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WOLBACHIA INFECTED AND UNINFECTED MOSQUITO GROWTH MODEL USING THE LESLIE MULTISPECIES MATRIX MODEL

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Abstract: Currently, there is a new strategy for stopping the spread of dengue, namely infecting wild mosquitoes with Wolbachia bacteria. Wolbachia is a bacteria that can lower the rate of infection in a mosquito's body, which lowers the risk of the virus being transmitted when it bites susceptible people. Our aim in this paper is to find the equilibrium points and perform local stability analysis for each equilibrium point of a growth model of uninfected and Wolbachia-infected *Aedes aegypti* mosquitoes. The growth model of the two types of mosquitoes was modeled using the Leslie multispecies matrix model. We assume that the first species is Wolbachia-uninfected mosquitoes and the second species is Wolbachia-infected mosquitoes. In this study, we obtained four equilibrium points. Then, we obtain asymptotically local stability conditions at the four equilibrium points. Based on these results, this study provides conditions that guarantee that efforts to use Wolbachia can suppress dengue disease.

Keywords: Wolbachia; aedes aegypti; Leslie matrix; dengue; local stability.

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1. INTRODUCTION

Dengue fever is a disease that is currently being highlighted in public health issues. The number of cases that occur due to dengue disease ranges from 390 million per year [1]. In general, the disease is transmitted through the *Aedes aegypti* mosquito infected with the dengue virus. Currently, dengue disease consists of DENV-1 to DENV-2 where a person is likely to be infected with these four serotypes. Generally, dengue fever is often found in tropical areas in Southeast Asia, including Indonesia, East Asia, namely Hong Kong, South Asia, namely India, and many more.

Due to the high number of dengue cases, several strategies are sought to suppress or eliminate the spread of dengue disease. Some of these strategies include the use of mosquito repellents, insecticides, vaccines, and so on. The use of mosquito repellents and insecticides has a very negative effect on humans, animals, and the environment. In particular, the use of insecticides is only able to last in the short term, and over time mosquitoes are already susceptible to these insects in the long term [2], [3]. Then, the use of vaccines also still needs further development. The first official vaccine is the Dengvaxia vaccine, which is reserved for ages 9-45 years. The use of vaccines is considered safe in patients who have been infected, while patients who have never been infected are at risk of infection [4]. Then, the efficacy of the vaccine was still in the 54% to 77% interval. The research results of Ndii et al.[5] with low vaccine efficacy shows that if seronegative individuals are vaccinated, there will be an increase in the number of secondary infections. Therefore, the use of vaccines is considered not to be effective and it is also possible to require a lot of costs for its development.

Currently, there is a new effort to overcome the spread of dengue disease, namely Wolbachia. Wolbachia is found in many species of anthropods which are intracellular parasites of these arthropods [6]. The use of Wolbachia is considered to be able to inhibit virus replication or competence in *Aedes aegypti* mosquitoes [7]. The use of Wolbachia in Yogyakarta obtained good results and there was a decrease in cases of dengue disease [8].

Several researchers have researched models of controlling the spread of dengue disease with various strategies. Ndii et al.[5] examined the use of vaccines to see which serotypes should be vaccinated. Ndii et al.[9] investigated the effect of vaccines on single and two serotype models validated with data in the city of Kupang. Mentuda [4] compared optimal control of the use of vaccines with general vector control and a combination of both where vector control requires a relatively short time to suppress mosquito populations. Zhang and Lui [10], Cardona-Salgado et al.[11], and Li and Liu [12] investigated the use of Wolbachia where releasing Wolbachia-infected mosquitoes could reduce the spread of dengue disease.

Some Wolbachia strains shorten the mosquito's life span, which limits the percentage of alive insects when the incubation phase is over [6]. Therefore, this raises the question of whether mosquitoes infected with Wolbachia will be able to survive in their ecosystem. Several researchers have researched to determine the population dynamics of mosquitoes with Wolbachia infection and without Wolbachia infection. Bliman et al.[6] studied a simple dynamic model based on Wolbachia and no Wolbachia mosquitoes, each of which was divided into two phases, namely the aquatic phase (eggs and larvae) and the posterior air phase (pupae, juvenile, and larval). mature). Then, Bliman et al.[13] studied further the model he worked on in his research [6]. Almeida et al.[14] conducted a study on the optimal control of infected mosquitoes in wild mosquito populations. Bliman [15] developed a study conducted on the research of Bliman et al.[13] with the same model. Li et al.[16] studied the growth model between Wolbachia mosquitoes and wild mosquitoes using the Leslie-Gower model.

In this paper, we formulate a growth model of the interaction of uninfected and Wolbachia-infected wild mosquitoes in an ecosystem with a new approach using the Leslie matrix model for multispecies. Several studies related to the population growth model using the Leslie matrix model, including Travis et al.[17], and Kon [18], [19], [20]. Based on the multispecies case, we assumed that the infected and non-infected Wolbachia mosquitoes were two different species even though the mosquitoes were the same species. Our focus on the model is to find the equilibrium points and perform an asymptotically local stability analysis of the model. In particular, in this study, we found conditions in which Wolbachia could survive in the long term so that suppression of the spread of dengue was achieved.

2. MATERIALS AND METHODS

2.1. Multispecies Leslie Matrix Model of Infected and Uninfected Wolbachia Mosquitoes

In this section, we present the Leslie matrix model of infected and uninfected mosquitoes by Wolbachia bacteria which is the mosquito that causes dengue disease. The Leslie matrix model was introduced by [21]. The Leslie matrix model is a female population growth model based on age class categories which assume each class or category has different characteristics such as birth rate, survival rate, and so on.

The problems that we examine in this study are presented in (1). The model in (1) is constructed based on the Leslie multispecies matrix model studied by [17] which was then modified to suit the environmental characteristics that occur in mosquitoes. We divided the population of mosquitoes both infected and uninfected with Wolbachia into two phases, namely the aquatic phase (A) and the posterior air phase (P) as performed by Bliman et al.[6]. We assumed the existence of intraspecific (competition between the same mosquito) and interspecific (competition between different mosquitoes) that occurred only in the aquatic phase. To describe the effect of competition, we use the Beverton-Holt equation which is widely used, especially in the research of Li et al.[16]. Then, we assume that the level of competition for the same mosquito is the same as the parameter $a > 0$ and the level of competition between different mosquitoes is the same value as the parameter $b > 0$. Furthermore, birth rates were assumed to occur only in the posterior air phase in both mosquito species. In addition, we considered Wolbachia for maternal vertical transmission and the mosquito population CI mechanism denoted by c where $0 \leq c \leq 1$ [22]. Therefore, if $c = 1$ then there will be no birth in the mosquito population that is not infected with Wolbachia. Here we present this problem model in the following discrete equation system:

$$
\begin{aligned}
M_A(t+1) &= f_{M_P}(1-c)M_P(t) \\
M_P(t+1) &= \frac{s_{M_A}}{1 + aM_A(t) + bW_A(t)} M_A(t) \\
W_A(t+1) &= f_{W_P} W_P(t) \\
W_P(t+1) &= \frac{s_{W_A}}{1 + aW_A(t) + bM_A(t)} W_A(t)
\end{aligned}
\tag{1}
$$

where $M_A(t)$ and $M_P(t)$ respectively represent the total population of mosquitoes not infected with Wolbachia in phases A and P at time t. On the other hand, $W_A(t)$ and $W_P(t)$, respectively, represent the total population of mosquitoes infected with Wolbachia in phases A and P at time t. Parameters $f_{M_P} > 0$ and $f_{W_P} > 0$ represents the birth rate which refers to M_P and W_P respectively. Then, the parameters $0 < s_{M_A} \leq 1$ and $0 < s_{W_A} \leq 1$ represents the survival rates of M_A to M_P and W_A to W_P, respectively.

The model in (1) in the first equation means that the total population of the aquatic phase at time $t + 1$ is obtained from the number of populations born to the posterior air phase population which is influenced by the CI mechanism. Then, the second equation of the model in (1) means that the total population of the posterior air phase at time $t + 1$ is obtained from the number of populations that survive from the aquatic phase with the influence of competition that occurs in the aquatic phase. Furthermore, the third and fourth equations have almost the same meaning as the first and second equations of the model in (1). The difference between the third and fourth equations with the first and second of the model in (1) is the focus on the Wolbachia-infected mosquito population. Then, the third equation of the model in (1) is not affected by the CI mechanism.

2.2. Asymptotically Local Stability Criteria Using M-Matrix Theory

One of the things that are studied in a model of the study, especially discrete dynamic systems is to determine the stability of the model. Stability studies can be in the form of local and global stability. In this case, this research is focused on studying the local stability of a model, especially the study of asymptotically local stability. Solutions must approach an equilibrium point under initial circumstances near the equilibrium point for local asymptotic stability [23]. The asymptotically stable condition of the discrete model occurs if the dominant eigenvalue of the Jacobian matrix of the model is less than one. In the case of analysis, however, it is very difficult to determine local stability asymptotically. Therefore, in this study, we use another criterion in determining local stability asymptotically, namely the M-Matrix theory used in the study of Travis et al.[17].

Definition 1 [17]:

Suppose a square matrix A has size n. $A_{n\times n}$ matrix is said to be M-Matrix if it satisfies two conditions, including i) element $a_{ij} \leq 0$ $i \neq j$. ii) one of these five conditions is met, namely:

i) All minor principles of matrix A have positive values.

ii) All real parts of the eigenvalues of matrix A have positive values.

iii) Matrix A is a nonsingular matrix and the inverse of matrix A is a positive matrix

iv) There is a vector $v > 0$ so that $Av > 0$ is satisfied.

v) There is a vector $w > 0$ so that $A^T w > 0$ is satisfied.

Theorem 1 [17]:

Suppose the matrix J is as follows:

$$J = \begin{bmatrix} A_{k\times k} & B_{k\times l} \\ C_{l\times k} & D_{l\times l} \end{bmatrix}.$$

If $G = I - SJS^{-1}$ is an M-Matrix with

$$\begin{array}{ll} S = I_{k+l} & \text{if } B_{k\times l} \text{ and } C_{l\times k} \geq 0 \\ S = \begin{bmatrix} I_{k\times k} & O_{k\times l} \\ O_{l\times k} & I_{l\times l} \end{bmatrix} & \text{if } B_{k\times l} \text{ and } C_{l\times k} \leq 0' \end{array}$$

where I is the symbol of the Identity matrix and O is the symbol of the zero matrix, then J is locally asymptotically stable.

In the results and discussion section, Definition 1 and Theorem 1 will be the two modalities for determining the asymptotically local stability of the model in (1). Then, the J matrix in Theorem 1 will be applied to the Jacobian matrix of the model in (1).

3. MAIN RESULTS

3.1. Inherent Net Reproduction Number

One of the most important and widely used quantities in the Leslie matrix model is the inherent net reproduction number. The definition of the inherent net reproduction number is the expected number of offspring per individual per lifetime. To put it simply, the inherent net reproduction number is the dominant eigenvalue of the $F(I-T)^{-1}$ matrix at the extinction equilibrium point, namely $x^* = 0$ where F, I, and T are the fertility matrix, identity matrix, and matrix respectively. transition (see details in [24]). In the Leslie multispecies matrix model, there are two inherent net reproductions, namely for the first and second species (see Travis et al.[17], Kon [17], [18], [19]). In this case, the first species is assumed to be a Wolbachia-uninfected mosquito, and the second species is assumed to be a Wolbachia-infected mosquito.

First, the fertility matrix F and the transition matrix T from (1) for mosquitoes that are not infected with Wolbachia are

$$F_M = \begin{bmatrix} 0 & f_{M_P}(1-c) \\ 0 & 0 \end{bmatrix}$$

and

$$T_M = \begin{bmatrix} \dfrac{s_{M_A}}{1 + aM_A + bW_A} & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we get the matrix $(I_2 - T_M)$ and the inverse of the matrix $(I_2 - T_M)$ are

$$I_2 - T_M = \begin{bmatrix} 1 & 0 \\ -\dfrac{s_{M_A}}{1 + aM_A + bW_A} & 1 \end{bmatrix}$$

and

$$(I_2 - T_M)^{-1} = \begin{bmatrix} 1 & 0 \\ \dfrac{s_{M_A}}{1 + aM_A + bW_A} & 1 \end{bmatrix}.$$

Then, we get the matrix

$$F_M(I_2 - T_M)^{-1}(\boldsymbol{x} = \boldsymbol{0}) = \begin{bmatrix} f_{M_P}(1-c)s_{M_A} & f_{M_P}(1-c) \\ 0 & 0 \end{bmatrix}$$

and the eigenvalues of $F_M\,(I_2 - T_M)^{-1}$ are $\lambda_1 = f_{M_P}\,s_{M_A}(1-c)$ and $\lambda_2 = 0$. Therefore, the dominant eigenvalue of $F_M(I_2 - T_M)^{-1}$ is $\lambda_1 = f_{M_A}s_{M_A}(1-c)$ then we symbolize it as R_M.

Second, the fertility matrix F and the transition matrix T from (1) for Wolbachia-infected mosquitoes are

$$F_W = \begin{bmatrix} 0 & f_{W_P} \\ 0 & 0 \end{bmatrix}$$

and

$$T_W = \begin{bmatrix} \dfrac{s_{W_A}}{1 + bM_A + aW_A} & 0 \\ 0 & 0 \end{bmatrix}.$$

By using the same method as for mosquitoes that were not infected with Wolbachia, we obtained

$$F_W(I - T_W)^{-1}(\boldsymbol{x} = \boldsymbol{0}) = \begin{bmatrix} f_{W_A} s_{W_A} & 0 \\ 0 & 0 \end{bmatrix}$$

and the eigenvalues of the matrix $F_W(I - T_W)^{-1}(\boldsymbol{x} = \boldsymbol{0})$, namely $\lambda_1 = f_{W_A} s_{W_A}$ and $\lambda_2 = 0$ then we symbolize it as R_W.

3.2. Equilibrium Points of the Model

In this section, we present the equilibrium points of the model in (1). The initial step taken to determine the equilibrium points is to form an equilibrium model from the model at (1) depending on time t. The following is the equilibrium model in (1).

$$M_A(t) = f_{M_P}(1 - c)M_P(t)$$

$$M_P(t) = \frac{s_{M_A}}{1 + aM_A(t) + bW_A(t)} M_A(t)$$

$$W_A(t) = f_{W_P} W_P(t)$$

$$W_P(t) = \frac{s_{W_A}}{1 + aW_A(t) + bM_A(t)} W_A(t)$$

The next step is to determine the solution of (2) and we obtain four equilibrium points. Here are the four equilibrium points:

i) The extinction equilibrium point for the two species is $E_0 = [0,0,0,0]^T$.

ii) The equilibrium point with mosquitoes not infected with Wolbachia exists when $R_M > 1$ while mosquitoes infected with Wolbachia become extinct, i.e.

$$E_M = \left[\frac{R_M - 1}{a}, \frac{R_M - 1}{af_{M_P}(1 - c)}, 0,0\right]^T.$$

iii) The equilibrium point with Wolbachia-infected mosquitoes exists when $R_W > 1$ while non-Wolbachia-infected mosquitoes are extinct, i.e.

$$E_W = \left[0,0, \frac{R_W - 1}{a}, \frac{R_W - 1}{af_{W_P}}\right]^T.$$

iv) The equilibrium point for the two types of mosquitoes that exist is

$$E_{MW} = \begin{bmatrix} \dfrac{a(R_M - 1) - b(R_W - 1)}{a^2 - b^2} \\ \dfrac{a(R_M - 1) - b(R_W - 1)}{f_{M_P}(1 - c)(a^2 - b^2)} \\ \dfrac{a(R_W - 1) - b(R_M - 1)}{a^2 - b^2} \\ \dfrac{a(R_W - 1) - b(R_M - 1)}{f_{W_P}(a^2 - b^2)} \end{bmatrix}$$

when $A_M > 0$, $A_W > 0$, and $C > 0$ or $A_M < 0$, $A_W < 0$, and $C < 0$ with

$$A_M = a(R_M - 1) - b(R_W - 1),$$

$$A_W = a(R_W - 1) - b(R_M - 1),$$

and

$$C = a^2 - b^2.$$

3.3. Asymptotically Local Stability Analysis for Each Equilibrium Point

In the previous section, we presented the equilibrium point of the model in (1). In this section, we present a theorem regarding the asymptotically local stability for each of the equilibrium points of (1). The asymptotically local stability of the model in (1) is presented in Theorem 1 below. Endemic equilibrium represents that the epidemic occurs. Based on (1), an endemic equilibrium point is obtained as follows:

Theorem 2

For the model in (1) that

i) If $R_M < 1$ and $R_W < 1$, then the equilibrium point E_0 is locally stable asymptotically.

ii) If $R_M > 1$ and $a(R_W - 1) < b(R_M - 1)$, then the equilibrium point E_M is locally stable asymptotically.

iii) If $R_W > 1$ and $a(R_M - 1) < b(R_W - 1)$, then the equilibrium point E_W is locally stable asymptotically.

iv) If $a > b$, $a(R_M - 1) > b(R_W - 1)$, and $a(R_W - 1) > b(R_M - 1)$, then the local equilibrium point E_{MW} is asymptotically stable.

Proof. In this case, we use the M-Matrix theory to prove the local stability condition asymptotically in Theorem 2. Simply put, the steps taken to determine the local stability asymptotically from the model in (1), include:

i) Determine the Jacobian matrix from the model in (1) which then this matrix is the J matrix in Theorem 1.
ii) Determine the matrix S that corresponds to the conditions in Theorem 1.
iii) Determine the Jacobian matrix for the equilibrium point of the model in (1) by substituting it in the Jacobian matrix obtained in step i).
iv) Transform the Jacobian matrix in step iii into a matrix $G = I - SJ(E^*)S^{-1}$.
v) Determine the conditions that satisfy that G is an M-Matrix in Definition 1.
vi) Step iii) is repeated as many times as the equilibrium point of the model in (1).

Based on these steps, the Jacobian matrix of the model in (1) is

$$J(E^*) = J\left(\begin{bmatrix} M_A^* \\ M_P^* \\ W_A^* \\ W_P^* \end{bmatrix}\right) = \begin{bmatrix} 0 & f_{M_P}(1-c) & 0 & 0 \\ P_M(1+bW_A^*) & 0 & -P_M b M_A^* & 0 \\ 0 & 0 & 0 & f_{W_P} \\ -P_W b W_A^* & 0 & P_W(1+bM_A^*) & 0 \end{bmatrix} \qquad (3)$$

where

$$P_M = \frac{s_{M_A}}{(aM_A^* + bW_A^* + 1)^2} \text{ and } P_W = \frac{s_{W_P}}{(aM_A^* + bW_A^* + 1)^2}.$$

After the Jacobian matrix of the model in (1) is obtained, the next thing that must be obtained is to determine the S matrix that corresponds to the $J(E^*)$ matrix in (3) based on the M-Matrix theory. Since the guaranteed equilibrium point is non-negative, so the matrix elements $J(E^*)$ in the i-the row for $i = 1,2$ with the j-the column for $j = 3,4$ and the i-the row for $i = 3,4$ with the j-the column for $j = 1,2$ is greater than or equal to zero. Based on Theorem 1, the selected matrix S is

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The next step is to determine the Jacobian matrix for each equilibrium point of the model in (1). Each equilibrium point obtained from the model substituted in (3). Asymptotically local stability analysis for each equilibrium point is presented as follows:

i) The first is to determine the local stability asymptotically from the equilibrium point E_0. The Jacobian matrix for the equilibrium point E_0, i.e.

$$J(E_0) = \begin{bmatrix} 0 & f_{M_P}(1-c) & 0 & 0 \\ s_{M_A} & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{W_P} \\ 0 & 0 & s_{W_A} & 0 \end{bmatrix}.$$

After obtaining the Jacobian matrix for E_0, based on M-Matrix theory, the next step is to transform the Jacobian matrix for E_0 or $J(E_0)$ into a matrix $G = I - SJ(E_0)S^{-1}$. The obtained matrix G, that is

$$G = \begin{bmatrix} 1 & -f_{M_P}(1-c) & 0 & 0 \\ -s_{M_A} & 1 & 0 & 0 \\ 0 & 0 & 1 & -f_{W_P} \\ 0 & 0 & -s_{W_A} & 1 \end{bmatrix}.$$

Note that all values of $g_{ij} \leq 0$ for $i \neq j$ therefore the first condition is said to be M-Matrix in Definition 1 is met. The next step is to show that all the minor principles of G are positive. Based on the calculations obtained,

$$PM_1 = |g_{11}| = 1, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = 1 - R_M, \qquad PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = 1 - R_M,$$

and

$$PM_4 = |G| = (1 - R_M)(1 - R_W).$$

In all minor principals of G, only PM_1 is positive while the others cannot be guaranteed to be positive. Note that PM_2 and PM_3 will be positive if $R_M < 1$. Since $R_M < 1$, consequently PM_4 will be positive if $R_W < 1$. Therefore, all minor principles of G will be positive if $R_M < 1$ and $R_W < 1$. So, G is said to be an M-Matrix, and the local equilibrium point E_0 is asymptotically stable if $R_M < 1$ and $R_W < 1$ are satisfied.

ii) The second is to determine the local stability asymptotically from the equilibrium point E_M. The Jacobian matrix for the equilibrium point E_M, i.e.

$$J(E_M)=\begin{bmatrix} 0 & f_{M_P}(1-c) & 0 & 0 \\ \dfrac{1}{f_{M_P}(1-c)R_M} & 0 & -\dfrac{bs_{M_A}(R_M-1)}{af_{M_P}(1-c)R_M} & 0 \\ 0 & 0 & 0 & f_{W_P} \\ 0 & 0 & -\dfrac{as_{W_A}}{(R_M-1)b+a} & 0 \end{bmatrix}.$$

After obtaining the Jacobian matrix for E_M, based on the M-Matrix theory, the next step is to transform the Jacobian matrix for E_M or $J(E_M)$ into a $G = I - SJ(E_M)S^{-1}$ matrix. The obtained matrix G, that is

$$G=\begin{bmatrix} 1 & -f_{M_P}(1-c) & 0 & 0 \\ -\dfrac{1}{f_{M_P}(1-c)R_M} & 1 & -\dfrac{bs_{M_A}(R_M-1)}{aR_M^2} & 0 \\ 0 & 0 & 1 & -f_{W_P} \\ 0 & 0 & -\dfrac{as_{W_A}}{(R_M-1)b+a} & 1 \end{bmatrix}.$$

Note that, since $R_M > 1$ consequently all values $g_{ij} \leq 0$ for $i \neq j$ are satisfied, the first condition is said to be M-Matrix in Definition 1 is satisfied. The next step is to show that all the minor principles of G are positive. Based on the calculations obtained,

$$PM_1 = 1, \qquad PM_2 = PM_3 = \frac{R_M-1}{R_M},$$

and

$$PM_4 = |G| = -\frac{(R_M-1)(A_W)}{R_M\big(a+b(R_M-1)\big)}.$$

PM_1, PM_2, and PM_3 are positive because $R_M > 1$. Since $R_M > 1$ consequently PM_4 will be positive if $a(R_W-1) - b(R_M-1) < 0$ or $a(R_W-1) < b(R_M-1)$. Therefore, all minor principles of G will be positive if $R_M > 1$ and $a(R_W-1) < b(R_M-1)$. So, G is said to be an M-Matrix, and the equilibrium point E_M is locally stable asymptotically if it is satisfied that $R_M > 1$ and $a(R_W-1) < b(R_M-1)$.

iii) The third is to determine the local stability asymptotically from the equilibrium point E_W. The Jacobian matrix for the equilibrium point E_W, i.e.

$$J(E_W)=\begin{bmatrix} 0 & f_{M_P}(1-c) & 0 & 0 \\ \dfrac{as_{M_A}}{a+b(R_W-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{W_P} \\ -\dfrac{b(R_W-1)}{af_{W_P}R_W} & 0 & \dfrac{1}{f_{W_P}R_W} & 0 \end{bmatrix}.$$

After obtaining the Jacobian matrix for E_W, based on M-Matrix theory, the next step is to transform the Jacobian matrix for E_W or $J(E_W)$ into a matrix $G=I-SJ(E_W)S^{-1}$. The obtained matrix G, that is

$$G=\begin{bmatrix} 1 & -f_{M_P}(1-c) & 0 & 0 \\ -\dfrac{as_{M_A}}{a+b(R_W-1)} & 1 & 0 & 0 \\ 0 & 0 & 1 & -f_{W_P} \\ -\dfrac{b(R_W-1)}{af_{W_P}R_W} & 0 & -\dfrac{1}{f_{W_P}R_W} & 1 \end{bmatrix}.$$

Note that, since $R_W>1$ consequently all values of $g_{ij}\le 0$ for $i\ne j$ are satisfied, the first condition is said to be M-Matrix in Definition 1 is satisfied. The next step is to show that all the minor principles of G are positive. Based on the calculations obtained,

$$PM_1=1,\qquad PM_2=PM_3=-\frac{A_M}{a+b(R_W-1)},$$

and

$$PM_4=-\frac{(R_W-1)A_M}{R_W\big(a+b(R_W-1)\big)}.$$

It is clear that $PM_1>0$. Because $R_W>1$ consequently PM_2 and PM_3 are positive if $a(R_M-1)-b(R_W-1)<0$ or $a(R_M-1)<b(R_W-1)$. Because $R_W>1$ and $a(R_M-1)<b(R_W-1)$ consequently PM_4 is positive. Therefore, all minor principles of G will be positive if $R_W>1$ and $a(R_M-1)<b(R_W-1)$. So, G is said to be an M-Matrix, and the equilibrium point E_W is locally stable asymptotically if it is satisfied that $R_W>1$ and $a(R_M-1)<b(R_W-1)$.

iv) The fourth or last is to determine the local stability asymptotically from the equilibrium point E_{MW}. The Jacobian matrix for the equilibrium point E_{MW}, i.e.

$$J(E_{MW}) = \begin{bmatrix} 0 & f_{M_P}(1-c) & 0 & 0 \\ \frac{A_1}{CR_M f_{M_P}(1-c)} & 0 & -\frac{bA_M}{CR_M f_{M_P}(1-c)} & 0 \\ 0 & 0 & 0 & f_{W_P} \\ -\frac{bA_W}{CR_W f_{W_P}} & 0 & \frac{A_2}{CR_W f_{W_P}} & 0 \end{bmatrix},$$

where

$$A_1 = a^2 + b(a(R_W - 1) - bR_M) \text{ and } A_2 = a^2 + b(a(R_M - 1) - bR_W).$$

After obtaining the Jacobian matrix for E_{MW}, based on M-Matrix theory, the next step is to transform the Jacobian matrix for E_{MW} or $J(E_{MW})$ into a matrix $G = I - SJ(E_{MW})S^{-1}$. The obtained matrix G, that is

$$G = \begin{bmatrix} 1 & -f_{M_P}(1-c) & 0 & 0 \\ -\frac{A_1}{CR_M f_{M_P}(1-c)} & 1 & -\frac{bA_M}{CR_M f_{M_P}(1-c)} & 0 \\ 0 & 0 & 1 & -f_{W_P} \\ -\frac{bA_W}{CR_W f_{W_P}} & 0 & -\frac{A_2}{CR_W f_{W_P}} & 1 \end{bmatrix}.$$

The next step is to ensure that all non-diagonal elements of the G matrix are non-positive. The existence condition of the equilibrium point E_{MW}, namely A_M, A_W, and C has the same sign (positive/negative) so that g_{23} and g_{31} are negative. Then, so that $g_{21} \leq 0$ and $g_{43} < 0$ must be fulfilled, A_1, A_2, and C must have the same sign (non-positive/non-negative). Therefore, these conditions result in the fulfillment of all values of $g_{ij} \leq 0$ for $i \neq j$ so that the first condition is said to be M-Matrix in Definition 1 is fulfilled. The next step is to show that all the minor principles of G are positive. Based on the calculations obtained,

$$PM_1 = 1, \qquad PM_2 = PM_3 = \frac{aA_M}{R_M C},$$

and

$$PM_4 = \frac{A_M A_W}{R_M R_W C}.$$

Note that PM_1 is positive. Then, for PM_2, PM_3, and PM_4 to be positive, it must be fulfilled $A_M > 0$, $A_W > 0$, and $C > 0$ or $a(R_M - 1) > b(R_M - 1)$, $a(R_W - 1) > b(R_M - 1)$, and $a^2 > b^2$ or $a > b$. Therefore, all minor principals of G will be positive if $a > b$, $a(R_M - 1) > b(R_M - 1)$, and $a(R_W - 1) > b(R_M - 1)$.

Recall that the first and second terms of the M-Matrix must be considered or related to each other. Because in the second condition (all minor principals of G) namely $a > b$, consequently A_1 and A_2 in the first condition (all elements of $g_{ij} \leq 0$) must be positive. Note that

$$A_1 = a^2 + b(a(R_W - 1) - bR_M) > 0 \rightarrow a^2 > -b(a(R_W - 1) - bR_M) \quad (4)$$

Then, because $a(R_W - 1) > b(R_M - 1)$ consequently

$$a(R_W - 1) - bR_M > -b \quad (5)$$

Therefore, if equation (5) is substituted into equation (4) and $a > b$, it is clear that $A_1 > 0$. Besides that, because $a > b$ and $a(R_M - 1) > b(R_W - 1)$ then with the same treatment as in A_1, it is clear that $A_2 > 0$. Therefore, G is said to be an M-Matrix if $a > b$, $a(R_M - 1) > b(R_W - 1)$, and $a(R_W - 1) > b(R_M - 1)$. Thus, the equilibrium point E_{MW} will be locally stable asymptotically if $a > b$, $a(R_M - 1) > b(R_W - 1)$, and $a(R_W - 1) > b(R_M - 1)$ are satisfied. This completes the proof

□

4. NUMERICAL SOLUTIONS

In this section, we present a numerical simulation of the analysis carried out in Section 5 regarding the local stability analysis for each of the equilibrium points obtained in Section 4. This section aims to provide proof of Theorem 2 which is then interpreted in the form of images. The figures will show the conditions under which the system is asymptotically stable for each equilibrium point of the model in (1) based on the given parameters. Since this simulation only aims to provide numerical proof of Theorem 2, the parameters presented in the simulation are only conjectural parameters.

The numerical simulation of the model in (1) is divided into 4 cases according to the number of equilibrium points. In this case, we assumed that the level of intraspecific competition between mosquitoes without Wolbachia and with Wolbachia was $a = 0.05$ and the level of interspecific competition between the two mosquitoes was $b = 0.01$. Then, we assumed the level of maternal vertical transmission and the mosquito population CI mechanism was $c = 0.35$. Furthermore, other parameters for this simulation are presented as follows.

i) $f_{M_P} = 3,\ s_{M_A} = 0.5,\ f_{W_P} = 1,\ s_{W_A} = 0.9$, so that $R_M = 0.975$ and $R_W = 0.9$.

ii) $f_{M_P} = 10,\ s_{M_A} = 0.8,\ f_{W_P} = 1,\ s_{W_A} = 0.9$, so that $R_M = 5.2$ and $R_W = 0.9$.

iii) $f_{M_P} = 2,\ s_{M_A} = 0.7,\ f_{W_P} = 10,\ s_{W_A} = 0.9$, so that $R_M = 0.91$ and $R_W = 9$.

iv) $f_{M_P} = 20,\ s_{M_A} = 0.7,\ f_{W_P} = 20,\ s_{W_A} = 0.6$, so that $R_M = 9.1$ and $R_W = 12$.

FIGURE 1. Population growth of each age phase in case i.

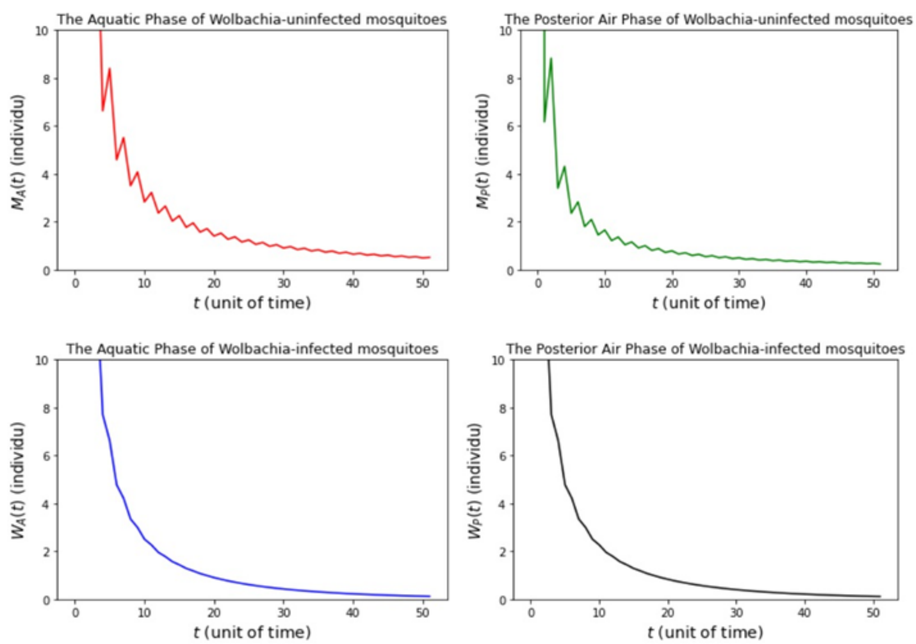
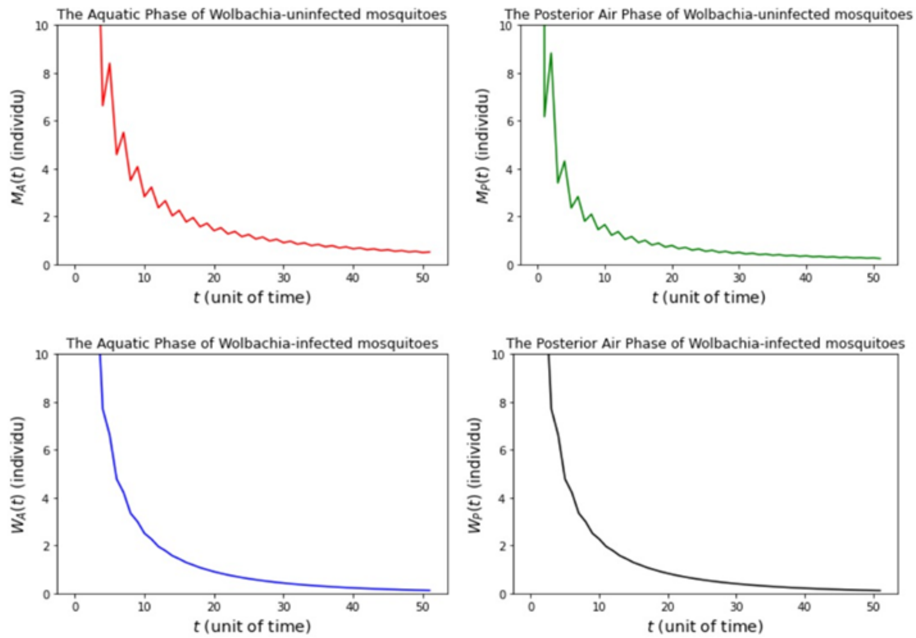


FIGURE 2. Population growth of each age phase in case ii.



The results of the simulation are presented graphically in Figure 1-Figure 4. Figure 1 shows that the local stable system is asymptotically towards the equilibrium point E_0 if it satisfies condition i of Theorem 2. In that sense, the growth of both mosquito species without Wolbachia infection and with Wolbachia infection is the same. equal to extinction. Then**,** Figure 2 shows that the local stable system asymptotically goes to the equilibrium point $E_M = [84,12.92,0]^T$ if it satisfies condition ii of Theorem 2. In a sense, in an ecosystem, mosquitoes infected with Wolbachia are extinct while mosquitoes without infected with surviving Wolbachia. As a result, prevention of dengue disease cannot be achieved. Furthermore, Figure 3 shows that the local stable system is asymptotically toward the equilibrium point $E_W = [0,0,160,16]^T$. In a sense, mosquitoes without Wolbachia infection are extinct while mosquitoes infected with Wolbachia survive in a particular ecosystem. The good side is that dengue disease can be reduced while the bad side is that we can no longer find wild (pure) mosquitoes that are not infected with Wolbachia. Last but not least, Figure 4 shows that the local stable system is asymptotically toward the equilibrium point $E_{MW} = [122.92.9.45,195.42.9.77]^T$. In a sense, both mosquitoes can coexist which makes it possible that dengue disease can be controlled and wild mosquitoes are not extinct.

FIGURE 3. Population growth of each age phase in case iii.

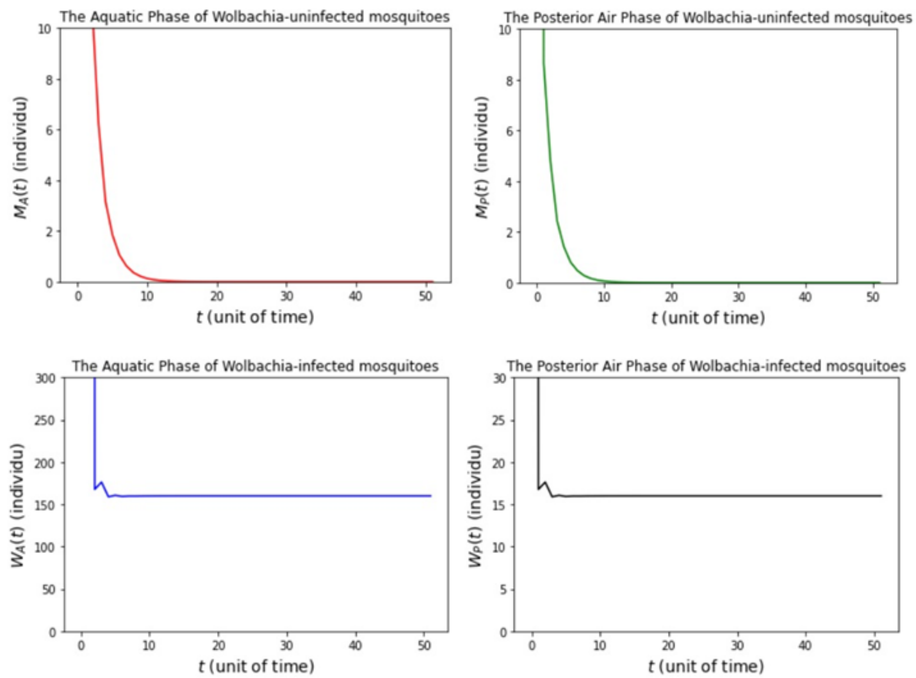
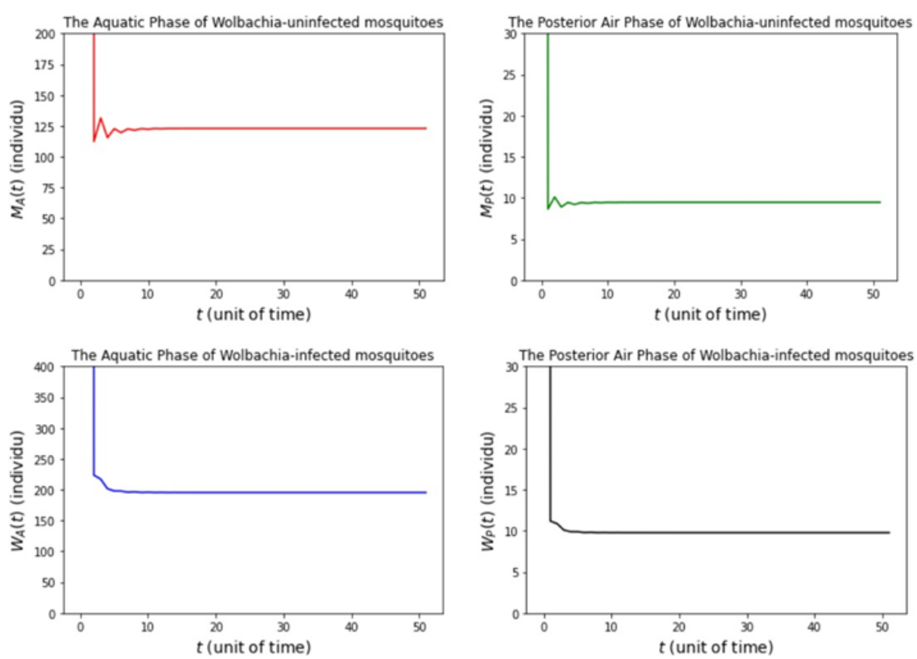


FIGURE 4. Population growth of each age phase in case iv.



5. CONCLUSION

In this paper, we have formulated a population growth model for the *Aedes aegypti* mosquito using the Leslie multispecies matrix model. In the model, the first species is assumed to be a population of *Aedes aegypti* mosquitoes not infected with Wolbachia and the second species is assumed to be a population of *Aedes aegypti* mosquitoes infected with Wolbachia. The model was formulated to be studied to obtain equilibrium points of equilibrium points and to analyze local stability asymptotically for equilibrium points using M-Matrix theory. The results show that there are four equilibrium points, including an equilibrium point where both mosquitoes experience extinction, an equilibrium point where only wild mosquitoes that are not infected with Wolbachia survive, an equilibrium point where only wild mosquitoes infected with Wolbachia survive, and an equilibrium point where both types of mosquitoes survive. The determination of the existing conditions of the equilibrium points and the asymptotically local stability of the equilibrium points of the model are influenced by each inherent net reproduction and the level of competition for both types of mosquitoes.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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