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DYNAMICS OF A FRACTIONAL-ORDER ECO-EPIDEMIC MODEL WITH ALLEE EFFECT AND REFUGE ON PREY

NURHALIS HASAN, AGUS SURYANTO*, TRISILOWATI

Department of Mathematics, Faculty of Mathematics and Natural Sciences,
University of Brawijaya, Jl. Veteran Malang 65145, Indonesia

Abstract. In this article, a fractional-order eco-epidemic model with Allee effect and prey refuge is studied. First we prove the existence and the uniqueness of solution of the proposed model. The non-negativity and the boundedness solutions are also shown. It is found that the model has four equilibrium points, namely the extinction of both prey and predator point (E_0), the disease-prey-free and predator-free point (E^*), the predator-free point ($\tilde{E}$), and the coexistence point ($\hat{E}$). All equilibrium points are locally and globally asymptotically stable with conditions. Those analytical results are confirmed by our numerical simulations. As expected, our analytical and numerical simulations show that the Allee effect may induced the extinction of prey population. The prey refuge, particularly for the case of weak Allee effect, may reduce the possibility of prey extinction.

Keywords: eco-epidemiology; fractional-order; equilibrium point; local stability ;global stability.
2010 AMS Subject Classification: 34A08, 37M05, 37M10, 37N25, 92B05, 92D25.

1. Introduction

In this era, mathematical modeling is often used as a scientific tool in studying most biological processes. One of interesting biological processes is the interaction between prey and predator. A model describing the interaction between predator dan prey was first introduced

*Corresponding author
E-mail address: suryanto@ub.ac.id
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by Lotka-Volterra in 1925. Furthermore, Gause (1934) and Leslie-Gower (1948) developed the predator-prey model by applying logistics growth to their models [1]. The development of predator-prey model does not only stop at considering logistic growth, but also continue to include other effects such as Allee effect [2, 3, 4, 5], refuge on a population [6, 7, 8, 9], and so on. The addition of these assumptions can certainly affect the population density of the interaction between the two populations. The spread of disease in an interaction between populations can also occur, which can certainly affect the population density. To study the spread of disease in the predator-prey populations, it is necessary to link the fields of ecology and epidemiology, which is know as eco-epidemiology. Eco-epidemiology was first introduced by Anderson and May (1986) [10] who explained the existence of a disturbance that occurred in the predator-prey equation system and found a case study of disease control in the predator-prey equation system. Several studies related to the spread of disease in predator-prey systems can be seen in [11, 12, 13]. There are many kinds of rate of disease spread, namely the standard disease spread rate $\frac{aSI}{N}$, bilinear aSI, and non-linear incidance rate $\frac{aSI}{1+cI}$. Nonlinear incidance rate were first introduced by Capasso and Serio (1978) [15] who reviewed the Kermack and Mckendrick (1927) epidemic model [14] in the case of a cholera outbreak in 1973. They added assumption of an increased risk of infection disease due to the number of individuals infected with the disease. Research that examines the rate of non-linear incidance rate can be seen for example in [16, 17].

In this study we consider an eco-epidemic model adapted from Moustafa et al (2021), [19] by assuming that the prey experiences Allee effect and the prey makes refuge to keep away from predation. Hence, we propose the following model

$$\begin{aligned}
\frac{dS}{d\hat{t}} &= rS\left(1-\frac{S+I}{K}\right)-\frac{mS}{S+b}-\frac{aSI}{1+cS},\\
\frac{dI}{d\hat{t}} &= \frac{aSI}{1+cS}-e_1I-f(1-\theta)IP,\\
\frac{dP}{d\hat{t}} &= g(1-\theta)IP-e_2P,
\end{aligned}\tag{1}$$

where S is susceptible prey, I is infected prey, and P is predator with initial conditions $S(0)\geq 0, I(0)\geq 0, P(0)\geq 0$. All parameters $r,K,m,b,a,c,\theta,f,g,e_1$, and e_2 are positive constant and they are described in Table 1.

TABLE 1. The description of biological parameters

Parameter	Biological interpretation
r	The intrinsic growth rate of prey
K	The environmental carrying capacity of prey
m and b	The Allee parameters: (i) the weak Allee effect if $0 < m < b$, and (ii) the strong Allee effect if $m > b$
a	The disease spread rate from susceptible prey to infected prey
c	The half-saturated constant for the disease spread of prey
θ	The coefficient of refuge, where $\theta \in (0,1]$
f	The predation rate of predators
g	The conversion of predatory prey to infected prey
e_1 and e_2	The natural death of infected prey and predator, respectively

To ease the analysis process, the model (1) are simplified by introducing variable transformation $(S,I,P,\hat{t}) \to \left(Kx, Ky, \frac{r}{f}z, \frac{1}{t}\right)$ to get the following non-dimensional system.

$$
\begin{aligned}
\dot{x} &= x(1-(x+y)) - \frac{\xi x}{x+\varepsilon} - \frac{\hat{\beta} xy}{1+\eta x},\\
\dot{y} &= \frac{\hat{\beta} xy}{1+\eta x} - \hat{\gamma} y - (1-\theta) yz,\\
\dot{z} &= \hat{\omega}(1-\theta) yz - \hat{\delta} z,
\end{aligned} \tag{2}
$$

where $\xi = \dfrac{m}{rK}, \hat{\beta} = \dfrac{aK}{r}, \eta = cK, \hat{\gamma} = \dfrac{e_1}{r}, \hat{\omega} = \dfrac{gK}{r}, \varepsilon = \dfrac{b}{K}$, and $\hat{\delta} = \dfrac{e_2}{r}$. The prey has a weak Allee effect if $0 < \xi < \varepsilon$ and strong Allee effect if $\xi > \varepsilon$ [18].

Fractional calculus is often used in describing complex ecological phenomena involving memory and genetically related biological properties that are hereditary, see references [20, 21, 22]. The derivative of fractional-order is a theory of fractional calculus that has non-local properties, where the state of the next model does not only depend on the current state, but also depends on all previous state [23]. As a result, the application of fractional-order derivatives in eco-epidemiological models is required for the expansion of the stability region [24].

To include the memory effect in model (2), we apply fractional-order derivative. To be consistent with the physical dimension, we introduce new parameters $\beta = \hat{\beta}^{\alpha}$, $\gamma = \hat{\gamma}^{\alpha}$, $\omega = \hat{\omega}^{\alpha}$, and $\delta = \hat{\delta}^{\alpha}$, to get the following model

$$\begin{aligned} {}^{C}D_t^{\alpha} x &= \left[1 - (x+y) - \frac{\xi}{x+\varepsilon} - \frac{\beta y}{1+\eta x}\right] x, \\ {}^{C}D_t^{\alpha} y &= \left[\frac{\beta x}{1+\eta x} - \gamma - (1-\theta) z\right] y, \\ {}^{C}D_t^{\alpha} z &= \left[\omega(1-\theta) y - \delta\right] z, \end{aligned} \tag{3}$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order-α with $0 < \alpha \leq 1$. The Caputo fractional derivative of order-α of function $f(t)$ is defined by

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \tag{4}$$

where $t \geq 0$, $f \in C^n([0,+\infty),\mathbb{R})$, and Γ is the Gamma function.

In this article we study the dynamics of the model (3). The structure of this article is as follows. In Section 2 we investigate the properties of solution of the model (3). The investigation includes the existence, uniqueness, non-negativity, and boundedness of the solutions. The existence of equilibrium points, local stability, and global stability are discussed in Section 3. Furthermore, in Section 4, several numerical simulations of the model (3) are presented. Finally, we present some conclusions in Section 5.

2. Theoretical Analysis

2.1. Existence and Uniqueness. Let $\Omega := \left\{(x,y,z) \in \mathbb{R}^3 : \max\{|x|,|y|,|z|\} \leq M\right\}$ and $\mathbb{R}^3_+ := \left\{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, x,y,z \in \mathbb{R}\right\}$. The existence and uniqueness of the solution of model (3) is explained in the following theorem.

Theorem 1. *For each initial value $\{x_0,y_0,z_0\} \in \mathbb{R}^3_+$ in Ω, model* (3) *has a unique solution in $\Omega \times (0,\infty]$ for all $t > 0$.*

Proof. Consider a mapping $H(X) = \big(H_1(X), H_2(X), H_3(X)\big)$. For a $X = (x,y,z)$, $\bar{X} = (\bar{x},\bar{y},\bar{z})$, with $X, \bar{X} \in \Omega$ then from model (3) we get

$$
\begin{aligned}
||H(X) - H(\bar{X})|| &= |H_1(X) - H_1(\bar{X})| + |H_2(X) - H_2(\bar{X})| + |H_3(X) - H_3(\bar{X})| \\
&= \Bigg|(x-\bar{x}) - (x+\bar{x})(x-\bar{x}) - y(x-\bar{x}) - \bar{x}(y-\bar{y}) \\
&\quad - \Big(\frac{\varepsilon\xi}{(x+\varepsilon)(\bar{x}+\varepsilon)}\Big)(x-\bar{x}) - \left(\frac{\beta y}{(1+\eta x)(1+\eta\bar{x})}\right)(x-\bar{x}) \\
&\quad - \left(\frac{\beta\bar{x}}{(1+\eta x)(1+\eta\bar{x})}\right)(y-\bar{y}) - \left(\frac{\beta\eta x\bar{x}}{(1+\eta x)(1+\eta\bar{x})}\right)(y-\bar{y})\Bigg| \\
&\quad + \Bigg|\frac{\beta y}{(1+\eta x)(1+\eta\bar{x})}(x-\bar{x}) + \frac{\beta\bar{x}}{(1+\eta x)(1+\eta\bar{x})}(y-\bar{y}) \\
&\quad + \frac{\beta\eta x\bar{x}}{(1+\eta x)(1+\eta\bar{x})}(y-\bar{y}) - \gamma(y-\bar{y}) - \big(1-\theta\big)z(y-\bar{y}) - \big(1-\theta\big)\bar{y}(z-\bar{z})\Bigg| \\
&\quad + \Big|\omega\big(1-\theta\big)z(y-\bar{y}) + \omega\big(1-\theta\big)\bar{y}(z-\bar{z}) - \delta(z-\bar{z})\Big| \\
&\le |x-\bar{x}| + 2M|x-\bar{x}| + M|x-\bar{x}| + M|y-\bar{y}| + \frac{\xi}{\varepsilon}|x-\bar{x}| + \beta M|x-\bar{x}| \\
&\quad + \beta M|y-\bar{y}| + \beta\eta M^2|y-\bar{y}| + \beta M|x-\bar{x}| + \beta M|y-\bar{y}| + \beta\eta M^2|y-\bar{y}| \\
&\quad + \gamma|y-\bar{y}| + (1-\theta)M|y-\bar{y}| + (1-\theta)M|z-\bar{z}| + \omega(1-\theta)M|y-\bar{y}| \\
&\quad + \omega(1-\theta)M|z-\bar{z}| + \delta|z-\bar{z}| \\
&= \left(1 + (3+2\beta)M + \frac{\xi}{\varepsilon}\right)|x-\bar{x}| + \Big(\big(1 + (1-\theta) + \omega(1-\theta) + 2\beta\big)M \\
&\quad + 2\beta\eta M^2 + \gamma\Big)|y-\bar{y}| + \Big(\big((1-\theta) + \omega(1-\theta)\big)M + \delta\Big)|z-\bar{z}| \\
&\le L||X-\bar{X}||
\end{aligned}
$$

where $L = \max\Big\{1 + (3+2\beta)M + \frac{\xi}{\varepsilon}, \big[1 + (1+\omega)(1-\theta) + 2\beta + 2\beta\eta M\big]M + \gamma, \big((1+\omega)(1-\theta)\big)M + \delta\Big\}$. Thus, $H(X)$ satisfies Lipschitz's condition. Based on Lemma 2 in [13], then for each initial value of $(x_0, y_0, z_0) \in \mathbb{R}^3_+$ in Ω, there is a unique solution in Ω of model (3) for all $t > 0$. □

2.2. Non-Negativity and Boundedness of Solutions. In this section, non-negativity and boundedness of the solution of model (3) will be studied. They are given in some theorem below.

Theorem 2. *For any initial values* $(x_0, y_0, z_0) \in \mathbb{R}^3_+$, *solutions of model* (3) *are non-negative and uniformly bounded.*

Proof. First, we will show that $(x_0, y_0, z_0) \in \mathbb{R}^3_+$ then $x(t) \geq 0$ along $t \to \infty$ by using contradiction. Suppose the statement is not true then there is $t_1 > 0$ so that

$$\begin{cases} x(t), y(t), z(t) & > 0, \quad 0 \leq t < t_1, \\ x(t_1), y(t_1), z(t_1) & = 0, \\ x(t_1^+), y(t_1^+), z(t_1^+) & < 0. \end{cases} \tag{5}$$

Using equations (5) and the first equation of the model (3) we get

$$^{C}D_t^{\alpha} x(t_1)|_{x(t_1)} = 0.$$

Based on Lemma 1 in [13], we get $x(t_1^+) = 0$, which contradicts the fact that $x(t_1^+) < 0$. Then it should be $x(t) \geq 0$ for any $t \to \infty$. Using the same method, it can be shown that $y(t) \geq 0$ and $z(t) \geq 0$ are for any $t \to \infty$.

Next, we will show the boundedness of the solution of model (3). First define the function

$$V(t) = x(t) + y(t) + \frac{1}{\omega} z(t).$$

Then, for every $\zeta > 0$, we get

$$\begin{aligned} ^{C}D_t^{\alpha} V(t) + \zeta V(t) &= \left(x\Big(1 - (x+y)\Big) - \frac{\xi x}{x+\varepsilon} - \frac{\beta xy}{1+\eta x} \right) + \left(\frac{\beta xy}{1+\eta x} - \gamma y - (1-\theta) yz \right) \\ &\quad + \frac{1}{\omega}\Big(\omega(1-\theta)yz - \delta z\Big) + \zeta\left(x + y + \frac{1}{\omega} z\right) \\ &= -x^2 + (1+\zeta)x - xy - \frac{\xi x}{x+\varepsilon} + (\zeta - \gamma) y + \Big(\frac{\zeta - \delta}{\omega}\Big) z. \end{aligned}$$

By choosing $\zeta < \min\{\gamma, \delta\}$, we get

$$\begin{aligned} ^{C}D_t^{\alpha} V(t) + \zeta V(t) &\leq -x^2 + (1+\zeta)x \\ &= -x^2 + (1+\zeta) + \frac{(1+\zeta)^2}{4} - \frac{(1+\zeta)^2}{4} \end{aligned}$$

$$\begin{aligned} &= -\left(x^2-(1+\zeta)x+\frac{(1+\zeta)^2}{4}\right)+\frac{(1+\zeta)^2}{4} \\ &\leq \frac{(1+\zeta)^2}{4} \\ {}^{C}D_t^{\alpha}V(t) &\leq -\zeta V(t)+\frac{(1+\zeta)^2}{4}. \end{aligned}$$

Based on Lemma 3 in [13], we get

$$V(t)=\left(V(0)-\frac{(1+\zeta)^2}{4\zeta}\right)E_{\alpha}[-\zeta t^{\alpha}]+\frac{(1+\zeta)^2}{4\zeta}.$$

If $t\to\infty$, then $V(t)\to\frac{(1+\zeta)^2}{4}$. Therefore, all solutions of model (3) are confined to the region ψ where

$$\psi:=\left\{(x,y,z)\in\mathbb{R}^3_+\ :\ x+y+\frac{1}{\omega}z\leq\varphi\right\} \tag{6}$$

with $\varphi=\frac{(1+\zeta)^2}{4}+\kappa$ and $\kappa>0$. Thus, we get a complete proof for the Theorem 2. □

3. Equilibrium Point, Local Stability, and Global Stability

In this section, the dynamics of the proposed model under study will be explained. Model (3) has several equilibrium points and their existence is described as follows

(1) The extinction of both prey and predator point $E_0(0,0,0)$, which is always defined in $\mathbb{R}^3_+$.

(2) The disease-prey-free and predator-free point $E^*(x^*,0,0)$, where x^* is

$$\begin{aligned} x^*=x_{1,2} &= \frac{(1-\varepsilon)\pm\sqrt{Dx^*_{E^*}}}{2}, \\ Dx^*_{E^*} &= (1-\varepsilon)^2-4(\xi-\varepsilon). \end{aligned}$$

The existence of the equilibrium point E^* is described in Lemma 3.

Lemma 3. *The equilibrium point $E^*\in\mathbb{R}^3_+$ if*

(i). $\xi=\frac{(1-\varepsilon)^2+4\varepsilon}{4}$ and $1>\varepsilon$. This condition produces a point of equilibrium E^*. Or,

(ii). $\xi>\frac{(1-\varepsilon)^2+4\varepsilon}{4}$ and $1>\varepsilon$. This condition produces two equilibrium points E^*.

(3) The predator-free point $\tilde{E}(\tilde{x},\tilde{y},0)$, where

$$\begin{aligned}\tilde{x} &= \frac{\gamma}{\beta-\gamma\eta},\\ \tilde{y} &= \frac{\left[(1-\tilde{x})(\tilde{x}+\varepsilon)-\xi\right](1+\eta\tilde{x})}{(\tilde{x}+\varepsilon)(1+\eta\tilde{x}+\beta)}.\end{aligned}$$

The equilibrium point $\tilde{E}\in\mathbb{R}^3_+$, if $\beta>\eta\gamma$ and $\xi<(\tilde{x}+\varepsilon)(1-\tilde{x})$.

(4) The coexistence point $\hat{E}(\hat{x},\hat{y},\hat{z})$, where

$$\begin{aligned}\hat{x} &\triangleq x^3+3\mu_1x^2+\mu_2x+\mu_3=0,\\ \hat{y} &= \frac{\delta}{\omega(1-\theta)},\\ \hat{z} &= \frac{\beta\hat{x}-\gamma(1+\eta\hat{x})}{(1+\eta\hat{x})(1-\theta)}.\end{aligned}$$

where $\mu_1=\frac{1}{\eta}+\hat{y}+\varepsilon-1$, $\mu_2=\frac{\hat{y}\beta}{\eta}+\hat{y}\varepsilon+\frac{\hat{y}}{\eta}+\xi+\frac{\varepsilon}{\eta}-\left(\frac{1}{\eta}+\varepsilon\right)$, and $\mu_3=\frac{\hat{y}\varepsilon+\xi+\hat{y}\beta\varepsilon-\varepsilon}{\eta}$.

To determine the existence of the equilibrium point $\hat{E}$, first we need to know the positive real roots of the cubic equation $\hat{x}$. In [25], it has been explained in Lemma 3.1 and will be rewritten in the following lemma.

Lemma 4. *Let $\hat{x}$ be the positive real root of the cubic equation. Using the $w=x+\mu_1$ transformation, the $\hat{x}$ equation becomes*

$$h(w)=w^3+3pw+q=0, \tag{7}$$

where $p=\mu_2-\mu_1^2$ and $q=\mu_3+2\mu_1^3-3\mu_1\mu_2$. We get the existence of a positive real root of the following Equation (7):

(i). If $q<0$, then Equation (7) has one positive real root.

(ii). If $q>0$ and $p<0$, then:

a. if $q^2+4p^3=0$, then Equation (7) has two positive twin roots.

b. if $q^2+4p^3<0$, then Equation (7) has two positive roots.

(iii). if $q=0$ and $p<0$, then Equation (7) has a unique positive root.

Therefore, we get the existence of the equilibrium point $\hat{E}$ described in Lemma 5.

Lemma 5. *The equilibrium point $\hat{E}\in\mathbb{R}^3_+$ if satisfied one of Lemma 4 and $\beta>\frac{\gamma(1+\eta\hat{x})}{\hat{x}}$.*

Now, we investigate the local stability and global stability of the equilibrium points of model (3), which is explained in the following theorems.

Theorem 6. *The equilibrium point E_0 is:*

(i). local asymptotically stable if $\xi > \varepsilon$.

(ii). saddle point if $\xi < \varepsilon$.

Proof. The Jacobian matrix of model (3) which has been substituted for E_0 is

$$J(E_0) = \begin{bmatrix} \dfrac{\varepsilon-\xi}{\varepsilon} & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & -\delta \end{bmatrix}$$

The eigenvalues of this Jacobian matrix are $\lambda_1 = \dfrac{\varepsilon-\xi}{\varepsilon}$, $\lambda_2 = -\gamma$, and $\lambda_3 = -\delta$. Get $|\arg(\lambda_{2,3})| = \pi > \frac{\alpha\pi}{2}$. Therefore, the stability of the equilibrium point E_0 depends on λ_1.

(i). If $\xi > \varepsilon$, then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. According to Matignon's condition (see Theorem 2 in [13]), E_0 is locally asymptotically stable.

(ii). If $\xi < \varepsilon$, then $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$. According to Matignon's condition (see Theorem 2 in [13]), E_0 is a saddle point.

□

Theorem 7. *The equilibrium point E^* is:*

(i). locally asymptotically stable if $\xi < (x^*+\varepsilon)^2$ and $\gamma > \frac{\beta x^*}{\eta x^*+1}$.

(ii). saddle point if $\xi > (x^*+\varepsilon)^2$ or $\gamma < \frac{\beta x^*}{\eta x^*+1}$.

Proof. The Jacobian matrix of model (3) evaluated at E^* is

$$J(E^*) = \begin{bmatrix} -x^* + \dfrac{\xi x^*}{(x^*+\varepsilon)^2} & -x^* - \dfrac{\beta x^*}{\eta x^*+1} & 0 \\ 0 & \dfrac{\beta x^*}{\eta x^*+1} - \gamma & 0 \\ 0 & 0 & -\delta \end{bmatrix}$$

The eigenvalues of this Jacobian matrix are $\lambda_1 = \dfrac{[\xi-(x^*+\varepsilon)^2]x^*}{(x^*+\varepsilon)^2}$, $\lambda_2 = \dfrac{\beta x^*}{\eta x^*+1} - \gamma$, and $\lambda_3 = -\delta$. Get $|\arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$. Therefore, the stability of the equilibrium point E^* depends on $\lambda_{1,2}$.

(i). If $\xi < (x^*+\varepsilon)^2$ and $\gamma > \dfrac{\beta x^*}{\eta x^*+1}$, then $|\arg(\lambda_1)| = \pi > \dfrac{\alpha\pi}{2}$ and $|\arg(\lambda_2)| = \pi > \dfrac{\alpha\pi}{2}$. According to Matignon's condition (see Theorem 2 in [13]), E^* is locally asymptotically stable.

(ii). If any of the conditions are not fulfilled, then $|\arg(\lambda_1)| = \pi < \dfrac{\alpha\pi}{2}$ or $|\arg(\lambda_2)| = \pi < \dfrac{\alpha\pi}{2}$. According to Matignon's condition (see Theorem 2 in [13]), E^* is a saddle point.

□

Theorem 8. *Suppose that:*

$$
\begin{aligned}
k_1 &= \tilde{x} - \left(\frac{\xi\tilde{x}}{(\tilde{x}+\varepsilon)^2} + \frac{\beta\eta\tilde{x}\tilde{y}}{(1+\eta\tilde{x})^2}\right),\\
k_2 &= \frac{\beta\tilde{x}\tilde{y}}{(1+\eta\tilde{x})^2}\left(1+\frac{\beta}{1+\eta\tilde{x}}\right).
\end{aligned} \tag{8}
$$

The equilibrium point $\tilde{E}$ is said to be locally asymptotically stable if it satisfies:

(i). $\omega < \dfrac{\delta}{(1-\theta)\tilde{y}}$, $k_1 > 0$ *and* $k_2 > 0$. *Or,*

(ii). $\omega < \dfrac{\delta}{(1-\theta)\tilde{y}}$, $k_1 < 0$, $4k_2 > k_1^2$, *and* $\tan^{-1}\left(\frac{\sqrt{4k_2-k_1^2}}{k_1}\right) > \frac{\alpha\pi}{2}$.

Proof. The Jacobian matrix of model (3) which has been substituted for $\tilde{E}$ is

$$
J(\tilde{E}) = \begin{bmatrix}
-\tilde{x} + \dfrac{\xi\tilde{x}}{(\tilde{x}+\varepsilon)^2} + \dfrac{\beta\eta\tilde{x}\tilde{y}}{(\eta\tilde{x}+1)^2} & -\tilde{x}\left(1+\dfrac{\beta}{\eta\tilde{x}+1}\right) & 0\\
\dfrac{\beta\tilde{y}}{\eta\tilde{x}+1}\left(1-\dfrac{\eta\tilde{x}}{\eta\tilde{x}+1}\right) & 0 & -(1-\theta)\tilde{y}\\
0 & 0 & \omega(1-\theta)\tilde{y}-\delta
\end{bmatrix}
$$

One of the eigenvalue og Jacobian matrix is $\lambda_1 = \omega(1-\theta)\tilde{y} - \delta$ and the two other eigenvalues are the solution of $\lambda^2 + k_1\lambda + k_2 = 0$. If $\omega < \dfrac{\delta}{(1-\theta)\tilde{y}}$, then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. Furthermore, by using the Routh-Hurwitz criteria for a fractional order dynamic system (See Proposition 1 in [26]), the two other eigenvalues satisfy $|\arg(\lambda)| = \pi > \frac{\alpha\pi}{2}$ if: (i) $k_1 > 0$ and $k_2 > 0$ or, (ii) $k_1 < 0$, $4k_2 > k_1^2$, and $\tan^{-1}\left(\frac{\sqrt{4k_2-k_1^2}}{k_1}\right) > \frac{\alpha\pi}{2}$. Therefore, $\tilde{E}$ is locally asymptotically stable if conditions (i) or (ii) is satisfied. □

Theorem 9. *Suppose that:*

$$
\begin{aligned}
o_1 &= \hat{x} - \left(\frac{\xi\hat{x}}{(\hat{x}+\varepsilon)^2} + \frac{\beta\eta\hat{x}\hat{y}}{(1+\eta\hat{x})^2}\right),\\
o_2 &= \omega(1-\theta)^2\hat{y}\hat{z} + \frac{\beta\hat{x}\hat{y}}{(1+\eta\hat{x})^2}\left(1+\frac{\beta}{\eta\hat{x}+1}\right),\\
o_3 &= \omega(1-\theta)^2\hat{y}\hat{z}\left(\hat{x} - \left[\frac{\xi\hat{x}}{(\hat{x}+\varepsilon)^2} + \frac{\beta\eta\hat{x}\hat{y}}{(1+\eta\hat{x})^2}\right]\right),\\
\Delta^* &= 18o_1o_2o_3 + (o_1o_2)^2 - 4o_3o_1^3 - 4o_2^3 - 27o_3^2.
\end{aligned}
\tag{9}
$$

The equilibrium point $\hat{E}$ is said to be locally asymptotically stable if it satisfies:

(i). $\Delta^* > 0, o_1 > 0$, $o_3 > 0$, *and* $o_1o_2 > o_3$. *Or,*
(ii). $\Delta^* < 0$, $o_1 \geq 0$, $o_2 \geq 0$, $o_3 > 0$, *and* $\alpha < \frac{2}{3}$. *Or,*
(iii). $\Delta^* < 0$, $o_1 < 0$, $o_2 < 0$, *and* $\alpha > \frac{2}{3}$. *Or,*
(iv). $\Delta^* < 0$, $o_1 > 0$, $o_2 > 0$, $o_1o_2 = o_3$ *for all* $0 < \alpha < 1$.

Proof. The Jacobian matrix of model (3) evaluated at $\hat{E}$ is

$$
\hat{J}(\hat{E}) = \begin{bmatrix}
-\hat{x} + \frac{\xi\hat{x}}{(\hat{x}+\varepsilon)^2} + \frac{\beta\eta\hat{x}\hat{y}}{(\eta\hat{x}+1)^2} & -\hat{x}\left(1+\frac{\beta}{\eta\hat{x}+1}\right) & 0\\
\frac{\beta\hat{y}}{\eta\hat{x}+1}\left(1-\frac{\eta\hat{x}}{\eta\hat{x}+1}\right) & 0 & -(1-\theta)\hat{y}\\
0 & \omega(1-\theta)\hat{z} & 0
\end{bmatrix}
$$

We get the characteristic equation $\lambda^3 + o_1\lambda^2 + o_2\lambda + o_3 = 0$. The equilibrium $\hat{E}$ is locally asymptotically stable, if the roots of the caracteristic equation satisfy the Routh-Hurwitz criteria for a fractional order dynamic system (see Proposition 1 in [26]). $\square$

Theorem 10. *The equilibrium point E_0 is said to be globally asymptotically stable if it satisfies $\xi > \varphi + \varepsilon$.*

Proof. First define the Lyapunov function

$$V_0(x,y,z) = x + y + \frac{1}{\omega}z.$$

Next, by following Lemma 4 in [13], we have

$$
\begin{aligned}
{}^C D_t^\alpha V_0(x,y,z) &\leq {}^C D_t^\alpha x + {}^C D_t^\alpha y + \frac{1}{\omega}{}^C D_t^\alpha z\\
&= \left[1 - x - y - \frac{\xi}{x+\varepsilon} - \frac{\beta y}{1+\eta x}\right]x + \left[\frac{\beta x}{1+\eta x} - \gamma - (1-\theta)z\right]y
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\omega}[\omega(1-\theta)y-\delta]z \\
=\;& x-x^2-xy-\frac{\xi x}{x+\varepsilon}-\gamma y-\frac{\delta z}{\omega} \\
\leq\;& \Big[1-\frac{\xi}{x+\varepsilon}\Big]x
\end{aligned}
$$

Equation (6) says that $x<\varphi$, then

$$
{}^{C}D_t^{\alpha}V_0(x,y,z)\leq\Big[1-\frac{\xi}{\varphi+\varepsilon}\Big]x.
$$

We can see ${}^{C}D_t^{\alpha}V_0(E_0)\leq 0$ if $\xi>\varphi+\varepsilon$. Thus, based on Lemma 5 in [13], the point E_0 is globally asymptotically stable if $\xi>\varphi+\varepsilon$. □

Theorem 11. *The equilibrium point E^* is said to be globally asymptotically stable if it satisfies $\xi<\varepsilon x^*$ and $\gamma>(1+\beta)x^*$.*

Proof. Consider a Lyapunov function

$$
V_1(x,y,z)=\Big[x-x^*-x^*\ln\frac{x}{x^*}\Big]+y+\frac{1}{\omega}z.
$$

According to Lemma 4 in [13], we get

$$
\begin{aligned}
{}^{C}D_t^{\alpha}V_1(x,y,z)\;&\leq\;\Big(\frac{x-x^*}{x}\Big){}^{C}D_t^{\alpha}x+{}^{C}D_t^{\alpha}y+\frac{1}{\omega}{}^{C}D_t^{\alpha}z \\
&=\;(x-x^*)\Big[1-x-y-\frac{\xi}{x+\varepsilon}-\frac{\beta y}{1+\eta x}\Big]+\Big[\frac{\beta x}{1+\eta x}-\gamma-(1-\theta)z\Big]y \\
&\quad+\frac{1}{\omega}[\omega(1-\theta)y-\delta]z \\
&=\;-(x-x^*)^2+\frac{\xi(x-x^*)^2}{(x^*+\varepsilon)(x+\varepsilon)}-xy+x^*y+\frac{\beta x^* y}{1+\eta x}-\gamma y-\frac{\delta z}{\omega} \\
&\leq\;-\Big[1-\frac{\xi}{x^*\varepsilon}\Big](x-x^*)^2+\big[(1+\beta)x^*-\gamma\big]y
\end{aligned}
$$

It is clear that ${}^{C}D_t^{\alpha}V_1(E^*)\leq 0$ if $\xi<\varepsilon x^*$ and $\gamma>(1+\beta)x^*$. Consequently, if $\xi<\varepsilon x^*$ and $\gamma>(1+\beta)x^*$ the point E^* is globally asymptotically stable (see Lemma 5 in [13]). □

Theorem 12. *The equilibrium point $\tilde{E}$ is said to be globally asymptotically stable if it satisfies $\omega<\dfrac{\delta}{(1-\theta)\tilde{y}}$ and $\max\Big\{\dfrac{\beta\varepsilon\tilde{y}+\xi}{\varepsilon},\gamma+\varphi\Big\}<\tilde{x}<\dfrac{\gamma}{1+\beta}$.*

Proof. We define a Lyapunov function

$$
V_2(x,y,z)=\Big[x-\tilde{x}-\tilde{x}\ln\frac{x}{\tilde{x}}\Big]+\Big[y-\tilde{y}-\tilde{y}\ln\frac{y}{\tilde{y}}\Big]+\frac{1}{\omega}z.
$$

Using the same argument as in proof of Theorem 10, we have

$$\begin{aligned}
{}^{C}D_t^{\alpha}V_2(x,y,z) &\leq \Big(\frac{x-\tilde{x}}{x}\Big){}^{C}D_t^{\alpha}x+\Big(\frac{y-\tilde{y}}{y}\Big){}^{C}D_t^{\alpha}y+\frac{1}{\omega}{}^{C}D_t^{\alpha}z\\
&=(x-\tilde{x})\Big[1-x-y-\frac{\xi}{x+\varepsilon}-\frac{\beta y}{1+\eta x}\Big]+(y-\tilde{y})\Big[\frac{\beta x}{1+\eta x}-\gamma-(1-\theta)z\Big]\\
&\quad+\frac{1}{\omega}[(1-\theta)y-\delta]z\\
&=-(x-\tilde{x})^2-xy+\tilde{x}y+\tilde{y}x-\tilde{x}\tilde{y}+\frac{\xi(x-\tilde{x})^2}{(\tilde{x}+\varepsilon)(x+\varepsilon)}+\frac{\beta\eta\tilde{y}(x-\tilde{x})^2}{(1+\eta\tilde{x})(1+\eta x)}\\
&\quad+\frac{\beta\tilde{x}y}{1+\eta x}-\frac{\beta\tilde{x}\tilde{y}}{1+\eta x}-\gamma y+\gamma\tilde{y}+(1-\theta)\tilde{y}z-\frac{\delta z}{\omega}\\
&\leq-\left[1-\Big(\frac{\xi}{\varepsilon\tilde{x}}+\frac{\beta\tilde{y}}{\tilde{x}}\Big)\right](x-\tilde{x})^2+\big[(1+\beta)\tilde{x}-(\varphi+\gamma)\big]y+\Big[(1-\theta)\tilde{y}-\frac{\delta}{\omega}\Big]z\\
&\quad+\left[x+\gamma-\left(1+\frac{\beta}{1+\eta x}\right)\tilde{x}\right]\tilde{y}\\
&\leq-\left[1-\Big(\frac{\xi}{\varepsilon\tilde{x}}+\frac{\beta\tilde{y}}{\tilde{x}}\Big)\right](x-\tilde{x})^2+\big[(1+\beta)\tilde{x}-(\varphi+\gamma)\big]y+\Big[(1-\theta)\tilde{y}-\frac{\delta}{\omega}\Big]z\\
&\quad+\left[\varphi+\gamma-\left(\frac{1+\eta\varphi+\beta}{1+\eta\varphi}\right)\tilde{x}\right]\tilde{y}.
\end{aligned}$$

Obviously, ${}^{C}D_t^{\alpha}V_2(\tilde{E})\leq 0$ when $\omega<\dfrac{\delta}{(1-\theta)\tilde{y}}$ and $\max\left\{\dfrac{\beta\varepsilon\tilde{y}+\xi}{\varepsilon},\dfrac{(\gamma+\varphi)(1+\eta\varphi)}{1+\eta\varphi+\beta}\right\}<\tilde{x}<\dfrac{\varphi+\gamma}{1+\beta}$. It follows from Lemma 5 in [13], the point $\tilde{E}$ is globally asymptotically stable if $\omega<\dfrac{\delta}{(1-\theta)\tilde{y}}$ and $\max\left\{\dfrac{\beta\varepsilon\tilde{y}+\xi}{\varepsilon},\dfrac{(\gamma+\varphi)(1+\eta\varphi)}{1+\eta\varphi+\beta}\right\}<\tilde{x}<\dfrac{\varphi+\gamma}{1+\beta}$. □

Theorem 13. *The equilibrium point $\hat{E}$ is said to be globally asymptotically stable if it satisfies* $\max\left\{\dfrac{\beta\varepsilon\hat{y}+\xi}{\varepsilon},\dfrac{\omega(\varphi+\gamma)\hat{y}+\delta\hat{z}}{\omega\hat{y}}\right\}<\hat{x}<\dfrac{\gamma+(1-\theta)\hat{z}}{\beta+1}$ *and* $\omega<\dfrac{\delta}{(1-\theta)\hat{y}}$.

Proof. Defined a positive Lyapunov function

$$V_3(x,y,z)=\Big[x-\hat{x}-\hat{x}\ln\frac{x}{\hat{x}}\Big]+\Big[y-\hat{y}-\hat{y}\ln\frac{y}{\hat{y}}\Big]+\frac{1}{\omega}\Big[z-\hat{z}-\hat{z}\ln\frac{y}{\hat{z}}\Big]$$

We using the same argument as in the proof of Theorem 10, we have

$$\begin{aligned}
{}^{C}D_t^{\alpha}V_3(x,y,z) &\leq \Big(\frac{x-\hat{x}}{x}\Big){}^{C}D_t^{\alpha}x+\Big(\frac{y-\hat{y}}{y}\Big){}^{C}D_t^{\alpha}y+\Big(\frac{z-\hat{z}}{z}\Big)\frac{1}{\omega}{}^{C}D_t^{\alpha}z\\
&=(x-\hat{x})\Big[1-x-y-\frac{\xi}{x+\varepsilon}-\frac{\beta y}{1+\eta x}\Big]+(y-\hat{y})\Big[\frac{\beta x}{1+\eta x}-\gamma-(1-\theta)z\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\omega}(z-\hat{z})[(1-\theta)y-\delta] \\
=&-(x-\hat{x})^2-xy+\hat{x}y+\hat{y}x-\hat{x}\hat{y}+\frac{\xi(x-\hat{x})^2}{(\hat{x}+\varepsilon)(x+\varepsilon)}+\frac{\beta\eta\hat{y}(x-\hat{x})^2}{(1+\eta\hat{x})(1+\eta x)} \\
&+\frac{\beta\hat{x}y}{1+\eta x}-\frac{\beta\hat{x}\hat{y}}{1+\eta x}-\gamma y+\gamma\hat{y}+(1-\theta)\hat{y}z-(1-\theta)\hat{z}y-\frac{\delta z}{\omega}-\frac{\delta\hat{z}}{\omega} \\
\leq&-\left[1-\left(\frac{\xi}{\varepsilon\hat{x}}+\frac{\beta\hat{y}}{\hat{x}}\right)\right](x-\hat{x})^2+\big[(\beta+1)\hat{x}-(x+\gamma+(1-\theta)\hat{z})\big]y \\
&+\left[(1-\theta)\hat{y}-\frac{\delta}{\omega}\right]z+\left[\frac{\omega(x+\gamma)\hat{y}+\delta\hat{z}}{\omega\hat{y}}-\left(1+\frac{\beta}{1+\eta x}\right)\hat{x}\right]\hat{y} \\
\leq&-\left[1-\left(\frac{\xi}{\varepsilon\hat{x}}+\frac{\beta\hat{y}}{\hat{x}}\right)\right](x-\hat{x})^2+\big[(\beta+1)\hat{x}-(\varphi+\gamma+(1-\theta)\hat{z})\big]y \\
&+\left[(1-\theta)\hat{y}-\frac{\delta}{\omega}\right]z+\left[\frac{\omega(\varphi+\gamma)\hat{y}+\delta\hat{z}}{\omega\hat{y}}-\left(\frac{1+\eta\varphi+\beta}{1+\eta\varphi}\right)\hat{x}\right]\hat{y}
\end{aligned}
$$

Thus ${}^{C}D_t^{\alpha}V_3(\hat{E})\leq 0$ if $\max\left\{\frac{\beta\varepsilon\hat{y}+\xi}{\varepsilon},\frac{\big[\omega(\varphi+\gamma)\hat{y}+\delta\hat{z}\big](1+\eta\varphi)}{\omega\hat{y}(1+\eta\varphi+\beta)}\right\}<\hat{x}<\frac{\varphi+\gamma+(1-\theta)\hat{z}}{\beta+1}$ and $\omega<\frac{\delta}{(1-\theta)\hat{y}}$. According to Lemma 5 in [13], the point $\hat{E}$ is globally asymptotically stable. □

4. NUMERICAL SIMULATIONS

In this section, we present some numerical simulation results of the model (3) using the Caputo fractional-order predictor-corrector scheme developed by Diethelm et al (2002) [27]. The parameter values in this simulation are hypothetical parameter values, because field data are not available.

First, a numerical simulation is performed using the following parameter values:

$$
\xi=0.8,\ \beta=0.25,\ \varepsilon=0.3,\ \eta=0.5,\ \gamma=0.2,\ \theta=0.2,\ \omega=0.1,\ \delta=0.1,\ \alpha=0.9. \tag{10}
$$

Using these parameter values, the model (3) has equilibrium point E_0. Since $\xi=0.8>\varepsilon=0.3$, the prey experiences a strong Allee effect and thus E_0 is locally asymptotically stable. This behavior can be seen in Figure 1. If we take $\xi=0.2$, we have a case of week Allee effect and the model (3) have two equilibrium points i.e., E^* which is locally asymptotically stable because $\xi=0.2<(x^*+\varepsilon)^2\approx 1.2582$ and $\gamma=0.1>\frac{\beta x^*}{1+\eta x^*}\approx 0.0058$ and E_0 which is unstable saddle point. Such behavior is shown in Figure 2. Next, we takee $\xi=0.2$ and $\beta=0.4$. Here, we

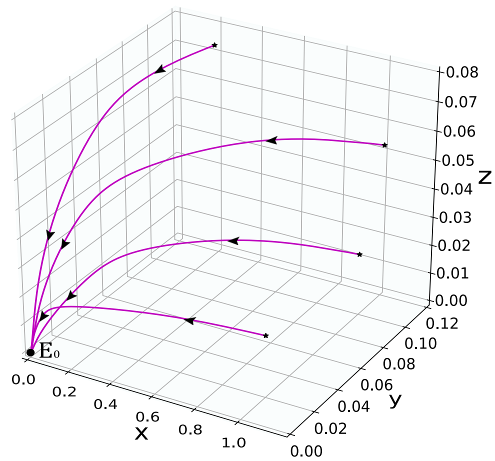

FIGURE 1. Phase portrait of system (3) with parameter $\xi = 0.8$, $\beta = 0.25$, $\varepsilon = 0.3$, $\eta = 0.5$, $\gamma = 0.2$, $\theta = 0.2$, $\omega = 0.1$, $\delta = 0.1$, and $\alpha = 0.9$.

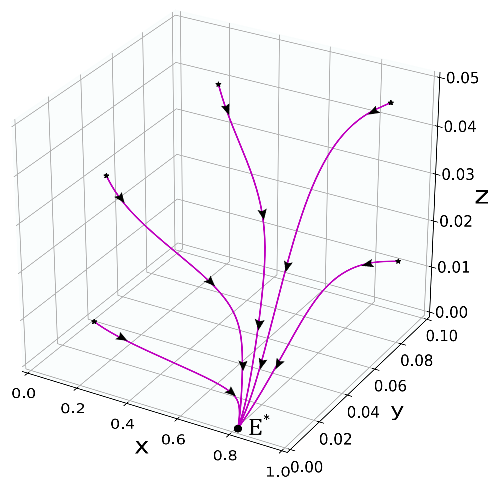

FIGURE 2. Phase portrait of system (3) with parameter $\xi = 0.2$, $\beta = 0.25$, $\varepsilon = 0.3$, $\eta = 0.5$, $\gamma = 0.2$, $\theta = 0.2$, $\omega = 0.1$, $\delta = 0.1$, and $\alpha = 0.9$.

still have a case of weak Allee effect, but the model (3) now has three equilibrium points i.e., unstable saddle points E_0 and E^*, and locally asymptotically stable $\tilde{E}$. $\tilde{E}$ is stable because k_2 is always positive, $\omega = 0.1 < \dfrac{\delta}{(1-\theta)\tilde{y}} \approx 1.2852$, and $k_1 \approx 0.5166 > 0$. Those properties are clearly seen in Figure 3.

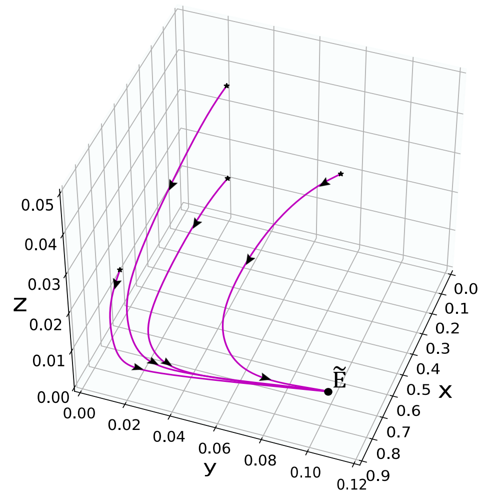

FIGURE 3. Phase portrait of system (3) with parameter $\xi = 0.2$, $\beta = 0.4$, $\varepsilon = 0.3$, $\eta = 0.5$, $\gamma = 0.2$, $\theta = 0.2$, $\omega = 0.1$, $\delta = 0.1$, and $\alpha = 0.9$.

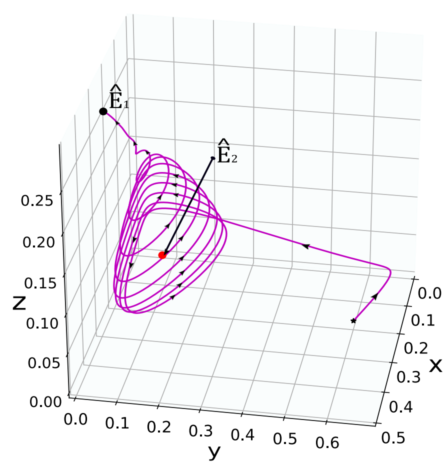

FIGURE 4. Phase portrait of system (3) with parameter $\xi = 0.38$, $\beta = 0.65$, $\varepsilon = 0.42$, $\eta = 0.5$, $\gamma = 0.01$, $\theta = 0.3$, $\omega = 0.2$, $\delta = 0.01$, and $\alpha = 0.95$.

We now perform a numerical simulation using the following parameter values:

$$\xi = 0.38,\ \beta = 0.65,\ \varepsilon = 0.42,\ \eta = 0.5,\ \gamma = 0.01,\ \theta = 0.3,\ \omega = 0.2,\ \delta = 0.01,\ \alpha = 0.95. \tag{11}$$

By using these parameter values, we have a case of week Allee effect and the model (3) has two co-existence points i.e., unstable $\hat{E}_2$ and locally asymptotically stable $\hat{E}_1$. $\hat{E}_1$ is stable because $o_1 \approx 0.2251 > 0$, o_2 is always positive, and $o_3 \approx 0.0007 > 0$. Such behavior is shown in Figure 4.

Finally, a numerical simulation is performed using the following parameter values:

$$\xi = 0.63,\ \beta = 0.1,\ \varepsilon = 0.6,\ \eta = 0.3,\ \gamma = 0.2,\ \theta = 0.2,\ \omega = 0.1,\ \delta = 0.2,\ \alpha = 0.9, \tag{12}$$

and

$$\xi = 0.3,\ \beta = 0.2,\ \varepsilon = 0.2,\ \eta = 0.1,\ \gamma = 0.1,\ \theta = 0.2,\ \omega = 0.3,\ \delta = 0.1,\ \alpha = 0.9. \tag{13}$$

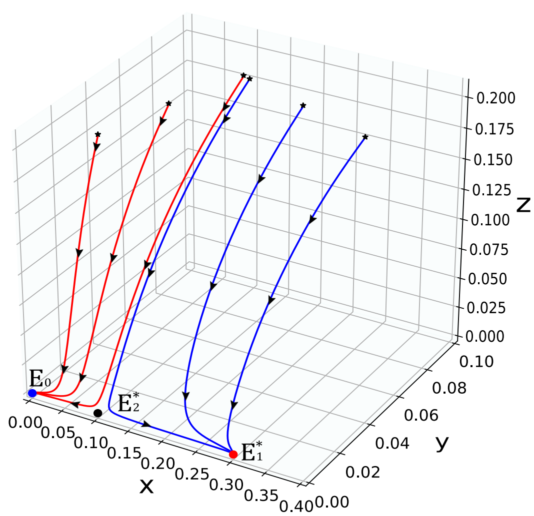

FIGURE 5. Phase portrait of system (3) with parameter $\xi = 0.63$, $\beta = 0.1$, $\varepsilon = 0.6$, $\eta = 0.3$, $\gamma = 0.2$, $\theta = 0.2$, $\omega = 0.1$, $\delta = 0.2$, and $\alpha = 0.9$.

The parameter values in (12) and (13) represent the cases of Allee effect becuase $\xi > \varepsilon$. The model (3) with parameter values (12) has three equilibrium points i.e., E_0 and E_1^* which are locally asymptotically stable; and E_2^* which is unstable saddle point. Since there are two locally stable equilibrium points, the model (3) shows a bistability phenomenon, see Figure 5. On the other hand, the model (3) with parameter values (13) has four equilibrium points E_0, E_1^*, E_2^*, and $\tilde{E}$. Because $\xi = 0.3 > \varepsilon = 0.2$, $\omega = 0.3 < \dfrac{\delta}{(1-\theta)\tilde{y}} \approx 2.4529$, $k_1 \approx 0.2265 > 0$, and k_2 is always positive, both E_0 and $\tilde{E}$ are locally asymptotically stable, while E_1^* and E_2^* are saddle. Hence, we also have a bistability phenomenon for this case, see Figure 6.

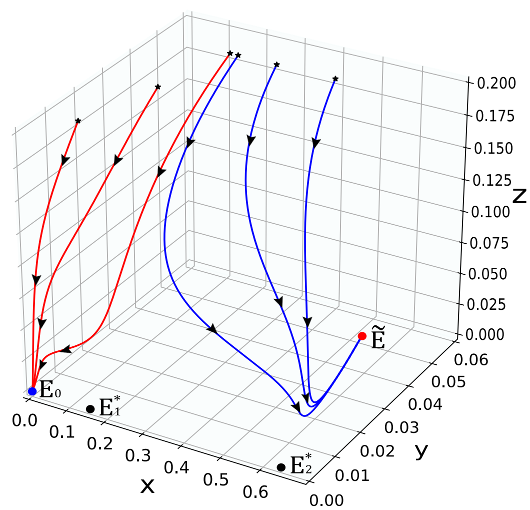

FIGURE 6. Phase portrait of system (3) with parameter $\xi = 0.3$, $\beta = 0.2$, $\varepsilon = 0.2$, $\eta = 0.1$, $\gamma = 0.1$, $\theta = 0.2$, $\omega = 0.3$, $\delta = 0.1$, and $\alpha = 0.9$.

5. CONCLUSIONS

In this article, we consider a fractional-order eco-epidemiological model with Allee effects and prey refuge. The solution properties of the model (3) including existence, uniqueness, non-negative, and boundedness solutions have been studied. The model (3) has four equilibrium points in $\mathbb{R}_+^3$. The local and global stabilities of each equilibrium point have been described in several theorems. Our numerical simulations confirm the analytical finding and also show the existence of bistability phenomenon. Furthermore, our numerical observations show that a weak Allee effect ($\xi < \varepsilon$) can reduce the risk of extinction of population, while strong Allee effect ($\xi > \varepsilon$) leads to the larger possibility of the population extinction. Moreover, the strong Allee effect may cause a bistability phenomenon.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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