Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2023, 2023:14
https://doi.org/10.28919/cmbn/7864
ISSN: 2052-2541

A FRACTIONAL SITR MODEL FOR DYNAMIC OF TUBERCULOSIS SPREAD

MUHAFZAN*, ZULAKMAL, NARWEN, AHMAD IQBAL BAQI, A. G. LESTARI, M. OKTAVIANI

Department of Mathematics and Data Science, Faculty of Mathematics and Natural Science, Andalas University, Padang 25163, Indonesia

Abstract. This work presents a fractional SITR mathematical model that investigates the Tuberculosis (TB) spread in a human population. It was shown that disease-free and endemic equilibrium stability depended on the basic reproduction number. These results are in accordance with the epidemic theory. A numerical example is given to demonstrate the validity of the results. The results show that the infected subpopulation increases in the absence of special treatment.

Keywords: fractional-order derivative; SITR model; basic reproduction number; equilibrium.

2020 AMS Subject Classification: 49L20, 92D25.

1. Introduction

Tuberculosis (TB) is an infectious disease caused by Mycobacterium tuberculosis, and is usually acquired through airborne infection from active TB cases [1, 2]. According to the World Health Organization, one third of the world's population is infected with tuberculosis either latently or actively. Despite effective antimicrobial chemotherapy, tuberculosis infection remains a leading cause of death from an infectious disease [3].

---

*Corresponding author

E-mail address: muhafzan@sci.unand.ac.id

Received December 21, 2022

Mathematical modelling has been proven to be important in better understanding the transmission dynamics of TB and evaluating the effectiveness of various control and prevention strategies. Several studies on the spread of TB disease using mathematical models have been carried out by many researchers, see [4, 5, 6, 7] for models in form of the nonlinear differential equations.

The $SITR$ model is one of among the models of the spread of tuberculosis in the form of non-linear differential equations that are widely discussed [8, 9]. In this model, the observed human population (N) is divided into fourth epidemiological sub-compartments denoted by susceptible $S(t)$, TB active (infected) $I(t)$, under treatment $T(t)$, and recovered individuals after treatment $R(t)$ as described in the compartment diagram in Figure 1. Based on the Figure 1, the transmission model for TB dynamics is given by the system of non-linear differential equations (1) [9].

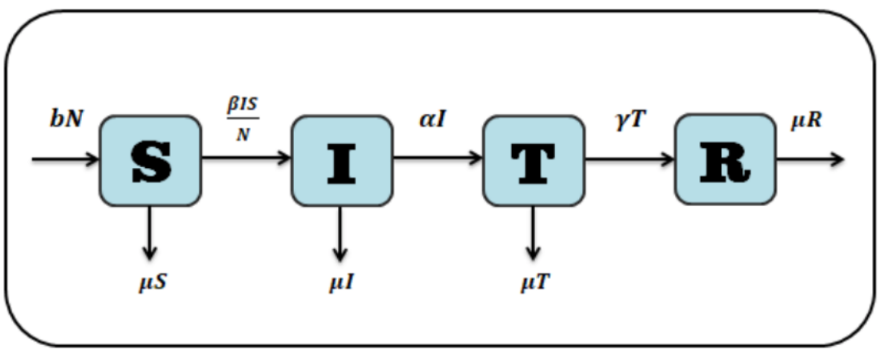

FIGURE 1. Compartment diagram for SITR model

$$
\begin{aligned}
\dot{S} &= bN - \beta \frac{I}{N} S - \mu S \\
\dot{I} &= \beta \frac{I}{N} S - \alpha I - \mu I \\
\dot{T} &= \alpha I - \gamma T - \mu T \\
\dot{R} &= \gamma T - \mu R,
\end{aligned}
\tag{1}
$$

with the initial conditions $S(0) = S_0, I(0) = I_0, T(0) = T_0, R(0) = R_0$ and the involve various parameters in (1) are described in Table 1. The population total is $N = S + I + T + R$. Along

TABLE 1. Parameter with description occuring in the model (1).

Parameter	**Description**
b	Birth rate
β	Transmission rate
μ	Natural death rate
α	Progression rate from I to T
γ	Rate at which treated people leave T class

with the development of the fractional-order differential equation, recently the issue on development of mathematical models in form of the fractional-order non-linear differential equation are widely discussed by many researchers, see [10, 11, 12, 13, 14, 15, 16]. In this paper, we modify the model (1) by replacing usual derivative into fractional-order derivative such that the model (1) can be written as a following new model:

$$\begin{aligned} \Delta^{(\delta)}S &= bN - \beta\frac{I}{N}S - \mu S \\ \Delta^{(\delta)}I &= \beta\frac{I}{N}S - \alpha I - \mu I \\ \Delta^{(\delta)}T &= \alpha I - \gamma T - \mu T \\ \Delta^{(\delta)}R &= \gamma T - \mu R, \end{aligned} \tag{2}$$

where $\Delta^{(\delta)}$ is the Caputo fractional derivative operator of oder δ with $0 < \delta < 1$. As a new $SITR$ model, we study the stability of the disease-free equilibrium and endemic equilibrium of the model (2). To the best of the author's knowledge, this issue has not been solved yet to date. Therefore the results of this work constitute a novelties at once a new contribution in the field of fractional-order epidemic dynamic.

The paper is organized as follows: Section 2 considers some useful results about Caputo fractional derivative and stability of the fractional-order nonlinear system. The main result of this article is presented in the section 3. Section 4 concludes the paper.

2. Some Useful Results

In this section we recall several mathematical tools used in this study. The Caputo fractional derivative of order δ with $\delta \in (k-1,k)$, $k \in \mathbb{N}$ for the integrable vector function $\mathbf{x} : [0,\infty) \to \mathbb{R}^n$, is defined by

$$\Delta^{(\delta)}\mathbf{x}(t) = \frac{1}{\Gamma(k-\delta)} \int_0^t (t-\tau)^{k-\delta-1} \Delta^{(k)}\mathbf{x}(\tau) d\tau \tag{3}$$

where $\Gamma(.)$ is the Euler Gamma function [17], and $\Delta^{(k)}\mathbf{x}(.)$ is the usual kth derivative of function $\mathbf{x}(.)$.

Let us consider the fractional-order nonlinear system involving Caputo derivative

$$\Delta^{(\delta)}\mathbf{x}(t) = \mathbf{f}(t,\mathbf{x}(t)) \tag{4}$$

with suitable initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector of the system (4), $\mathbf{f} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$. If $\mathbf{f}$ is linear, the system (4) can be written as

$$\Delta^{(\delta)}\mathbf{x}(t) = A\mathbf{x}(t)), \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$. The point $\mathbf{x}^*$ is said the equilibrium point of the system (4) if $\mathbf{f}(t,\mathbf{x}^*) = \mathbf{0}$.

Theorem 2.1. [18] *The fractional-order linear system (5) with $\delta \in (0,1)$, is asymptotically stable if*

$$|\arg(\lambda_j)| > \frac{\delta\pi}{2}, \tag{6}$$

and λ_j, $j = 1,2,\cdots,n$ are eigenvalues of the matrix A.

Theorem 2.2. [18] *Let $\mathbf{x} = \mathbf{x}^*$ is an equilibrium of the the fractional-order system (4) with $\delta \in (0,1)$. The equilibrium point $\mathbf{x} = \mathbf{x}^*$ is asymptotically stable if*

$$|\arg(\lambda)| > \frac{\delta\pi}{2}, \tag{7}$$

for all roots λ of the equation

$$|J_{\mathbf{x}^*} - \lambda I| = 0 \tag{8}$$

where $J_{\mathbf{x}^*}$ is the Jacobian matrix of system (4) at the equilibrium $\mathbf{x}^*$.

3. Stability Analysis

The dynamical behavior of the model can be classified by the basic reproductive number [6]. By applying the next generation technique presented in [6], the basic reproduction number, denoted by $\mathbb{R}_0$, for the model (2) is

$$\mathbb{R}_0 = \frac{\beta b}{\mu(\mu+\alpha)}. \tag{9}$$

The equilibrium points of the model (2) is evaluated by solving the following equations:

$$\Delta^{(\delta)}S = \Delta^{(\delta)}I = \Delta^{(\delta)}T = \Delta^{(\delta)}R = 0. \tag{10}$$

The disease-free equilibrium, denoted by $\mathbb{K}_0$, of the fractional order TB model (2) is obtained by assuming $I = 0$, such that the disease-free equilibrium is

$$\mathbb{K}_0 = \left(\frac{bN}{\mu}, 0, 0, 0\right). \tag{11}$$

The endemic equilibrium, denoted by $\mathbb{K}_1$, of the fractional order TB model (2) exists if $\mathbb{R}_0 > 1$. Thus the endemic equilibrium of the model (2) is $\mathbb{K}_1 = (S^*, I^*, T^*, R^*)$, where

$$\begin{aligned} S^* &= \frac{(\mu+\alpha)N}{\beta}, \\ I^* &= \frac{(\beta b - \mu(\mu+\alpha))N}{\beta(\mu+\alpha)}, \\ T^* &= \frac{(\beta b - \mu(\mu+\alpha))\alpha N}{\beta(\mu+\alpha)(\gamma+\mu)}, \\ R^* &= \frac{\gamma\alpha N(\beta b - \mu(\mu+\alpha))}{\beta(\mu+\alpha)(\gamma+\mu)}. \end{aligned} \tag{12}$$

We will analyze the stability of these two equilibrium points. First of all, the Jacobian matrix of the vector field corresponding to model (2) is

$$J = \begin{bmatrix} -\frac{\beta I}{N} - \mu & -\frac{\beta S}{N} & 0 & 0 \\ \frac{\beta I}{N} & \frac{\beta S}{N} - \mu - \alpha & 0 & 0 \\ 0 & \alpha & -\gamma - \mu & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix}. \tag{13}$$

The stability of the disease-free equilibrium $\mathbb{K}_0$ is given in the following theorem.

Theorem 3.1. *If $\mathbb{R}_0 < 1$, then $\mathbb{K}_0$ is asymptotically stable and becomes unstable when $\mathbb{R}_0 \geq 1$.*

Proof. For $\mathbb{K}_0 = (\frac{bN}{\mu}, 0, 0, 0)$, the characteristic equation of (13) is given by

$$|J_{\mathbb{K}_0} - \lambda I| = \left(\frac{\beta b}{\mu} - \mu - \alpha - \lambda\right)(-\mu - \lambda)(-\gamma - \mu - \lambda)(-\mu - \lambda) = 0. \tag{14}$$

The equation (14) shows that the eigenvalues of $J_{\mathbb{K}_0}$ are $\frac{\beta b}{\mu} - \mu - \alpha, -(\gamma + \mu)$ and $-\mu$. One can see that all eigenvalues of (14) satisfy $|\arg(\lambda_j)| > \frac{\delta\pi}{2}$ if $\mathbb{R}_0 < 1$ and at least one eigenvalue satisfy $|\arg(\lambda_j)| < \frac{\delta\pi}{2}$ when $\mathbb{R}_0 > 1$. Hence, $\mathbb{K}_0$ is locally asymptotically stable if $\mathbb{R}_0 < 1$ and becomes unstable if $\mathbb{R}_0 > 1$. □

We now consider the stability of the endemic equilibrium $\mathbb{K}_1$. The Jacobian matrix of (13) at $\mathbb{K}_1$ becomes

$$J_{\mathbb{K}_1} = \begin{bmatrix} \frac{-\beta b}{\mu+\alpha} & -\mu - \alpha & 0 & 0 \\ \frac{\beta b}{\mu+\alpha} - \mu & 0 & 0 & 0 \\ 0 & \alpha & -\gamma - \mu & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix} \tag{15}$$

The stability of the endemic equilibrium $\mathbb{K}_1$ is given in the following theorem.

Theorem 3.2. *If $\mathbb{R}_0 > 1$, then $\mathbb{K}_1$ is asymptotically stable and becomes unstable when $\mathbb{R}_0 < 1$.*

Proof. The characteristic equation of (15) is given by

$$|J_{\mathbb{K}_1} - \lambda I| = (-\mu - \lambda)(-\gamma - \mu - \lambda)\left(\lambda^2 + \left(\frac{\beta b}{\mu + \alpha}\right)\lambda + \beta b - \mu(\mu + \alpha)\right) = 0. \tag{16}$$

The eigenvalues of $J_{\mathbb{K}_1}$ are

$$\lambda_1 = -\mu, \quad \lambda_2 = -\mu - \gamma,$$

and $\lambda_{3,4}$ that constitute roots of the equation

$$\lambda^2 + \left(\frac{\beta b}{\mu+\alpha}\right)\lambda + (\beta b - \mu(\mu+\alpha)) = 0. \tag{17}$$

Observe that if $\mathbb{R}_0 > 1$, then λ_i satisfies $|\arg(\lambda_i)| > \frac{\delta\pi}{2}$, for $i = 1,2,3,4$, thus $\mathbb{K}_1$ is locally asymptotically stable. Otherwise, $\mathbb{K}_1$ is unstable. □

In order to show the validity of the results, let us consider the following numerical example. For the model (2), let $b = 0.3, \mu = 0.1, \beta = 0.5, \alpha = 0.3, \gamma = 0.85$ and $N = 1$. The initial conditions are $S_0 = 0.2921, I(0) = 0.2921, T(0) = 0.2921$ and $R(0) = 0.1237$. Base on these parameter values, we find the basic reproduction number $\mathbb{R}_0 = 3.75$, thus the equiblirium point is endemic is $\mathbb{K}_1 = (0.8, 0.55, 0.1737, 0.1476)$. Graph of the susceptible subpopulation, infected subpopulation, treatment subpopulation and recovered subpopulation for several fractional order, respectively, are given in the Figure 2, Figure 3, Figure 4 and Figure 5. One can see that the infected subpopulation increases in the absence of special treatment.

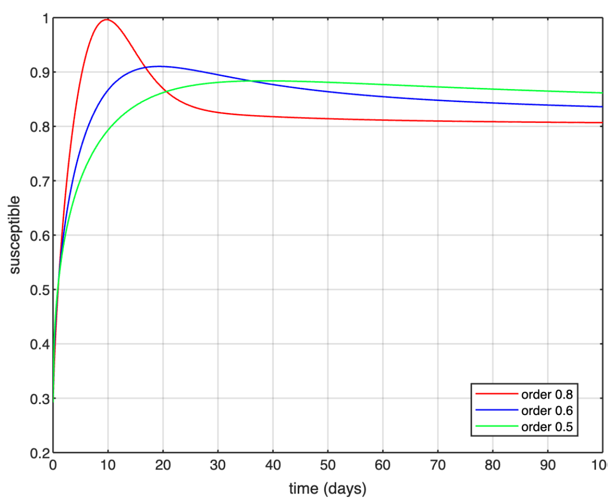

FIGURE 2. Susceptible Subpopulation

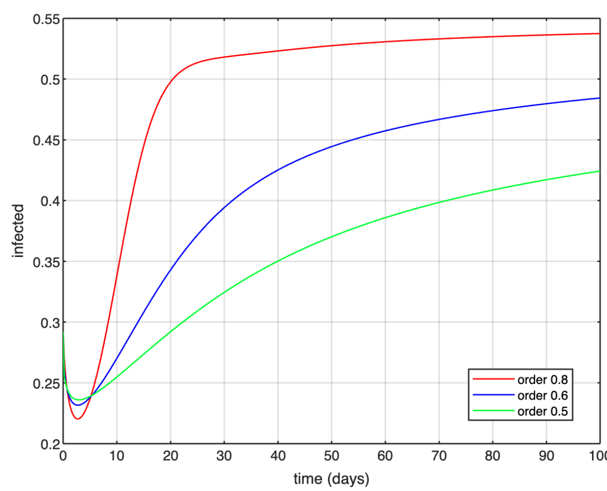

FIGURE 3. Infected Subpopulation

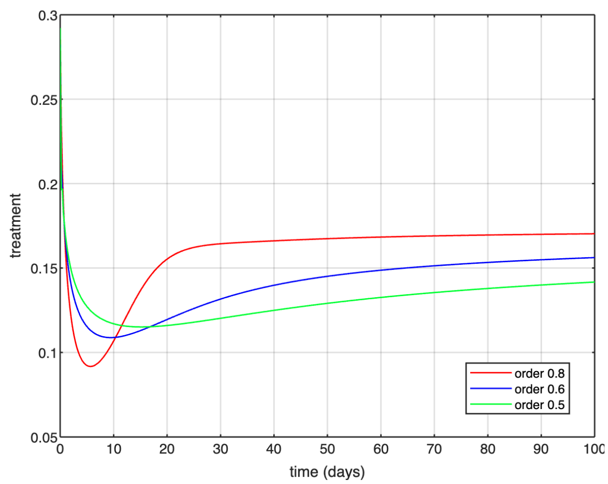

FIGURE 4. Treatment Subpopulation

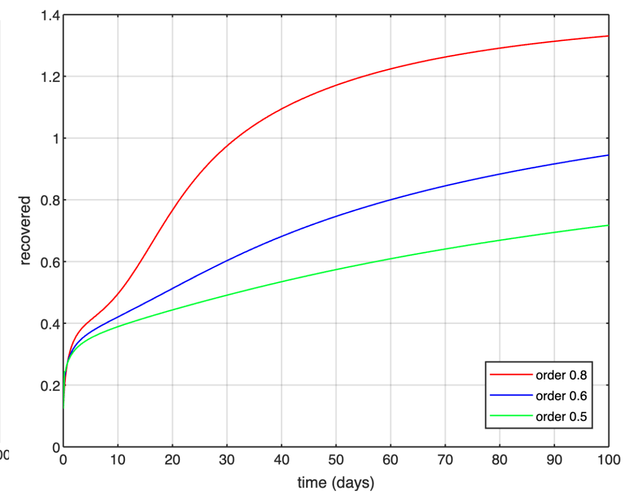

FIGURE 5. Recovered Subpopulation

4. Conclusion

We have find the fractional $SITR$ model for dynamic of tuberculosis spread. An example that illustrating the result has been presented. The analysis show that the $SITR$ model give the adequate information about spread of tuberculosis.

Acknowledgment

This work was supported by Universitas Andalas under Grant Riset Publikasi Terindeks, No. T/20/UN.16.17/PT.01.03/IS-RPT/ 2022.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] G. Bjune, Tuberculosis in the 21st century: an emerging pandemic? Norsk Epidemiol. 15 (2009), 133–139. https://doi.org/10.5324/nje.v15i2.211.

[2] T.F. Brewer, To control and beyond: moving towards eliminating the global tuberculosis threat, J. Epidemiol. Community Health. 58 (2004), 822–825. https://doi.org/10.1136/jech.2003.008664.

[3] W.H.O. Report, Global tuberculosis control: epidemiology, strategy, financing, World Health Organization, Geneva, (2009).

[4] A.O. Egonmwan, D. Okuonghae, Analysis of a mathematical model for tuberculosis with diagnosis, J. Appl. Math. Comput. 59 (2018), 129–162. https://doi.org/10.1007/s12190-018-1172-1.

[5] S. Whang, S. Choi, E. Jung, A dynamic model for tuberculosis transmission and optimal treatment strategies in South Korea, J. Theor. Biol. 279 (2011), 120–131. https://doi.org/10.1016/j.jtbi.2011.03.009.

[6] R. Xu, J. Yang, X. Tian, J. Lin, Global dynamics of a tuberculosis model with fast and slow progression and age-dependent latency and infection, J. Biol. Dyn. 13 (2019), 675–705. https://doi.org/10.1080/17513758.2019.1683628.

[7] S. Kasereka Kabunga, E.F. Doungmo Goufo, V. Ho Tuong, Analysis and simulation of a mathematical model of tuberculosis transmission in Democratic Republic of the Congo, Adv. Differ. Equ. 2020 (2020), 642. https://doi.org/10.1186/s13662-020-03091-0.

[8] R. Jerubet, G. Kimathi, M. Wanaina, Analysis and modeling of Tuberculosis transmission dynamics, J. Adv. Math. Computer Sci. 32 (2019), 1–14.

[9] D. Pan, Y. Zhang, Analysis of information propagation and control of a layered SITR model in complex networks, Front. Phys. 10 (2022), 985517. https://doi.org/10.3389/fphy.2022.985517.

[10] S. Ullah, M. Altaf Khan, M. Farooq, A fractional model for the dynamics of TB virus, Chaos, Solitons Fractals. 116 (2018), 63–71. https://doi.org/10.1016/j.chaos.2018.09.001.

[11] S. Ullah, M.A. Khan, M. Farooq, et al. A fractional model for the dynamics of tuberculosis infection using Caputo-Fabrizio derivative, Discrete Contin. Dyn. Syst. Ser. S. 13 (2020), 975-993.

[12] M.U. Rahman, M. Arfan, Z. Shah, et al. Nonlinear fractional mathematical model of tuberculosis (TB) disease with incomplete treatment under Atangana-Baleanu derivative, Alexandria Eng. J. 60 (2021), 2845–2856. https://doi.org/10.1016/j.aej.2021.01.015.

[13] L. Yulianti, A. Nazra, Zulakmal, et al. On discounted LQR control problem for disturbanced differential-algebraic system, Arch. Control Sci. 29 (2019), 147-156. https://doi.org/10.24425/acs.2019.127528.

[14] Muhafzan, A. Nazra, L. Yulianti, et al. On positive normalization of fractional order continuous disturbance singular system, Cybern. Phys. 8 (2019), 34-48.

[15] A. Nazra, Zulakmal, L. Yulianti, Linear quadratic optimization for fractional-order differential algebraic system of Riemann-Liouville type, Cybern. Phys. 9 (2020), 192-197.

[16] Muhafzan, A. Nazra, L. Yulianti, et al. On LQ optimization problem subject to fractional order irregular singular systems, Arch. Control Sci. 30 (2020), 745-756. https://doi.org/10.24425/acs.2020.135850.

[17] I. Podlubny, Fractional differential equations, Academic Press, California, 1999.

[18] Y. Ji, J. Lu, J. Qiu, Stability of equilibrium points for incommensurate fractional-order nonlinear systems, in: 2016 35th Chinese Control Conference (CCC), IEEE, Chengdu, 2016: pp. 10453–10458. https://doi.org/10.1109/ChiCC.2016.7555012.