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TUBERCULOSIS WITH CONTAMINATION BY THE CONSUMPTION OF UNPASTEURIZED DAIRY PRODUCTS: MATHEMATICAL MODELLING AND NUMERICAL SIMULATIONS

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Abstract. In this article, we present a mathematical model of transmission tuberculosis which takes account the contamination by consuming unpasteurised dairy products. We have determined the equilibrium points, whose local stability is guaranteed by Lyapunov's indirect method using the Routh-Hurwitz stability criterion. The results obtained show that, even if there are no bacteria in the environment, the probability of infection with tuberculosis remains high, and this is due to the consumption of unpasteurised dairy products. Finally, we introduce some numerical simulations graphics to validate our results.

Keywords: tuberculosis; unpasteurised dairy products; stability analysis; numerical simulation.
2020 AMS Subject Classification: 34D20, 92B05, 92-10.

1. INTRODUCTION

Tuberculosis is a disease that has been present since the dawn of humanity. It is a potentially fatal infectious disease caused by a bacterium called Mycobacterium tuberculosis or Koch's bacillus (BK), named by the doctor who discovered it in 1882, Robert Koch . This bacterium is an immobile, straight or slightly curved aerobic bacillus, its average length is $2\mu m$ to $4\mu m$ for

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a diameter of $0.2\mu m$ to $0.5\mu m$ [1].

This infectious disease is the 13^{th} leading cause of death and the second due to an infectious disease. In fact, 1.5 million people died of tuberculosis in 2020 and 214 000 of whom was also infected by the human immunodeficiency virus. According to the World Health Organization, 9.9 million people have developed tuberculosis worldwide, between 5.5 million men, 3.3 million women and 1.1 million children. 86% of new tuberculosis cases recorded in 2020 occurred in the 30 countries with the highest disease burden. Two-thirds of those cases are concentrated in eight countries, with India leading, followed by China, Indonesia, the Philippines, Pakistan, Nigeria, Bangladesh and South Africa [2]. Concerning Morocco, 29 000 cases of tuberculosis were detected and treated in 2020 [3]. The disease is concentrated in urban areas and has particularly affected disadvantaged neighborhoods in large cities, which is linked to unsanitary housing, overcrowding, malnutrition, precariousness and poverty. And, despite the great efforts made to prevent and control tuberculosis, which is still among the main priorities of the Ministry of Health and Social Welfare in Morocco, the disease continues to spread and settle in various parts of the country. As a result, this old global scourge claimed the lives of around 3000 person in 2020 in Morocco [2].

Tuberculosis often affects the lungs, this form is called pulmonary tuberculosis. Moreover, there is the extra-pulmonary tuberculosis which can attack other organs. It is transmitted from person to person through the air. When a person with pulmonary tuberculosis coughs, sneezes, or spits, he project tubercle bacillus into the air, then, an individual may become infected with tuberculosis if he or she inhales those tubercle bacillus[2]. In addition, tuberculosis can also be bovine, it is an infectious disease transmissible to humans caused mainly by the bacterium Mycobacterium bovis (M. bovis). Cattle are considered to be the host range of M. bovis, and are the principle source of infection for humans. Nevertheless, the disease has been reported in many other domesticated and non-domesticated animals [4]. So, an individual can contract this disease by inhaling infectious droplets, by contact with infected tissues in slaughterhouses and butcher shops, or by consuming raw milk from infected cows and unpasteurized dairy products sold by merchants walkers without any respect for health prevention conditions. According to

estimates by the World Organization for Animal Health, in some countries up to 10% of human tuberculosis cases are of bovine origin [5].

People infected with the tubercle bacillus have a 5 to 10% risk of developing the disease. This risk is much higher in people with a weakened immune system, especially those living with HIV, suffering from malnutrition or diabetes, smokers...[6]. Tuberculosis then becomes active, contagious and symptomatic. Symptoms such as cough with sometimes bloody sputum, chest pain, weakness, weight loss, fever, and night sweats may remain mild for many months. The disease is remediable provided an early diagnosis, and its treatment is based on combinations of antibiotics given for at least 6 months, sometimes longer, accompanied by patient support from a health worker or trained volunteer. Compliance with the protocol is absolutely necessary, otherwise drug resistance will appear. According to statistics, the diagnosis and treatment of tuberculosis has saved 66 million lives since 2000 [2].

Mathematical models appears to be a good tool for understanding the spread of infectious diseases. Several models exist in the literature to model the transmission of tuberculosis[7],[8]. The first one is built by the statistician Waaler in 1962. He used an unknown function of the number of infectious individuals to formulate the infection rate, and he predicted that the time trend of tuberculosis is improbable to increase, but his linear model did not model all the mechanics of transmission [9]. In 1967, Brogger developed a model based on Waaler's model. He changed the method used for calculating infection rates. His objective was to compare different control strategies such as treating more cases, the vaccination, and mass roentgenography. In the same year, ReVelle modeled the tuberculosis dynamics using a nonlinear system of ordinary differential equations. His aim is to develop an optimization model to select control strategies that could be carried out at a minimal cost [9]. Incomplete treatment, wrong therapy, and co-infection with other diseases, like HIV, may develop a new form of tuberculosis known as multi-drug resistant. Many models that include this type of tuberculosis have been developed [10], [11]. Most recently, an age-structured tuberculosis model is constructed to look at optimal vaccination strategy problems. The basic reproductive number is calculated and used to study cost related optimization problems [12].

In the present work, we model, analyze and simulate a mathematical model of the dynamics of tuberculosis with contamination by the consumption of unpasteurized dairy products. The purpose is to study the role of some control measures available in the event of an epidemic. This choice is motivated by an administrative note issued on 9 June 2022, written by the provincial delegate of the Ministry of Health of the Casablanca-Settat region, concerning an upsurge in cases of tuberculosis. He linked this significant increase in cases of tuberculosis to the consumption of dairy products sold by vendors in the streets and around mosques without any respect for hygiene conditions. First, we present the description of the model proposed, then we present its mathematical analysis. Here we show the positivity of the solution and its boundedness, the computation of different equilibrium points and the analysis of their stability. Numerical simulations are presented and their results are discussed.

2. Formulation of the Epidemic Model

The aim of this article is to study the role of some control measures available in an epidemic. These measures include reducing the rate of infection, eliminating dairy products contaminated with tuberculosis, and observing hygiene measures in the manufacturing process of these products.

We present, in this article, a mathematical model for the tuberculosis transmission. In this model, we are interested in three main components, namely: the human population, Mycobacterium tuberculosis and dairy products.

The human population comprises three compartments such that at the instant $t \geq 0$: $S(t)$ are the susceptible individuals to be infected, $I(t)$ are the infected individuals with tuberculosis, and $R(t)$ are the recovered individuals from tuberculosis. Thus, the size of the human population is given by: $N(t) = S(t) + I(t) + R(t)$.

The model diagram of tuberculosis transmission taking account consuming dairy products unpasteurised is as follow:

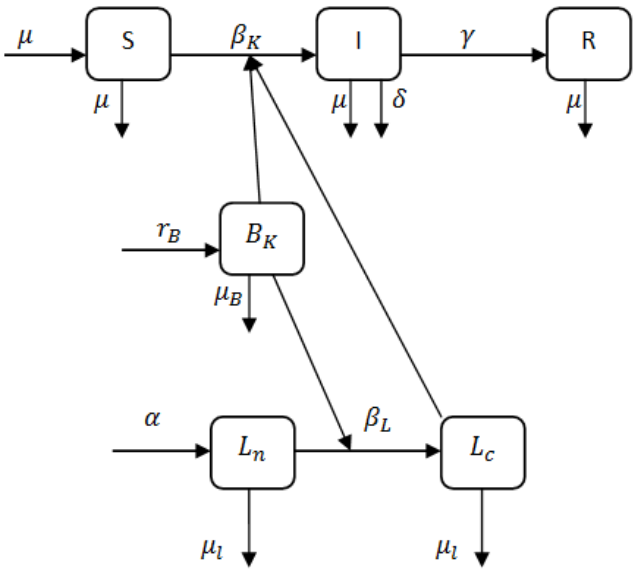

FIGURE 1. Compartmental model for the transmission dynamics of Tuberculosis.

Susceptible individuals grow at the rate μ, proportional to the total population N, and die at rate μ, proportional to the susceptible population S. Susceptible individuals can be infected with a force of infection β_K, that will be defined later. In fact, there are two modes of transmission of tuberculosis: the direct mode, that is to say when the bacterium is transmitted by close contact between an infected subject and a susceptible host, and the indirect mode, when the transmission takes place through a food such as unpasteurized dairy products. An infected individual may recover at the rate γ, die naturally at the rate μ, or die from tuberculosis at the rate δ.

The bacteria population obey the logistical law with a carrying capacity of K. Its growth rate is r_b and its death rate is μ_b.

As for dairy products, a distinction is made between non-contaminated products $L_n(t)$ and contaminated products $L_c(t)$. It is supposed that the dairy products $L(t)$ are produced at the rate α and that the uncontaminated ones become contaminated, either by direct contact with the bacteria or by contact of the contaminated products with a force of infection β_L, that will be defined later, and eliminated by a rate of μ_l. See Table 1 for a clear description of the parameters used.

Parameter	Description
r_B	Rate of bacterial growth
μ_B	Rate of bacterial death
μ	Recruitment or natural mortality rate
α	Production rate of dairy products
μ_l	Elimination rate of dairy products
γ	Recovery rate
δ	Disease caused death rate
β_K	Force of infection at which the susceptible individuals become infected
β_L	Force of infection at which the dairy products become infected

TABLE 1. Description of parameters used in the tuberculosis model

Based on FIGURE 1, and the description of the model, we have the following system of non linear ordinary differential equations:

$$\begin{cases} \frac{dS}{dt} = \mu N - (\mu + \beta_K)S \\ \frac{dI}{dt} = \beta_K S - (\mu + \delta + \gamma)I \\ \frac{dR}{dt} = \gamma I - \mu R \\ \frac{dB}{dt} = r_B B\left(1 - \frac{B}{K}\right) - \mu_B B \\ \frac{dL_n}{dt} = \alpha L - (\mu_l + \beta_l)L_n \\ \frac{dL_c}{dt} = \beta_L L_n - \mu_l L_c \end{cases} \tag{1}$$

where:

$$\begin{cases} \beta_K(t) = \beta_1 B + \beta_2 L_c \\ \beta_L(t) = \beta_3 B + \beta_4 L_c. \end{cases}$$

Noted that:

- β_1 is the rate at which susceptible individuals become infected by the bacteria in the environnement.
- β_2 is the rate at which susceptible individuals become infected by consuming contaminated dairy products.

- β_3 is the rate at which non contaminated dairy products become infected by contacting the bacteria.
- β_4 is the rate at which non contaminated dairy products become infected by contacting the contaminated one.

To reduce the number of variables, we apply the following change of variables:

$$s=\frac{S}{N},\quad i=\frac{I}{N},\quad r=\frac{R}{N},\quad b=\frac{B}{K},\quad l_n=\frac{L_n}{L},\quad l_c=\frac{L_c}{L}$$

so,

$$\widehat{\beta}_1=\beta_1 K,\quad \widehat{\beta}_2=\beta_2 L,\quad \widehat{\beta}_3=\beta_3 K,\quad \widehat{\beta}_4=\beta_4 L.$$

On the other hand, we have: $S+I+R=N$, thus $r=1-s-i$,

and $L_n+L_c=L$, thus $l_n=1-l_c$.

So, the reduced system is given as:

$$\begin{cases}\frac{ds}{dt}=\mu-(\mu+\widehat{\beta}_K)s\\ \frac{di}{dt}=\widehat{\beta}_K s-(\mu+\delta+\gamma)i\\ \frac{dl_c}{dt}=\widehat{\beta}_L-(\widehat{\beta}_L+\mu_l)l_c\\ \frac{db}{dt}=r_B b(1-b)-\mu_B b\end{cases}\tag{2}$$

where

$$\begin{cases}\widehat{\beta}_K(t)=\widehat{\beta}_1 b+\widehat{\beta}_2 l_c\\ \widehat{\beta}_L(t)=\widehat{\beta}_3 b+\widehat{\beta}_4 l_c\end{cases}$$

taking into account the following initial conditions:

$$s(0)>0,\quad i(0)\geq 0,\quad l_c(0)\geq 0,\quad b(0)\geq 0,$$

in order to be biologically meaningful.

3. Mathematical Analysis

In this section, the model is analysed. We show the positivity and boundedness of the solution and we calculate the equilibrium points.

3.1. Positivity and boundedness of the system solution. The formulated model will be epidemiologically meaningful if all its variables are positive at any time t.

Theorem 3.1. *The solutions of the system* $(s(t),\ i(t),\ l_c(t),\ b(t))$ *for all* $t \geq 0$ *are bounded in the set* Ω*, which is given by*

$$\Omega = \left\{(s(t),i(t),l_c(t),b(t)) \in \mathbb{R}^4_+; 0 < s+i \leq 1, 0 \leq l_c \leq 1, 0 \leq b \leq 1\right\},$$

taking into account the following initial conditions:

$$s(0) > 0, \quad i(0) \geq 0, \quad l_c(0) \geq 0, \quad b(0) \geq 0.$$

Proof 3.1. *Let* $(s(t),i(t),l_c(t),b(t))$ *be a solution of the system* (2) *with the previous initial conditions.*

From the first equation of the system (2)*, we can state that:*

$$\frac{ds}{dt} \geq -(\mu + \widehat{\beta}_K)s$$

$$\frac{ds}{s} \geq -(\mu + \widehat{\beta}_K)dt$$

$$ln\left|\frac{s(t)}{s(0)}\right| \geq -\mu t - \int_0^t \widehat{\beta}_K(\tau)d\tau$$

then

$$s(t) \geq s(0)e^{-\mu t - \int_0^t \widehat{\beta}_K(\tau)d\tau}.$$

Since $s(0) > 0$*, then* $s(t) > 0$ *for all* $t \geq 0$*.*

The second equation of the system (2) *gives:*

$$\frac{di}{dt} \geq -(\mu + \delta + \gamma)i$$

$$\frac{di}{i} \geq -(\mu + \delta + \gamma)dt$$

$$ln\left|\frac{i(t)}{i(0)}\right| \geq -(\mu + \delta + \gamma)t$$

then

$$i(t) \geq i(0)e^{-(\mu+\delta+\gamma)t}.$$

Since $i(0) \geq 0$, then $i(t) \geq 0$ for all $t \geq 0$.

Also, by setting $n = s + i$, the variation of the total population is given by:

$$\frac{dn}{dt} = \frac{ds}{dt} + \frac{di}{dt} = \mu - \mu s - \mu i - \delta i - \gamma i = \mu - \mu n - (\delta + \gamma) i$$

so:

$$\frac{dn}{dt} \leq \mu(1-n) \tag{3}$$

Integrating the inequation (3) from 0 to t, we obtain :

$$n(t) \leq 1 + C_0 e^{-\mu t}$$

where C_0 is a constant.

Since $\lim_{t \to +\infty} 1 + C_0 e^{-\mu t} = 1$, $s(t) > 0$ and $i(t) \geq 0$, for all $t \geq 0$, we have:

$$\forall t \geq 0,\ 0 < n(t) \leq 1.$$

The third equation of the system (2) gives:

$$\frac{dl_c}{dt} \geq -(\widehat{\beta}_L + \mu_l) l_c$$

by similar ideas, it yields :

$$l_c(t) \geq l_c(0) e^{-\mu_l t - \int_0^t \widehat{\beta}_l(\tau) d\tau}.$$

Since $l_c(0) \geq 0$, then $l_c(t) \geq 0$ for all $t \geq 0$.

On the other side, from the equation:

$$\frac{dl_c}{dt} = \widehat{\beta}_L - (\widehat{\beta}_L + \mu_l) l_c$$

we get:

$$\frac{dl_c}{dt} \leq \widehat{\beta}_L - \widehat{\beta}_L l_c$$

$$\frac{dl_c}{dt} \leq \widehat{\beta}_L (1 - l_c)$$

Applying integration, we prove that:

$$l_c(t) \leq 1 + C_1 e^{-\int_0^t \widehat{\beta}_L(\tau) d\tau}.$$

Since

$$\lim_{t\to+\infty} 1+C_1 e^{-\int_0^t \widehat{\beta}_L(\tau)d\tau} = 1$$

and

$$\forall t \geq 0,\ l_c(t) \geq 0,$$

we get:

$$\forall t \geq 0,\ 0 \leq l_c(t) \leq 1.$$

From the equation that governs the variations of the bacteria, we get:

$$\frac{db}{dt} = (r_B - \mu_B)b - r_B b^2 \tag{4}$$

thus:

$$\frac{1}{b^2}\frac{db}{dt} = (r_B - \mu_B)\frac{1}{b} - r_B$$

putting:

$$z = \frac{1}{b}, \tag{5}$$

the equation (4) *becomes:*

$$\frac{dz}{dt} = -(r_B - \mu_B)z + r_B. \tag{6}$$

The solution of the equation (6) *is:*

$$z(t) = ke^{-(r_B-\mu_B)t} + \frac{r_B}{r_B - \mu_B}$$

where k is a constant.

Substituting the expression of $z(t)$ *in the equation* (5)*, it yields:*

$$b(t) = \frac{1}{ke^{-(r_B-\mu_B)t} + \frac{r_B}{r_B-\mu_B}}.$$

As a result,

$$\lim_{t\to+\infty} b(t) = \frac{1}{\frac{r_B}{r_B-\mu_B}} = 1 - \frac{\mu_B}{r_B}$$

However, the existence of the bacteria requires that its mortality rate must be lower than its growth rate, so $\frac{\mu_B}{r_B} < 1$*, and* $1 - \frac{\mu_B}{r_B} < 1$*, thus*

$$\forall t \geq 0,\ 0 \leq b(t) \leq 1.$$

3.2. Equilibrium Points of the system. The equilibrium points of the dynamical system (2) is obtained by resolving the system:

$$\begin{cases} \mu - (\mu + \widehat{\beta}_K)s = 0 \\ \widehat{\beta}_K s - (\mu + \delta + \gamma)i = 0 \\ \widehat{\beta}_L - (\widehat{\beta}_L + \mu_l)l_c = 0 \\ r_B b(1-b) - \mu_B b = 0 \end{cases} \tag{7}$$

The fourth equation of the system (7) yields:

$$b(r_B(1-b) - \mu_B) = 0$$

which gives:

$$b_1 = 0 \qquad ou \qquad b_2 = 1 - \frac{\mu_B}{r_B}.$$

Knowing that $\forall t \geq 0, b(t) \geq 0$, we can state that b_2 exists only if $\mu_B < r_B$.

- Case 1:

  If $b_1 = 0$, then $\widehat{\beta}_L(t) = \widehat{\beta}_4 l_c$.

  The third equation becomes:

  $$\widehat{\beta}_4 l_c - \widehat{\beta}_4 l_c^2 - \mu_l l_c = 0$$

  $$l_c(\widehat{\beta}_4(1 - l_c) - \mu_l) = 0$$

  Thus,

  $$l_{c_1} = 0 \qquad ou \qquad l_{c_2} = 1 - \frac{\mu_l}{\widehat{\beta}_4}.$$

  Since $\forall t \geq 0, l_c(t) \geq 0$, then l_{c_2} exists only if $\mu_l < \widehat{\beta}_4$.

  - Case 1.1:

    If $b_1 = 0$ and $l_{c_1} = 0$, the second equation of the system (7) yields $i = 0$, and its first equation gives $s = 1$.

    Then, the non-endemic equilibrium point of the system (7) is given by:

    $$P_0 = (s, i, l_{c_1}, b_1) = (1, 0, 0, 0)$$

- Case 1.2:

  If $b_1 = 0$ and $l_{c_2} = 1 - \frac{\mu_l}{\widehat{\beta}_4}$, we obtain: $\widehat{\beta}_K = \widehat{\beta}_2 l_{c_2}$.

  So, the first equation of the system (7) gives:

$$s_1 = \frac{\mu}{\mu + \widehat{\beta}_2 l_{c_2}}.$$

  Substituting the expression of s_1 in the second equation of the system (7), it yields:

$$i_1 = \frac{\mu \widehat{\beta}_2 l_{c_2}}{(\mu + \delta + \gamma)(\mu + \widehat{\beta}_2 l_{c_2})}.$$

  Thus, the endemic equilibrium point with contamination only by consuming unpasteurized dairy products is given by:

$$P_1 = (s_1, i_1, l_{c_2}, b_1) = \left(\frac{\mu}{\mu + \widehat{\beta}_2 l_{c_2}}, \frac{\mu \widehat{\beta}_2 l_{c_2}}{(\mu + \delta + \gamma)(\mu + \widehat{\beta}_2 l_{c_2})}, 1 - \frac{\mu_l}{\widehat{\beta}_4}, 0 \right)$$

- Case 2:

If $b_2 = 1 - \frac{\mu_b}{r_b}$, then:

$$\begin{cases} \widehat{\beta}_K(t) = \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3} \\ \widehat{\beta}_L(t) = \widehat{\beta}_3 b_2 + \widehat{\beta}_4 l_{c_3} \end{cases}$$

the third equation of the system can be written as:

$$\widehat{\beta}_L(1 - l_{c_3}) - \mu_l l_{c_3} = 0$$

with the expression of $\widehat{\beta}_L$, it becomes:

$$\widehat{\beta}_4 l_{c_3}^2 + (\widehat{\beta}_3 b_2 - \widehat{\beta}_4 + \mu_l) l_{c_3} - \widehat{\beta}_3 b_2 = 0.$$

By setting:

$$\begin{cases} \Psi_0 = -\widehat{\beta}_3 b_2 \\ \Psi_1 = \widehat{\beta}_3 b_2 - \widehat{\beta}_4 + \mu_l \\ \Psi_2 = \widehat{\beta}_4 \end{cases}$$

we get:

$$\Psi_2 l_{c_3}^2 + \Psi_1 l_{c_3} + \Psi_0 = 0$$

whose discriminant is given by: $\Delta = \Psi_1^2 - 4\Psi_2\Psi_0$,

since, $\Psi_0 < 0$ and $\Psi_2 > 0$, then $\Delta > 0$.

Thus:

$$l_{c_3} = \frac{-\Psi_1 + \sqrt{\Psi_1^2 - 4\Psi_2\Psi_0}}{2\Psi 2}, \quad l_{c_3} = \frac{-\Psi_1 - \sqrt{\Psi_1^2 - 4\Psi_2\Psi_0}}{2\Psi 2}$$

We have already demonstrated that $l_{c_3} > 0$ for all positive initial condition, so we will choose l_{c_3} which is in $\mathbb{R}_+^*$, that it will be noted $l_{c_3}^+$.

Hence, it is easy to get:

$$s_2 = \frac{\mu}{\mu + \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+},$$

and from the second equation of the system (7), we obtain:

$$i_2 = \frac{\mu(\widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+)}{(\mu + \delta + \gamma)(\mu + \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+)}.$$

Consequently, the endemic equilibrium point is given by:

$$P_2 = (s_2, i_2, l_{c_3}^+, b_2) = \left(\frac{\mu}{\mu + \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+}, \frac{\mu(\widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+)}{(\mu + \delta + \gamma)(\mu + \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+)}, l_{c_3}^+, 1 - \frac{\mu_B}{r_B} \right)$$

In the next section, we demonstrate the local stability of the different equilibrium points.

4. Local Stability of the Equilibrium Points

Using the theorem of Poincarré-Lyapunov in [13], [14], an equilibrium point is locally asymptotically stable if and only if all the eigenvalues of the Jacobian matrix of the dynamical system evaluated at this equilibrium point are strictly negatives. In this regard, we will use the Routh-Hurwitz criterion for the second-degree polynomial [13]. It shows that if all the coefficients of the characteristic polynomial are strictly positives, then it does not admit a positive root.

Proposition 4.1. *The non-endemic equilibrium point P_0 is locally asymptotically stable if $\widehat{\beta}_4 < \mu_l$ and $r_B < \mu_B$.*

Proof 4.1. *The Jacobian matrix of the dynamical system* (2) *at* $P=(s,i,l_c,b)$ *is given by:*

$$J(P)=\begin{pmatrix} -\mu-\widehat{\beta}_K & 0 & -\widehat{\beta}_2 s & -\widehat{\beta}_1 s \\ \widehat{\beta}_K & -(\mu+\delta+\gamma) & \widehat{\beta}_2 s & \widehat{\beta}_1 s \\ 0 & 0 & \widehat{\beta}_4-\widehat{\beta}_3 b-\mu_l-2\widehat{\beta}_4 l_c & \widehat{\beta}_3-\widehat{\beta}_3 l_c \\ 0 & 0 & 0 & r_B-\mu_B-2r_B b \end{pmatrix}$$

The Jacobian matrix on the point P_0 *is given by:*

$$J(P_0)=\begin{pmatrix} -\mu & 0 & -\widehat{\beta}_2 & -\widehat{\beta}_1 \\ 0 & -(\mu+\delta+\gamma) & \widehat{\beta}_2 & \widehat{\beta}_1 \\ 0 & 0 & \widehat{\beta}_4-\mu_l & \widehat{\beta}_3 \\ 0 & 0 & 0 & r_B-\mu_B \end{pmatrix}$$

Its eigenvalues are:

$$\begin{cases} \lambda_1=-\mu, \\ \lambda_2=-(\mu+\gamma+\delta), \\ \lambda_3=\widehat{\beta}_4-\mu_l, \\ \lambda_4=r_B-\mu_B. \end{cases}$$

Since all the parameters of the system are strictly positives, λ_1 *and* λ_2 *are strictly negatives. Therefore, The nonendemic equilibrium point* P_0 *is locally asymptotically stable if and only if* λ_3 *and* λ_4 *are strictly negatives, that is if* $\widehat{\beta}_4<\mu_l$ *and* $r_B<\mu_B$.

Proposition 4.2. *The endemic equilibrium point with contamination only by consuming unpasteurized dairy products* P_1 *is locally asymptotically stable if and only if* $\mu_l<\widehat{\beta}_4$ *and* $r_B<\mu_B$.

Proof 4.2. *The Jacobian matrix on the point* P_1 *is given by:*

$$J(P_1)=\begin{pmatrix} -\mu-\widehat{\beta}_2 l_{c_2} & 0 & -\widehat{\beta}_2 s_1 & -\widehat{\beta}_1 s_1 \\ \widehat{\beta}_2 l_{c_2} & -(\mu+\delta+\gamma) & \widehat{\beta}_2 s_1 & \widehat{\beta}_1 s_1 \\ 0 & 0 & \widehat{\beta}_4-\mu_l-2\widehat{\beta}_4 l_{c_2} & \widehat{\beta}_3-\widehat{\beta}_3 l_{c_2} \\ 0 & 0 & 0 & r_B-\mu_B \end{pmatrix}$$

Putting:

$$J(P_1) = \begin{pmatrix} J_1(P_1) & J_2(P_1) \\ J_3(P_1) & J_4(P_1) \end{pmatrix}$$

such as:

$$J_1(P_1) = \begin{pmatrix} -\mu - \widehat{\beta}_2 l_{c_2} & 0 \\ \widehat{\beta}_2 l_{c_2} & -(\mu+\delta+\gamma) \end{pmatrix} \qquad J_2(P_1) = \begin{pmatrix} -\widehat{\beta}_2 s_1 & -\widehat{\beta}_1 s_1 \\ \widehat{\beta}_2 s_1 & \widehat{\beta}_1 s_1 \end{pmatrix}$$

$$J_3(P_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad J_4(P_1) = \begin{pmatrix} \widehat{\beta}_4 - \mu_l - 2\widehat{\beta}_4 l_{c_2} & \widehat{\beta}_3 - \widehat{\beta}_3 l_{c_2} \\ 0 & r_B - \mu_B \end{pmatrix}.$$

The characteristic polynomial associated to the matrix $J_1(P_1)$ is:

$$P(\lambda) = \begin{vmatrix} -\mu - \widehat{\beta}_2 l_{c_2} - \lambda & 0 \\ \widehat{\beta}_2 l_{c_2} & -(\mu+\delta+\gamma) - \lambda \end{vmatrix} = \lambda^2 + \lambda\Phi_1 + \Phi_0$$

where:

$$\begin{cases} \Phi_1 = 2\mu + \widehat{\beta}_2 l_{c_2} + \delta + \gamma > 0 \\ \Phi_0 = (\mu+\delta+\gamma)(\mu + \widehat{\beta}_2 l_{c_2}) > 0. \end{cases}$$

It's clear that the coefficients of $P(\lambda)$ are strictly positives. Thus, by Routh-Hurwitz Criterion, its eigenvalues have a strictly negative real part.

As for $J_4(P_1)$, its eigenvalues are given by:

$$\begin{aligned} \lambda_1 &= \widehat{\beta}_4 - \mu_l - 2\widehat{\beta}_4 l_{c_2} \\ &= \widehat{\beta}_4 - \mu_l - 2\widehat{\beta}_4 \left(1 - \frac{\mu_l}{\widehat{\beta}_4}\right) \\ &= \widehat{\beta}_4 - \mu_l - 2\widehat{\beta}_4 + 2\mu_l \\ &= \mu_l - \widehat{\beta}_4, \end{aligned}$$

and

$$\lambda_2 = r_B - \mu_B.$$

$\lambda_1 < 0$ if and only if $\mu_l < \widehat{\beta}_4$, and $\lambda_2 < 0$ if and only if $r_B < \mu_B$.

Therefore, The endemic equilibrium point with contamination only by consuming unpasteurized dairy products P_1 is locally asymptotically stable if and only if $\mu_l < \widehat{\beta}_4$ and $r_B < \mu_B$.

Proposition 4.3. *Thus, The endemic equilibrium point P_2 is locally asymptotically stable if and only if $\widehat{\beta}_4 < C$ and $r_B < 2r_B b_2 + \mu_B$ where $C = \widehat{\beta}_3 b_2 + \mu_l + 2\widehat{\beta}_4 l_{c_3}^+$.*

Proof 4.3. *The Jacobian matrix on the point P_2 is given by:*

$$J(P_2) = \begin{pmatrix} -\mu - \widehat{\beta}_1 b_2 - \widehat{\beta}_2 l_{c_3}^+ & 0 & -\widehat{\beta}_2 s_2 & -\widehat{\beta}_1 s_2 \\ \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+ & -(\mu+\delta+\gamma) & \widehat{\beta}_2 s_2 & \widehat{\beta}_1 s_2 \\ 0 & 0 & \widehat{\beta}_4 - \widehat{\beta}_3 b_2 - \mu_l - 2\widehat{\beta}_4 l_{c_3}^+ & \widehat{\beta}_3 - \widehat{\beta}_3 l_{c_3}^+ \\ 0 & 0 & 0 & r_B - \mu_B - 2r_B b_2 \end{pmatrix}$$

Putting:

$$J(P_2) = \begin{pmatrix} J_1(P_2) & J_2(P_2) \\ J_3(P_2) & J_4(P_2) \end{pmatrix}$$

such as:

$$J_1(P_2) = \begin{pmatrix} -\mu - \widehat{\beta}_1 b_2 - \widehat{\beta}_2 l_{c_3}^+ & 0 \\ \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+ & -(\mu+\delta+\gamma) \end{pmatrix} \qquad J_2(P_2) = \begin{pmatrix} -\widehat{\beta}_2 s_2 & -\widehat{\beta}_1 s_2 \\ \widehat{\beta}_2 s_2 & \widehat{\beta}_1 s_2 \end{pmatrix}$$

$$J_3(P_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad J_4(P_2) = \begin{pmatrix} \widehat{\beta}_4 - \widehat{\beta}_3 b_2 - \mu_l - 2\widehat{\beta}_4 l_{c_3}^+ & \widehat{\beta}_3 - \widehat{\beta}_3 l_{c_3}^+ \\ 0 & r_B - \mu_B - 2r_B b_2 \end{pmatrix}$$

The characteristic polynomial associated to the matrix $J_1(E_2)$ is given by:

$$\begin{aligned} P(\lambda) &= \begin{vmatrix} -\mu - \widehat{\beta}_1 b_2 - \widehat{\beta}_2 l_{c_3}^+ - \lambda & 0 \\ \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+ & -(\mu+\delta+\gamma) - \lambda \end{vmatrix} \\ &= \lambda^2 + \Upsilon_1 \lambda + \Upsilon_0; \end{aligned}$$

where:

$$\Upsilon_1 = \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+ + 2\mu + \delta + \gamma > 0$$

$$\Upsilon_2 = (\mu + \widehat{\beta}_1 b_2 + \widehat{\beta}_2 l_{c_3}^+)(\mu + \delta + \gamma) > 0.$$

Since all the coefficients of $P(\lambda)$ are strictly positives, then $P(\lambda)$ does not have positive roots. Therfore, by Routh-Hurwitz Criterion, its eigenvalues have a strictly negative real part.

As for the matrix $J_4(P_2)$, its eigenvalues are:

$$\begin{aligned}\lambda_1 &= \widehat{\beta}_4 - \widehat{\beta}_3 b_2 - \mu_l - 2\widehat{\beta}_4 l_{c_3}^+ \\ &= \left(\widehat{\beta}_3 b_2 + \mu_l + 2\widehat{\beta}_4 l_{c_3}^+\right)\left(\frac{\widehat{\beta}_4}{\widehat{\beta}_3 b_2 + \mu_l + 2\widehat{\beta}_4 l_{c_3}^+} - 1\right) \\ &= C\left(\frac{\widehat{\beta}_4}{C} - 1\right)\end{aligned}$$

where: $C = \widehat{\beta}_3 b_2 + \mu_l + 2\widehat{\beta}_4 l_{c_3}^+$ is a positive constant, and $\lambda_2 = r_B - \mu_B - 2r_B b_2$. So, $\lambda_1 < 0$ if and only if $\widehat{\beta}_4 < C$, and $\lambda_2 < 0$ if and only if $r_B < 2r_B b_2 + \mu_B$. Thus, the endemic equilibrium point P_2 is locally asymptotically stable if and only if $\widehat{\beta}_4 < C$ and $r_B < 2r_B b_2 + \mu_B$.

5. NUMERICAL SIMULATION

Numerical simulations for the tuberculosis model given by the system (2) are done using Matlab, whose objective is the description of the behavior of the system solutions over time, and the affirmation of the results obtained.

For the simulation of the nonendemic equilibrium point, we start with initial conditons: $s_0 = 0.4$, $i_0 = 0.1$, $l_{c_0} = 0.2$, $b_0 = 0.4$. We will run simulation, in an interval of 150 days, and we obtain the numerical solution for the system (2). We defined the values of the parameters in Table 2.

Parameter	Values
r_B	0.18
μ_B	0.4
μ	0.028
μ_l	0.5
γ	0.08
δ	0.1
$\widehat{\beta}_1$	0.004
$\widehat{\beta}_2$	0.006
$\widehat{\beta}_3$	0.03
$\widehat{\beta}_4$	0.05

TABLE 2. Parameters values used in the the nonendemic model

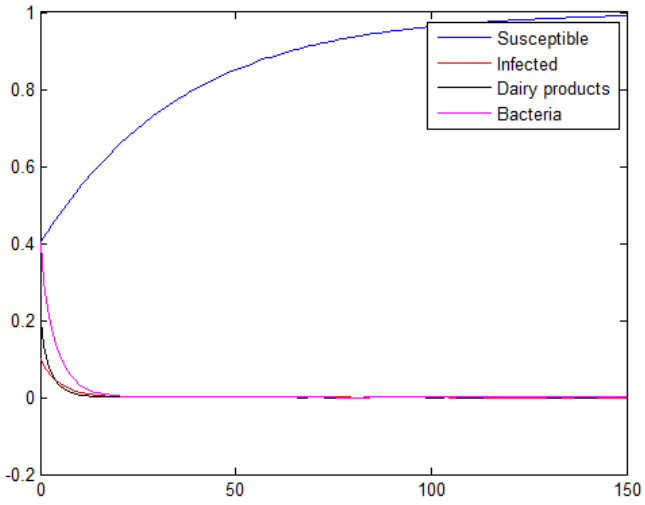

FIGURE 2. The dynamic behavior of compartments s, i, l_c and b for the nonendemic equilibrium point

For the simulation of the endemic equilibrium point with contamination only by consuming unpasteurized dairy products, we start with initial conditons: $s_0 = 0.5$, $i_0 = 0.1$, $l_{c_0} = 0.1$, $b_0 = 0.1$. We will run simulation, in an interval of 150 days, and we obtain the numerical solution for the system (2). We defined the values of the parameters used in Table 3.

Parameter	Values
r_B	0.05
μ_B	0.089
μ	0.028
μ_l	0.1
γ	0.004
δ	0.0035
$\widehat{\beta}_1$	0.01
$\widehat{\beta}_2$	0.07
$\widehat{\beta}_3$	0.004
$\widehat{\beta}_4$	0.2

TABLE 3. Parameters values used in the endemic model with contamination only by consuming unpasteurized dairy products

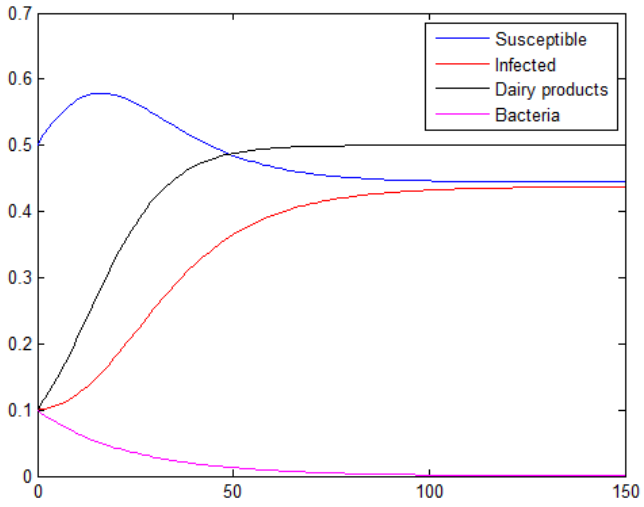

FIGURE 3. The dynamic behavior of compartments s, i, l_c and b for the endemic equilibrium point with contamination only by consuming unpasteurized dairy products

For the simulation of the endemic equilibrium point, we start with initial conditons: $s_0 = 0.5$, $i_0 = 0.1$, $l_{c_0} = 0.2$, $b_0 = 0.0005$. We will run simulation, in an interval of 150 days, and we obtain the numerical solution for the system (2). We defined the values of the parameters used in Table 4.

Parameter	Values
r_B	0.32
μ_B	0.2
μ	0.09
μ_l	0.06
γ	0.07
δ	0.004
$\widehat{\beta}_1$	0.15
$\widehat{\beta}_2$	0.20
$\widehat{\beta}_3$	0.004
$\widehat{\beta}_4$	0.1

TABLE 4. Parameters values used in the simulation of the endemic equilibrium point

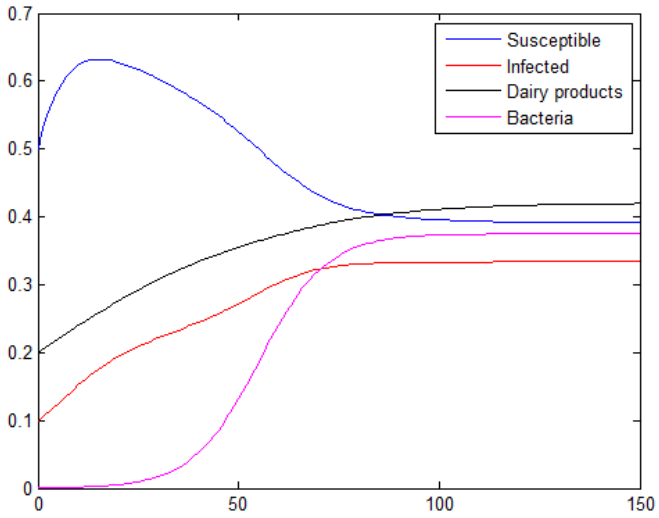

FIGURE 4. The dynamic behavior of compartments s, i, l_c and b for the endemic equilibrium point

6. CONCLUSION

We have presented in this article a compartmental mathematical model of the dynamics of tuberculosis in Morocco, that took in consideration the infection by consumption of unpasteurised dairy products. We have calculate the equilibrium points and studied their local stability using eigenvalues analysis. In the end, numerical simulations are presented to illustrate the results obtained. This research confirms that the elimination of unpasteurized dairy products allows the reduction in the rate of infection by tuberculosis, and consequently the number of people infected with this bacillus. In addition, it is recommended to know some practices aimed at ensuring food safety, including washing hands regularly, separating raw dairy products from cooked dairy products, and the pasteurization which remains the most effective measure to prevent the food transmission of pathogens, including M. Bovis, to humans.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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