows various types of attractors of the system (2) when the parameters is varying. It is observed also that, the parameters u_{10} has similar influence on the dynamic of the system (2) as that shown for the parameter u_6. Now, the influence of altering the parameters u_{11}, and u_{12} on the behavior of the system (2) is explained in Figs. (10) and (11) respectively.



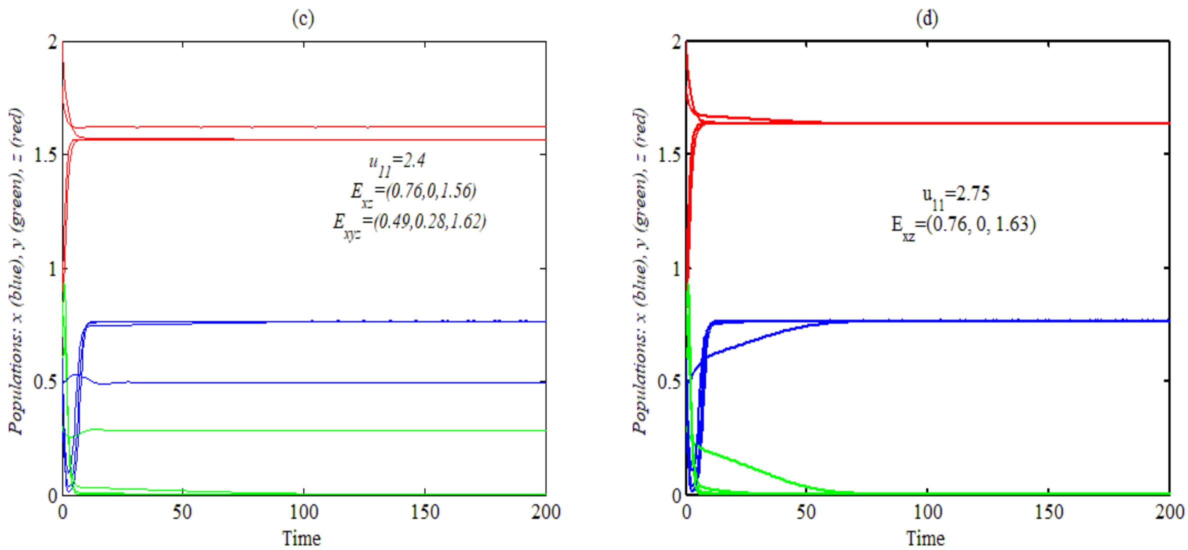
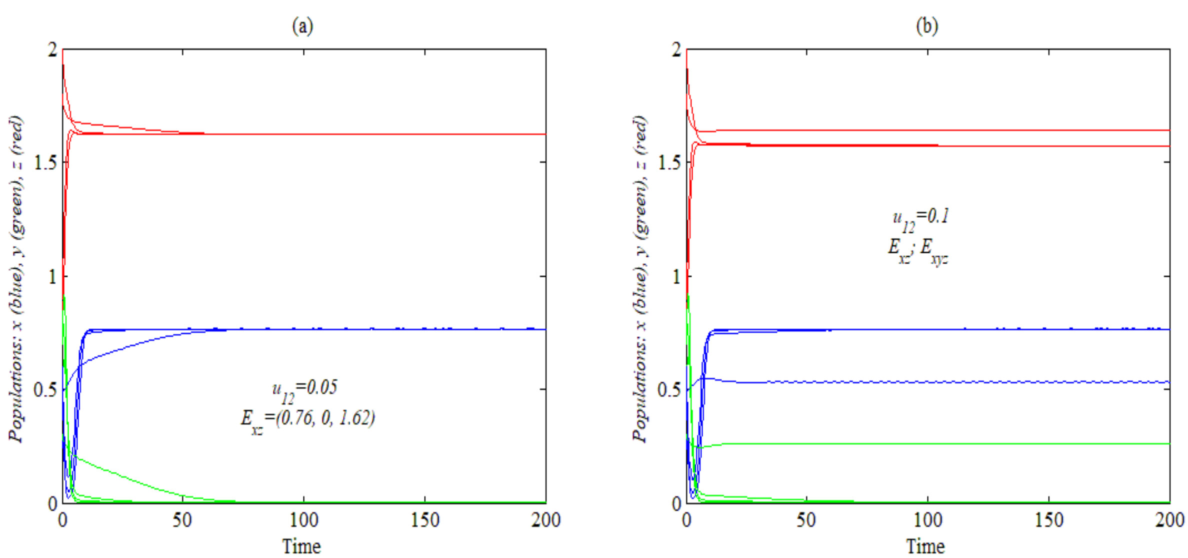


Figure 10: Time-dependent population trajectories of the system (2) utilizing data set (43) with various starting positions and various values of u_{11}, shows that: (a) Stable 3D limit cycle when $u_{11} = 1.75$. (b) Asymptotic stable SSSP when $u_{11} = 2.15$. (c) Bi-stable case between ZFSSP and SSSP when $u_{11} = 2.4$. (d) Asymptotic stable ZFSSP when $u_{11} = 2.75$.

In addition to the explained results given in Fig. (10), which indicates the huge influence of the varying in the parameter u_{11} on the system's (2) dynamic, it is observed that the system has asymptotic stable SSSP for the range $u_{11} \in (0,0.09]$, after that it becomes unstable and the system approaches asymptotically to a stable 3D limit cycle.



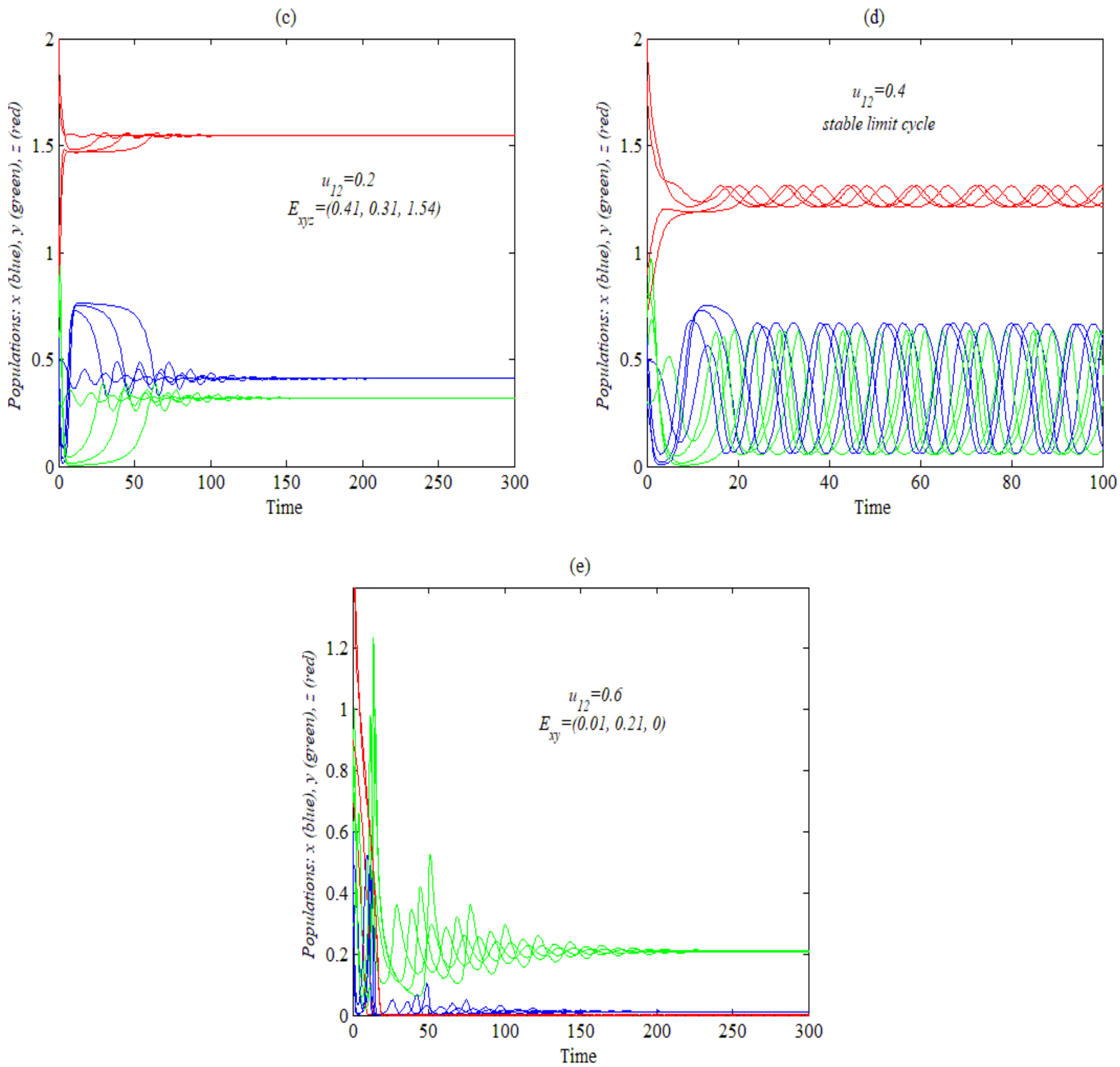


Figure 11: Time-dependent population trajectories of the system (2) utilizing data set (43) with various starting positions and various values of u_{12}, shows that: (a) Asymptotic stable ZFSSP when $u_{12} = 0.05$. (b) Bi-stable case between ZFSSP and SSSP when $u_{12} = 0.$. (c) Asymptotic stable SSSP when $u_{12} = 0.2$. (d) Stable 3D limit cycle when $u_{12} = 0.4$. (e) Asymptotic stable FFSSP when $u_{12} = 0.6$.

Different types of attractors are shown in Fig. (11) depending on varying in the parameter u_{12}. Furthermore, it is obtained that, system (2) approaches asymptotically to the SSSP for the range $u_{13} \in (0,0.98]$, while it goes to the ZFSSP for the range $u_{13} \geq 0.99$. On the other hand, the parameter u_{14} has a similar influence on the dynamic of the system (2) as that shown with u_6 and u_{10}.

8. CONCLUSION

In this paper, the aquatic food chain model consisting of harmful phytoplankton, zooplankton, and fish living in the contaminated environment is formulated mathematically. In these food webs, humans are a key player as one of the top predators. Therefore, take care must be taken to protect the ocean from chemicals that bioaccumulate in food webs and maintain sustainable fisheries. Accordingly, the toxin produced by harmful phytoplankton as a defensive property and the existence of contamination in the environment is considered in the formulation of the model. On the other hand, the influence of fear and harvesting on the dynamic of the food web is also included. A set of nonlinear ordinary differential equations was used to simulate the dynamics of such an aquatic food web system.

All the qualitative properties of the solution of the proposed model were studied. The biologically possible steady-state points were determined. The local, as well as global stability analysis of the model, was studied. The conditions that guarantee the survival of all populations as time becomes large were determined. The influence of altering the values of the parameters on the dynamics of the model was investigated using the Sotomayor theorem for local bifurcation. Finally, a numerical example for the proposed model was given and then solved numerically to confirm the obtained theoretical results and understand the influence of the parameters on the system's dynamics.

According to the numerical example of the food web model, the following results are obtained.

Lowering the half-saturation constant of the zooplankton destabilizes the system and the solution approaches a stable 3D limit cycle instead, which indicates obtaining a Hopf bifurcation. However, rising its value causes destabilizing of the SSSP, and then the system transfers to ZSSP by passing through a bi-stable behavior between SSSP and ZFSSP. The external toxic substances coefficient that affects the phytoplankton population, the external toxic substances coefficient that affects the zooplankton population and the conversion rate of the food to zooplankton have a similar impact on the system's dynamic as that obtained with the half-saturation constant of the zooplankton.

Increasing the level of fear in zooplankton gradually leads to instability of the SSSP and the system loses its stability and goes to ZFSSP by passing through the bi-stable behavior between SSSP and

ZFSSP. The liberation rate of toxic substances by the harmful phytoplankton, the harvest effort and the zooplankton death rate have a similar impact on the system's dynamic as that obtained with the level of fear in zooplankton.

Lowering the half-saturation constant of the fish leads the SSSP to be unstable and the system to lose its persistence and approaches asymptotically to the ZFSSP by passing through bi-stable behavior between SSSP and ZFSSP. While increasing its value makes the SSSP be unstable too and a 3D stable limit cycle takes place. The external toxic substances coefficient that affects the fish population and the fish's preference rate of zooplankton have a similar impact on the system's dynamic as that obtained with the half-saturation constant of the fish.

Decreasing the intrinsic growth rate of fish makes the SSSP be unstable and a 3D stable limit cycle takes place first and then an extinction in the fish population occurs and the solution of the system (2) approaches asymptotically FFSSP. On the other hand, rising its value leads the SSSP to be unstable and the system to lose its persistence and approaches asymptotically to the ZFSSP by passing through bi-stable behavior between SSSP and ZFSSP.

Decreasing the carrying capacity of the fish in the absence of zooplankton makes the SSSP unstable and a 3D stable limit cycle takes place. However, increasing its value makes the SSSP unstable and the system loses its persistence and approaches asymptotically to the ZFSSP by passing through bi-stable behavior between SSSP and ZFSSP.

Lowering the catchability rate of the fish leads the SSSP to be unstable and the system to lose its persistence and approaches asymptotically to the ZFSSP by passing through bi-stable behavior between SSSP and ZFSSP. While increasing its value makes the SSSP unstable too and a 3D stable limit cycle takes place first then with further increase the system faces extinction in the fish population and the solution approaches asymptotically to FFSSP.

Keeping the above in mind, system (2) is very sensitive to varying in its parameters set and undergoes different types of attraction including limit cycle and bi-stable behavior.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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