Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2022, 2022:95
https://doi.org/10.28919/cmbn/7679
ISSN: 2052-2541

STABILITY PROPERTY OF THE PREDATOR-FREE EQUILIBRIUM OF A PREDATOR-PREY-SCAVENGER MODEL WITH FEAR EFFECT AND QUADRATIC HARVESTING

SIJIA LIN, QIANQIAN LI, QUN ZHU, FENGDE CHEN*

College of Mathematics and Statistics, Fuzhou University, Fuzhou, Fujian, 350116, China

Abstract. This article investigates the stability property of predator-free equilibrium of a predator-prey-scavenger model with fear effect and quadratic harvesting. The model was proposed by Mohammed Abdellatif Ahmed and Dahlia Khaled Bahlool (The influence of fear on the dynamics of a prey-predator-scavenger model with quadratic harvesting, Commun. Math. Biol. Neurosci., 2022, 2022: 62). By applying the standard comparison theorem and fluctuation lemma, we show that the conditions which ensure the locally asymptotically stable of predator-free equilibrium are enough to ensure global attractivity. Our result complements and supplements one of the main results of Mohammed Abdellatif Ahmed and Dahlia Khaled Bahlool.

Keywords: predator; prey; equilibrium; harvesting; stability.

2010 AMS Subject Classification: 34C25, 92D25, 34D20, 34D40.

1. Introduction

Mohammed Abdellatif Ahmed and Dahlia Khaled Bahlool [1] proposed the following prey-predator-scavenger model with quadratic harvesting:

$$\frac{dX}{dT} = \frac{rX}{1+f(Y+Z)} - bX^2 - \frac{a_1XY}{b_1+X} - \frac{a_2XZ}{b_2+X}, \tag{1}$$

*Corresponding author
E-mail address: fdchen@263.net

Received August 16, 2022

$$\frac{dY}{dT} = \frac{a_3XY}{b_1+X} - d_1Y - q_1E_1Y^2,$$
$$\frac{dZ}{dT} = \frac{a_4XZ}{b_2+X} + a_5YZ - d_2Z - q_2E_2Z^2,$$

where $X(t)$, $Y(t)$, and $Z(t)$ denote the population density of prey, predator and scavenger at time T, respectively. All the coefficients are positive constants. One could refer to [1] for a more detailed formulation of the model. The non-dimensional model that corresponds to the system (1) takes the form

$$\begin{aligned} \frac{dx}{dt} &= x\Big[\frac{1}{1+w_0(w_1y+z)} - x - \frac{y}{w_2+x} - \frac{z}{w_3+x}\Big], \\ \frac{dy}{dt} &= y\Big[\frac{w_4x}{w_2+x} - w_5 - w_6y\Big], \\ \frac{dz}{dt} &= z\Big[\frac{w_7x}{w_3+x} + w_8y - w_9 - w_{10}z\Big]. \end{aligned} \tag{2}$$

The model always exists a predator-free equilibrium $P_x(1,0,0)$. Concerned with this equilibrium's local and global stability, the authors obtained the following results.

Theorem A *P_x is locally asymptotically stable if and only if the following conditions are met:*

$$w_4 < w_5(w_2+1), \tag{3}$$

$$w_7 < w_9(w_3+1). \tag{4}$$

Theorem B *Suppose that P_x is locally asymptotically stable, then it is globally asymptotically stable if the following conditions are met:*

$$\frac{w_5}{w_4} > w_0w_1 + \frac{1}{w_2}, \tag{5}$$

$$\frac{w_9}{w_7} > \frac{w_8}{w_7}\beta_1 + w_0 + \frac{1}{w_3}, \tag{6}$$

where

$$\beta_1 = \frac{w_4(1+w_5)}{w_5}.$$

Now, let us consider the following example.

Example 1.1. Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{1}{1+3(y+z)} - x - \frac{y}{1+x} - \frac{z}{1+x}\Big), \\ \frac{dy}{dt} &= y\Big(\frac{x}{1+x} - 3 - y\Big), \\ \frac{dz}{dt} &= z\Big(\frac{x}{1+x} + y - 3 - z\Big). \end{aligned} \tag{7}$$

Here, corresponding to system (2), we choose $w_0 = w_5 = w_9 = 3, w_1 = w_2 = w_3 = w_4 = w_6 = w_7 = w_8 = w_{10} = 1$. By simple computation, we have

$$w_4 = 1 < 6 = w_5(w_2+1), \tag{8}$$

$$w_7 = 1 < 6 = w_9(w_3+1), \tag{9}$$

$$\frac{w_5}{w_4} = 3 < 3+1 = w_0 w_1 + \frac{1}{w_2}, \tag{10}$$

$$\frac{w_9}{w_7} = 3 < \frac{4}{3} + 3 + 1 = \frac{w_8}{w_7}\beta_1 + w_0 + \frac{1}{w_3}. \tag{11}$$

That is, conditions (3) and (4) in Theorem A are satisfied, while none of the conditions (5) and (6) in Theorem B are met. It follows from Theorem A that the predator-free equilibrium $P_x(1,0,0)$ is locally asymptotically stable. However, since the conditions of Theorem B are not met, we have no idea of the global stability property of the equilibrium $P_x(1,0,0)$. Numeric simulations (Figures 1–3) show that in this case, $P_x(1,0,0)$ is globally asymptotically stable.

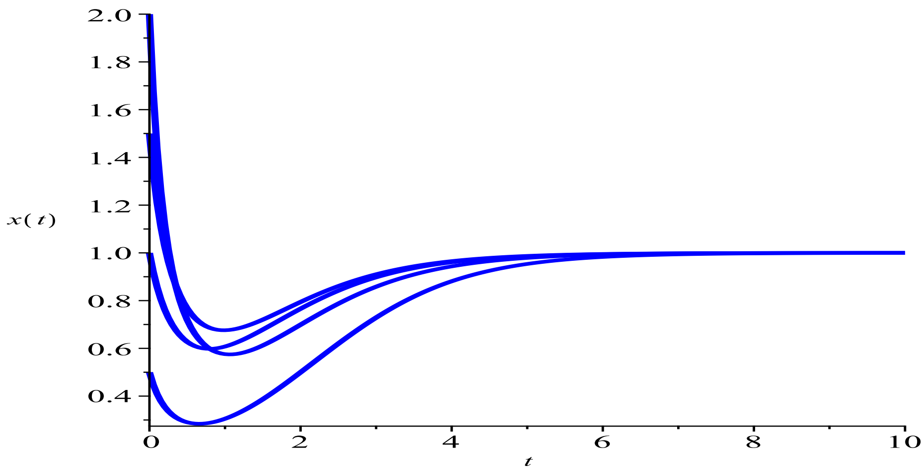

FIGURE 1. Dynamic behaviors of the first component x in system (7) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 1.5)$, $(1, 0.5, 1)$, $(1.5, 1.5, 0.5)$, and $(2, 2, 2)$, respectively.

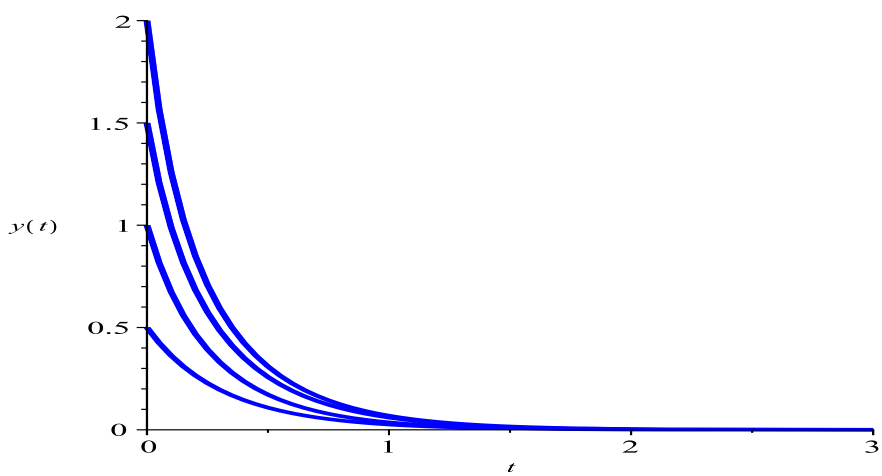

FIGURE 2. Dynamic behaviors of the second component y in system (7) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 1.5)$, $(1, 0.5, 1)$, $(1.5, 1.5, 0.5)$, and $(2, 2, 2)$, respectively.

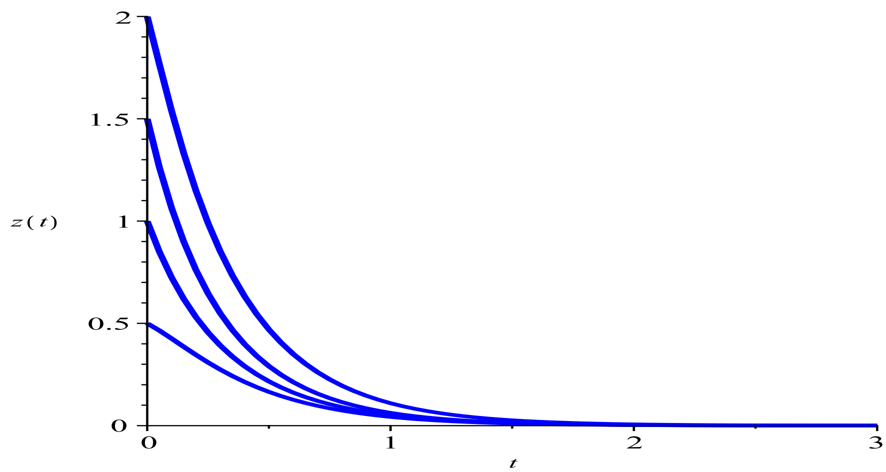

FIGURE 3. Dynamic behaviors of the third component z in system (7) with the initial conditions $(x(0),y(0),z(0)) = (0.5,1,1.5)$, $(1,0.5,1)$, $(1.5,1.5,0.5)$, and $(2,2,2)$, respectively.

Above example showed that in the system (2), conditions (5) and (6) in Theorem B are not the essential ones to ensure the globally asymptotically stable of the predator-free equilibrium P_x.

Now, one genuine issue is finding suitable sufficient conditions to ensure the globally asymptotically stable of equilibrium $P_x(1,0,0)$. This paper aims to put forward some studies on this direction. Indeed, we will prove the following result.

Theorem 1.1 *Assume that (3) and (4) are satisfied, then the predator-free equilibrium* $P_x(1,0,0)$ *is globally attractive.*

Remark 1.1. Theorem 1.1 shows that the conditions (5) and (6) in Theorem B are both redundant. The conditions which ensure the local stability of the predator-free equilibrium are enough to ensure its globally attractive.

The rest of the paper is organized as follows. We will prove Theorem 1.1 in the next section. We end this work with a brief discussion. For more works on the predator-prey system with fear effect, one could refer to [1]-[23] and the references cited therein.

2. PROOF OF THEOREM 1.1

Lemma 2.1[11] *If $a > 0, b > 0$ and $\dot{x} \geq b - ax$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}.$$

If $a > 0, b > 0$ and $\dot{x} \leq b - ax$, when $t \geq 0$ and $x(0) > 0$, we have

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}.$$

Lemma 2.2[12]*[Fluctuation Lemma] Let $x(t)$ be a bounded differentiable function on $[\alpha, \infty)$. Then there exist sequences $\tau_n \to \infty$ and $\sigma_n \to \infty$ such that*

(i) $x^{'}(\tau_n) \to 0$ *and* $x(\tau_n) \to \limsup_{t\to\infty} x(t) = \overline{x}$ *as* $n \to \infty$*;*

(ii) $x^{'}(\sigma_n) \to 0$ *and* $x(\sigma_n) \to \liminf_{t\to\infty} x(t) = \underline{x}$ *as* $n \to \infty$*.*

Proof of Theorem 1.1. For $\varepsilon > 0$ enough small, without loss of generality, from (3) and (4) we may assume that

$$\frac{w_7(1+\varepsilon)}{w_3+1+\varepsilon} + w_8\varepsilon < w_9, \tag{12}$$

$$\frac{w_4(1+\varepsilon)}{w_2+1+\varepsilon} < w_5 \tag{13}$$

and

$$\frac{1}{1+w_0(w_1\varepsilon+\varepsilon)} - \frac{\varepsilon}{w_2} - \frac{\varepsilon}{w_3} > 0 \tag{14}$$

hold.

From the positivity of the solution of system (2) and the first equation of system (2), it follows that

$$\begin{aligned} \frac{dx}{dt} &= x\Big[\frac{1}{1+w_0(w_1y+z)} - x - \frac{y}{w_2+x} - \frac{z}{w_3+x}\Big] \\ &\leq x\Big[\frac{1}{1+w_0(w_1y+z)} - x\Big] \\ &\leq x\Big[1-x\Big]. \end{aligned} \tag{15}$$

Applying Lemma 2.1 to the above inequality leads to

$$\limsup_{t\to+\infty} x(t) \le 1. \tag{16}$$

Hence, for aforementioned $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$x(t) < 1+\varepsilon \qquad \text{for all} \qquad t > T_1. \tag{17}$$

From the second equation of (2) and (17), it follows that

$$\begin{aligned} \frac{dy}{dt} &= y\Big[\frac{w_4 x}{w_2+x} - w_5 - w_6 y\Big] \\ &\le y\Big[\frac{w_4(1+\varepsilon)}{w_2+1+\varepsilon} - w_5 - w_6 y\Big] \\ &\le y\Big[\frac{w_4(1+\varepsilon)}{w_2+1+\varepsilon} - w_5\Big]. \end{aligned} \tag{18}$$

Hence, it follows from (13) and (18) that

$$y(t) < y(T_1)\exp\Big\{\Big(\frac{w_4(1+\varepsilon)}{w_2+1+\varepsilon} - w_5\Big)(t-T_1)\Big\} \to 0 \quad \text{as} \quad t\to+\infty. \tag{19}$$

For aforementioned $\varepsilon > 0$, from (19), there exists a $T_2 > T_1$ such that

$$y(t) < \varepsilon \qquad \text{for all} \qquad t > T_2. \tag{20}$$

For $t > T_2$, from the third equation in (2), (17), and (20), we have

$$\begin{aligned} \frac{dz}{dt} &= z\Big[\frac{w_7 x}{w_3+x} + w_8 y - w_9 - w_{10} z\Big]. \\ &\le z\Big[\frac{w_7(1+\varepsilon)}{w_3+1+\varepsilon} + w_8\varepsilon - w_9 - w_{10} z\Big] \\ &\le z\Big[\frac{w_7(1+\varepsilon)}{w_3+1+\varepsilon} + w_8\varepsilon - w_9\Big]. \end{aligned} \tag{21}$$

Hence, it follows from (12) and (21) that

$$z(t) \le z(T_2)\exp\Big\{\Big(\frac{w_7(1+\varepsilon)}{w_3+1+\varepsilon} + w_8\varepsilon - w_9\Big)(t-T_2)\Big\} \to 0 \qquad as \qquad t\to+\infty. \tag{22}$$

For aforementioned $\varepsilon > 0$, from (22), there exists a $T_3 > T_2$ such that

$$z(t) < \varepsilon \qquad \text{for all} \qquad t > T_3. \tag{23}$$

For $t > T_3$, it follows from the first equation of (2), (17), (20), and (23) that

$$\begin{aligned} \frac{dx}{dt} &= x\Big[\frac{1}{1+w_0(w_1 y+z)} - x - \frac{y}{w_2+x} - \frac{z}{w_3+x}\Big] \\ &\geq x\Big[\frac{1}{1+w_0(w_1\varepsilon+\varepsilon)} - x - \frac{\varepsilon}{w_2} - \frac{\varepsilon}{w_3}\Big]. \end{aligned} \tag{24}$$

Applying Lemma 2.1 to (24), leads to

$$\liminf_{t\to+\infty} x(t) \geq \frac{1}{1+w_0(w_1\varepsilon+\varepsilon)} - \frac{\varepsilon}{w_2} - \frac{\varepsilon}{w_3} > 0. \tag{25}$$

It follows from (17) that $x(t)$ is bounded. Let $\underline{x} = \liminf_{t\to+\infty} x(t)$, then from (25) we have $\underline{x} > 0$. According to Fluctuation Lemma there exist sequences $\sigma_n \to +\infty$ such that $x'(\sigma_n) \to 0$ and $x(\sigma_n) \to \liminf_{t\to\infty} x(t) = \underline{x}$ as $n \to \infty$. From (19) and (22) one could easily see that $y(\sigma_n) \to 0, z(\sigma_n) \to 0$ as $n \to \infty$. From the first equation of (2), we have

$$0 = \lim_{n\to+\infty} x'(\sigma_n) = \lim_{n\to+\infty} x(\sigma_n)\Big(\frac{1}{1+3(y(\sigma_n)+z(\sigma_n))} - x(\sigma_n) - \frac{y(\sigma_n)}{1+x(\sigma_n)} - \frac{z(\sigma_n)}{1+x(\sigma_n)}\Big). \tag{26}$$

Since $\underline{x} > 0$, above equality is equivalent to

$$\lim_{n\to+\infty}\Big(\frac{1}{1+3(y(\sigma_n)+z(\sigma_n))} - x(\sigma_n) - \frac{y(\sigma_n)}{1+x(\sigma_n)} - \frac{z(\sigma_n)}{1+x(\sigma_n)}\Big) = 0. \tag{27}$$

Therefore,

$$\underline{x} = 1. \tag{28}$$

(28) together with (16) leads to

$$1 = \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq 1. \tag{29}$$

That is,

$$\lim_{t\to+\infty} x(t) = 1. \tag{30}$$

(19), (22), and (30) shows that $P_x(1,0,0)$ is globally attractive. This ends the proof of Theorem 1.1. □

3. Discussion

Recently, Mohammed Abdellatif Ahmed and Dahlia Khaled Bahlool [1] proposed a predator-prey-scavenger model with fear effect and quadratic harvesting. The authors investigated the local and global stability of the equilibria; However, probably due to the complexity of the system, the conditions obtained by the authors to ensure the global stability of the equilibrium points of the system are complicated and difficult to verify. We observe that the authors investigated the global stability of the equilibrium points by constructing some suitable Lyapunov functions. This method makes it possible obtain sufficient conditions to guarantee the global stability of the equilibrium points, but such conditions may not be optimal. Some additional requirements are unavoidable, but for the system itself, it can indeed be redundant. By applying the differential inequality and fluctuation Lemma, we can show that the conditions in [1] to ensure the global stability of the predator-free equilibrium are redundant. It seems that this is the first time that the fluctuation Lemma be applied to the predator-prey system with fear effect. We hope we can do more work in this direction.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to thank two anonymous reviewers for their valuable comments, which greatly improve the final expression of the paper.

Funding

The research was supported by the Natural Science Foundation of Fujian Province (2020J01499).

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M.A. Ahmed, D.K. Bahlool, The influence of fear on the dynamics of a prey-predator-scavenger model with quadratic harvesting, Commun. Math. Biol. Neurosci. 2022 (2022), 62. https://doi.org/10.28919/cmbn/7506.

[2] Y. Huang, Z. Zhu, Z. Li, Modeling the Allee effect and fear effect in predator–prey system incorporating a prey refuge, Adv. Differ. Equ. 2020 (2020), 321. https://doi.org/10.1186/s13662-020-02727-5.

[3] Z. Xiao, Z. Li, Stability analysis of a mutual interference predator-prey model with the fear effect, J. Appl. Sci. Eng. 22 (2019), 205-211. https://doi.org/10.6180/jase.201906_22(2).0001.

[4] L. Lai, X. Yu, M. He, Z. Li, Impact of Michaelis–Menten type harvesting in a Lotka–Volterra predator–prey system incorporating fear effect, Adv. Differ. Equ. 2020 (2020), 320. https://doi.org/10.1186/s13662-020-02724-8.

[5] M. He, Z. Li, Stability of a fear effect predator-prey model with mutual interference or group defense, J. Biol. Dyn. 16 (2022), 480–498. https://doi.org/10.1080/17513758.2022.2091800.

[6] T. Liu, L. Chen, F. Chen, Z. Li, Stability analysis of a Leslie–Gower model with strong Allee effect on prey and fear effect on predator, Int. J. Bifurcation Chaos. 32 (2022), 2250082. https://doi.org/10.1142/s0218127422500821.

[7] Y. Huang, Z. Li, The stability of a predator-prey model with fear effect in prey and square root functional response, Ann. Appl. Math. 36 (2020), 186-194. http://global-sci.org/intro/article_detail/aam/18096.html.

[8] L. Lai, Z. Zhu, F. Chen, Stability and bifurcation in a predator–prey model with the additive Allee effect and the fear effect, Mathematics. 8 (2020), 1280. https://doi.org/10.3390/math8081280.

[9] J. Chen, X. He, F. Chen, The influence of fear effect to a discrete-time predator-prey system with predator has other food resource, Mathematics. 9 (2021), 865. https://doi.org/10.3390/math9080865.

[10] Z. Zhu, R. Wu, L. Lai, et al. The influence of fear effect to the Lotka–Volterra predator–prey system with predator has other food resource, Adv. Differ. Equ. 2020 (2020), 237. https://doi.org/10.1186/s13662-020-02612-1.

[11] F. Chen, Z. Li, Y. Huang, Note on the permanence of a competitive system with infinite delay and feedback controls, Nonlinear Anal., Real World Appl. 8 (2007), 680–687. https://doi.org/10.1016/j.nonrwa.2006.02.006.

[12] F. Chen, Y. Chen, J. Shi, Stability of the boundary solution of a nonautonomous predator–prey system with the Beddington–DeAngelis functional response, J. Math. Anal. Appl. 344 (2008), 1057–1067. https://doi.org/10.1016/j.jmaa.2008.03.050.

[13] Z. Zhu, Y. Chen, Z. Li, F. Chen, Stability and bifurcation in a Leslie-Gower predator-prey model with Allee effect, Int. J. Bifurcation Chaos. 32 (2022), 2250040. https://doi.org/10.1142/s0218127422500407.

[14] L. Chen, T. Liu, F. Chen, Stability and bifurcation in a two-patch model with additive Allee effect, AIMS Math. 7 (2021), 536–551. https://doi.org/10.3934/math.2022034.

[15] X. Wang, L. Zanette, X. Zou, Modelling the fear effect in predator–prey interactions, J. Math. Biol. 73 (2016), 1179-1204. https://doi.org/10.1007/s00285-016-0989-1.

[16] S.K. Sasmal, Population dynamics with multiple Allee effects induced by fear factors - A mathematical study on prey-predator interactions, Appl. Math. Model. 64 (2018), 1–14. https://doi.org/10.1016/j.apm.2018.07.021.

[17] S. Pal, N. Pal, S. Samanta, et al. Effect of hunting cooperation and fear in a predator-prey model, Ecol. Complex. 39 (2019), 100770. https://doi.org/10.1016/j.ecocom.2019.100770.

[18] J. Wang, Y. Cai, S. Fu, W. Wang, The effect of the fear factor on the dynamics of a predator-prey model incorporating the prey refuge, Chaos. 29 (2019), 083109. https://doi.org/10.1063/1.5111121.

[19] X. Li, M. Zhang, Integrability and multiple limit cycles in a predator-prey system with fear effect, J. Funct. Spaces. 2019 (2019), 3948621. https://doi.org/10.1155/2019/3948621.

[20] H. Zhang, Y. Cai, S. Fu, et al. Impact of the fear effect in a prey-predator model incorporating a prey refuge, Appl. Math. Comput. 356 (2019), 328–337. https://doi.org/10.1016/j.amc.2019.03.034.

[21] S. Pal, N. Pal, S. Samanta, et al. Fear effect in prey and hunting cooperation among predators in a Leslie-Gower model, Math. Biosci. Eng. 16 (2019), 5146-5179. https://doi.org/10.3934/mbe.2019258.

[22] X. Wang, X. Zou, Modeling the fear effect in predator-prey interactions with adaptive avoidance of predators, Bull. Math. Biol. 79 (2017), 1325-1359. https://doi.org/10.1007/s11538-017-0287-0.

[23] X. Wang, Y. Tan, Y. Cai, W. Wang, Impact of the fear effect on the stability and bifurcation of a Leslie-Gower predator-prey model, Int. J. Bifurcation Chaos. 30 (2020), 2050210. https://doi.org/10.1142/S0218127420502107.