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MODELING AND ANALYSIS OF CORRUPTION DYNAMICS INCORPORATING MEDIA COVERAGE

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Abstract. This paper presents and analyzes a mathematical model that takes into account media coverage to explain how corruption spreads and is controlled. The model solution's positivity and boundedness are established, and the fundamental reproduction number is determined. We also examine the local and global stability of the model's endemic and corruption-free equilibrium points, as well as their corruption equilibria. According to the study, the free-corruption equilibrium is locally and globally asymptotically stable if the basic reproduction number is less than one. When the basic reproduction number is greater than one, the endemic equilibrium point is asymptotically stable both locally and globally. To verify the study findings, numerical simulations were performed using MATLAB software's ode45.

Keywords: modeling; corruption; basic reproduction number; stability; media coverage; numerical simulation.
2010 AMS Subject Classification: 00A71.

1. Introduction

The word "corrupt" comes from the Latin word "corruptus," which means "to disturb or damage" [1]. It is an illegal practice carried out for private benefit by public or private officials abusing their position or power, [2]. Corruption is non-compliance, a gain that breaks down

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any law or rule or community action [3]. It's a worm inside society's body, and it's a cancer for economic, social, and political reform [4]. Many nations round the sector be afflicted by deep-seated corruption that hampers financial improvement, undermines democracy, harms social justice and the guideline of thumb of regulation [5]. It is one of the reasons of instability and battle seen withinside the present day state of affairs in Ethiopia [6].

Nowadays, different epidemiological models are used to study the transmission dynamics of corruption as a disease. [7] have proposed and analyzed a easy social contagion version that depicts the dynamics of social impacts amongst politicians in an artificially corrupt parliament. The outcomes display that there may be a essential density of such zealot above which sincere deputies can not live to tell the tale as sincere after an extended time, whilst zealot density under the stated essential density of sincere deputies coexist with corrupt ones. [1] models an epidemiological version of corruption of immunity clauses in Nigeria. The simulation end result indicates that managing corruption withinside the presence of an immunity clause regime may be a tough conflict to win if the clause persists.

Nathan and Jackob [8] divided the overall population into three classes: susceptible, corrupt, and politically corrupt. They mainly target those who exploit officials and politicians. Prevention and disengagement strategies are modeled using model parameters and strategies are evaluated to counteract the vice of corruption. A model of corruption dynamics was proposed by [9]. Using an approximation of the honest population, an epidemiological corruption threshold was updated and developed. [4] provided a mathematical model for the transmission dynamics of corruption as a disease with a constant recruiting rate and a current occurrence. The version answer's positivity and boundedness are tested. In addition, the primary duplicate number (R_0), corruption unfastened and the endemic equilibrium factor had been determined.

Lemecha [6] proposed and analyzed a deterministic model for the spread of corruption like a disease. The version has comparable residences to [4] besides that someone who loses the immunity received thru the council in prison does now no longer immediately be a part of the corrupt elegance however is prone because of human behavior. And they're involved approximately the effect of the notice created with the aid of using anti-corruption.

Cuervo-Cazurra [10] proposed and analyzed a deterministic mathematical model for the transmission of corruption dynamics. [11] developed differential equation-based models for the spread of corruption that represented either growth or decay norms. To measure the level of corruption dynamics, [12] developed a difference equation-based approach. [5] developed and assessed a mathematical model with a standard incidence for studying corruption dynamics as a disease. Local stability analyses of endemic corruption and corruption-free equilibrium were investigated. The importance of the parameters in the model's long-term trajectory is also explained via numerical simulations.

By identifying and measuring the drivers that explain political corruption, [13] model and quantify the population at risk of political corruption in Spain. Once the problem has been quantified, the model allows parameters to be modified and fiscal, economic and legal measures to be simulated in order to quantify and better understand the impact on Spanish citizens. The results suggest strengthening women's leadership positions to mitigate this problem, in addition to changes in the law governing political parties in Spain and an increase in the budget of the judiciary system.

Mokaya et al.[14] developed and studied a deterministic model for the spread of corrupt morals that involves a group of people who are going through a counseling and guidance procedure. They determined theoretical results by computing the basic reproduction number, R_0, studying the presence and stability of equilibria, and doing numerical simulations. The findings reveal that an integrated control strategy is the most effective way to counteract moral corruption transmission. [15] used Lotka Volterra, Predator-Prey equations to broaden a version to explain corruption in better training institutions. Corrupt students and staff act as predators even as their non-corrupt friends act as prey withinside the newspaper. The numerical simulation indicates a growing trend towards corruption. The trend of staff and student corruption is declining, concluding that more emphasis should be placed on staff than on students to curb the spread of corruption.

[16] created a mathematical model of corruption dynamics in the presence of management controls. The model was simulated in MATLAB using RungeKutta's fourth-order technique, and the findings suggest that combining mass education with religious education reduces corruption

in a timely manner as compared to utilizing each control strategy independently.

Bonyah [17] formulated the fractional optimal control model of corruption dynamics. The results of the numerical solution show that the best strategy to control corruption in society is to optimize all controls at the same time. [18], broaden a deterministic version of the unfold of corruption and its evaluation the use of differential equations. Then the version changed into prolonged to optimum manage and the end result suggests that the extent of corruption in society may be decreased if anti-corruption efforts are made via the consequences are improved and positioned into practice. [3] developed and tested a deterministic mathematical model to describe the dynamics of corruption transmission while taking into account the social consequences for honest people. The basic features of the model are determined, including the basic reproduction number, corruption equilibria, and local and global asymptotic stability. The model was then extended to optimal control, and the results reveal that the prevention and punishment technique is the most effective way to limit corruption's transmission dynamics.

Recently, [2] developed a non-linear deterministic mathematical model for the dynamics of corruption by dividing the total population into susceptible (S), exposed (E), corrupt (C), recovered (R) and honest (H) classes. Furthermore, the model was extended to the optimal control problem and the author concluded that the integrated control strategy should be used to fight corruption. Media coverege has been known to greatly influence an individuals behaviour as well as government policies on prevention and control of infectious disease [19].

In this study, we modify the model proposed by [2] by adding the jail class to the existing model. Moreover, in our present model, we have taken into account that the impact of media coverege for the transmission and control of corrruption dynamics. This study is organized as follows. In the second part, a new mathematical model for the spread of corruption dynamics is constructed. In part three, we look at the existence and stability of corruption equilibria, as well as the positivity and boundedness of solutions. Part four deals about numerical simulation. Finally, conclusions are given in part five.

2. Model Formulation

In this section, we subdivided the total population into six classes:- Susceptible(S(t)), Exposed(E(t)), Corrupted(C(t)), Jailed(J(t)), Recovered(R(t)) and Honest(H(t)) individuals and we assumed that:

- Susceptible are individuals not involved in corrupt activities, or individuals who are imprisoned and then released after taking the advice;
- Exposed are individuals who are exposed to corruption but are not involved in corrupt activities;
- Individuals who are involved in corrupt activities are considered to as corrupted;
- Jailed are those found guilty by law of corrupt activities;
- Recovered are an individuals who are stopped to do any corrupt activities;
- An honest person is a sincere person who can never corrupt and
- An individuals can engage in corrupt activities only by coming into contact with corrupt people.

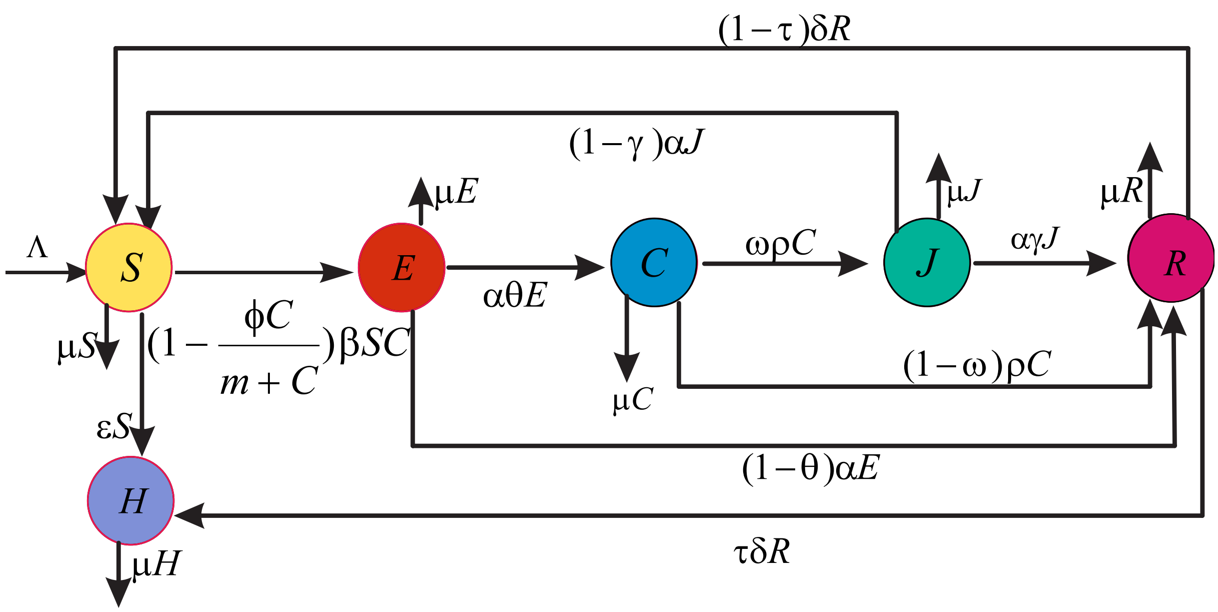


Figure 1. Flow Chart of the Model.

Parameter	Description
Λ	Recruitment rate.
β	Effective corruption contact rate.
ϕ	Measures the efficacy of media coverage.
m	Media coverage.
α	Rate at which exposed individuals become corrupted.
θ	Proportion of individuals that joins to corrupt class from exposed class.
ρ	Rate at which corrupted individuals become jailed.
ω	Proportion of individuals that joins to jailed class from corrupted class.
γ	Proportion of individuals that joins to recovered class from jailed class.
σ	Rate at which jailed individuals become recovered.
δ	Rate at which recovered individuals become honest.
τ	Proportion of individuals that joins to honest class become recovered class.
ε	Rate at which susceptible individuals become honest.
μ	Natural death rate.

TABLE 1. Parameters of the model

In addition, we assumed that there is a positive recruitment rate Λ in the susceptible class and a positive natural mortality rate μ for all classes. The susceptible individuals can be joined to exposed class by the reduced rate of contact with corrupt individuals due to media coverage $\frac{\phi\beta C}{m+C}$; where m, is media coverege, $\phi(0<\phi<1)$ is efficacy of media coverege, the function $\frac{C}{m+C}$ is a continuous bounded function which has taken corruption saturation into account. And, the remaining individuals from susceptible class are joined to honest class with ε proportion. The exposed individuals are joined to corrupt class with the rate of $\alpha\theta$. And due to their corrupt activities,these corrupt individuals are joined to jailed class with the rate of $\omega\rho$. Also, the exposed and corrupt individuals can be joined to recovered class with the rate of $(1-\theta)\alpha$ and $(1-\omega)\rho$ respectively and then, they become more honest with the rate of $\gamma\delta$ as a result of anti-corruption or therapy in jail, as well as moral or religious beliefs. In Table 1, all of the parameter descriptions are presented. We can derive the following from the model's assumptions and flow chart in figure 1:

We can derive the following from the model's assumptions and flow chart in figure 1.

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \left(1-\frac{\phi C}{m+C}\right)\beta SC + (1-\gamma)\sigma J + (1-\tau)\delta R - (\mu+\varepsilon)S,\\
\frac{dE}{dt} &= \left(1-\frac{\phi C}{m+C}\right)\beta SC - (\mu+\alpha)E,\\
\frac{dC}{dt} &= \alpha\theta E - (\mu+\rho)C,\\
\frac{dJ}{dt} &= \omega\rho C - (\mu+\sigma)J,\\
\frac{dR}{dt} &= (1-\theta)\alpha E + (1-\omega)\rho C + \gamma\sigma J - (\mu+\delta)R,\\
\frac{dH}{dt} &= \varepsilon S + \delta\tau R - \mu H,
\end{aligned} \tag{1}
$$

with

(2)

$S(0) = S_0 > 0,\ E(0) = E_0 \geq 0\ , C(0) = C_0 \geq 0,\ J(0) = J_0 \geq 0,\ R(0) = R_0 \geq 0,\ H(0) = H_0 \geq 0,$

and

$$0 < \gamma < 1, 0 < \theta < 1, 0 < \tau < 1, 0 < \omega < 1.$$

3. Model Analysis

In this part, we look at the epidemiologically feasible region's solution to equation (1).

$$\Omega = \left\{(S,E,C,J,R,H) \in R_+^6 : 0 \leq S(t)+E(t)+C(t)+J(t)+R(t)+H(t) \leq \frac{\Lambda}{\mu}\right\}. \tag{3}$$

3.1. Positivity and boundedness of the solution.

The following theorem is used to demonstrate the positivity and boundedness of the system of equation (1).

Theorem 1:

Let the initial data be $\{S(0) > 0, E(0) \geq 0, C(0) \geq 0, J(0) \geq 0, R(0) \geq 0, H(0) \geq 0\} \in \Omega$. Then, the solution set $\{S(t), E(t), C(t), J(t), R(t), H(t)\}$ of the system of equation (1) is non-negative for all $t \geq 0$.

Proof:

Let r = sup $\{t > 0 : S_0(u) > 0, E_0(u) \geq 0, C_0(u) \geq 0, J_0(u) \geq 0, R_0(u) \geq 0, H_0(u) \geq 0, u \in [0,t]\}$.

Since, from equation (2), all initial data are non negative, then $r > 0$. But, if $r < 0$, then either of the initial data is zero at r. Now, from the first equation of system (1), we can that:

$$\frac{dS}{dt} = \Lambda - \left(1 - \frac{\phi C}{m+C}\right)\beta SC + (1-\gamma)\sigma J + (1-\tau)\delta R - (\mu+\varepsilon)S. \tag{4}$$

Then, using the variation of constant formula, the solution of equation (4) at t is given by:

$$\begin{aligned} S(r) =& S(0)\exp\left[-\int_0^T \left(\left(1-\frac{\phi C}{m+C}\right)\beta C + (\mu+\varepsilon)\right) sds\right] \\ &+ \int_0^t (\Lambda + (1-\gamma)\sigma J + (1-\tau)\delta R) \\ &.\exp\left[-\int_s^t \left(\left(1-\frac{\phi C}{m+C}\right)\beta C + (\mu+\varepsilon)\right) udu\right] ds > 0. \end{aligned} \tag{5}$$

Furthermore, since all the state variables are positive in $[0,r]$, thus $S(r) > 0$. Similarly, it can be shown that $E(r) \geq 0, C(r) \geq 0, J(r) \geq 0, R(r) \geq 0$ and $H(r) \geq 0$ which is a contradiction. Therefore, $r = 0$. Hence, all the solution sets are positive for $t \geq 0$.

Theorem 2: Assume that all the initial conditions are non-negative in R^6_+ for the system

$$\Omega = \left\{(S,E,C,J,R,H) \in R^6_+; 0 \leq S(t)+E(t)+C(t)+J(t)+R(t)+H(t) : N(t) \leq \frac{\Lambda}{\mu}\right\}, \tag{6}$$

then the region Ω is positively invariant.

Proof:- Let N be the total population, $N(t) = S(t)+E(t)+C(t)+J(t)+R(t)+H(t)$. Then, by differentiating N(t) with respect to time and substituting equation (1), we can obtain that:

$$\frac{dN}{dt} = \Lambda - \mu N. \tag{7}$$

After some simplification,

$$N(t) = \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu}\right)e^{-\mu t}. \tag{8}$$

Then, $\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}$. As a result, is invariant in a positive sense. As a result, all solutions of the system of equations (1) with initial conditions in Ω continue to be in Ω.

3.2. Corruption Free Equilibrium Point of the Model. The model's corruption-free equilibrium points, E^0, are stationary solutions in which society is devoid of corruption. It is obtained by equating equation (1) to zero and using $E=0$, $C=0$, $J=0$ and $R=0$. Then, corruption free equilibrium points, E^0 of our model equation (1) is given by:

$$E^0=(\widehat{S},\widehat{E},\widehat{C},\widehat{J},\widehat{R},\widehat{H})=\left(\frac{\Lambda}{(\mu+\varepsilon)},0,0,0,0,\frac{\varepsilon\Lambda}{(\mu+\varepsilon)\mu}\right). \tag{9}$$

The basic reproduction number R_0: The basic reproduction number R0 measures the expected quantity of secondary infections so one can end result from a newly inflamed man or woman delivered right into a susceptible population. Rewriting the model equation (1) starting with newly corrupt classes and using the next generation matrix method, the basic reproduction number R_0 is obtained as follows:

$$\begin{aligned}
\frac{dE}{dt}&=\left(1-\frac{\phi C}{m+C}\right)\beta SC-(\mu+\alpha)E,\\
\frac{dC}{dt}&=\alpha\theta E-(\mu+\rho)C,\\
\frac{dJ}{dt}&=\omega\rho C-(\mu+\sigma)J,\\
\frac{dR}{dt}&=(1-\theta)\alpha E+(1-\omega)\rho C+\gamma\sigma J-(\mu+\delta)R,\\
\frac{dH}{dt}&=\varepsilon S+\delta\tau R-\mu H,
\end{aligned} \tag{10}$$

the basic reproduction number R_0 is its own dominant value of the FV^{-1} generation matrix or spectral radio FV^{-1} where

$$F=\begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu+\varepsilon} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, and\ V=\begin{pmatrix} \mu+\alpha & 0 & 0 & 0 & 0\\ -\alpha\theta & \mu+\rho & 0 & 0 & 0\\ 0 & \omega\rho & \mu+\sigma & 0 & 0\\ -(1-\theta)\alpha & -(1-\omega)\rho & -\gamma\delta & \mu+\delta & 0\\ 0 & 0 & 0 & -\delta\tau & \mu \end{pmatrix}.$$

As a result, the basic number reproduction ratio R_0 is given by:

$$R_0=\frac{\beta\alpha\theta\Lambda}{(\mu+\alpha)(\mu+\rho)(\mu+\varepsilon)}. \tag{11}$$

3.3. Local Stability of the Corruption Free Equilibrium Point. The Jacobian matrix of the system equation (1) at E^0 can be used to discuss the local stability of the model's corruption-free equilibrium point, E^0.

Theorem 3: Corruption free equilibrium point. E^0 of system of equation (1) is locally asymptotically stable, if $R_0 < 1$.

Proof. The Jacobian matrix of system of equation (1) is $J =$

(12)
$$\begin{pmatrix} -\left(1-\frac{\phi C}{m+C}\right)\beta C-(\mu+\varepsilon) & 0 & -\beta S+\left(\frac{2m+C}{(m+C)^2}\right)\phi\beta SC & (1-\gamma)\sigma & (1-\tau)\delta & 0 \\ \left(1-\frac{\phi C}{m+C}\right)\beta C & -(\mu+\alpha) & \beta S-\left(\frac{2m+C}{(m+C)^2}\right)\phi\beta SC & 0 & 0 & 0 \\ 0 & \alpha\theta & -(\mu+\rho) & 0 & 0 & 0 \\ 0 & 0 & \omega\rho & -(\mu+\sigma) & 0 & 0 \\ 0 & (1-\theta)\alpha & (1-\omega)\rho & \gamma\sigma & -(\mu+\delta) & 0 \\ \varepsilon & 0 & 0 & 0 & \delta\tau & -\mu \end{pmatrix}$$

The characteristic equation of Jacobian matrix of equation (10) at corruption free equilibrium point, E^0 is $|J(E^0)-\lambda I_4| = 0$. That is

(13)
$$\begin{vmatrix} -(\mu+\varepsilon)-\lambda & 0 & -\frac{\beta\Lambda}{\mu+\varepsilon} & (1-\gamma)\sigma & (1-\tau)\delta & 0 \\ 0 & -(\mu+\alpha)-\lambda & \frac{\beta\Lambda}{\mu+\varepsilon} & 0 & 0 & 0 \\ 0 & \alpha\theta & -(\mu+\rho)-\lambda & 0 & 0 & 0 \\ 0 & 0 & \omega\rho & -(\mu+\sigma)-\lambda & 0 & 0 \\ 0 & (1-\theta)\alpha & (1-\omega)\rho & \gamma\sigma & -(\mu+\delta)-\lambda & 0 \\ \varepsilon & 0 & 0 & 0 & \delta\tau & -\mu-\lambda \end{vmatrix} = 0$$

(14)
$$(-(\mu+\sigma)-\lambda)(-(\mu+\delta)-\lambda)(-\mu-\lambda)(\lambda^2+c_1\lambda+c_2) = 0.$$

Clearly,

$$\lambda_1 = -(\mu+\sigma) < 0,$$

$$\lambda_2 = -(\mu+\delta) < 0$$

$$\lambda_3 = -\mu < 0$$

Here, $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_3 < 0$. Moreover, using Routh Hurwitz criteria, the last equation of (14) has strictly negative real part since $c_1 = (2\mu+\alpha+\rho) > 0$ and $c_2 = (\mu+\alpha)(\mu+\rho)\left(1-\frac{\beta\alpha\theta\Lambda}{(\mu+\alpha)(\mu+\rho)(\mu+\varepsilon)}\right) = (\mu+\alpha)(\mu+\rho)(1-R_0) > 0$ if $R_0 < 1$. As a result, with $R_0 < 1$, our model equation (1) at E^0 offers all eigenvalues with a negative real part, and so it is locally asymptotically stable.

3.4. Global Stability of the Corruption Free Equilibrium Point.

We used the method proposed by [20] to investigate global stability. We have expressed the system of equation (1) in the following form, based on [20]: To investigate the global stability, we applied the method proposed by [20]. Based on [20], we have written the system of equation (1) in the following form:

$$\begin{aligned} \frac{dX}{dt} &= A(X - X_{E^0,n}) + A_1 Y, \\ \frac{dY}{dt} &= A_2 Y, \end{aligned} \tag{15}$$

where $X = (S,H)$ represent the number of uncorrupted individuals, while, $Y = (E,C,J,R)$ represent the number of corrupted individuals and $X_{E^0,n}$ is a vector at corruption free equilibrium point of the same vector length as X. Based on [20], the corruption free equilibrium point E^0 is globaly asymptotically stable if the following conditions are fulfilled:

(1) A should be a matrix with real negative eigenvalues.
(2) A_2 should be a Metzler matrix.

Theorem 4: The corruption free equilibrium point, E^0 is globally asymptotically stable if $R_0 < 1$.

Proof: From our model of equation (1),

$$X = (S,H)^T, Y = (E,C,J,R)^T, E^0 = \left(\frac{\Lambda}{(\mu+\varepsilon)}, 0,0,0,0, \frac{\varepsilon\Lambda}{(\mu+\varepsilon)\mu}\right)^T, X_{E^0,n} = \left(\frac{\Lambda}{(\mu+\varepsilon)}, \frac{\varepsilon\Lambda}{(\mu+\varepsilon)\mu}\right)^T$$

The system of equation (1) together with equation (15) can be written as

$$\begin{pmatrix} \Lambda - \left(1-\frac{\phi C}{m+C}\right)\beta SC + (1-\gamma)\sigma J + (1-\tau)\delta R - (\mu+\varepsilon)S \\ \varepsilon S + \delta\tau R - \mu H \end{pmatrix}$$

$$= A \begin{pmatrix} S - \frac{\Lambda}{(\mu+\varepsilon)} \\ H - \frac{\varepsilon\Lambda}{(\mu+\varepsilon)\mu} \end{pmatrix} + A_1 \begin{pmatrix} E \\ C \\ J \\ R \end{pmatrix},$$

$$\begin{pmatrix} \left(1 - \frac{\phi C}{m+C}\right)\beta SC - (\mu+\alpha)E \\ \alpha\theta E - (\mu+\rho)C \\ \omega\rho C - (\mu+\sigma)J \\ (1-\theta)\alpha E + (1-\omega)\rho C + \gamma\sigma J - (\mu+\delta)R \end{pmatrix} = A_2 \begin{pmatrix} E \\ C \\ J \\ R \end{pmatrix},$$

Using uncorrupt entry of Jacobian matrix of sytem of equation (1) and the representation in equation (13), the matrix A, A_1, A_2 are:

$$A = \begin{pmatrix} -(\mu+\varepsilon) & 0 \\ \varepsilon & -\mu) \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -\beta S + \left(\frac{2m+C}{(m+C)^2}\right)\phi\beta SC & (1-\gamma)\sigma & (1-\tau)\delta \\ 0 & 0 & 0 & \delta\tau \end{pmatrix}.$$

$$A_2 = \begin{pmatrix} -(\mu+\alpha) & \beta S - \left(\frac{2m+C}{(m+C)^2}\right)\phi\beta SC & 0 & 0 \\ \alpha\theta & -(\mu+\rho) & 0 & 0 \\ 0 & \omega\rho & -(\mu+\sigma) & 0 \\ (1-\theta)\alpha & (1-\omega)\rho & \gamma\sigma & -(\mu+\delta) \end{pmatrix}.$$

This implies that the sufficient conditions are satisfied. Thus, the corruption free equilibrium point, E^0, is globally asymptotically stable if $R_0 < 1$.

3.5. Corruption Endemic Equilibrium Point of the Model.

The endemic equilibrium point for corruption, E_1, is a steady-state solution in which corruption endures in the population. The corruption endemic equilibrium point, $E_1 = (S^*, E^*, C^*, J^*, R^*, H^*)$, is obtained by equating each equation in (1) to zero:

$$\Lambda - \left(1 - \frac{\phi C^*}{m+C^*}\right)\beta S^* C^* + (1-\gamma)\sigma J^* + (1-\tau)\delta R^* - (\mu+\varepsilon)S^* = 0,$$

$$\left(1 - \frac{\phi C^*}{m+C^*}\right)\beta S^* C^* - (\mu+\alpha)E^* = 0,$$

$$\alpha\theta E^* - (\mu+\rho)C^* = 0,$$

$$\omega\rho C^* - (\mu+\sigma)J^* = 0, \tag{16}$$

$$(1-\theta)\alpha E^* + (1-\omega)\rho C^* + \gamma\sigma J^* - (\mu+\delta)R^* = 0,$$

$$\varepsilon S^* + \delta\tau R^* - \mu H^* = 0.$$

Then, after some simplification, we can obtain corruption endemic equilibrium point, E_1 interms of C^* as

$$\begin{aligned}
S^* =& \frac{(\mu+\alpha)(\mu+\rho)}{\left(1-\frac{\phi C^*}{m+C^*}\right)\beta\alpha\theta},\\
E^* =& \frac{(\mu+\rho)}{\alpha\theta}C^*,\\
J^* =& \frac{\omega\rho}{\mu+\sigma}C^*,\\
R^* =& \frac{((\mu+\sigma)(1-\theta)\alpha(\mu+\rho)+\alpha\theta(\mu+\sigma)(1-\omega)\rho+\gamma\sigma\omega\rho\alpha\theta)}{(\mu+\delta)\alpha\theta(\mu+\sigma)}C^*,\\
H^* =& \frac{\varepsilon(\mu+\alpha)(\mu+\rho)}{\left(1-\frac{\phi C^*}{m+C^*}\right)\beta\mu\alpha\theta} + \frac{\delta\tau((\mu+\sigma)(1-\theta)\alpha(\mu+\rho)+\alpha\theta(\mu+\sigma)(1-\omega)\rho+\gamma\sigma\omega\rho\alpha\theta)}{(\mu+\delta)\alpha\theta(\mu+\sigma)\mu}C^*,
\end{aligned}$$

where C^* is positive solution of

$$d_2 {C^*}^2 + d_1 C^* + d_0, \tag{17}$$

$$\begin{aligned}
d_2 =& \alpha\theta\omega\rho(\mu+\delta)(1-\gamma)\sigma\beta(1-\phi) + (1-\tau)\delta((\mu+\sigma)(\mu+\rho)(1-\theta)\alpha + \alpha\theta(\mu+\sigma)\\
& (1-\omega)\rho + \gamma\sigma\omega\rho\alpha\theta)\beta(1-\phi) - (\mu+\alpha)(\mu+\rho)(\mu+\delta)(\mu+\sigma)\beta(1-\phi),\\
d_1 =& (R_0-1)(\mu+\alpha)(\mu+\rho)(\mu+\delta)(\mu+\sigma)(\mu+\varepsilon) - \beta\alpha\theta\Lambda\phi(\mu+\rho)(\mu+\sigma)+\\
& \beta\sigma m(1-\tau)((\mu+\sigma)(\mu+\rho)(1-\theta)\alpha + \alpha\theta(\mu+\sigma)(1-\omega)\rho + \gamma\sigma\omega\rho\alpha\theta)\\
& m\omega\rho\beta\alpha\theta(1-\gamma)(\mu+\delta) - \beta m(\mu+\alpha)(\mu+\rho)(\mu+\sigma)(\mu+\delta),\\
d_0 =& (R_0-1)(\mu+\alpha)(\mu+\rho)(\mu+\delta)(\mu+\sigma)(\mu+\varepsilon)m.
\end{aligned}$$

3.6. Local Stability of the Corruption Endemic Equilibrium Point.

We used the Jacobian stability approach to prove the local stability of the corruption endemic equilibrium state in this section.

Theorem 5: When $R_0 > 1$, the model's endemic equilibrium point, E_1, is locally asymptotically stable.

Proof: The local stability of the corruption endemic equilibrium, E_1, is determined based on the signs of the eigenvalues of the Jacobian matrix which is computed at the corruption endemic equilibrium, E_1. Now, the Jacobian matrix of the our model at E_1 is given by:

$$J = \begin{pmatrix} -\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* - (\mu+\varepsilon) & 0 & -\beta S^* + d & (1-\gamma)\sigma & (1-\tau)\delta & 0 \\ \left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* & -(\mu+\alpha) & \beta S^* - d & 0 & 0 & 0 \\ 0 & \alpha\theta & -(\mu+\rho) & 0 & 0 & 0 \\ 0 & 0 & \omega\rho & -(\mu+\sigma) & 0 & 0 \\ 0 & (1-\theta)\alpha & (1-\omega)\rho & \gamma\sigma & -(\mu+\delta) & 0 \\ \varepsilon & 0 & 0 & 0 & \delta\tau & -\mu \end{pmatrix}, \tag{18}$$

where $d = \left(\frac{2m+C^*}{(m+C^*)^2}\right)\phi\beta S^* C^*$. After simplification, the characteristic polynomial is obtained as:

$$g(\lambda) = (\mu+\lambda)\left[a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0\right], \tag{19}$$

where;

$$\begin{aligned} a_5 =& 1, a_4 = 5\mu+\alpha+\rho+\sigma+\delta+\varepsilon+\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*, a_3 = \left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* + (\mu+\varepsilon)\right) \\ & (4\mu+\alpha+\rho+\sigma+\delta) + (\mu+\alpha)(3\mu+\rho+\sigma+\delta) + (\mu+\rho)(2\mu+\sigma+\delta) + (\mu+\sigma)(\mu+\delta), \\ a_2 =& \left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* + (\mu+\varepsilon)\right)(\mu+\alpha)(3\mu+\rho+\sigma+\delta) + (\mu+\rho)(2\mu+\sigma+\delta) \\ & \left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* + 2\mu+\varepsilon+\alpha\right) + (\mu+\sigma)(\mu+\delta)\left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* + 3\mu+\alpha+\varepsilon+\rho\right) \\ & + \left(1-\left(\frac{2m+C^*}{(m+C^*)^2}\right)\phi C^*\right)\alpha\theta\beta^2 S^* C^* - (1-\tau)(1-\theta)\alpha\delta\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*, \\ a_1 =& (2\mu+\sigma+\delta)\left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* + (\mu+\varepsilon)\right)(\mu+\alpha)(\mu+\rho) + \alpha\theta\left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*\right) \\ & \left(1-\left(\frac{2m+C^*}{(m+C^*)^2}\right)\phi C^*\right)\beta S^*(2\mu+\sigma+\delta) + (\mu+\sigma)(\mu+\delta)\left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* + (\mu+\varepsilon)\right) \\ & (2\mu+\alpha+\rho) - \alpha\theta\left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*\right)((1-\gamma)\sigma\omega\rho + (1-\tau)(1-\omega)\sigma\rho) - (1-\tau)(1-\theta)\delta\alpha \\ & \left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*\right)(2\mu+\rho+\sigma), a_0 = \left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^* + (\mu+\varepsilon)\right)(\mu+\alpha)(\mu+\rho) + \\ & \alpha\theta\left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*\right)\left(1-\left(\frac{2m+C^*}{(m+C^*)^2}\right)\phi C^*\right)\beta S^*(\mu+\sigma)(\mu+\delta) - (1-\gamma)\sigma\alpha\rho\theta\omega(\mu+\delta) \\ & \left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*\right) - (1-\tau)\delta\left(\left(1-\frac{\phi C^*}{m+C^*}\right)\beta C^*\right)(\alpha\theta(\gamma\sigma\rho\omega + (1-\omega)\rho(\mu+\sigma)) \\ & + (1-\theta)\alpha(\mu+\rho)(\mu+\sigma)). \end{aligned}$$

From the characteristic polynomial of equation (19), one of the eigenvalues of $J(E_1)$ is $\lambda_l = -\mu < 0$ and the others five roots of equation (19) are analyzed by Routh-Hurwitz criteria. The coefficients $a_5, a_4, a_3, a_2, a_1, a_0$ of the characteristic polynomial are real positive. As a result, the required conditions for the stability of the endemic corruption equilibrium point have been met. The following are the required conditions for system stability using the Hurwitz array for the characteristic polynomial:

$$\begin{array}{c|ccc} \lambda^5 & a_5 & a_3 & a_1 \\ \lambda^4 & a_4 & a_2 & a_0 \\ \lambda^3 & b_1 & b_2 & b_3 \\ \lambda^2 & c_1 & c_2 & c_3 \\ \lambda^1 & d_1 & d_2 & d_3 \\ \lambda^0 & e_1 & e_2 & e_3 \end{array}$$

where $a_5, a_4, a_3, a_2, a_1, a_0$ are the coefficients of the characteristic polynomial and the remaining elements in the array are determined as follows:

$$b_1 = -\frac{1}{a_4}\begin{vmatrix} a_5 & a_3 \\ a_4 & a_2 \end{vmatrix} = \frac{a_3a_4 - a_2}{a_4} > 0, b_2 = -\frac{1}{a_4}\begin{vmatrix} a_5 & a_1 \\ a_4 & a_0 \end{vmatrix} = \frac{a_1a_4 - a_0}{a_4}, b_3 = -\frac{1}{a_4}\begin{vmatrix} a_5 & 0 \\ a_4 & 0 \end{vmatrix} = 0,$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_4 & a_2 \\ b_1 & b_2 \end{vmatrix} = \frac{a_2b_1 - a_4b_2}{b_1} > 0, c_2 = -\frac{1}{b_1}\begin{vmatrix} a_4 & a_0 \\ b_1 & b_3 \end{vmatrix} = \frac{a_0b_1 - a_4b_3}{b_1} = a_0, c_3 = -\frac{1}{b_1}\begin{vmatrix} a_4 & 0 \\ b_1 & 0 \end{vmatrix} = 0,$$

$$d_1 = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \frac{b_2c_1 - b_1c_2}{c_1} > 0, d_2 = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = \frac{b_3c_1 - b_1c_3}{c_1} = 0, d_3 = -\frac{1}{c_1}\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix} = 0,$$

$$e_1 = -\frac{1}{d_1}\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} = \frac{c_2d_1 - c_1d_2}{d_1} = c_2 = a_0 > 0, e_2 = -\frac{1}{d_1}\begin{vmatrix} c_1 & c_3 \\ d_1 & d_3 \end{vmatrix} = 0, e_3 = -\frac{1}{d_1}\begin{vmatrix} c_1 & 0 \\ d_1 & 0 \end{vmatrix} = 0.$$

The coefficients of the characteristic polynomial $a_5, a_4, a_3, a_2, a_1, a_0$ are real positive and the first column of the Routh-Hurwitz array have the same positive sign. Therefore, by Routh-Hurwitz's criteria, all eigenvalues of the characteristic polynomial are negative. Hence, the corruption endemic equilibrium point E_1 is locally asymptotically stable if $R_0 > 1$.

3.7. Global Stability of Corruption Endemic Equilibrium Point. The following results are used to study the global asymptotical stability of the corruption endemic equilibrium point E_1 of the model of equation (1):

$$\begin{aligned}
&\Lambda = \left(1-\frac{\phi C^*}{m+C^*}\right)\beta S^*C^* - (1-\gamma)\sigma J^* - (1-\tau)\delta R^* + (\mu+\varepsilon)S^*,\\
&\left(1-\frac{\phi C^*}{m+C^*}\right)\beta S^*C^* = (\mu+\alpha)E^*,\\
&\alpha\theta E^* = (\mu+\rho)C^*,\\
&\omega\rho C^* = (\mu+\sigma)J^*,\\
&(1-\theta)\alpha E^* + (1-\omega)\rho C^* + \gamma\sigma J^* = (\mu+\delta)R^*,\\
&\varepsilon S^* + \delta\tau R^* = \mu H^*.
\end{aligned} \tag{20}$$

The following theorem can then be given and proved.

Theorem 6: For $R_0 > 1$, then the model equation of (1) at E_1 is global asymptotical stable.

Proof: Using the method proposed by [3], we consider the following Lyapunov function for model of equation (1):

$$\begin{aligned}
L(t) =& S - S^* - S^*\ln\frac{S}{S^*} + E - E^* - E^*\ln\frac{E}{E^*} + \frac{\mu+\alpha}{\alpha\theta}\\
&\left(C - C^* - C^*\ln\frac{C}{C^*}\right) + \frac{\mu+\alpha}{\omega\rho}\left(J - J^* - J^*\ln\frac{J}{J^*}\right).
\end{aligned} \tag{21}$$

By differentiating (21) with respect to time, we have

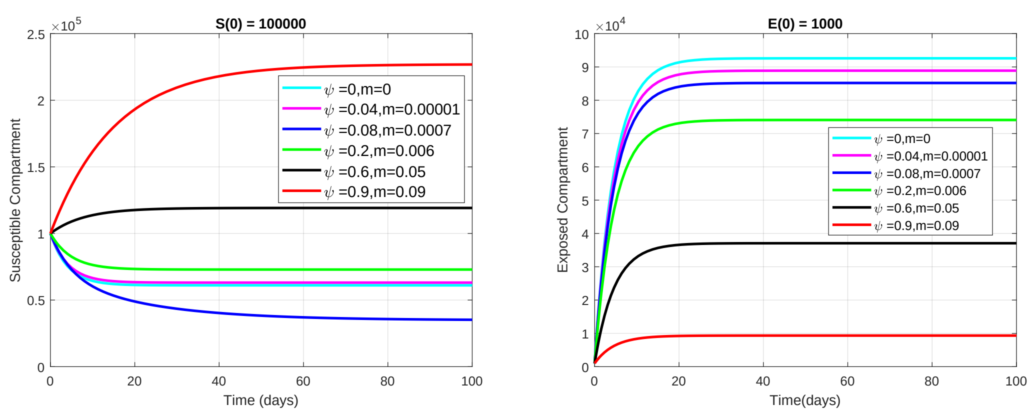
$$\begin{aligned}
\frac{dL}{dt} &= \left(1-\frac{S^*}{S}\right)\frac{dS}{dt} + \left(1-\frac{E^*}{E}\right)\frac{dE}{dt} + \frac{\mu+\alpha}{\alpha\theta}\left(1-\frac{C^*}{C}\right)\frac{dC}{dt} + \frac{\mu+\alpha}{\omega\rho}\left(1-\frac{J^*}{J}\right)\frac{dJ}{dt}\\
&= \left(1-\frac{S^*}{S}\right)\left[\Lambda - \left(1-\frac{\phi C}{m+C}\right)\beta SC + (1-\gamma)\sigma J + (1-\tau)\delta R - (\mu+\varepsilon)S\right]\\
&+ \left(1-\frac{E^*}{E}\right)\left[\left(1-\frac{\phi C}{m+C}\right)\beta SC - (\mu+\alpha)E\right] + \frac{\mu+\alpha}{\alpha\theta}\left(1-\frac{C^*}{C}\right)[\alpha\theta E - (\mu+\rho)C]\\
&+ \frac{\mu+\alpha}{\omega\rho}\left(1-\frac{J^*}{J}\right)[\omega\rho C - (\mu+\sigma)J].
\end{aligned} \tag{22}$$

Using equation (20), equation (22) can becomes,

$$\frac{dL}{dt} = \left(1-\frac{S^*}{S}\right)\left[\left(1-\frac{\phi C^*}{m+C^*}\right)\beta S^* C^* - (1-\gamma)\sigma J^* - (1-\tau)\delta R^* + (\mu+\varepsilon)S^*\right]$$
$$\left(1-\frac{S^*}{S}\right)\left[-\left(1-\frac{\phi C}{m+C}\right)\beta SC + (1-\gamma)\sigma J + (1-\tau)\delta R - (\mu+\varepsilon)S\right]$$
$$+\left(1-\frac{E^*}{E}\right)[(\mu+\alpha)E^* - (\mu+\alpha)E] + \frac{\mu+\alpha}{\alpha\theta}\left(1-\frac{C^*}{C}\right)[(\mu+\rho)C^* - (\mu+\rho)C]$$
$$+\frac{\mu+\alpha}{\omega\rho}\left(1-\frac{J^*}{J}\right)[(\mu+\sigma)J^* - (\mu+\sigma)J], \tag{23}$$
$$= -\frac{(m+(1-\phi)C^*)}{m+C^*}\beta S^* C^* \left[-1+\frac{SC(m+(1-\phi)C)(m+C^*)}{S*C*(m+C)(m+(1-\phi)C^*)}\right]\left(1-\frac{S^*}{S}\right)$$
$$-\left[(\mu+\varepsilon)S\left(1-\frac{S^*}{S}\right) + (1-\gamma)\sigma J^*\left(1-\frac{J}{J^*}\right) + (1-\tau)\delta R^*\left(1-\frac{R}{R^*}\right)\right]\left(1-\frac{S^*}{S}\right)$$
$$-(\mu+\alpha)\left[E\left(1-\frac{E^*}{E}\right)^2 + \frac{(\mu+\rho)C}{\alpha\theta}\left(1-\frac{C^*}{C}\right)^2 + \frac{(\mu+\sigma)J}{\omega\rho}\left(1-\frac{J^*}{J}\right)^2\right]. \tag{24}$$

Here, from equation (24), we observe that $\frac{dL}{dt} < 0$. Therefore, using [21], E_1 is global asymptotical stable whenever $R_0 > 1$.

Bifurcation: The central manifold theory is used to establish the equilibrium point stability behavior at $R_0 = 1$. At $R_0 = 1$, the uncorrupted equilibrium point shifts from stable to unstable, and when R_0 passes one, there is a positive equilibrium. $E_0 = E_1$ as well as $R_0 = 1$. As a result, at the bifurcation point $R_0 = 1$, a transcritical bifurcation occurs in the model.



(a)

(b)

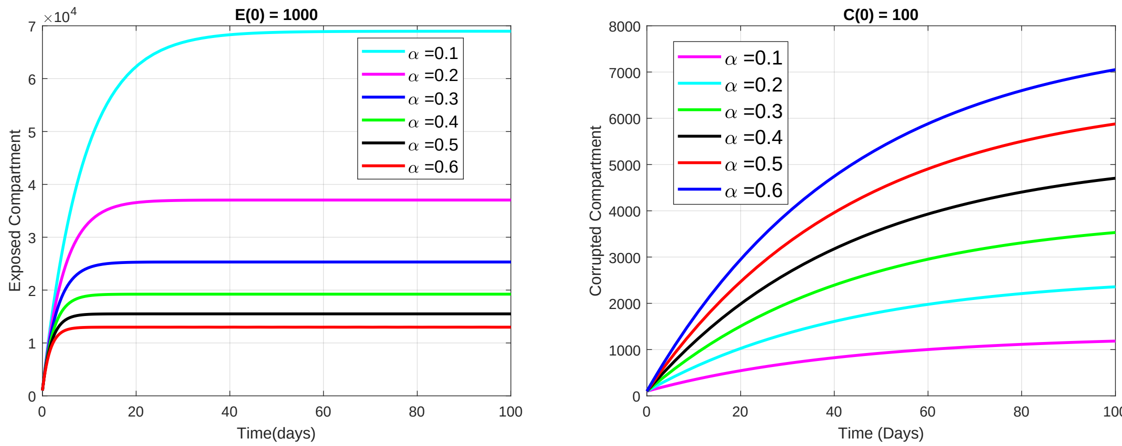
FIGURE 2. Susceptible population (S) and exposed population (E) w.r.t. time t for different values of ϕ and m.

4. Numerical Simulation

In this section, we used MATLAB ode45 solvers to numerically verify our work. Our simulations look at the impact of various model parameter combinations on the transmission and control of corruption dynamics. The simulation is carried out with a variety of parameter values. The source of the set of parameter values are mainly from literature review as well as some assumption to investigate the effect of the awarness of the media coverege in the transmission of corruption model. The relevant initial circumstances are used in the simulations and analyses: $S(0) = 100000, E(0) = 1000, C(0) = 100, J(0) = 0, R(0) = 1300, H(0) = 13000$ and the parameters values are displayed in Table 2.

Parameter	Value	Source
Λ	15000	Assumption
β	0.002	Assumption
ϕ	0.6	varies
μ	0.016	[6]
ε	0.03	[6]
m	0.00001	varies
σ	0.19	Assumption
α	0.2	[2]
θ	0.3	[6]
ρ	0.007	[2]
ω	0.06	Assumption
γ	0.125	[4]
δ	0.35	[2]
τ	0.1	[6]

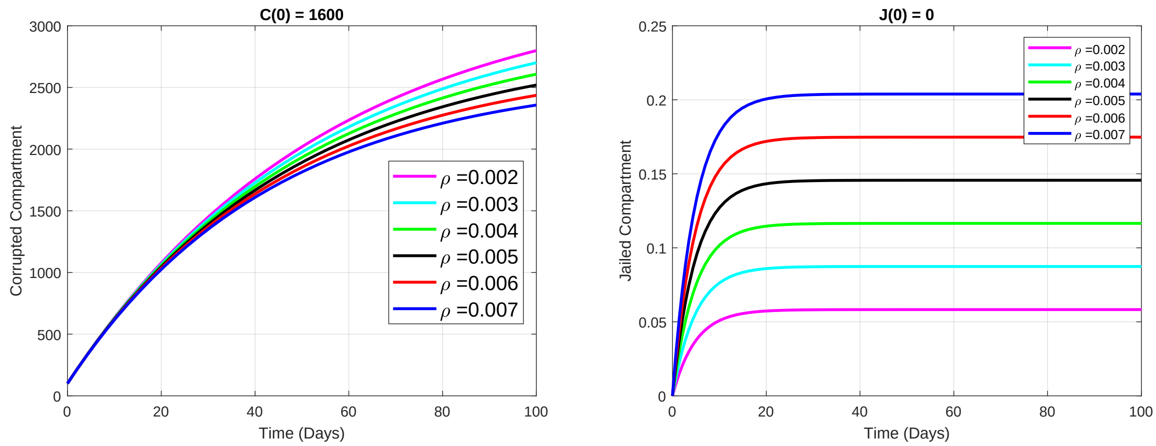
TABLE 2. The parameter values of model.



(a) (b)

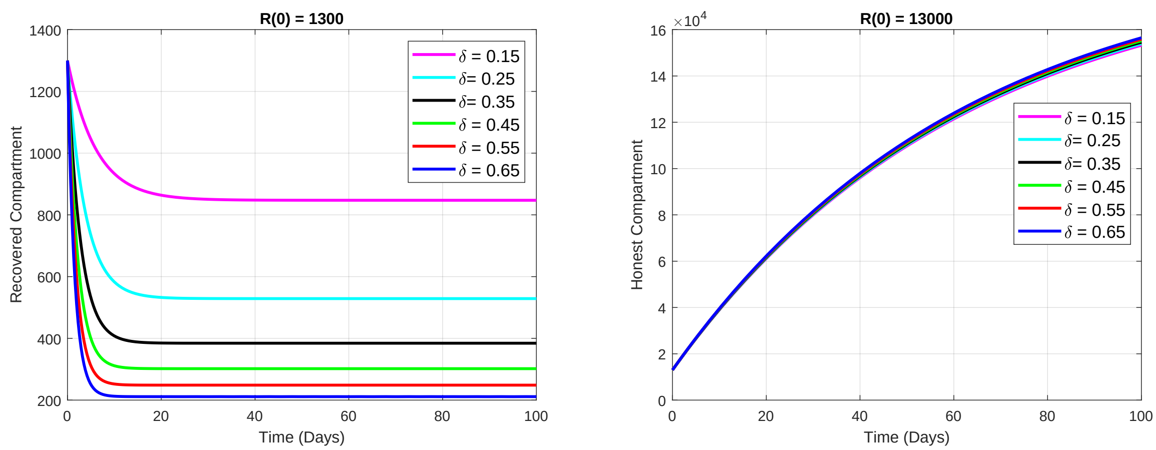
FIGURE 3. Exposed population (E) and Corrupted population (C) w.r.t. time t for different values of α.

Susceptible population and exposed population with respect to time t for different values of ϕ and m is shown in fig.2. From figure 2a and 2b, we observe that as media coverage increases susceptible population increases while exposed population decrease as a result the corrupt population removes out from the community. Also, exposed population and corrupted population with respect to time t for different values of α is shown in fig.3. The figure shows that as the value of α increases the exposed population decreases, while the corrupt population increases. From figures 4a and 4b we observe that as the value of ρ increases the corrupt population decreases, while the jail population increases. From figures 5a and 5b we observe that as the value of δ increases the recovered population decreases, while the honest population increases. Furthermore, figures 6a and 6b we observe that as the value of ε increases the susceptible population decreases, while the honest population increases.



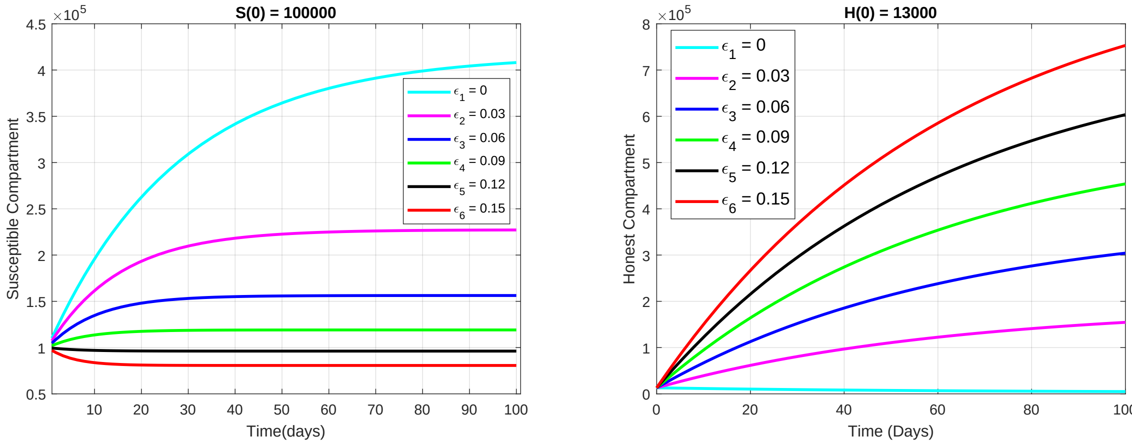
(a) (b)

FIGURE 4. Corrupted population (C) and jailed population (J) w.r.t. time t for different values of ρ.



(a) (b)

FIGURE 5. Recovered population (R) and honest population (H) w.r.t. time t for different values of δ.



(a) (b)

FIGURE 6. Susceptible population (S) and honest population (H) w.r.t. time t for different values of ε.

5. CONCLUSION

A mathematical model for the dynamics of corruption transmission, including media coverage, has been developed in this work. The model's well-posedness, as well as its positivity and boundedness, are examined. The basic reproduction number, as well as the stability analysis of the model's corruption equilibria, were investigated. According to the study, if the basic reproduction number is less than one, the corruption-free equilibrium is locally and globally asymptotically stable, however if the basic reproduction number is more than one, the endemic equilibrium is locally asymptotically stable. The numerical simulation shows that in the presence of media coverage, the susceptible individuals increase, while the exposed as well as the corrupted individuals decreases. This implies that in the presence of media coverage, corruption is removed out faster while ineffective media reporting on the transmission and control of the corruption measures greatly increases the number of corrupt individuals in the population. Our model has not conducted out optimal control and cost effectiveness of different corruption intervention strategies, which can be investigated in future to find out which strategy is the best in the control of the corruption.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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