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A DISCRETE MATHEMATICAL MODEL SEIR WITH THE EVOLUTION OF THE REGIONS

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Abstract. In this study, we provide a discrete mathematical *SEIR* model that depicts the evolution of an infectious disease while introducing the novel idea of taking regional infection spread into account. To reduce the disease's ability to spread among people and places, we suggest three control measures. The optimal controls are defined using the Pontryagin maximum principle, and the optimality system is solved using an iterative method. Finally, MATLAB-based numerical simulations are performed to check the results of the theoretical analysis.

Keywords: mathematical model; discrete-time systems; optimal control; contagious virus; SEIR; Pontryagin maximum.

2010 AMS Subject Classification: 92C60.

1. INTRODUCTION

Infectious diseases existed in the Stone Age, during the time of hunters and gatherers. Yet, the transition to a sedentary lifestyle has given rise not only to socio-economic progress [1], but also created an ideal environment for the development of epidemics. Researchers suggest that it was then that malaria, leprosy, tuberculosis, smallpox, diphtheria, measles, plagues and flu

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appeared [2]. And the more civilization developed, the larger cities became and the closer trade relations between different countries became, the closer the threat of a pandemic approached humanity [3]. An epidemic is considered to occur when a contagious disease affects one country or region [4], while in a pandemic, the disease crosses borders and affects many countries or regions [5]. In the history of mankind, epidemics and pandemics covered many countries, claimed thousands to millions of lives, and involved a large number of patients who needed care and treatment.

Historical studies have been provided plenty of spatio-geographical epidemic spread. According to researchers, there have been six pandemics in the last two centuries. In 1918-1919 occurred the infamous Spanish flu measuring 21 millions of deaths worldwide [6]. In 1957-1958 the Asian flu pandemic which appeared within a short space of time in widely separated countries, and claimed about 2 million lives [7]. In 2009 WHO declared a pandemic of H1N1 influenza (swine flu) that spread fast around the world, asserted the lives of about 450 thousand people [8]. In 2002-200", 8,5 thousand people were infected with SARS across the continents [9]. Over decades, HIV steadily expanded across Africa and later into other parts of the world. However, it is known that the virus has existed in the US since at least the mid to late 1970s [10]. The recent virus Covid-19 known as severe acute respiratory syndrome–coronavirus is a new instance of pandemics that have the ability to spread through continents, the pandemic was reported firstly in Wuhan, China, then the outbreak increased greatly and moved to other Chinese cities and multiple countries, moving to other continents [11, 12]. In Europe, France was reported as the first official case of the virus, followed by Germany and Finland, in a short period of time all 27 countries of the European Union were affected [13]. Observations state that more than 4 millions of people were infected in more than 200 countries. This epidemic can cause serious respiratory symptoms and severe diseases such as organ failure and death [14, 15].

Therefore, epidemiological modeling is globally used for detailing the process of epidemic dissemination and providing further apprehension of numerical mechanisms of disease transmission and spread [16]. Thus, the latter came with the need for evaluating intervention strategies for newly emerging and re-emerging pathogens. Mathematical modeling describes a mathematical framework utilizing variables and their interrelationships in order to emphasize certain phenomena or predict future events [17].. This methodology, is used as a tool in scientific disciplines, yet, it doesn't compete with other subjects, physics, or biology; it is used systematically in all spheres of creative activity. Fundamentally, many epidemiological models are recognized to control the development of infectious diseases such as SARS, HIV, Ebola, Influenza, Tuberculosis, Cholera, Measles and others [18, 19][20, 21]. Beside the discrete-time SIR model presented in various geographical regions, to control the spatio-temporal propagation of an epidemic. In contrast to prior models that have primarily concentrated on the optimization of one single region, the multi-regional discrete-time SIR model focuses on the intervention of numerous regions. [22].

Epidemiology, due to some combination of circumstances, has become very popular over the past years. Many people have become interested in modeling epidemics, and more and more people are already aware of mathematical and epidemiological models [23]. SEIR model is considered as a modification of SIR. It represents a whole class of models, which are called "compartment epidemiological models". These models assume that each individual in a population can be in one of several states and move from one state to another over time, and are based on the creation of differential equations describing the dependence of the number of infected individuals on time [24, 25].

The SEIR model devises the population into four classes S-Susceptible (class consists of people who does not receive any infection yet, thus the possibility to get infected is strong when making contact with other members of the population), this model takes into account the incubation period (E-exposed, individuals get sick, but are not contagious and will eventually become completely ill), I-Infected (individuals who have the virus and the capacity to spread it to other), R-Recovered (with immunity those users who were released from the disease, either because of immunity or because of death) [26, 27]. In such model, infection of susceptible

individuals occurs in the same way as in the SIR model, but such individuals do not fall into group I, but into group E. And from E, with a certain probability (the number is the inverse of the incubation period), the transition occurs already in I. During the latency period E, the node is considered infected, hence does not spread the virus. After some time, it becomes capable of infecting other hosts I and then becomes R [28]. The model behaves radically differently depending on the indicator representing the average number of people that one infected person manages to infect during the time until he himself recovers. If this indicator number is less than one, the epidemic subsides; if the indicator is greater than one, a significant part of the population becomes infected. The value of the indicator depends on the characteristics of the virus, the proportion of the population that has received immunity (as a result of vaccination or a past illness), as well as measures to suppress the epidemic (various forms of quarantine) [29, 25]. Modeling shows how fast the epidemic widespread, how many people will be infected, and patients in critical condition. The latter indicator can be compared with the capacity of the medical system and determine whether it is able to cope with the influx of patients in need of specialized care: in the case of COVID-19 for instance, this is resuscitation, artificial ventilation, etc [30].

The combination of various scientifically based methods of epidemic control measures insures the prevention of development of infectious diseases among the most vulnerable groups of the population, reduce the overall incidence in many countries, and even eliminate individual diseases. In case of the COVID-19 epidemic spread, quarantine measures have been implemented around the world beside other control strategies as vaccination, and non-pharmaceutical interventions such as social distancing, public education, and staging of medical equipment. Chinazzi et al proved that travel restrictions introduced in China 2020 slowed rates of epidemic dissemination by 3 to 5 days. However, the most remarkable effect was on the international scale [31]. In [32] a deterministic SEIR model was developed to evaluate the impact of international air travel restrictions in the influenza pandemic. Other models have shown the role of Travel restrictions in decreasing the influx of new infected cases [33][34]. Brownstein, et al. [35] have organized supporting evidence, revealing that the grounding of airliners in the US after September 11/2001 retarded epidemic dynamics during the period of 2001 to 2002 season

by nearly 2 weeks.This control strategy's primary goal is to stop the spread of the infectious disease until specific medical and other control strategies can be created and implemented. Some control systems can be found in the following references [36, 37, 38].

The remaining parts of the paper are arranged as follows. In Section 2, we provide the discrete-time mathematical model *SEIRZSZIZR* that depicts the evolution of a contagious virus and accounts for its spread among people and geographic regions. In addition, we provide a numerical simulation without control to our model. The optimal control problem of the considered model is studied in Section 3. Results and discussion are provided to ensure the effectiveness of the control strategies in Section 4. To conclude our paper, a conclusion is given in Section 5.

2. Presentation of the Model

We consider a simple *SEIR* model [39], with the innovation of taking into account the evolution of regions. The considered model is divided in two parts. The first part *SEIR* describes the evolution of individuals during an infectious disease, where S represents the number of susceptible, E the exposed individuals (Individuals get sick, but are not contagious and will eventually become completely ill), I the infected, and R the recovered. The second part Z represents the different types of regions, Z^S represents the number of susceptible regions, where there are only susceptible individuals, after visiting an infected person, a susceptible region is likely to be infected, which we will note Z^I, the last compartment Z^R designates the infected regions, which are recovered. We obtain the following two models: 1 for the evolution of individuals, and 2 for the evolution of regions.

$$\begin{cases} S_{i+1} &= \Lambda + S_i + - \alpha S_i I_i - \mu S_i \\ E_{i+1} &= E_i + \alpha S_i I_i - (\mu + \delta) E_i \\ I_{i+1} &= I_i + \delta E_i - (\mu + d) I_i - r I_i \\ R_{i+1} &= -\mu R_i + r I_i + R_i \end{cases} \tag{1}$$

$$\begin{cases} Z^S_{i+1} &= Z^S_i - \beta Z^S_i I_i + \theta Z^R_i \\ Z^I_{i+1} &= Z^I_i + \beta Z^S_i I_i - \gamma Z^I_i \\ Z^R_{i+1} &= Z^R_i + \gamma Z^I_i - \theta Z^R_i \end{cases} \tag{2}$$

with initial conditions $S_0 \geq 0, E_0, I_0 \geq 0, R_0 \geq 0, Z_0^S \geq 0, Z_0^I \geq 0$ and $Z_0^R \geq 0$ and where $i \in \{0, 1, ..., N-1\}$.

Parameters description can be found in Table 1.

Parameter	Physical interpretation
Λ	The incidence of susceptible
α	The rate of people who were infected by contact with infected.
β	The rate of regions which become infected since infected people.
δ	The rate of individual exposures will have symptoms and become infected.
θ	The rate of convertion from recovered regions to susceptible regions.
μ	The natural death rate.
d	Mortality since the virus
r	The rate of individuals who were recovered from the disease.

TABLE 1. List of all parameters of systems (1) and (2)

2.1. Simulation without control. In this section, we are interested in giving a simulation of the people and regions infected and recovered in 50 weeks.

Through Figures 1, 3, 2 and 4, we see a significant increase in the number of infected people and infected regions, and a slight increase in the number of those recovered, which will make us propose three control strategies in the next section.

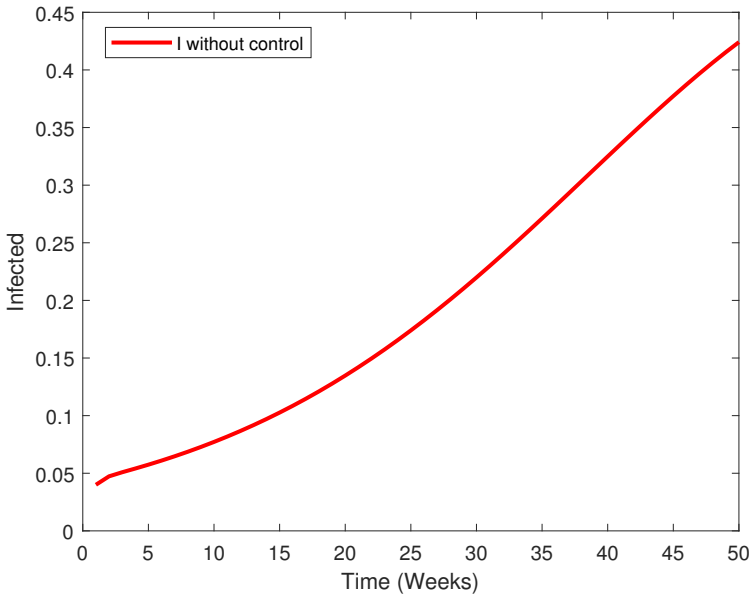

FIGURE 1. Infected individuals without controls

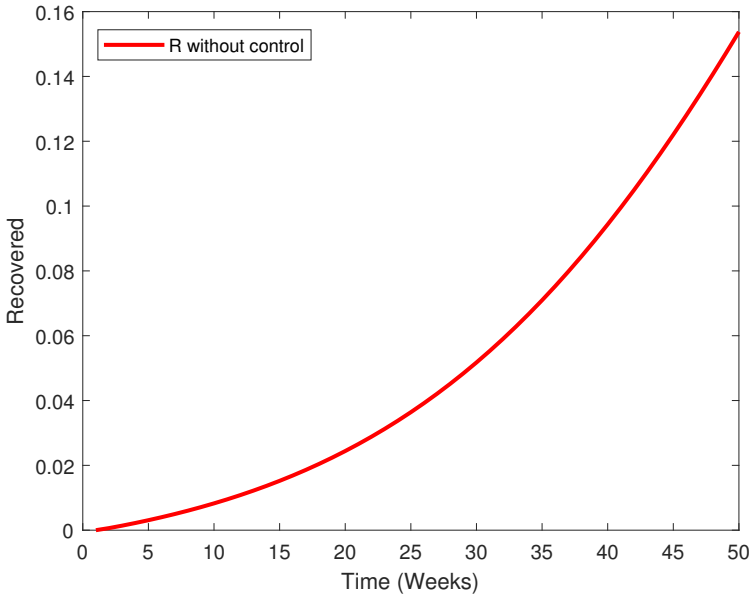

FIGURE 2. Recovered individuals without controls

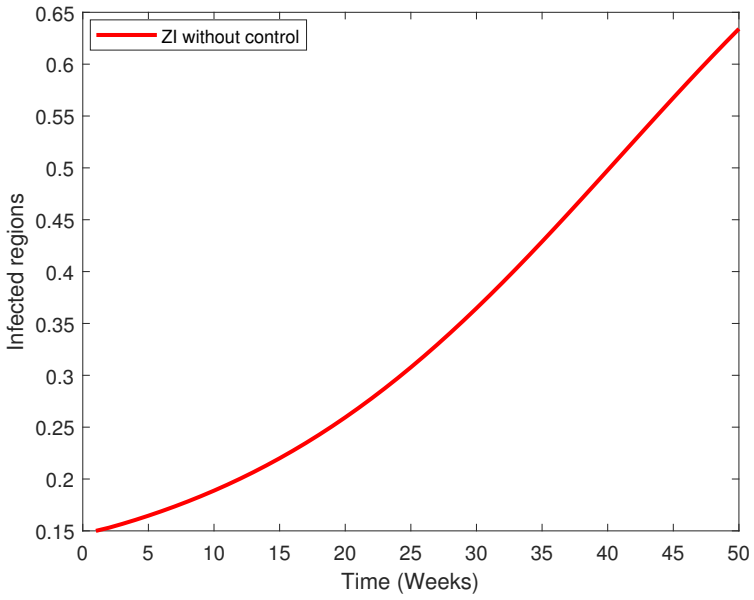

FIGURE 3. Infected regions without without controls

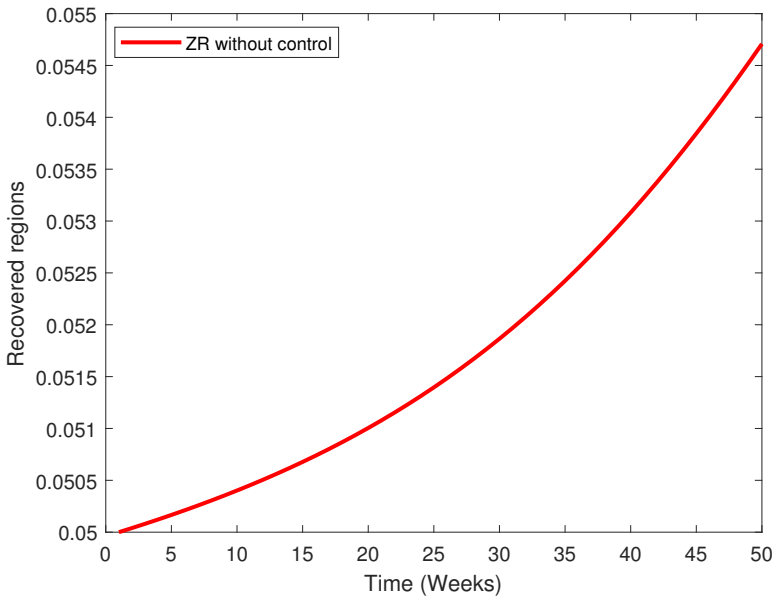

FIGURE 4. Recovered regions without controls

3. THE OPTIMAL CONTROL PROBLEM

3.1. Presentation of the controls. The discrete-time controlled systems associated with (1) and (2) are given as follows

$$\begin{cases} S_{i+1} &= \Lambda + S_i + -\alpha S_i I_i - \mu S_i - v_i S_i \\ E_{i+1} &= E_i + \alpha S_i I_i - (\mu + \delta) E_i \\ I_{i+1} &= I_i + \delta E_i - (\mu + d) I_i - r I_i - u_i J_i \\ R_{i+1} &= -\mu R_i + r I_i + u_i I_i + v_i S_i + R_i \end{cases} \tag{3}$$

$$\begin{cases} Z^s_{i+1} &= Z^s_i - \beta w_i Z^S_i I_i + \theta Z^R_i \\ Z^i_{i+1} &= Z^I_i + \beta w_i Z^S_i I_i - \gamma Z^I_i \\ Z^R_{i+1} &= Z^R_i + \gamma Z^I_i - \theta Z^R_i \end{cases} \tag{4}$$

with initial conditions $S_0 \geq 0, I_0 \geq 0, R_0 \geq 0, Z^S_0 \geq 0, Z^I_0 \geq 0$ and $Z^R_0 \geq 0$ and where $i \in \{0,1,...,N-1\}$.

The control strategy u represents treatment and medical care for infected people. The control strategy v represents vaccination of susceptible individuals and thus protects them from the virus. The control strategy w represents the blocking of travel where we block the meeting between susceptible and infected individuals and the visit of infected individuals to susceptible areas.

3.2. Objective functional. The goal is to minimize the objective function $\mathscr{J}(u,v,w)$ defined by

$$\mathscr{J}(u,v,w) = (xI_N - yR_N + zZ^I_N) + \sum_{i=0}^{N-1} \left(xI_i - yR_i + zZ^I_i + \frac{A}{2}(u_i)^2 + \frac{B}{2}(v_i)^2 + \frac{C}{2}(w_i)^2 \right) \tag{5}$$

where $A > 0, B > 0, C > 0, x > 0, y > 0, z > 0$ are the weight constants of controls, $u = (u_0,...,u_{N-1})$,$v = (v_0,...,v_{N-1})$ and $w = (w_0,...,w_{N-1})$, and N is the final time of our strategy of control. Our objectives are to reduce the number of infected people and infected areas, reduce the cost of applying controls, and increase the number of removed. To put it another

way, we're looking for optimal controls u^*, v^* and w^* such that

$$\mathscr{J}(u^*,v^*,w^*) = min\{\mathscr{J}(u,v,w)/u \in U, v \in V, w \in W\} \tag{6}$$

where U, V and W are the control sets defined by

$$U = \{u/u_{min} \leq u_i \leq u_{max}, i = 0,...,N-1\} \tag{7}$$

$$V = \{v/v_{min} \leq v_i \leq v_{max}, i = 0,...,N-1\} \tag{8}$$

$$W = \{w/w_{min} \leq w_i \leq w_{max}, i = 0,...,N-1\} \tag{9}$$

such that $0 < u_{min} < u_{max} < 1$, $0 < v_{min} < v_{max} < 1$ and $0 < w_{min} < w_{max} < 1$.

3.3. Sufficient conditions. In this paragraph we present the theorem that ensures the existence of an optimal control.

Theorem 3.1. *There exists an optimal control* $(u^*,v^*,w^*) \in U \times V \times W$ *such that*

$$J(u^*,v^*,w^*) = min\{\mathscr{J}(u,v,w)/u \in U, v \in V, w \in W\}$$

subject to the controlled systems (3) , (4) and initial conditions.

Proof. Since the parameters of the system are bounded and there are a finite number of time steps, that is S, E, I, R, Z^S, Z^I and Z^R are uniformly bounded for all (u,v,w) in the control set $U \times V \times W$, thus $J(u,v,w)$ is also bounded for all $(u,v,w) \in U \times V \times W$. Which implies that $\inf_{(u,v,w)\in U\times V\times W} J(u,v,w)$ is finite, and there exists a sequence $(u^n,v^n,w^n) \in U \times V \times W$ such that

$$\lim_{n\to+\infty} J(u^n,v^n,w^n) = \inf_{(u,v,w)\in U\times \mathscr{V}\times \mathscr{W}} J(u,v,w)$$

and corresponding sequences of states I^n, S^n, R^n and Z^{Sn}, Z^{In}, Z^{Rn}. Since there is a finite number of uniformly bounded sequences, there exists $(u^*,v^*,w^*) \in U \times V \times W$ and S^*, E^*, I^* R^* and Z^{S*}, Z^{I*}, Z^{R*} such that, on a sequence,

$$(u^n,v^n,w^n) \to (u^*,v^*,w^*)$$

$$S^n \to S^*$$

$$E^n \to E^*$$

$$I^n \to I^*$$

$$R^n \to R^*$$

$$Z^{Sn} \to Z^{S*}$$

$$Z^{In} \to Z^{I*}$$

$$Z^{Rn} \to Z^{R*}.$$

Finally, due to the finite dimensional structure of the systems (3), (4) and the objective function $J(u,v,w)$, (u^*,v^*,w^*) is an optimal control with corresponding states I^*, S^*, R^*,Z^{S*},Z^{I*} and Z^{R*}. Which complete the proof □

3.4. Necessary conditions. By using a discrete version of the Pontryagin's maximum principle [40], we extract the necessary conditions for our optimal controls. For this goal, we define the Hamiltonian as

$$\begin{aligned}\mathscr{H}_i &= xI_i - yR_i + zZ^I{}_i + \frac{1}{2}Au_i{}^2 + \frac{1}{2}Bv_i{}^2 + \frac{1}{2}Cw_i{}^2 + \lambda_1^{i+1}\left(-\alpha S_iI_i - \mu S_i - v_iS_i + \Lambda + S_i\right)\\ &+\lambda_2^{i+1}\left(\alpha S_iI_i - (\mu+\delta)E_i + E_i\right) + \lambda_3^{i+1}\left(I_i + E_i\delta - (\mu+d)I_i - rI_i - u_iI_i\right)\\ &+\lambda_4^{i+1}\left(-\mu R_i + rI_i + u_iI_i + v_iS_i + R_i\right) + \lambda_5^{i+1}\left(-\beta w_iZ^S{}_iI_i + \theta Z^R{}_i + Z^S{}_i\right)\\ &+\lambda_6^{i+1}\left(\beta w_iZ^S{}_iI_i - \gamma Z^I{}_i + Z^I{}_i\right) + \lambda_7^{i+1}\left(\gamma Z^I{}_i - \theta Z^R{}_i + Z^R{}_i\right)\end{aligned}$$

Theorem 3.2. *Considering the optimal controls u^*, v^*, w^* and solutions S^*, E^* I^*, R^*, Z_i^{S*},Z_i^{I*} and Z_i^{R*}, there exists λ_k^i, $i = 1...N$, $k = 1,2,\cdots,7$ the adjoint variables satisfying the following equations*

$$\begin{aligned}\Delta\lambda_1^i &= -\left[\lambda_1^{i+1}\left(-\alpha I_i - \mu - v_i + 1\right) + \lambda_2^{i+1}\alpha I_i + v_i\lambda_4^{i+1}\right]\\ \Delta\lambda_2^i &= -\left[\lambda_2^{i+1}\left(-\mu - \delta + 1\right) + \lambda_3^{i+1}\delta\right]\\ \Delta\lambda_3^i &= -\left[x - \alpha S_i\lambda_1^{i+1} + \alpha S_i\lambda_2^{i+1} + \lambda_3^{i+1}\left(-d - r - \mu - u_i + 1\right)\right.\\ &\quad\left. + \lambda_4^{i+1}\left(r + u_i\right) - \beta Z^S{}_i\lambda_5^{i+1} + \beta Z^S{}_i\lambda_6^{i+1}w_i\right]\\ \Delta\lambda_4^i &= -\left[-y + \lambda_4^{i+1}\left(-\mu + 1\right)\right]\\ \Delta\lambda_5^i &= -\left[\lambda_5^{i+1}\left(-\beta I_i + 1\right) + \lambda_6^{i+1}\beta I_iw_i\right]\end{aligned}$$

$$\begin{aligned}\Delta\lambda_6^i &= -\left[z+\lambda_6^{i+1}(-\gamma+1)+\gamma\lambda_7^{i+1}\right]\\ \Delta\lambda_7^i &= -\left[\lambda_5^{i+1}\theta+\lambda_7^{i+1}(-\theta+1)\right]\end{aligned}$$

where $\lambda_1^N=0,\lambda_2^N=0,\lambda_3^N=x,\lambda_4^N=-y,\lambda_5^N=0,\lambda_6^N=z,\lambda_7^N=0$ *are the transversality conditions. In addition* $u^*=\left(u_0^*,...,u_{N-1}^*\right)$, $v^*=\left(v_0^*,...,v_{N-1}^*\right)$ *and* $w^*=\left(w_0^*,...,w_{N-1}^*\right)$ *are given by*

$$\begin{aligned}u_i^* &= min\left\{max\left\{u_{min},\frac{I_i\left(\lambda_3^{i+1}-\lambda_4^{i+1}\right)}{A}\right\},u_{max}\right\}\\ &\quad i=0,...,N-1\end{aligned}$$

$$\begin{aligned}v_i^* &= min\left\{max\left\{v_{min},\frac{S_i\left(\lambda_1^{i+1}-\lambda_4^{i+1}\right)}{B}\right\},v_{max}\right\}\\ &\quad i=0,...,N-1\end{aligned}$$

$$\begin{aligned}w_i^* &= min\left\{max\left\{\frac{\beta I_i Z^S{}_i\left(\lambda_5^{i+1}-\lambda_6^{i+1}\right)}{C}\right\},w_{max}\right\}\\ &\quad i=0,...,N-1\end{aligned}$$

Proof. Using the discrete version of the Pontryagin's maximum principle [40], we obtain the following adjoint equations:

$$\begin{aligned}\Delta\lambda_1^i &= -\frac{\partial\mathscr{H}}{\partial S_i}\\ &= -\left[\lambda_1^{i+1}(-\alpha I_i-\mu-v_i+1)+\lambda_2^{i+1}\alpha I_i+v_i\lambda_4^{i+1}\right]\\ \Delta\lambda_2^i &= -\frac{\partial\mathscr{H}}{\partial E_i}\\ &= -\left[\lambda_2^{i+1}(-\mu-\delta+1)+\lambda_3^{i+1}\delta\right]\\ \Delta\lambda_3^i &= -\frac{\partial\mathscr{H}}{\partial I_i}\\ &= -\Big[x-\alpha S_i\lambda_1^{i+1}+\alpha S_i\lambda_2^{i+1}+\lambda_3^{i+1}(-d-r-\mu-u_i+1)\\ &\quad +\lambda_4^{i+1}(r+u_i)-\beta Z^S{}_i\lambda_5^{i+1}+\beta w_i Z_i^S\lambda_6^{i+1}\Big]\\ \Delta\lambda_4^i &= -\frac{\partial\mathscr{H}}{\partial R_i}\\ &= -\left[-y+\lambda_4^{i+1}(-\mu+1)\right]\end{aligned}$$

$$\begin{aligned}
\Delta\lambda_5^i &= -\frac{\partial \mathscr{H}}{\partial Z_i^I} \\
&= -\left[\lambda_5^{i+1}(-\beta I_i + 1) + \lambda_6^{i+1}\beta I_i w_i\right] \\
\Delta\lambda_6^i &= -\frac{\partial \mathscr{H}}{\partial Z_i^I} \\
&= -\left[z + \lambda_6^{i+1}(-\gamma + 1) + \gamma\lambda_7^{i+1}\right] \\
\Delta\lambda_7^i &= -\frac{\partial \mathscr{H}}{\partial Z_i^R} = \\
&= -\left[\lambda_5^{i+1}\theta + \lambda_7^{i+1}(-\theta + 1)\right]
\end{aligned}$$

with $\lambda_3^N = x, \lambda_4^N = -y, \lambda_6^N = z$. To get the optimality conditions, we consider the variation with respect to controls (u_i, v_i, w_i) and set it to zero

$$\frac{\partial \mathscr{H}_i}{\partial u_i} = Au_i - I_i\lambda_3^{i+1} + \lambda_4^{i+1}I_i = 0$$

$$\frac{\partial \mathscr{H}_i}{\partial v_i} = Bv_i - S_i\lambda_1^{i+1} + \lambda_4^{i+1}S_i = 0$$

$$\frac{\partial \mathscr{H}_i}{\partial w_i} = -\lambda_5^{i+1}Z^S{}_i I_i\beta + \lambda_6^{i+1}Z^S{}_i I_i\beta + Cw_i = 0$$

Then we obtain the optimal control

$$u_i = \frac{I_i\left(\lambda_3^{i+1} - \lambda_4^{i+1}\right)}{A}$$

$$v_i = \frac{S_i\left(\lambda_1^{i+1} - \lambda_4^{i+1}\right)}{B}$$

$$w_i = \frac{\beta I_i Z^S{}_i\left(\lambda_5^{i+1} - \lambda_6^{i+1}\right)}{C}$$

By the bounds in U, V and W of the controls in the definitions (7),(8) and (9), it is easy to obtain u_i^*, v_i^* and w_i^* in the following form

$$\begin{aligned}
u_i^* &= min\left\{max\left\{u_{min}, \frac{I_i\left(\lambda_3^{i+1} - \lambda_4^{i+1}\right)}{A}\right\}, u_{max}\right\} \\
&\quad i = 0, \ldots, N-1
\end{aligned}$$

$$\begin{aligned}
v_i^* &= min\left\{max\left\{v_{min}, \frac{S_i\left(\lambda_1^{i+1} - \lambda_4^{i+1}\right)}{B}\right\}, v_{max}\right\} \\
&\quad i = 0, \ldots, N-1
\end{aligned}$$

$$w_i^* = min\left\{max\left\{\frac{\beta I_i Z^S{}_i\left(\lambda_5^{i+1}-\lambda_6^{i+1}\right)}{C}\right\}, w_{max}\right\}$$
$$i = 0,...,N-1$$

□

4. Results and Discussion

4.1. Numerical simulation. The optimal control problem involving the two systems 3 and 4 and the objective function $mathcalJ(u,v,w)$ will be numerically solved in this section. We write the code in MATLAB (See algorithm 1) and we simulate our results. The FBSM method-like discrete iterative discrete method is used to solve the optimality systems and converges after a proper test. The adjoint system is then solved backward in time due to the transversality conditions after the state system is first solved with the initial assumption forward in time. The state and co-state values obtained in the preceding steps are then used to update the optimal control values. The final step is to carry out the preceding actions up until the tolerance standard is reached.

Algorithm 1 Determination of states of the controlled system and controls u, v and w.

REQUIRE: S_0, E_0, I_0, R_0, Z_0^s, Z_0^I, Z_0^R, N, $u(0) = v(0) = w(0) = 0$, $\zeta_{1,N} = 0$, $\zeta_{2,N} = 0$, $\zeta_{3,N} = x^R$, $\zeta_{4,N} = -y$, $\zeta_{5,N} = \alpha^Z$, $\zeta_{6,N} = z$, $\zeta_{7,N} = 0$.

For $i = 1, ..., N-1$

$$\begin{cases} S_{i+1} &= \Lambda + S_i + -\alpha S_i I_i - \mu S_i - v_i S_i \\ E_{i+1} &= E_i + \alpha S_i I_i - (\mu + \delta) E_i \\ I_{i+1} &= I_i + \delta E_i - (\mu + d) I_i - r I_i - u_i J_i \\ R_{i+1} &= -\mu R_i + r I_i + u_i I_i + v_i S_i + R_i \\ Z^s_{i+1} &= Z^s_i - \beta Z^S_i I_i + \theta Z^R_i \\ Z^i_{i+1} &= Z^I_i + \beta w_i Z^S_i I_i - \gamma Z^I_i \\ Z^R_{i+1} &= Z^R_i + \gamma Z^I_i - \theta Z^R_i \end{cases}$$

$$\begin{aligned}
\lambda_1^i &= \left[\lambda_1^{i+1}(-\alpha I_i - \mu - v_i + 2) + \lambda_2^{i+1}\alpha I_i + v_i \lambda_4^{i+1}\right] \\
\lambda_2^i &= \left[\lambda_2^{i+1}(-\mu - \delta + 2) + \lambda_3^{i+1}\delta\right] \\
\lambda_3^i &= \left[x - \alpha S_i \lambda_1^{i+1} + \alpha S_i + \lambda_2^{i+1}\ \lambda_3^{i+1}(-d - r - \mu - u_i + 2)\ \lambda_4^{i+1}(r + u_i) - \beta Z^S{}_i \lambda_5^{i+1} + \beta Z^S{}_i \lambda_6^{i+1} w_i\right] \\
\lambda_4^i &= \left[-y + \lambda_4^{i+1}(-\mu + 2)\right] \\
\lambda_5^i &= \left[\lambda_5^{i+1}(-\beta I_i + 2) + \lambda_6^{i+1}\beta I_i w_i\right] \\
\lambda_6^i &= \left[z + \lambda_6^{i+1}(-\gamma + 2) + \gamma \lambda_7^{i+1}\right] \\
\lambda_7^i &= \left[\lambda_5^{i+1}\theta + \lambda_7^{i+1}(-\theta + 2)\right]
\end{aligned}$$

$$\begin{aligned}
u_i^* &= min\left\{max\left\{u_{min}, \frac{I_i\left(\lambda_3^{i+1} - \lambda_4^{i+1}\right)}{A}\right\}, u_{max}\right\} \\
&\quad i = 0, ..., N-1 \\
v_i^* &= min\left\{max\left\{v_{min}, \frac{S_i\left(\lambda_1^{i+1} - \lambda_4^{i+1}\right)}{B}\right\}, v_{max}\right\} \\
&\quad i = 0, ..., N-1 \\
w_i^* &= min\left\{max\left\{\frac{\beta I_i Z^S{}_i\left(\lambda_5^{i+1} - \lambda_6^{i+1}\right)}{C}\right\}, w_{max}\right\} \\
&\quad i = 0, ..., N-1
\end{aligned}$$

ENDFOR

4.2. Simulation with and without the control u and discussions on the results.

Strategy one: We use only the optimal control u.

Because of the risk of serious complications in people affected by the epidemic, we proposed a strategy based on home treatment for infected people with minor complications and hospitalisation for serious complications. From the figures 5 and 6, we can see that this strategy slightly reduces the number of infected patients and increases the number of recovered persons.

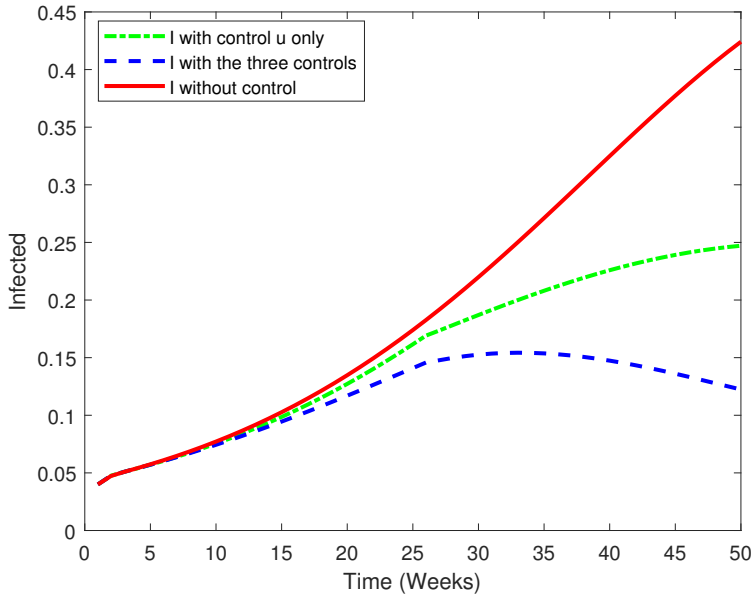

FIGURE 5. Infected individuals without and with the control u

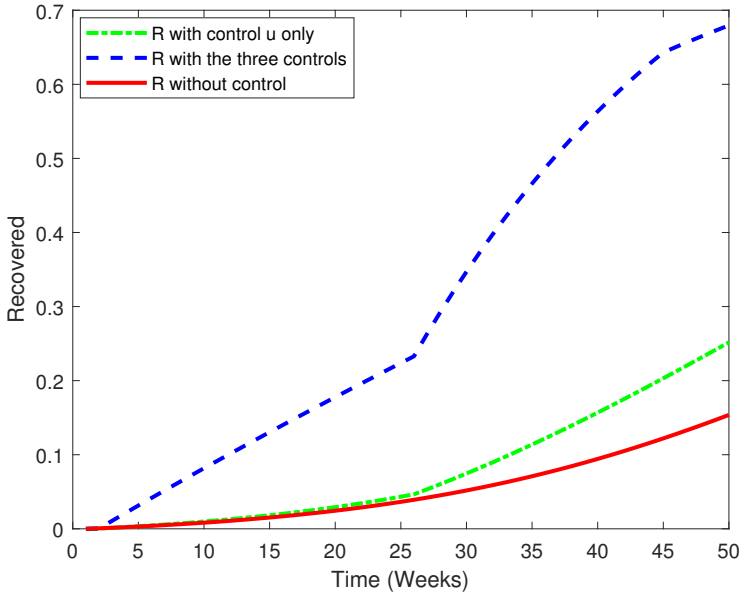

FIGURE 6. Recovered individuals without and with the control u

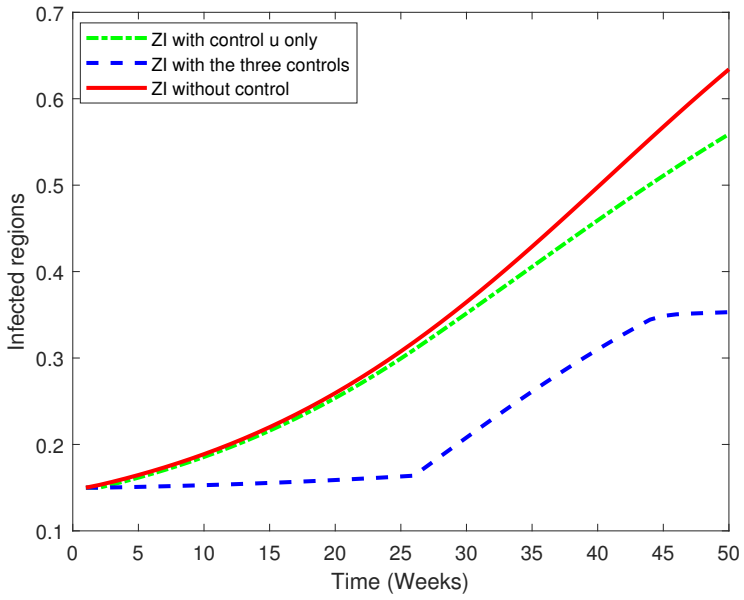

FIGURE 7. Infected regions without and with the control u

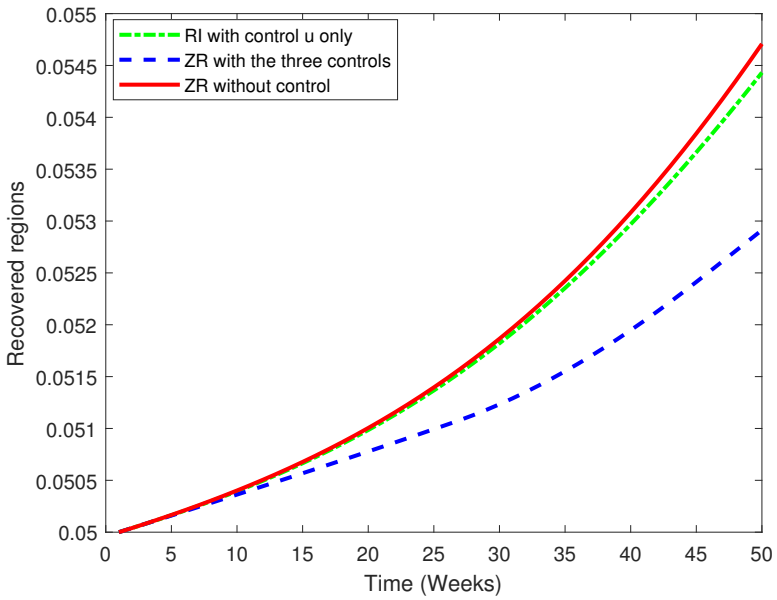

FIGURE 8. Recovered regions without and with the control u

4.3. Simulation with and without the control v and discussions on the results.

Strategy two: We use only the optimal control v.

In this strategy we propose a strategy based on vaccination of the population and protect them

from the virus. We can see from figures 9 and 10 that although this strategy increases the number of recovering and decreases the number of infected, it is still insufficient for reducing the number of infected areas. (see figure 11).

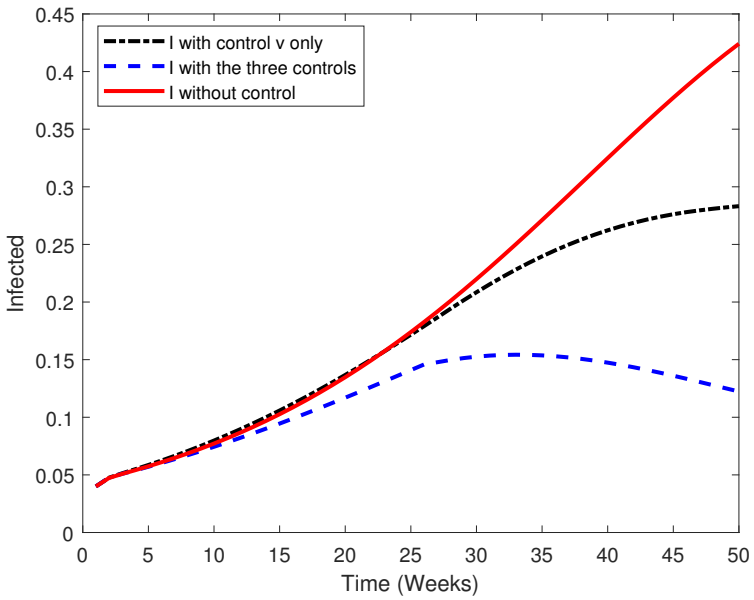

FIGURE 9. Infected individuals without and with the control v

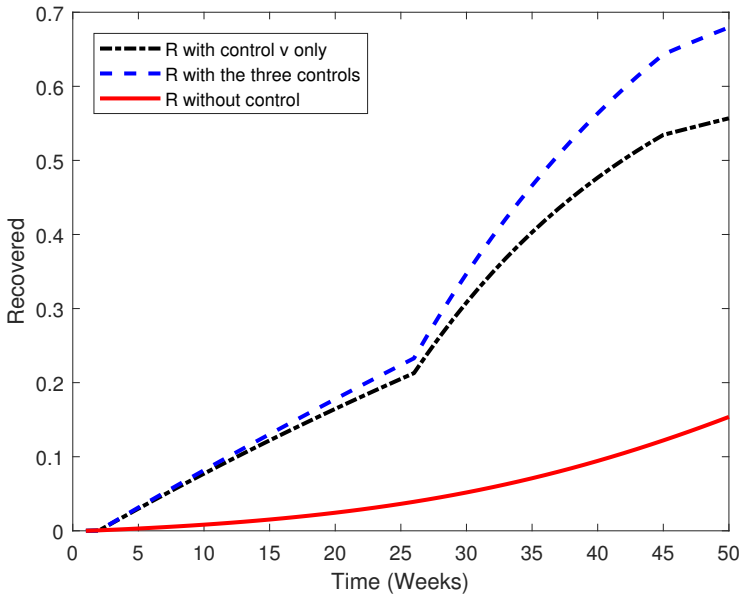

FIGURE 10. Recovered individuals without and with the control v

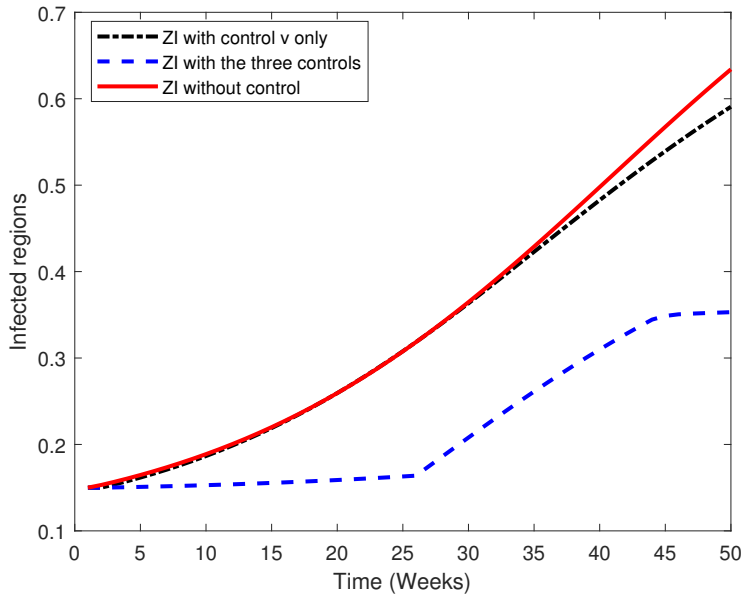

FIGURE 11. Infected regions without and with the control v

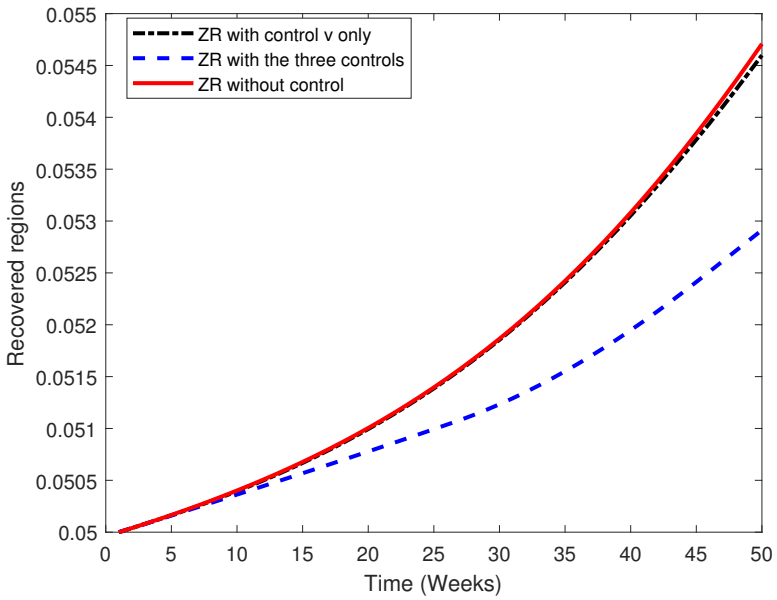

FIGURE 12. Recovered regions without and with the control v

4.4. Simulation with and without the control w and discussions on the results.

Strategy three: We use only the optimal control w.

The control strategy w represents movement blocking where we block the contact between susceptible and infected individuals and the visit of infected individuals to susceptible regions.This strategy proved to be effective in reducing the number of infected regions (see figure 15) the thing that is not obtained by the first two strategies

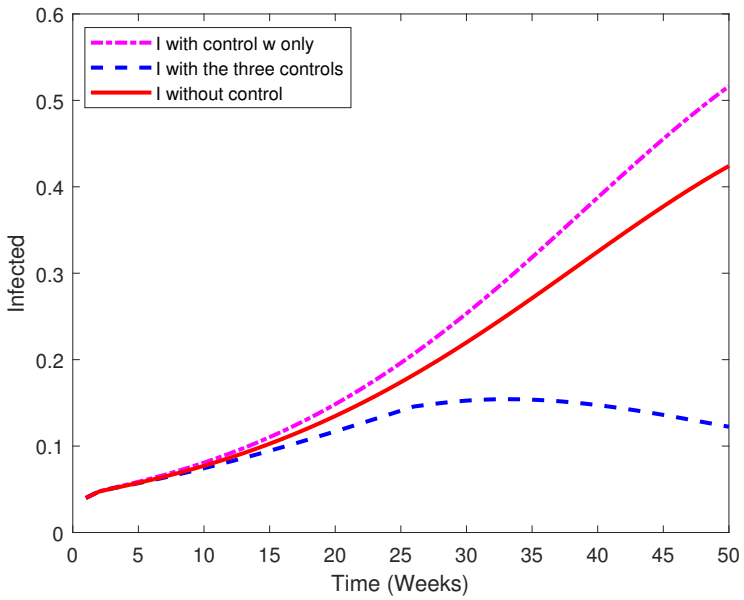

FIGURE 13. Infected individuals without and with the control w

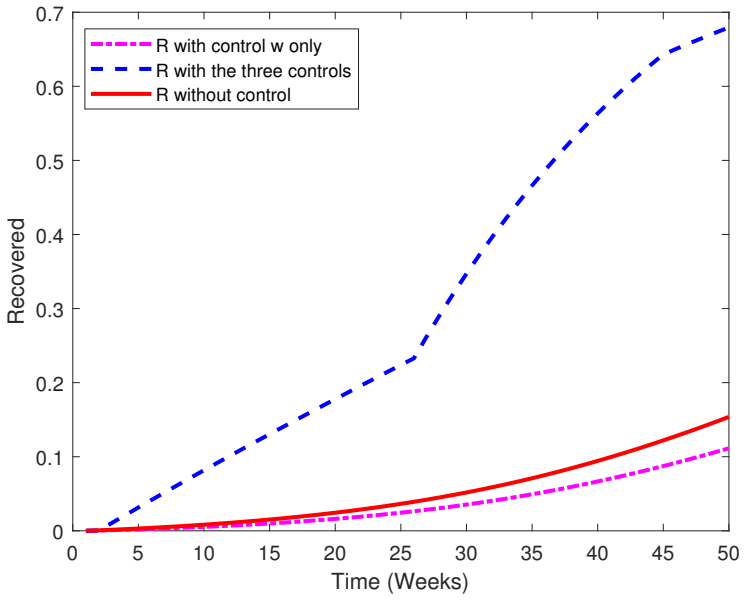

FIGURE 14. Recovered individuals without and with the control w

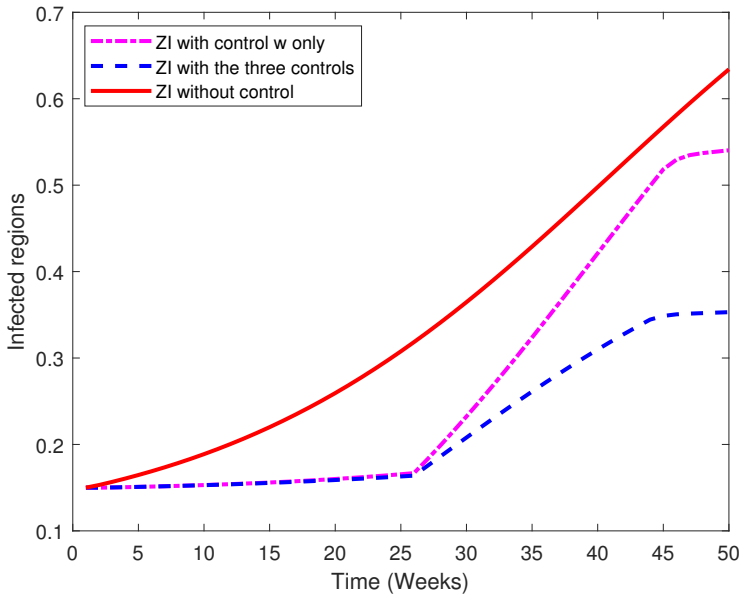

FIGURE 15. Infected regions without and with the control w

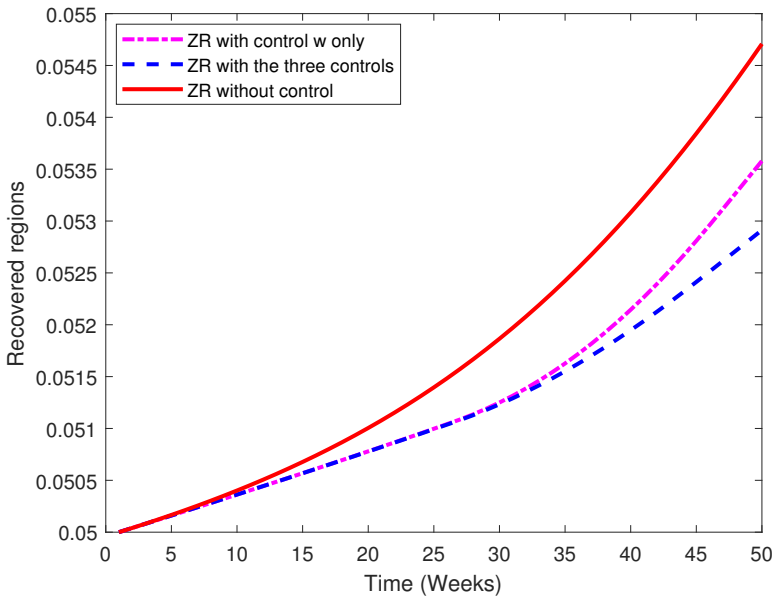

FIGURE 16. Recovered regions without and with the control w

4.5. Simulation with and without the three controls and discussions on the results.

Strategy four : Applying all controls u, v and w.

The objective of this strategy is to combine the previous strategies simultaneously.Figures 17,

18, and 19 show how this strategy decreased the number of infected people and the number of infected areas while increasing the number of recovered people enough to stop the epidemic's spread. This resulted from our strategy's effectiveness as well as our consideration of the (2) systems, which chart the evolution of the areas.

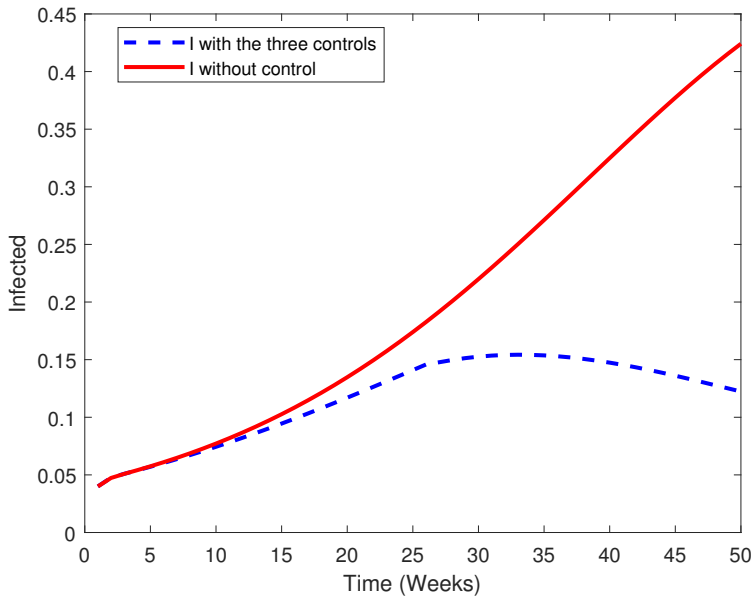

FIGURE 17. Infected individuals without and with the three controls

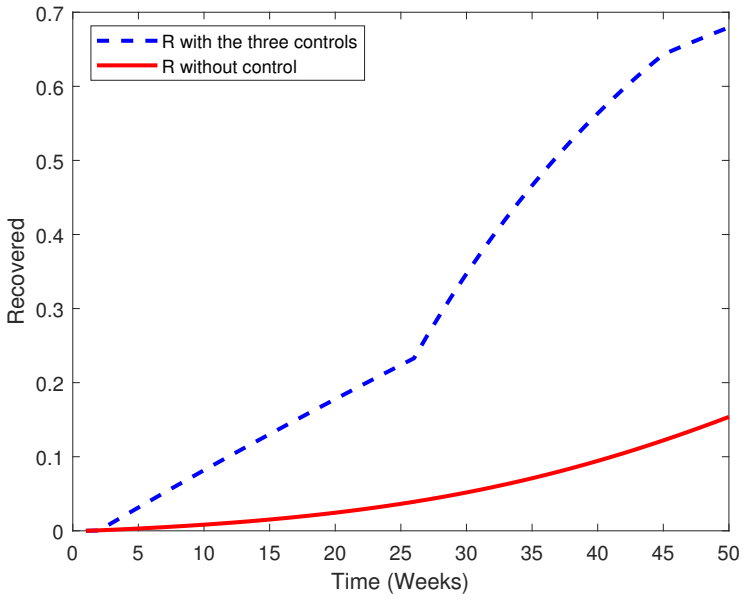

FIGURE 18. Recovered individuals without and with the three controls

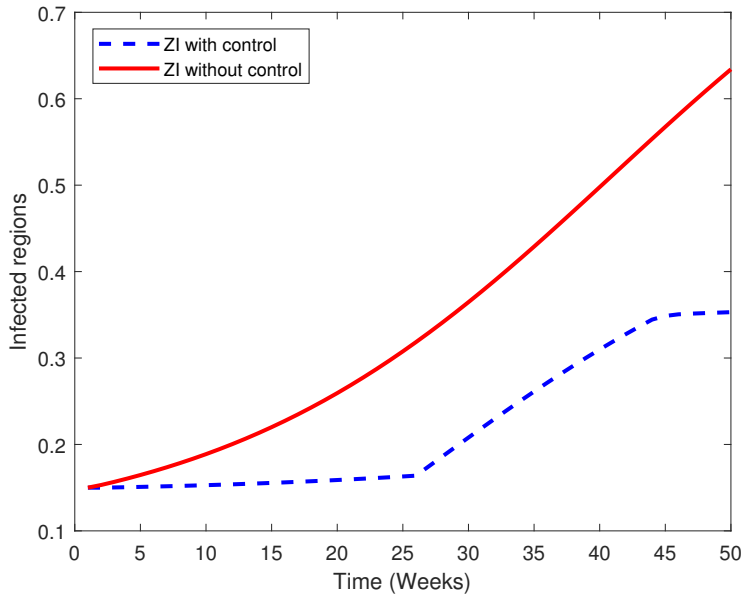

FIGURE 19. Infected regions without and with the three controls

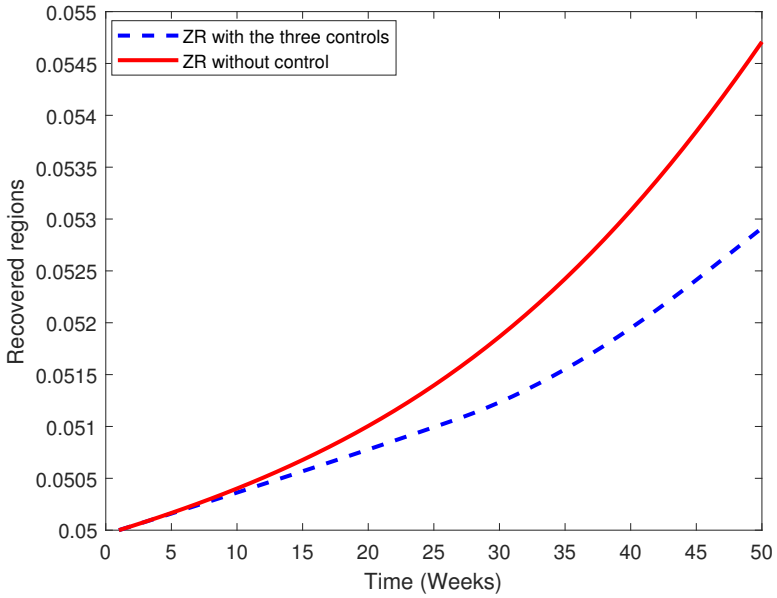

FIGURE 20. Recovered regions without and with the three controls

5. CONCLUSION

The discrete mathematical model *SEIR* that this paper presents explains the development of an infectious disease with the novel insight of examining the spread of the infection between

regions. To prevent the disease from spreading between people and areas, we suggest three different control measures. The optimal controls are defined according to the Pontryagin maximum principle, and an iterative approach is used to solve the optimality system. The effectiveness of our control strategies is finally demonstrated through numerical simulations, which also take the spread of the disease between regions into account.

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DATA AVAILABILITY

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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