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## DYNAMICS OF A DELAYED PREY-PREDATOR MODEL WITH HATTAF-YOUSFI FUNCTIONAL RESPONSE

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**Abstract.** In this paper, we propose a delayed prey-predator model with Hattaf-Yousfi functional response. We first show that our proposed model is mathematically and ecologically well-posed. The dynamical behaviors of the model are studied by establishing the local stability of equilibria and the existence of Hopf bifurcation. Furthermore, the theoretical results are validated by numerical simulations.

**Keywords:** ecology; prey-predator; stability; Hopf bifurcation; Hattaf-Yousfi functional response.

**2010 AMS Subject Classification:** 34D20, 37G10, 92D25.

### 1. INTRODUCTION

Ecology is a sub-discipline of environmental science that studies the interactions of living beings with each other and with their environment. The mathematical modeling in ecology can be describe the dynamics of these interactions in order to avoid the extinction of certain species as well as to protect ecosystems, biodiversity and the environment in general.

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Prey-predator models are widely used in ecology to describe the three main types of interactions that are predation, competition and mutualism or symbiosis. One of the first of these models was introduced by Lotka and Volterra [1, 2, 3]. Recently, Tripathi et al. [4] presented and studied a two-dimensional continuous time dynamical system modeling a predator-prey with discrete delay incorporating Crowley-Martin functional response. Garain et al. [5] considered a prey-predator model with logistic functional response in the prey growth, Beddington-DeAngelis functional response and density dependent death rate for the predator. In 2021, Garain and Mandal [6] provided a prey-predator system including both component Allee effect and density dependent death for predator population. The interaction between prey and predator was modeled in [6] by Holling II functional response. For more details about the three above functional responses, we refer the reader to the works [7, 8, 9, 10].

On the other hand, a recent generalized functional response introduced by Hattaf and Yousfi [11] includes the three functional responses of types Crowley-Martin, Beddington-DeAngelis and Holling II as well as the other functional responses existing in the literature. Therefore, in this paper, we propose a delayed prey-predator model with Hattaf-Yousfi functional response. This model is given by the following nonlinear system

$$(1) \quad \begin{cases} \frac{dX(t)}{dt} = rX(t)\left(1 - \frac{X(t)}{K}\right) - \frac{aX(t)Y(t)}{\alpha_0 + \alpha_1 X(t) + \alpha_2 Y(t) + \alpha_3 X(t)Y(t)}, \\ \frac{dY(t)}{dt} = \frac{abX(t-\tau)Y(t-\tau)}{\alpha_0 + \alpha_1 X(t-\tau) + \alpha_2 Y(t-\tau) + \alpha_3 X(t-\tau)Y(t-\tau)} - cY(t) - dY^2(t), \end{cases}$$

where  $X(t)$  and  $Y(t)$  denote the prey and predator densities at time  $t$ , respectively. The parameter  $r$  is the prey intrinsic growth rate;  $K$  is the environmental carrying capacity for prey;  $b$  is the conversion rate of prey to predator;  $c$  is the death rate of predator;  $d$  is the rate of competition between predators and  $\tau$  is a time delay that represents the gestation period of predators. The interaction between prey and predator is modeled by Hattaf-Yousfi functional response of the form  $\frac{aXY}{\alpha_0 + \alpha_1 X + \alpha_2 Y + \alpha_3 XY}$ , where  $a$  represents the rate of prey capture by the predator called also consumption rate, and  $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \geq 0$  are the saturation factors measuring the inhibitory or psychological effect.

It is important to note that our model (1) improves and generalizes two recent special cases existing in the literature. More precisely:

- When  $\alpha_3 = \alpha_1 \alpha_2$ , we get the model of Tripathi et al. [4] that used Crowley-Martin functional response.
- When  $\alpha_1 = 1$ ,  $\alpha_3 = 0$  and  $\tau = 0$ , we obtain the model of Garain et al. [5] that used Beddington-DeAngelis functional response.

The rest of this paper is organized as follows. The next section deals with the positivity and the boundedness of the solutions as well as the existence of equilibria. In section 3, we investigate the stability analysis and Hopf bifurcation. In section 4, we validate our theoretical results by numerical simulations. Finally, a brief conclusion ends the paper.

## 2. PROPERTIES OF SOLUTIONS AND EQUILIBRIA

In this section, we show the positivity and the boundedness of the solutions as well as the existence the equilibria of system (1).

**Theorem 1.** *All solutions of system (1) starting from nonnegative initial conditions remain positive and bounded for all  $t \geq 0$ .*

*Proof.* From (1), we have

$$\begin{aligned} X(t) &= X(0)e^{\left[\int_0^t \left\{ r \left( 1 - \frac{X(s)}{K} \right) - \frac{aY(s)}{\alpha_0 + \alpha_1 X(s) + \alpha_2 Y(s) + \alpha_3 X(s)Y(s)} \right\} ds\right]}, \\ Y(t) &= Y(0)e^{\left[\int_0^t \left\{ \frac{abX(s-\tau)Y(s-\tau)}{Y(s)(\alpha_0 + \alpha_1 X(s-\tau) + \alpha_2 Y(s-\tau) + \alpha_3 X(s-\tau)Y(s-\tau))} - c - dY(s) \right\} ds\right]}, \end{aligned}$$

which implies that  $X(t)$  and  $Y(t)$  are nonnegative.

To investigate the boundedness of solutions. We consider the following function

$$N(t) = \frac{1}{K}X(t - \tau) + \frac{1}{bK}Y(t).$$

Then

$$\begin{aligned} \frac{dN(t)}{dt} &= \frac{r}{K}X(t - \tau) \left( 2 - \frac{1}{K}X(t - \tau) \right) + \frac{d}{bK}Y(t) \left( \frac{r-c}{d} - Y(t) \right) \\ &\quad - r \left( \frac{1}{K}X(t - \tau) + \frac{1}{bK}Y(t) \right), \\ &\leq r \left( 1 + \frac{r(1 - \frac{c}{r})^2}{4dbK} \right) - rN(t). \end{aligned}$$

Hence,  $\limsup_{t \rightarrow +\infty} N(t) \leq 1 + \frac{r(1-\frac{\epsilon}{r})^2}{4dbK}$ , which implies that  $X(t)$  and  $Y(t)$  are bounded.  $\square$

Next, we discuss the existence of equilibria of system (1). It is not hard to see that system (1) has two equilibria  $E^0(0,0)$  and  $E^1(K,0)$ . Biologically,  $E^0(0,0)$  denotes the trivial equilibrium with absence of prey and predator and  $E^1(K,0)$  represents the predator free axial equilibrium called also the prey population reaches in the carrying capacity in the absence of predators. The remaining other equilibria satisfy the following equations:

$$(2) \quad r \left( 1 - \frac{X}{K} \right) - \frac{aY}{\alpha_0 + \alpha_1 X + \alpha_2 Y + \alpha_3 XY} = 0,$$

$$(3) \quad \frac{abX}{\alpha_0 + \alpha_1 X + \alpha_2 Y + \alpha_3 XY} - c - dY = 0.$$

From (2), we get

$$Y = \frac{r \left( 1 - \frac{1}{K} X \right) (\alpha_0 + \alpha_1 X)}{a - r \left( 1 - \frac{1}{K} X \right) (\alpha_2 + \alpha_3 X)}.$$

From (3), we obtain

$$(4) \quad (c + dY)(\alpha_0 + \alpha_1 X + \alpha_2 Y + \alpha_3 XY) = abX.$$

By replacing  $Y$  into Eq. (4), we get the following equation

$$(5) \quad AX^5 + BX^4 + CX^3 + DX^2 + EX + F = 0,$$

where

$$\begin{aligned} A &= \frac{ab\alpha_3^2 K^3}{r\alpha_0^3}, \\ B &= \frac{2ab\alpha_3 K^2}{r^2\alpha_0^3} (r\alpha_2 - a), \\ C &= \frac{a\alpha_1 K}{r^2\alpha_0^3} (d\alpha_1 K - c\alpha_3 K) + \frac{abK}{r^2\alpha_0^3} (r\alpha_2^2 + r\alpha_3^2 K^2 + 2a\alpha_3 K - 4r\alpha_2\alpha_3 K), \\ D &= \frac{a}{r^2\alpha_0^3} (-c\alpha_0\alpha_3 K + c\alpha_1 K(\alpha_3 K - \alpha_2)) - \frac{ad\alpha_1 K}{r^2\alpha_0^3} (\alpha_1 K + 2\alpha_0) \\ &\quad + \frac{2abK}{r^2\alpha_0^3} (a\alpha_2 - a\alpha_3 K - r\alpha_2^2 + r\alpha_2\alpha_3 K), \end{aligned}$$

$$\begin{aligned}
E &= \frac{ac}{r\alpha_0^3}(\alpha_3K - \alpha_2) + \frac{ac\alpha_1K}{r^3\alpha_0^3}(r\alpha_2 - a) - \frac{ad}{r^2\alpha_0^2}(\alpha_0 + 2\alpha_1K) \\
&\quad + \frac{abK}{r^3\alpha_0^3}(a^2 - r^2\alpha_2^2 - 2ra\alpha_2), \\
F &= \frac{ac}{r^3\alpha_0^2}(r\alpha_2 - a) - \frac{ad}{r^2\alpha_0}.
\end{aligned}$$

Consider the following function

$$f(X) = AX^5 + BX^4 + CX^3 + DX^2 + EX + F.$$

Since  $A > 0$ , we have  $\lim_{x \rightarrow +\infty} f(x) = +\infty$ . In addition, we have  $f(0) = F < 0$  if  $\alpha_2 < \frac{a}{r}$ . Then there exists a  $X^* \in (0, +\infty)$  such that  $f(X^*) = 0$ . Therefore, system (1) has a unique interior coexistence equilibrium  $E^*(X^*, Y^*)$ , where  $X^* \in (0, +\infty)$  and  $Y^* = \frac{r(1 - \frac{1}{K}X^*)(\alpha_0 + \alpha_1X^*)}{a - r(1 - \frac{1}{K}X^*)(\alpha_2 + \alpha_3X^*)}$ .

### 3. STABILITY ANALYSIS AND HOPF BIFURCATION

In this section, we investigate the stability analysis and the existence of Hopf bifurcation.

Let  $E(X, Y)$  be an arbitrary equilibrium of system (1). The characteristic equation at  $E$  is given by

$$(6) \quad \begin{vmatrix} r\left(1 - \frac{2X}{K}\right) - \frac{aY(\alpha_0 + \alpha_2Y)}{(\alpha_0 + \alpha_1X + \alpha_2Y + \alpha_3XY)^2} - \lambda & -\frac{aX(\alpha_0 + \alpha_1X)}{(\alpha_0 + \alpha_1X + \alpha_2Y + \alpha_3XY)^2} \\ \frac{abY(\alpha_0 + \alpha_2Y)}{(\alpha_0 + \alpha_1X + \alpha_2Y + \alpha_3XY)^2} e^{-\lambda\tau} & -c - 2dY + \frac{abX(\alpha_0 + \alpha_1X)}{(\alpha_0 + \alpha_1X + \alpha_2Y + \alpha_3XY)^2} e^{-\lambda\tau} - \lambda \end{vmatrix} = 0.$$

**Theorem 2.** *The equilibrium  $E^0(0, 0)$  is unstable.*

*Proof.* At  $E^0(0, 0)$ , the characteristic equation (6) reduces to

$$(7) \quad (\lambda - r)(c + \lambda) = 0,$$

where the roots of Eq. (7) are  $\lambda_1 = r > 0$  and  $\lambda_2 = -c < 0$ , which implies that  $E^0(0, 0)$  is unstable.  $\square$

**Theorem 3.** *Let  $R_0 = \frac{abK}{c(\alpha_0 + \alpha_1K)}$ . The equilibrium  $E^1(K, 0)$  is locally asymptotically stable for any time delay  $\tau \geq 0$  if  $R_0 < 1$ , and becomes unstable if  $R_0 > 1$ .*

*Proof.* At  $E^1(K, 0)$ , the characteristic equation (6) becomes

$$(8) \quad (r + \lambda)(\lambda + c(1 - R_0e^{-\lambda\tau})) = 0.$$

Obviously,  $\lambda = -r < 0$  is a root of (8). The remaining roots are provided by solving the following equation:

$$(9) \quad \lambda + c(1 - R_0 e^{-\lambda\tau}) = 0.$$

Assume  $R_0 < 1$ . For  $\tau = 0$ , we have  $\lambda = c(R_0 - 1) < 0$ . Then  $E^1(K, 0)$  is locally asymptotically stable.

For  $\tau \neq 0$ , let  $\lambda = i\omega$  ( $\omega > 0$ ) be a root of (8). Then

$$c + i\omega = cR_0 e^{-i\omega\tau},$$

which leads to

$$\begin{cases} c = cR_0 \cos \omega\tau, \\ \omega = -cR_0 \sin \omega\tau. \end{cases}$$

This implies that

$$(10) \quad \omega^2 + c^2(1 - R_0^2) = 0.$$

Thus, Eq. (10) has no positive root if  $R_0 < 1$ . Therefore,  $E^1(K, 0)$  is locally asymptotically stable for  $R_0 < 1$ .

For  $R_0 > 1$ , we consider the following function

$$g(\lambda) = \lambda + c(1 - R_0 e^{-\lambda\tau}).$$

We have  $g(0) = c(1 - R_0) < 0$  and  $\lim_{\lambda \rightarrow +\infty} g(\lambda) = +\infty$ . Then there exists a  $\bar{\lambda} \in (0, +\infty)$  such that  $g(\bar{\lambda}) = 0$ . Hence, Eq. (8) has at least one positive eigenvalue when  $R_0 > 1$ . Thus,  $E^1(K, 0)$  is unstable. This completes the proof.  $\square$

Now, for the positive equilibrium point  $E^*(X^*, Y^*)$ , the characteristic equation of (6) is given by

$$(11) \quad \lambda^2 + a_1\lambda + a_2 + (b_1\lambda + b_2)e^{-\lambda\tau} = 0,$$

where

$$\begin{aligned} a_1 &= -r \left( 1 - \frac{2X^*}{K} \right) + \frac{aY^*(\alpha_0 + \alpha_2Y^*)}{(\alpha_0 + \alpha_1X^* + \alpha_2Y^* + \alpha_3X^*Y^*)^2} + c + 2dY^*, \\ a_2 &= \left( -r \left( 1 - \frac{2X^*}{K} \right) + \frac{aY^*(\alpha_0 + \alpha_2Y^*)}{(\alpha_0 + \alpha_1X^* + \alpha_2Y^* + \alpha_3X^*Y^*)^2} \right) (c + 2dY^*), \\ b_1 &= \frac{-abX^*(\alpha_0 + \alpha_1X^*)}{(\alpha_0 + \alpha_1X^* + \alpha_2Y^* + \alpha_3X^*Y^*)^2}, \\ b_2 &= r \left( 1 - \frac{2X^*}{K} \right) \frac{abX^*(\alpha_0 + \alpha_1X^*)}{(\alpha_0 + \alpha_1X^* + \alpha_2Y^* + \alpha_3X^*Y^*)^2}. \end{aligned}$$

When  $\tau = 0$ , Eq. (11) reduces to

$$(12) \quad \lambda^2 + (a_1 + b_1)\lambda + a_2 + b_2 = 0.$$

Based on Routh-Hurwitz criterion, all the roots of equation (12) have negative real parts if and only if

$$(13) \quad a_1 + b_1 > 0 \quad \text{and} \quad a_2 + b_2 > 0.$$

Therefore, we have the following result.

**Lemma 1.** *For  $\tau = 0$ , the equilibrium  $E^*$  is locally asymptotically stable if the condition (13) holds.*

When  $\tau \neq 0$ , putting  $\lambda = i\omega$  into equation (11) and separating real and imaginary parts, we get

$$(14) \quad \omega^2 - a_2 = b_1 \omega \sin \omega \tau + b_2 \cos \omega \tau,$$

$$(15) \quad a_1 \omega = -b_1 \omega \cos \omega \tau + b_2 \sin \omega \tau,$$

which implies that

$$(16) \quad \omega^4 + (a_1^2 - 2a_2 - b_1^2)\omega^2 + a_2^2 - b_2^2 = 0.$$

Let  $z = \omega^2$ , Eq. (16) becomes

$$(17) \quad h(z) := z^2 + p_1z + p_0 = 0,$$

where  $p_1 = a_1^2 - 2a_2 - b_1^2$  and  $p_0 = a_2^2 - b_2^2$ .

Clearly, Eq. (17) has at least one positive root when  $p_0 < 0$ . Moreover, we have

- If  $p_0 \geq 0, \Delta = p_1^2 - 4p_0 \leq 0$  or  $p_1 > 0$ , then Eq. (17) has no positive roots.
- If  $p_0 \geq 0, \Delta = p_1^2 - 4p_0 \geq 0$  and  $p_1 < 0$ , then Eq. (17) has at least one positive root.

Summarizing the above discussions to the following lemma.

**Lemma 2.**

- (i) If  $p_0 < 0$ , then Eq. (17) has at least one positive root.
- (ii) If  $p_0 \geq 0, \Delta \leq 0$  or  $p_1 > 0$ , then Eq. (17) has no positive roots.
- (iii) If  $p_0 \geq 0, \Delta \geq 0$  and  $p_1 < 0$ , then Eq. (17) has at least one positive root.

Based on the above lemma, we consider the following conditions:

- (a)  $p_0 \geq 0, \Delta \leq 0$  or  $p_1 > 0$ ,
- (b)  $p_0 \geq 0, \Delta \geq 0, p_1 > 0$  and  $z^* \leq 0$ .

**Theorem 4.** *Assume conditions (13) holds. If one of the conditions (a) – (b) is satisfied, then the equilibrium  $E^*$  is locally asymptotically stable for any time delay  $\tau \geq 0$ .*

Next, we show under which conditions the system (1) undergoes a Hopf bifurcation by considering the delay  $\tau$  as a parameter of bifurcation. The necessary condition for a change in stability of the interior equilibrium  $E^*$  is that the characteristic equation (11) has purely imaginary roots. Therefore, to obtain the stability criterion, substituting  $\tau = \hat{\tau}$  and  $\omega = \hat{\omega}$  in (14) and (15), and solving these equations for  $\cos \hat{\omega} \hat{\tau}$  or  $\sin \hat{\omega} \hat{\tau}$ , we get

$$(18) \quad \hat{\tau}_n = \frac{1}{\hat{\omega}} \arccos \left[ \frac{b_2(\hat{\omega}^2 - a_2) - b_1 a_1 \hat{\omega}^2}{b_1^2 \hat{\omega}^2 + b_2^2} \right] + \frac{2\pi n}{\hat{\omega}},$$

where  $n \in \mathbf{N}$ . The transversality condition for the Hopf bifurcation at  $\tau = \hat{\tau}$  is  $\left[ \frac{d\mu}{d\tau} \right]_{\tau=\hat{\tau}} > 0$ .

Let  $\lambda = \mu + i\omega$  the root of equation (11) satisfying  $\mu(\hat{\tau}) = 0$  and  $\omega(\hat{\tau}) = \hat{\omega}$ .

Differentiating both sides of equation (11) with respect to  $\tau$ , we obtain

$$\begin{aligned} M_1 \left[ \frac{d\mu}{d\tau} \right]_{\tau=\hat{\tau}} + M_2 \left[ \frac{d\omega}{d\tau} \right]_{\tau=\hat{\tau}} &= M_3, \\ -M_2 \left[ \frac{d\mu}{d\tau} \right]_{\tau=\hat{\tau}} + M_1 \left[ \frac{d\omega}{d\tau} \right]_{\tau=\hat{\tau}} &= M_4, \end{aligned}$$



where

$$M_1 = a_1 - b_2 \hat{\tau} \cos \hat{\omega} \tau - b_1 \hat{\tau} \hat{\omega} \sin \hat{\omega} \hat{\tau} + b_1 \cos \hat{\omega} \hat{\tau},$$

$$M_2 = -2\hat{\omega} - b_2 \hat{\tau} \sin \hat{\omega} \hat{\tau} + b_1 \sin \hat{\omega} \hat{\tau} + b_1 \hat{\omega} \hat{\tau} \cos \hat{\omega} \hat{\tau},$$

$$M_3 = b_2 \hat{\omega} \sin \hat{\omega} \hat{\tau} - b_1 \hat{\omega} \hat{\tau} \cos \hat{\omega} \hat{\tau},$$

$$M_4 = b_1 \hat{\omega}^2 \sin \hat{\omega} \hat{\tau} b_2 \cos \hat{\omega} \hat{\tau}.$$

Then

$$(19) \quad \left[ \frac{d\mu}{d\tau} \right]_{\tau=\hat{\tau}} = \frac{M_3 M_1 - M_4 M_2}{M_1^2 + M_2^2}.$$

From (19), the transversality condition  $\left[ \frac{d\mu}{d\tau} \right]_{\tau=\hat{\tau}} > 0$  for the occurrence of Hopf bifurcation at  $\tau = \hat{\tau}$  is well satisfied provided  $M_3 M_1 - M_4 M_2 > 0$ . Therefore, we obtain the following result.

**Theorem 5.** *Assume that the condition (13) holds. If either  $p_0 < 0$  or  $p_0 \geq 0$ ,  $\Delta \geq 0$  and  $p_1 < 0$ , then the equilibrium  $E^*$  of system (1) is locally asymptotically stable for  $\tau < \hat{\tau}$  and becomes unstable when  $\tau > \hat{\tau}$ . Moreover, when  $\tau = \hat{\tau}$ , the system (1) undergoes a Hopf bifurcation at  $E^*$  provided  $M_3 M_1 - M_4 M_2 > 0$ .*

#### 4. NUMERICAL SIMULATIONS

In this section, we present the numerical simulations of system (1) to mainly illustrate our theoretical results and to better understand its dynamical behavior from an ecological point of view. We consider system (1) with different values initial conditions satisfying  $X(0), Y(0) > 0$ , and we choose the following data set of system (1):  $r = 1$ ,  $K = 1$ ,  $a = 1$ ,  $b = 1$ ,  $c = 0.2$ ,  $d = 5 \times 10^{-12}$ ,  $\tau = 0$ ,  $\alpha_0 = 1$ ,  $\alpha_1 = 2.1$ ,  $\alpha_2 = 1.1$  and  $\alpha_3 = 0.001$ . From these parameter values, the condition  $a_1 + b_1 = 0.0493 > 0$ ,  $a_2 + b_2 = 0.1371 > 0$  of Lemma 1 is satisfied. Then the interior coexistence equilibrium  $E^* = (0.0564, 0.2661)$  is locally asymptotically stable. Figure 1 demonstrates the above analysis.

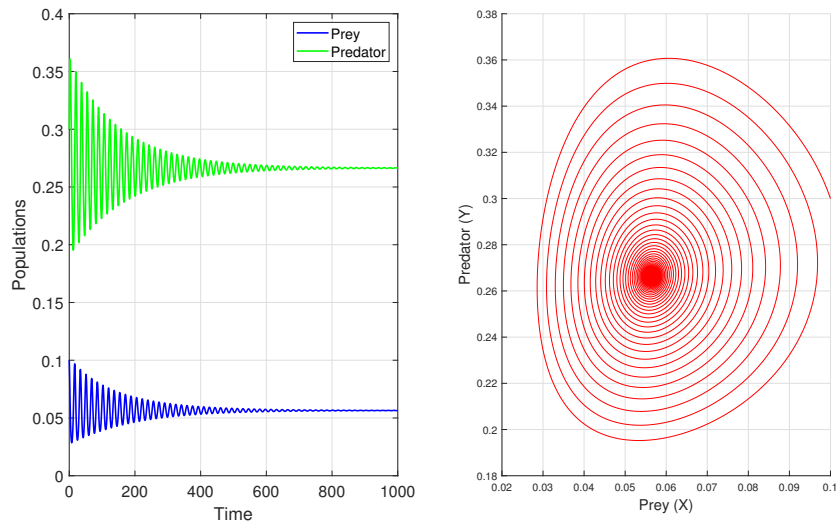


FIGURE 1. Dynamical behavior of system (1) around interior coexistence equilibrium  $E^*$  with  $a = 1$  and  $\tau = 0$ .

When  $a$  is chosen as 9 and other parameters are set as in Figure 1, condition (13) becomes unsatisfied ( $a_1 + b_1 = -0.4597 < 0$ ), while the coexistence equilibrium  $E^*$  loses its stability, Hopf bifurcation occurs, and system (1) exhibits a stable periodic solution. Figure 2 illustrates this result.

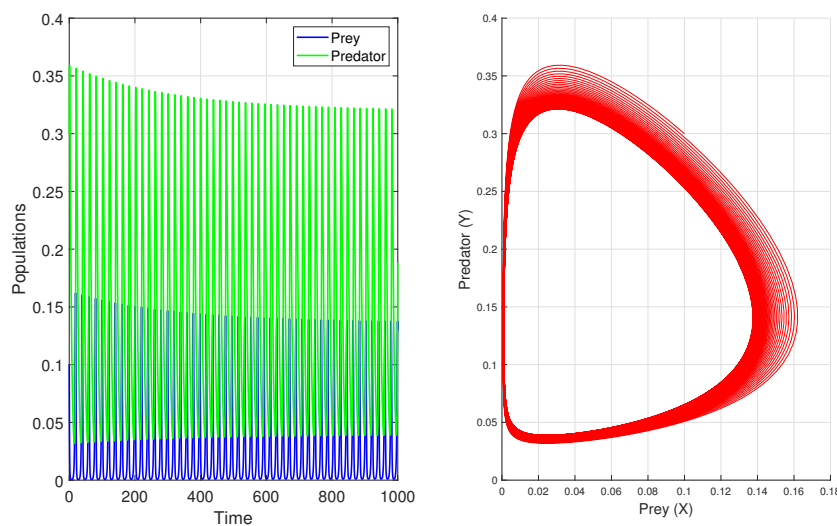


FIGURE 2. Dynamical behavior of system (1) around interior coexistence equilibrium  $E^*$  with  $a = 9$  and  $\tau = 0$ .

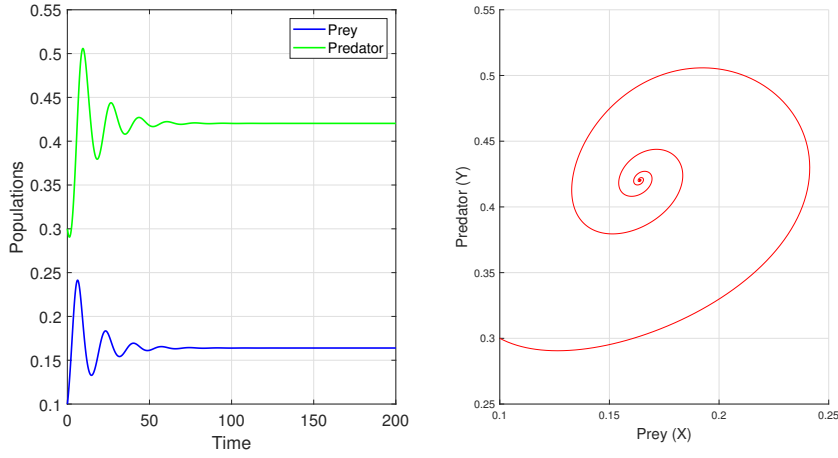


FIGURE 3. Dynamical behavior of system (1) around interior coexistence equilibrium  $E^*$  with  $\tau = 0.005 < \hat{\tau} = 1.9958$ .

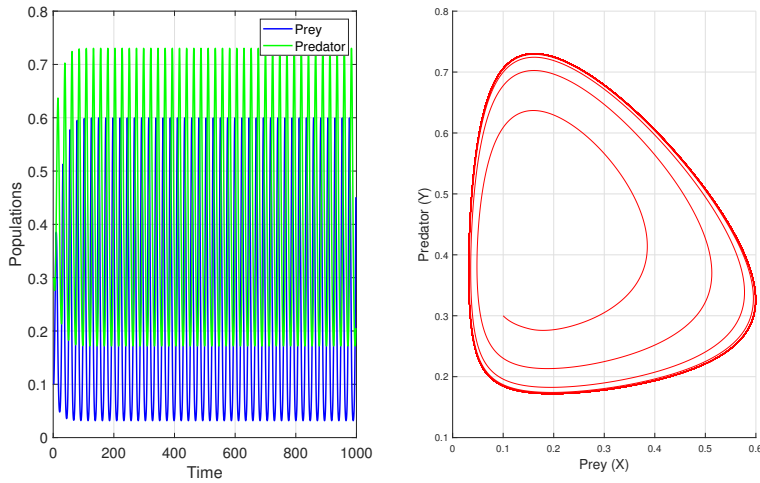


FIGURE 4. Dynamical behavior of system (1) around interior coexistence equilibrium  $E^*$  with  $\tau = 2 > \hat{\tau} = 1.9958$ .

Now, we change some parameter values as  $a = 4.99$ ,  $d = 0.3$ ,  $\alpha_1 = 3.09$ ,  $\alpha_2 = 1.2$ ,  $\alpha_3 = 0.0005$  and  $\tau = 0.005$ . From (16) and (18), we can easily compute  $\omega = 0.3488$  and the critical delay  $\hat{\tau} = 1.9958$ . In addition, conditions (13) and  $p_0 = -0.0328 < 0$  of Theorem 5 are satisfied, which implies that the equilibrium  $E^*$  of system (1) is locally asymptotically stable for  $\tau < \hat{\tau}$  (see, figure 3).

Choosing  $\tau = 2 > \hat{\tau} = 1.9958$ . According to theorem 5, the coexistence equilibrium  $E^* = (0.1639, 0.4203)$  becomes unstable (see, figure 4), and when  $\tau = \hat{\tau} = 1.9958$ , the system (1) undergoes a Hopf bifurcation at the equilibrium  $E^*$ .

## 5. CONCLUSION

In this work, we have proposed and analyzed a delayed prey-predator model with Hattaf-Yousfi functional response. Firstly, we proved that the proposed model is mathematically and ecologically well-posed, and discussed the existence of different possible stationary points (i.e. the trivial equilibrium  $E^0$ , predator free equilibrium  $E^1$  and interior coexistence equilibrium  $E^*$ ). Secondly, we discussed the local stability of the three equilibriums by analyzing the corresponding characteristic equations. We remind that the trivial equilibrium  $E^0$  represents the absence of prey and predator, then this equilibrium is not important in ecology since is always unstable. The predator free axial equilibrium  $E^1$  is locally asymptotically stable if  $R_0 < 1$  and it becomes unstable when  $R_0 > 1$ . In addition, we have established some sufficient conditions for the local asymptotic stability of the interior equilibrium  $E^*$ . Moreover, using the Hopf bifurcation theorem, we have shown that the delay disturbs the stability of  $E^*$  and causes the population to fluctuate. More precisely, from Theorem 5, there is a critical value  $\hat{\tau}$  such that  $E^*$  is conditionally stable in the range  $\tau \in (0, \hat{\tau})$  (see, figure 3), and it becomes unstable in the range  $\tau > \hat{\tau}$  (see, figure 4). However, the Hopf bifurcation is obtained and the periodic solution will occur when  $\tau = \hat{\tau}$ .

On the other hand, the memory is an important characteristic of ecological systems. Also, the Allee effect is a phenomenon in biology characterized by a correlation between population density and per-capita growth rate of a population or species [12]. In future work, both effects will be considered and investigated by means of the new generalized Hattaf fractional (GHF) derivative and its properties[13, 14].

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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