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STABILITY AND HOPF BIFURCATION OF AN EPIDEMIOLOGICAL MODEL WITH EFFECT OF DELAY THE AWARENESS PROGRAMS AND VACCINATION: ANALYSIS AND SIMULATION

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Abstract. The present research is concerned with studying the media coverage delay impact and vaccine impact to control outbreaks the infectious disease in the population, we suggested that our mathematical modeling divide the population into three classes: susceptible persons *S(t)*, vaccinated person *V(t)* and infected persons *I(t)*. The susceptible individuals are also divided due to media programs into awareness and unawareness of disease danger and its mode of transmission. We studied the influence of time delay in response to the media program on awareness of the epidemic. We first discussed the existence of the equilibrium points of the system and obtain sufficient conditions for the local asymptotic stability of all equilibrium points. Further, the existence of Hopf bifurcation is shown near endemic equilibrium. It is interesting to note that there exists at least one limit cycle around the unstable endemic equilibrium. In particular, sufficient conditions for a unique stable limit cycle have been presented. Finally, we verified the feasibility of the results by numerical simulation and give a brief conclusion to

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generalize that the delay effect of media and vaccination can cause unexpected dynamic predictions for interacting populations.

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1. INTRODUCTION

The study of epidemic systems using mathematical models becomes one of the most important research topics among applied mathematicians, biologists since the pioneering work of Kermark and Mckendrick [1]. It is a well-known fact, there are many viral diseases that infect humans and animals for example dengue is caused by a virus of the Flaviviridae family and there are four distinct. Also, influenza is caused by four serotypes of viruses. As for the COVID-19 virus being caused by the SARS-CoV-2 virus, many strains have now been detected. The presence of multiple strains of the pathogen complicates our ability to combat these diseases. Because different genetic strains may have different characteristics. For example, it may spread more easily, may cause more severe disease, or it may evade the host's immune response [2].

Recently, several epidemic models have been proposed with the aim to understand, describe and control the spread of the SARS-CoV-2 disease. These models analyze the evolution of the disease over time by dividing the communities into some compartments, which mainly include the susceptible class (S), the carrier (C), the exposed class (E), the infected class (I) and the removed class (R).

After the rapid spread of the epidemic, the World Health Organization (WHO) accelerated the first measures, which is awareness through media programs such as wearing masks, social distancing to 1.5 m, washing hands for 30 seconds, and others. On the other hand, vaccination is an important control measure to reduce the spread of infectious diseases. Thus, studying the effect of vaccines on the disease dynamics is crucial. Many studies on modeling have focused on the influence of media programs and vaccination such as Mohsen et al. [3], presented a mathematical model to study two different measures to control the COVID-19. The spread of SARS-CoV-2 in china by Ivorra et al. [4]. Zeb et al. [5] described a nonlinear SEIQR

COVID-19epidemic model. Hattaf et al. [6] studied the transmission dynamics of COVID-19 propagation due to environmental contamination and carrier effect. In [7], Mohsen et al., formulated a SARS-CoV-2 model to consider the effect of curfew strategy. Dehingia et al. [8] developed a mathematical model for within-host SARS-CoV-2. Bai et al. [9] have proposed and analyzed an epidemic-economic model for SARS-CoV-2 by mathematical model. Also, the following authors have formulated and explored some COVID-19 model see [10]-[20]. The remainder of this work is organized as follows. In Section 2, we introduce the proposed SARS-CoV-2 model with vaccination strategies and awareness measures. In section 3, we present the existence analysis of the equilibrium solutions. In Section 4, we study the dynamical behaviors and stability analysis of the proposed model. In Section 5, we discuss the properties of the Hopf bifurcation. In section 6, we describe the numerical simulations and sensitive analysis to validate the theoretical results. Finally, conclusions and discussions are presented in Section 7.

2. The Mathematical Model Involving Awareness Programs

In this work, we investigate the influence of both vaccination and awareness by media on the infectious diseases. So, it is formulated by the following nonlinear system of ordinary differential equations, where the 1st part susceptible unaware of the diseases and denoted by $S_u(t)$ with time t, the 2nd part is vaccinated individuals referred by $V(t)$ at time t, the 3rd part is susceptible of the diseases but has awareness about it and referred by $S_a(t)$ at time t, the 4th part is referred by $I(t)$ at time t represented by infected persons and the last part $M(t)$ is the level of awareness that population need. In addition to the above that the total population is $N(t)$ at time t and the natural birth of susceptible is Λ with fraction $0 < p < 1$ (i.e that is mean p the rate of vaccination from $S_u(t)$). μ the death rate of each population. We assume that the disease spread by contact between the susceptible population (unaware and aware) and infected population are denoted by β_i for i=1,3 respectively with it is noted that $\beta_3 < \beta_1$. As well as the unaware susceptible become aware due to of the awareness programs at rate β_2, with the constant q such that $0 \leq q < 1$, is the media-induced vaccination coverage. Moreover, v and d are the recovery rate and mortality due to infection respectively. Also, we have some people convert from awareness level to unaware level due to confuse from media programs at rate ρ. The influence rate of media campaigns is denoted by γ and the reduction rate is represented by

θ. finally, in some time when the transfer the unaware susceptible to aware susceptible due to any loss or random in awareness occurs of delay in time, we denoted for this delay by $t-\tau$ (for $\tau > 0$). Now, from the above facts we can write the model by the following system of nonlinear (DDE$_s$):

$$
\begin{aligned}
&\dot{S}_u(t) = (1-p)A - \beta_1 S_u I - \beta_2 S_u M(t-\tau) + \rho S_a - \mu S_u, \\
&\dot{V}(t) = pA + q\beta_2 S_u M(t-\tau) + \nu I - \mu V, \\
&\dot{S}_a(t) = (1-q)\beta_2 S_u M(t-\tau) - \beta_3 S_a I - \rho S_a - \mu S_a, \\
&\dot{I}(t) = \beta_1 S_u I + \beta_3 S_a I - \nu I - dI - \mu I, \\
&\dot{M}(t) = \gamma I - \theta M,
\end{aligned} \tag{1}
$$

where $S_u(0) > 0, V(0) > 0, S_a(0) \geq 0, I(0) \geq 0, M(\theta) \geq 0$ for $\theta \in [-\tau, 0]$ and all the variables and parameter are presumed to be positive, since (1) is a description of human population.

3. Preliminaries

3.1. Positive Invariance.

We study the positive invariance in the following theorem.

Theorem (1): All the solutions of (1) are nonnegative.

Proof: We begin the proof by contradiction(i.e suppose that the theorem is incorrect), then there exists $t^* > 0$ be the first time verifying $S_u(t^*) = 0, V(t^*) = 0, S_a(t^*) = 0, I(t^*) = 0$ and $M(t^*) = 0$. Now, by the 1st eq. of (1) we have

$$\dot{S}_u(t)\big|_{t=t^*} = \underbrace{(1-p)A}_{>0} - \underbrace{\beta_1 S_u I}_{=0} - \underbrace{\beta_2 S_u M(t-\tau)}_{=0} + \underbrace{\rho S_a}_{\geq 0} - \underbrace{\mu S_u}_{=0}.$$

Hence, we should have $S_u(t) < 0$ for all $t \in (t^* - \varepsilon, t^*)$ with $\varepsilon > 0$. This is contradicts that $S_u(t) > 0$ for al $t \in [0, t^*)$ we have $S_u(t) > 0$ for all $t > 0$. Also, from 2nd eq. we get:

$$\dot{V}(t)\big|_{t=t^*} = \underbrace{pA}_{>0} + \underbrace{q\beta_2 S_u M(t-\tau)}_{>0} + \underbrace{\nu I}_{\geq 0} - \underbrace{\mu V}_{=0}.$$

Clearly, by same way above and hence we obtain that $V(t) > 0$ for all $t > 0$. Now,from 3rd eq. of (1) we get:

$$\dot{S}_a(t)\big|_{t=t^*} = \underbrace{(1-q)\beta_2 S_u M(t-\tau)}_{>0} - \underbrace{\beta_3 S_a I}_{=0} - \underbrace{\rho S_a}_{=0} - \underbrace{\mu S_a}_{=0}.$$

Hence we obtain that $S_a(t) > 0$. Therefore, we have $I(t) > 0$ as shown in the following:

$$\dot{I}(t)\big|_{t=t^*} = \underbrace{\beta_1 S_u I}_{>0} + \underbrace{\beta_3 S_a I}_{>0} - \underbrace{(\mu + d + \nu) I}_{=0}.$$

Finally, from the last eq. in (1) and using the same way above we have

$$\dot{M}(t)\big|_{t=t^*} = \underbrace{\gamma I}_{\geq 0} - \underbrace{\theta M}_{=0}.$$

Consequently, $M(t) > 0$ for all $t > 0$.

3.2. Boundedness. Theorem(2): All the solutions of (1) with initial conditions are bounded.

Proof: Let$(S_u(t), V(t), S_a(t), I(t), M(t))$ be any solution of (1) with initial conditions, hence, by adding the equations is (1) to each other we get:

$$(S_u, V, S_a, I) = A - \mu S_u - \mu V - \mu S_a - dI - \mu I.$$

It easy see that, by fact the total population is denoted by $N(t)$ we get:

$$\dot{N} + \mu N \leq A$$

Then solving the above inequality with result of Gronwall lemma we get $N \leq A/\mu + \varepsilon$. Similarly, from the last eq. (media programs) of (1) we have:

$$\dot{M} = \gamma I - \theta M.$$

This implies that $\dot{M}(t) + \theta M(t) = \gamma I$.

So,

$$\dot{M}(t) + \theta M(t) \leq \gamma A/\mu.$$

Then, $\dot{M}(t)) \leq \gamma A/\mu\theta$.

3.3. The existence the all possible equilibrium points of System (1). In this subsection, we see that system (1) has only two equilibria:

(1) The first point is called the disease-free equilibrium point (DFE) and denoted by $E_\circ = (S_{u\circ}, V_\circ, 0, 0, 0)$.

(2) The second point is called the endemic equilibrium point (EE) and denoted by $E_1 = (S_u^*, V^*, S_a^*, I^*, M^*)$

Now, discussing the existence conditions of each equilibrium point of (1), and hence the system (1) has the (DFE) only in case $I=0$, then we can write it by $E_\circ = ((1-p)A/\mu, pA/\mu, 0, 0, 0)$, and it is exists when $\mathscr{R}_0 < 1$, where

$$\mathscr{R}_0 = \frac{(1-p)\beta_1 A}{\mu(d+\nu+\mu)}.$$

The threshold parameter $\mathscr{R}_0$ is called the basic reproduction number. On the other hand, if $I \neq 0$, we need to check the existence conditions of the (EE). Note that, S_u^*, V^*, S_a^*, I^* and M^* represented the positive solution of the following set of equations.

$$\begin{aligned}
&(1-p)A - \beta_1 S_u^* I^* - \beta_2 S_u^* I^* + \rho S_a^* - \mu S_u^* = 0,\\
&pA + q\beta_2 S_u^* M^* + \nu I^* - \mu V^* = 0,\\
&(1-q)\beta_2 S_u^* M^* - \beta_3 S_a^* I^* - \rho S_a^* - \mu S_a^* = 0,\\
&\beta_1 S_u^* I^* + \beta_3 S_a^* I^* - \nu I^* - dI^* - \mu I^* = 0,\\
&\gamma I^* - \theta M^* = 0.
\end{aligned}$$

After solving the above system we get value of last equation in above system in following

$$M^* = \frac{\gamma I^*}{\theta}. \tag{2}$$

Now, substituting M^* in the 1st, 2nd and 3rd equations in above system and solving its we get:

$$S_u^* = \frac{(1-p)(\beta_3 I^* + \rho + \mu)\theta A}{X}, \tag{3}$$

$$V^* = \frac{pAX + (1-p)(\beta_3 I^* + \rho + \mu)q\beta_2\gamma A I^* + \nu X I^*}{X}, \tag{4}$$

$$S_a^* = \frac{(1-q)(1-p)\beta_2\gamma A I^*}{X}, \tag{5}$$

where

$$X = \rho(\beta_1\theta I^* + \mu\theta + q\gamma\beta_2 I^*) + (\beta_3 I^* + \mu)(\beta_1\theta I^* + \mu\theta + \gamma\beta_2 I^*).$$

Now, we can simplify the 4th equation in the same system in following:

$$D_1(I^*)^2 + D_2 I^* + D_3 = 0. \tag{6}$$

where

$$D_1 = -(\nu + d + \mu)(\beta_1\theta + \beta_2\gamma)\beta_3 < 0,$$

$$D_2 = (1-p)\beta_3 A[\beta_1\theta + (1-q)\beta_2\gamma] - (\nu+d+\mu)[\rho(\beta_1\theta + q\beta_2\gamma) + \mu(\beta_3\theta + \beta_1\theta + \beta_2\gamma)],$$

$$D_3 = \theta(\rho+\mu)[(1-p)\beta_1 A - (\nu+d+\mu)\mu].$$

Clearly, by help the Descartes rule, (6) has unique positive real root I^* if the following condition holds

$$\mathscr{R}_0 > 1. \tag{7}$$

Hence, we can find the values of S_u^*, V^*, S_a^* and M^* if we know the value of I^*.

Remark. System(1) has more than one endemic equilibrium point (EE) if the (7) not be satisfied.

4. STABILITY ANALYSIS

In this section, to discuss the stability of the equilibrium points, we need the calculating the Jacobian matrix at each points (DFE) and (EE), thus, we state the following theorems. It is well known that, the general form Jacobian matrix for system (1) at any equilibrium point in the following:

$$J = \begin{bmatrix} -(\beta_1 I + \beta_2 M + \mu) & 0 & \rho & -\beta_1 S_u & -\beta_2 S_u e^{-\lambda\tau} \\ \beta_2 M & -\mu & 0 & \nu & q\beta_2 S_u e^{-\lambda\tau} \\ (1-q)\beta_2 M & 0 & -(\beta_3 I + \rho + \mu) & -\beta_3 S_a & (1-q)\beta_2 S_u e^{-\lambda\tau} \\ \beta_1 I & 0 & \beta_3 I & \beta_1 S_u + \beta_3 S_a - (\nu+d+\mu) & 0 \\ 0 & 0 & 0 & \gamma & -\theta \end{bmatrix}. \tag{8}$$

4.1. Stability of (DFE). Theorem(3)The (DFE) is locally asymptotically stable if:

$$\mathscr{R}_0 < 1. \tag{9}$$

Proof: The equation (8) at (DFE), we can be reduces to:

$$J(E_\circ) = \begin{bmatrix} -\mu & 0 & \rho & \frac{-(1-p)\beta_1 A}{\mu} & \frac{-(1-p)\beta_2 A}{\mu} \\ 0 & -\mu & 0 & \nu & \frac{-q(1-p)\beta_2 A}{\mu} \\ 0 & 0 & -(\rho+\mu) & 0 & \frac{(1-q)(1-p)\beta_2 A}{\mu} \\ 0 & 0 & 0 & \frac{(1-p)\beta_1 A}{\mu} - (\nu+d+\mu) & 0 \\ 0 & 0 & 0 & \gamma & -\theta \end{bmatrix}. \tag{10}$$

Then the characteristic equation of (10):

$$(\lambda_1+\mu)(\lambda_2+\mu)(\lambda_3+(\rho+\mu))(\lambda_4+\theta)[\lambda_5+(1-p)\beta_1 A-\mu(\mu+d+\mu)]=0. \tag{11}$$

Consequently, if (9) holds we get the each eigenvalues are always negative and given the (DFE) is asymptotic stable.

4.2. **Stability and Hopf Bifurcation of (EE).**

Now, we discuss the existence of Hopf bifurcation of the (EE) point of system(1), we can rewrite the matrix (8) with the characteristic equation around the (EE) in this following form.

$$J(E_1)=\begin{bmatrix} b_{11} & 0 & b_{13} & b_{14} & b_{15}e^{-\lambda\tau} \\ \beta_2 M & -\mu & 0 & \nu & q\beta_2 S_u e^{-\lambda\tau} \\ b_{31} & 0 & b_{33} & b_{43} & b_{35}e^{-\lambda\tau} \\ b_{41} & 0 & b_{43} & b_{44} & 0 \\ 0 & 0 & 0 & b_{54} & b_{55} \end{bmatrix}. \tag{12}$$

Here:

$$b_{11}=-(\beta_1 I^*+\beta_2 M^*+\mu);\ b_{13}=\rho;\ b_{14}=-(\beta_1 S_u^*);\ b_{15}=-(\beta_2 S_u^*)$$

$$b_{31}=(1-q)\beta_2 M^*;\ b_{33}=-(\beta_3 I^*+\rho+\mu);\ b_{34}=-(\beta_3 S_a^*);\ b_{35}=(1-q)\beta_2 S_u^*$$

$$b_{41}=\beta_1 I^*;\ b_{43}=\beta_3 I^*;\ b_{44}=\beta_1 S_u^*+\beta_3 S_a^*-(\nu+d+\mu);\ b_{54}=\gamma;\ b_{55}=-\theta.$$

Hence, the characteristic equation of (12) can be written by form:

$$(\lambda+\mu)[\lambda^4+B_1\lambda^3+B_2\lambda^2+B_3\lambda+B_4+(B_7\lambda+B_8)e^{-\lambda\tau}]. \tag{13}$$

Here:

$$B_1=-[b_{11}+b_{33}+b_{44}+b_{55}],$$

$$B_2=[b_{11}(b_{33}+b_{44})-b_{13}b_{31}-b_{14}b_{41}+b_{55}(b_{11}+b_{33})+b_{44}(b_{33}+b_{55})-b_{34}b_{43}],$$

$$B_3=-[(b_{33}+b_{55})(b_{11}b_{44}-b_{14}b_{41})+b_{33}b_{55}(b_{11}+b_{44})-b_{34}b_{43}(b_{11}+b_{55})$$
$$-b_{13}b_{31}(b_{44}+b_{55})+b_{13}b_{34}b_{41}+b_{31}b_{43}b_{14}],$$

$$B_4=[b_{33}b_{55}(b_{11}b_{44}-b_{14}b_{41})+b_{55}b_{13}(b_{34}b_{41}-b_{44}b_{31})+b_{55}b_{43}(b_{31}b_{14}-b_{11}b_{34})],$$

$B_7 = -[b_{54}(b_{14}b_{15} + b_{43}b_{35})]$,

$B_8 = [b_{15}b_{54}(b_{33}b_{41} - b_{31}b_{43}) + b_{54}b_{35}(b_{11}b_{43} - b_{13}b_{41})]$.

It is easy to verify that condition $b_{44} < 0$ guarantees that B_1 is positive and the condition $b_{13}b_{31} < b_{11}(b_{33} + b_{44})$ guarantees that B_2 is positive. as well, the conditions $b_{44}b_{31} < b_{34}b_{41}$ and $b_{11}b_{34} < b_{31}b_{14}$ guarantees that B_4 is positive. While, B_i, $\imath = 3, 8$ is always positive but, B_7 is always negative. Then to discuss the stability of (EE) without delay (i.e $\tau = 0$). Clearlly, we can rewrite the equation (13) in the following format:

$$(\lambda + \mu)[\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + (B_3 + B_7)\lambda + (B_4 + B_8)] = 0. \tag{14}$$

Then, according to Routh-Hurwtiz criterion, we see that all roots of equation (14), have negative real part. provided that $B_i > 0, i = 1,2$, $B_3 + B_7 > 0$, $B_4 + B_8 > 0$ and $B_1B_2(B_3 + B_7) - (B_3 + B_7)^2 - B_1^2(B_4 + B_8) > 0$. We obtain that the (EE) without delay is locally asymptotically stable.

Now, to discuss the stability of (EE) when $\tau > 0$ depends on the roots of equation (13). We assume that $\lambda = i\omega$ $(\omega > 0)$ in equation (13) we have:

$$\omega^4 - i\omega^3 B_1 - \omega^2 B_2 + i\omega B_3 + B_4 + (i\omega B_7 + B_8)(Cos\omega\tau - iSin\omega\tau) = 0.$$

Now, by isolation the real and imaginary parts, which gives:

$$\omega B_3 - \omega^3 B_1 = B_8 Sin\omega\tau - \omega B_7 Cos\omega\tau \omega^4 - \omega^2 B_2 + B_4 = -\omega B_7 Sin\omega\tau - B_8 Cos\omega\tau. \tag{15}$$

By squaring the two equation in (15) and adding to each other we get:

$$\omega^8 + \omega^6(B_1^2 - 2B_2) + \omega^4(2B_4 + B_2 - 2B_1B_3)\omega^2(B_3^2 - 2B_2B_4 - B_7^2) + B_4^2 - B_8^2 = 0, \tag{16}$$

put $h = \omega^2$, equation(16) becomes as follow:

$$h^4 + C_1h^3 + C_2h^2 + C_3h + C_4 = 0, \tag{17}$$

where

$$C_1 = B_1^2 - 2B_2;\ C_3 = B_3^2 - 2B_2B_4 - B_7^2,\ C_2 = 2B_4 + B_2 - 2B_2B_3;\ C_4 = (B_4 + B_8)(B_4 - B_8).$$

Now, it easy by according to Descarte rule of sign the equation (16) and (17) has a positive root denoted by $h = \omega_\circ$ provided that the following condition holds:

$$B_4 < B_8. \tag{18}$$

Clearly, we get the characteristic equation(13) has at least a pair of pure imaginary roots $\pm i\omega_\circ$, corresponding to the time delay τ. Hence, we can rewritten the system (15) but substituting $\omega_\circ$ instead of ω and solving the resulting system for τ, we have the following result.

$$\tau_j = \frac{1}{\omega_\circ} Cos^{-1} - \frac{[\omega_\circ^4(B_8 - B_1B_7) + \omega_\circ^2(B_3B_7 - B_2B_8) + B_4B_8]}{\omega_\circ^2 B_7^2 + B_8^2} + \frac{2j\pi}{\omega_\circ}, \tag{19}$$

where j=0,1,2,......

Now, since equation(13) has least two pure imaginary roots if the condition (18) is satisfing with τ_0 such that $\tau_0 = min.\tau_j$, but if the conditions of Routh-Hurwtiz criterion are hold. Then all the roots have negative real parts with $0 \le \tau < \tau_0$. And hence, let $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ be a root of equation (13) such that $\mu(\tau_0) = 0$ and $\omega(\tau_\circ) = \omega_\circ$. Then we get the following theorem.

Theorem(4) If the condition (18) holds we have the roots $\lambda(\tau)$ of equation (130 satisfy the following transversality condition.

$$\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} \neq 0. \tag{20}$$

Provided that

$$\begin{aligned} &\left(B_7^2\omega_\circ^2 + B_8^2\right)\left[(B_3 - 3B_1\omega_\circ)(B_3 - B_1\omega_\circ^3) + 2\omega_\circ(2 - B_2)(\omega_\circ^4 + B_4 - B_2\omega_\circ^2)\right] \\ &\qquad + B_7\left[(B_3 - B_1\omega_\circ^3)^2 + (\omega_\circ^4 + B_4 - B_2\omega_\circ^2)\right] \neq 0. \end{aligned} \tag{21}$$

Proof: By deriving equation (13) with respect to τ, the following is obtain the result:

$$\left\{4\lambda^3 + 3B_1\lambda^2 + 2B_2\lambda + B_3 + [B_7 - \tau(B_7\lambda + B_8)]e^{-\lambda\tau}\right\}\frac{d\lambda}{d\tau} = \lambda(B_7\lambda + B_8)e^{-\lambda\tau}. \tag{22}$$

Thus,

$$\left[\frac{\lambda}{d\tau}\right]^{-1} = \frac{(4\lambda^3 + 3B_1\lambda^2 + 2B_2\lambda + B_3)e^{\lambda\tau}}{\lambda(B_7\lambda + B_8)} + \frac{B_7}{\lambda(B_7\lambda + B_8)} - \frac{\tau}{\lambda}. \tag{23}$$

Since, $\lambda = i\omega_\circ$ at $\tau = \tau_0$, with using equation (13), we can rewrite (23) as following:

$$\left[\frac{\lambda}{d\tau}\right]_{\tau=\tau_0}^{-1} = \frac{-4i\omega_\circ - 3B_1\omega_\circ + 2B_2 i\omega_\circ + B_3}{\omega_\circ(B_3 - B_1\omega_\circ^3) - i\omega_\circ(\omega_\circ^4 + B_4 - B_2\omega_\circ^2)} + \frac{B_7}{-B_7\omega_\circ^2 + i\omega_\circ B_8} - \frac{\tau_0}{i\omega_\circ}.$$

Since

$$Sgn\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_0} = Sgn\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_0}. \tag{24}$$

We have:

$$\begin{aligned}
&Re\left[\frac{-4i\omega_\circ - 3B_1\omega_\circ + 2B_2 i\omega_\circ + B_3}{\omega_\circ(B_3 - B_1\omega_\circ^3) - i\omega_\circ(\omega_\circ^4 + B_4 - B_2\omega_\circ^2)}\right] \\
&= \frac{(B_3 - 3B_1\omega_\circ)(B_3 - B_1\omega_\circ^3) + 2\omega_\circ(2 - B_2)(\omega_\circ^4 + B_4 - B_2\omega_\circ^2)}{(B_3 - B_1\omega_\circ^3)^2 + (\omega_\circ^4 + B_4 - B_2\omega_\circ^2)^2}, \\
&Re\left[\frac{B_7}{-B_7\omega_\circ^2 + i\omega_\circ B_8}\right] = \frac{-B_7^2}{B_7^2\omega_\circ^2 + B_8^2}, \\
&Re\left[\frac{\tau_0}{i\omega_\circ}\right] = 0.
\end{aligned}$$

We get

$$\begin{aligned}
\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_0} &= \frac{\left(B_7^2\omega_\circ^2 + B_8^2\right)\left[(B_3 - 3B_1\omega_\circ)(B_3 - B_1\omega_\circ^3) + 2\omega_\circ(2 - B_2)(\omega_\circ^4 + B_4 - B_2\omega_\circ^2)\right]}{\left(B_7^2\omega_\circ^2 + B_8^2\right)\left[(B_3 - B_1\omega_\circ^3)^2 + (\omega_\circ^4 + B_4 - B_2\omega_\circ^2)\right]^2} \\
&+ \frac{B_7\left[(B_3 - B_1\omega_\circ^3)^2 + (\omega_\circ^4 + B_4 - B_2\omega_\circ^2)\right]}{\left(B_7^2\omega_\circ^2 + B_8^2\right)\left[(B_3 - B_1\omega_\circ^3)^2 + (\omega_\circ^4 + B_4 - B_2\omega_\circ^2)\right]^2}.
\end{aligned}$$

Hence,the transversality condition is satisfied if the condition (22) holds. Now, from the above result on the local stability and Hopf bifurcation at (EE), we have the following remark:

Remark. Let the condition (7) with conditions of Routh-Hurwtiz criterion and (18) are satisfied then:

- i The (EE) point is locally asymptotically stable for $\tau < \tau_0$.
- ii The (EE) is unstable point for $\tau > \tau_0$.
- iii Near (EE) point the Hopf bifurcation is occur in system (1) for $\tau = \tau_0$.

5. Direction of Hopf Bifurcation and Stability of Bifurcation Periodic Solution

In this section, we discuss the the direction and the stability of the Hopf bifurcation by using center manifold reduction. we first normalize the system (1) We let $Y_1(t) = S_u(\tau t) - S_u^*$, $Y_2(t) =$

$V(\tau t)-V^*$, $Y_3(t)=S_a(\tau t)-S_a^*$, $Y_4(t)=I(\tau t)-I^*$ and $Y_5(t)=M(\tau t)-M^*$. Then system (1) becomes

(25)

$$\begin{aligned}
\dot{Y}_1 &= \tau\left[(1-p)A-\beta_1\left(Y_1+S_u^*\right)\left(Y_4+I^*\right)-\beta_2\left(Y_1+S_u^*\right)\left(Y_5(t-\tau)+M^*\right)+\rho\left(Y_3+S_a^*\right)-\mu\left(Y_1+S_u^*\right)\right],\\
\dot{Y}_2 &= \tau\left[pA+q\beta_2\left(Y_1+S_u^*\right)\left(Y_5(t-\tau)+M^*\right)+\nu\left(Y_4+I^*\right)-\mu\left(Y_2+V^*\right)\right],\\
\dot{Y}_3 &= \tau\left[(1-q)\beta_2\left(Y_1+S_u^*\right)\left(Y_5(t-\tau)+M^*\right)-\beta_3\left(Y_3+S_a^*\right)\left(Y_4+I^*\right)-(\rho+\mu)\left(Y_3+S_a^*\right)\right],\\
\dot{Y}_4 &= \tau\left[\beta_1\left(Y_1+S_u^*\right)\left(Y_4+I^*\right)-\beta_3\left(Y_3+S_a^*\right)\left(Y_4+I^*\right)-(\nu+d+\mu)\left(Y_4+I^*\right)\right],\\
\dot{Y}_5 &= \tau\left[\gamma\left(Y_4+I^*\right)-\theta\left(Y_5+M^*\right)\right].
\end{aligned}$$

So, choose $\tau=\tau_\circ+\alpha$, and by linear of (25) at (0,0,0,0,0), we get

$$\begin{aligned}
\dot{Y}_1 &= (\tau+\alpha)\left[-(\beta_1 I^*+\beta_2 M^*+\mu)Y_1-\beta_1 S_u^* Y_4-\beta_2 Y_5(t-\tau)\right],\\
\dot{Y}_2 &= (\tau+\alpha)\left[q\beta_2 M^* Y_1+q\beta_2 Y_5(t-\tau)+\nu Y_4-\mu Y_2\right],\\
\dot{Y}_3 &= (\tau+\alpha)\left[(1-q)\beta_2 M^* Y_1+(1-q)\beta_2 S_u^* Y_5(t-\tau)-(\beta_3 I^*+\rho+\mu)Y_3-\beta_3 S^* Y_4\right],\\
\dot{Y}_4 &= (\tau+\alpha)\left[\beta_1 I^* Y_1+\beta_3 I^* Y_3+(\beta_1 S_u^*+\beta_3 S_a^*-(\nu+d+\mu))Y_4\right],\\
\dot{Y}_5 &= (\tau+\alpha)\left[\gamma Y_4-\theta Y_5\right].
\end{aligned} \tag{26}$$

Hence, the nonlinear term of (25) in following

$$f=(\tau_\circ+\alpha)\begin{bmatrix} -\beta_1 Y_1 Y_4-\beta_2 Y_1 Y_5(t-\tau)\\ q\beta_2 Y_1 Y_5(t-\tau)\\ (1-q)\beta_2 Y_1 Y_5(t-\tau)-\beta_3 Y_3 Y_4\\ \beta_1 Y_1 Y_4+\beta_3 Y_3 Y_4\\ 0\end{bmatrix}. \tag{27}$$

We consider the phase space $C=C\left([-1,0],\Re_+^5\right]$, then (25) can be written as

$$\dot{Y}(t)=L_\alpha(y_t)+f(\alpha,y_t), \tag{28}$$

where $Y(t)=\left(Y_1(t),Y_2(t),Y_3(t),Y_4(t),Y_5(t)\right)^T$, $Y_t=Y_t(\theta)=Y(t+\theta),\theta\in[-1,0]$, and $L_\alpha\varphi=H_1(\alpha)\varphi(0)+H_2(\alpha)\varphi(-1)$, $\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)^T\in C$. From (28), we can get

(29)
$$A_1(\alpha) = (\tau_\circ + \alpha)\begin{bmatrix} -(\beta_1 I^* + \beta_2 M^* + \mu) & 0 & 0 & -\beta_1 S_u^* & 0 \\ q\beta_2 M^* & -\mu & 0 & \nu & 0 \\ (1-q)\beta_2 M^* & 0 & -(\beta_3 I^* + \rho + \mu) & -\beta_3 S_a^* & 0 \\ \beta_1 I^* & 0 & \beta_3 I^* & \beta_1 S_u^* + \beta_3 S_a^* - (\nu + d + \mu) & 0 \\ 0 & 0 & 0 & \gamma & -\theta \end{bmatrix},$$

(30)
$$A_2(\alpha) = (\tau_\circ + \alpha)\begin{bmatrix} 0 & 0 & 0 & 0 & -\beta_2 S_u^* \\ 0 & 0 & 0 & 0 & q\beta_2 S_u^* \\ 0 & 0 & 0 & 0 & (1-q)\beta_2 S_u^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

And

(31)
$$f(\alpha + \varphi) = (\tau_\circ + \alpha)\cdot\begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \\ f_{51} \end{bmatrix},$$

where

$$\begin{aligned} f_{11} &= -\beta_1\varphi_1(0)\varphi_4(0) - \beta_2\varphi_1(0)\varphi_5(-1), \\ f_{22} &= q\beta_2\varphi_1(0)\varphi_5(-1), \\ f_{31} &= (1-q)\beta_2\varphi_1(0)\varphi_5(-1) - \beta_3\varphi_3(0)\varphi_4(0), \\ f_{41} &= \beta_1\varphi_1(0)\varphi_4(0) + \beta_3\varphi_3(0)\varphi_4(0). \end{aligned}$$

Clearly, L_α linear bounded operator in C. Now, by Riesz theorem, there exists a bounded variation function matrix $\eta(\theta,\alpha), \theta \in [-1,0]$, satisfying that $L_\alpha\varphi = \int_{-1}^{0}\varphi(\theta)d\eta(\theta,\alpha)$. Indeed, by choosing Dirac function δ where $\eta(\theta,\alpha) = A_1(\alpha)\delta(\theta) + A_2(\alpha)\delta(\theta+1)$. By $\varphi \in C$, we define

(32)
$$A(\alpha)\varphi = \begin{array}{l} \frac{d\varphi(\theta)}{d\theta},\ if\ \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(\xi,\alpha)\varphi(\xi),\ if\ \theta = 0 \end{array}$$

(33)
$$R(\alpha)\varphi = \begin{cases} 0,\ if\ \theta \in [-1,0), \\ f(\alpha,\varphi),\ if\ \theta = 0 \end{cases}$$

(28) can be expressed as

(34)
$$\dot{Y}(t) = A(\alpha)Y_t + R(\alpha)Y_t$$

Define $A(0) = A, R(0) = R, \eta(\theta,\alpha) = \eta(\theta) and A_i(0) = A_i, i = 1,2.$

Now, for $\varphi \in C^* = C\left([0,1],\Re_+^5\right)$, we denote

(35)
$$A^*\varphi(s) = \begin{cases} \frac{-d\varphi(s)}{ds},\ if\ s \in (0,1], \\ \int_{-1}^{0} d\eta^T(\xi,0)\varphi(-\xi),\ if\ s = 0, \end{cases}$$

where η^T is the transpose of η. The bilinear from can be defined as inner product as

(36)
$$\langle \psi(s),\varphi(\theta)\rangle = \bar{\psi}^T(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \bar{\psi}^T(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi,$$

where $\varphi \in C$, $\psi \in C^*$. On the other hand, $\langle \psi, A\varphi\rangle = \langle A^*\psi,\varphi\rangle$. Clearly, $\pm i\tau_\circ\omega_\circ$ are the eigenvector of (26), therefore we have the followed results

Theorem 5: Let $K(\theta) = (1,z_1,z_2,z_3,z_4)^T e^{i\theta\tau_\circ\omega_\circ}$ is the eigenvector corresponding to $i\tau_\circ\omega_\circ$,$K^*(s) = (1,z_1^*,z_2^*,z_3^*,z_4^*)^T e^{is\tau_\circ\omega_\circ}$, is eigenvector of A^* corresponding to $-i\tau_\circ\omega_\circ$, and $\langle K^*(s),K(\theta)\rangle = 1$,$\langle K^*(s),\bar{K}(\theta)\rangle = 0$ such that

$$\tau_\circ \begin{bmatrix} i\omega_\circ + \beta_1 I^* + \beta_2 M^* + \mu & 0 & 0 & \beta_1 S_u^* & \beta_2 S_u^* e^{-i\tau_\circ\omega_\circ} \\ -q\beta_2 M^* & i\omega_\circ + \mu & 0 & -\nu & \beta_2 S_u^* e^{-i\tau_\circ\omega_\circ} \\ -(1-q)\beta_2 M^* & 0 & i\omega_\circ + \beta_3 I^* + \rho + \mu & \beta_3 S_a^* & -(1-q)\beta_2 S_u^* e^{-i\tau_\circ\omega_\circ} \\ -\beta_1 I^* & 0 & -\beta_3 I^* & i\omega_\circ + \nu + d + \mu - (\beta_1 S_u^* + \beta_3 S_a^*) & 0 \\ 0 & 0 & 0 & -\gamma & i\omega_\circ + \theta \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

where

$$
\begin{aligned}
z_1 &= \frac{-q\beta_2 M^*(i\omega_\circ+\theta)-(i\omega_\circ+\theta+q\beta_2 S_u^* e^{-i\tau_\circ\omega_\circ})\gamma z_3}{(i\omega_\circ+\mu)(i\omega_\circ+\theta\mu)},\\
z_2 &= \frac{[i\omega_\circ+\nu+d+\mu-(\beta_1 S_u^*+\beta_3 S_a^*)]z_3-\beta_1 I^*}{\beta_3 I^*},\\
z_3 &= \frac{(i\omega_\circ+\theta)(i\omega_\circ+\beta_1 I^*+\beta_2 M^*+\mu)}{[\beta_1(i\omega_\circ+\theta)+\beta_2\gamma e^{-i\tau_\circ\omega_\circ}]S_u^*},\\
z_4 &= \frac{\gamma z_3}{i\omega_\circ+\theta}.
\end{aligned}
$$

And

$$
\tau_\circ \begin{bmatrix}
i\omega_\circ-(\beta_1 I^*+\beta_2 M^*+\mu) & q\beta_2 M^* & (1-q)\beta_2 M^* & \beta_1 I^* & 0\\
0 & i\omega_\circ-\mu & 0 & 0 & 0\\
0 & 0 & i\omega_\circ-(\beta_3 I^*+\rho+\mu) & \beta_3 I^* & 0\\
-\beta_1 S_u^* & \nu & -\beta_3 S_a^* & i\omega_\circ-(\nu+d+\mu)+(\beta_1 S_u^*+\beta_3 S_a^*) & \gamma\\
-\beta_2 S_u^* e^{i\tau_\circ\omega_\circ} & q\beta_2 S_u^* & (1-q)\beta_2 S_u^* & 0 & i\omega_\circ-\theta
\end{bmatrix}
$$

$$
\begin{bmatrix} 1\\ z_1^*\\ z_2^*\\ z_3^*\\ z_4^* \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix}.
$$

Where

$$
\begin{aligned}
z_1^* &= \frac{e^{i\tau_\circ\omega_\circ}(i\omega_\circ(\beta_3 I^*+\rho+\mu))-(1-q)\beta_3 I^* z_3^*}{q(i\omega_\circ-(\beta_3 I^*+\rho+\mu))},\\
z_2^* &= \frac{\beta_3 I^* z_3^*}{i\omega_\circ-(\beta_3 I^*+\rho+\mu)},\\
z_3^* &= \frac{-[i\omega_\circ-(\beta_3 I^*+\rho+\mu)]\left[q\beta_2 M^* e^{i\tau_\circ\omega_\circ}+i\omega_\circ-(\beta_1 I^*+\beta_2 M^*+\mu)\right]}{I^*\{\beta_1[i\omega_\circ-(\beta_3 I^*+\rho+\mu)]+(1-q)\beta_3(\beta_2 M^*-1)\}},\\
z_4^* &= \frac{\beta_1 S_u^*+\beta_3 S_a^* z_2^*-\nu z_1^*-[i\omega_\circ-(\nu+d+\mu)+(\beta_1 S_u^*+\beta_3 S_a^*)]}{\gamma}.
\end{aligned}
$$

Now, to find the constant D, and $\langle K^*, K\rangle = 1$, $\langle K^*, \bar{K}\rangle = 0$, from (40), we have

$$
\begin{aligned}
\langle K^*, K\rangle &= (\bar{K}^*)^T(0)K(0) - \int_{-1}^{0}\int_0^{\theta}(\bar{K}^*)^T(\xi-\theta)d\eta(\theta)K(\xi)d\xi\\
&= \bar{D}(1,\bar{z}_1^*,\bar{z}_2^*,\bar{z}_3^*,\bar{z}_4^*)^T\cdot(1,z_1,z_2,z_3,z_4) - \int_{-1}^{0}\int_0^{\theta}\bar{D}(1,\bar{z}_1^*,\bar{z}_2^*,\bar{z}_3^*,\bar{z}_4^*)^T e^{i\tau_\circ\omega_\circ(\theta-\xi)}d\eta(\theta)(1,z_1,z_2,z_3,z_4)^T e^{i\tau_\circ\omega_\circ\xi}d\xi
\end{aligned}
$$

$$
\bar{D}\left\{1+\bar{z}_1^* z_1+\bar{z}_2^* z_2+\bar{z}_3^* z_3+\bar{z}_4^* z_4+\tau_\circ(1,\bar{z}_1^*,\bar{z}_2^*,\bar{z}_3^*,\bar{z}_4^*)(-\beta_2 S_u^* z_4, q\beta_2 S_u^* z_4, (1-q)\beta_2 S_u^* z_4, 0, 0)^T e^{-i\tau_\circ\omega_\circ}\right\}. \tag{37}
$$

Thus

$$
D = \left[1+z_1^*\bar{z}_1+z_2^*\bar{z}_2+z_3^*\bar{z}_3+z_4^*\bar{z}_4+\tau_\circ\left\{-\beta_2 S_u^*\bar{z}_4+q\beta_2 S_u^* z_1^*\bar{z}_4+(1-q)\beta_2 S_u^* z_2^*\bar{z}_4\right\}e^{i\tau_\circ\omega_\circ}\right]^{-1}.
$$

Then, we obtain $\langle K^*, K\rangle = 1$.

In addition of above, we using the adjoint property $\langle \varphi, A\phi\rangle = \langle A^*\varphi, \phi\rangle$ it easy to show that

$$\begin{aligned} i\tau_\circ\omega_\circ \langle K^*, \bar{K}\rangle &= \langle -i\tau_\circ\omega_\circ K^*, \bar{K}\rangle \\ &= \langle A^*K^*, \bar{K}\rangle \\ &= \langle K^*, A\bar{K}\rangle \\ &= \langle K^*, i\tau_\circ\omega_\circ\bar{K}\rangle \\ &= -i\tau_\circ\omega_\circ \langle K^*, \bar{K}\rangle. \end{aligned}$$

Hence, we obtain $\langle K^*, \bar{K}\rangle = 0$. The proof completed. Next, we will compute the coordinates to describe the center manifold at $\alpha = 0$. Assuming Y_t be a solution of (28) when $\alpha = 0$, we define in following

$$\begin{aligned} &Z(t) = \langle K^*, Y_t\rangle, \\ &\textit{and} \\ &\quad W(z,\bar{z},\theta) = Y_t(\theta) - 2Re\{Z(t)K(\theta)\}. \end{aligned} \tag{38}$$

Now, From (28), with $\alpha = 0$ and equation (38), we get

$$\begin{aligned} \dot{Z}(t) &= \left\langle K^*, \dot{Y}_t\right\rangle \\ &= \langle K^*, AY_t\rangle + \langle K^*, RY_t\rangle \\ &= \langle A^*K^*, Y_t\rangle + \bar{K}^*(0)f(0,Y_t) \\ &= i\tau_\circ\omega_\circ z + \bar{K}^*(0)f(0, W(z,\bar{z},\theta) + 2Re\{Z(t)K(\theta)\}) \\ &= i\tau_\circ\omega_\circ z + \bar{K}^*(0)f_\circ(z,\bar{z}). \end{aligned} \tag{39}$$

On the center manifold, we get

$$W(t,\theta) = W(zz,\bar{z},\theta) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \dots. \tag{40}$$

Where, z and $\bar{z}$ are local coordinates for center manifold in the direction of K^* and $\bar{K}^*$. From (38), clearly we know that W is real if Y_t is real. If we consider only the real solutions, we can rewritten the (40) becomes

$$\dot{Z}(t) = i\tau_\circ\omega_\circ z + g(z,\bar{z}), \tag{41}$$

with

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + ... \tag{42}$$

By transformed (38) becomes

$$\begin{aligned} Y_t(\theta) &= W(z,\bar{z},\theta) + 2Re\{Z(t)K(\theta)\} \\ &= W(z,\bar{z},\theta) + Z(t)K(\theta) + \bar{Z}(t)\bar{K}(\theta) \end{aligned} \tag{43}$$

$$\begin{aligned}
&Y_1(t) = z(t) + \bar{z}(t) + W_{20}^{(1)}(0)\tfrac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_2(t) = z_1z(t) + \bar{z}_1\bar{z}(t) + W_{20}^{(2)}(0)\tfrac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_3(t) = z_2z(t) + \bar{z}_2\bar{z}(t) + W_{20}^{(3)}(0)\tfrac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_4(t) = z_3z(t) + \bar{z}_3\bar{z}(t) + W_{20}^{(4)}(0)\tfrac{z^2}{2} + W_{11}^{(4)}(0)z\bar{z} + W_{02}^{(4)}(0)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_5(t) = z_4z(t) + \bar{z}_4\bar{z}(t) + W_{20}^{(5)}(0)\tfrac{z^2}{2} + W_{11}^{(5)}(0)z\bar{z} + W_{02}^{(5)}(0)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_2(t-1) = z_1z(t)e^{-i\tau_\circ\omega_\circ} + \bar{z}_1\bar{z}(t)e^{i\tau_\circ\omega_\circ} + W_{20}^{(2)}(-1)\tfrac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_3(t-1) = z_2z(t)e^{-i\tau_\circ\omega_\circ} + \bar{z}_2\bar{z}(t)e^{i\tau_\circ\omega_\circ} + W_{20}^{(3)}(-1)\tfrac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_4(t-1) = z_3z(t)e^{-i\tau_\circ\omega_\circ} + \bar{z}_3\bar{z}(t)e^{i\tau_\circ\omega_\circ} + W_{20}^{(4)}(-1)\tfrac{z^2}{2} + W_{11}^{(4)}(-1)z\bar{z} + W_{02}^{(4)}(-1)\tfrac{\bar{z}^2}{2} + ...,\\
&Y_5(t-1) = z_4z(t)e^{-i\tau_\circ\omega_\circ} + \bar{z}_4\bar{z}(t)e^{i\tau_\circ\omega_\circ} + W_{20}^{(5)}(-1)\tfrac{z^2}{2} + W_{11}^{(5)}(-1)z\bar{z} + W_{02}^{(5)}(-1)\tfrac{\bar{z}^2}{2} + ...
\end{aligned}$$

Now from (40) and (41), we get

$$\begin{aligned}
g(z,\bar{z}) &= \bar{K}^*(0)f(0,W(z,\bar{z},\theta) + 2Re\{Z(t)K(\theta)\}) \\
&= \tau^*\bar{D}(1,z_1,z_2,z_3,z_4)\cdot\begin{bmatrix}1\\ d_{11}\\ d_{21}\\ d_{31}\\ d_{41}\\ d_{51}\end{bmatrix}
\end{aligned}$$

$$= \tau_\circ\bar{D}\left[(z_3-1)\beta_1Y_1Y_4 + (z_1q + (1-q)z_2 - 1)\beta_2Y_1Y_5(t-1) + (z_3 - z_2)\beta_3Y_3Y_4\right],$$

where

$$d_{11} = -\beta_1Y_1Y_4 - \beta_2Y_1Y_5(t-\tau),$$

$$d_{21} = q\beta_2Y_1Y_5(t-\tau),$$

$$d_{31} = (1-q)\beta_2 Y_1 Y_5(t-\tau) - \beta_3 Y_3 Y_4,$$

$$d_{41} = \beta_1 Y_1 Y_4 + \beta_3 Y_3 Y_4,$$

$$d_{41} = 0.$$

Comparing the rates with (42), then

$$g_{20} = K^* f_{z^2},$$

$$g_{02} = K^* f_{\bar{z}^2},$$

$$g_{11} = K^* f_{z\bar{z}},$$

$$g_{21} = K^* f_{z^2\bar{z}}.$$

From (38) and (40) we get

$$\dot{W} = \dot{Y}_t - \dot{Z}q(\theta) - \dot{\bar{Z}}\bar{q}(\theta)$$

$$= AY_t + RY_t - i\tau_\circ\omega_\circ zq - (\bar{q}^*)^T f_\circ(z,\bar{z})q + i\tau_\circ\omega_\circ\bar{z}\bar{q} - (q^*)^T \bar{f}_\circ(z,\bar{z})\bar{q}$$

$$\dot{W} = \begin{array}{l} AW - 2Re\left\{(q^*)^T(0) f_\circ(z,\bar{z})q(\theta)\right\},\ if\ \theta \in [-1,0), \\ AW + f_\circ(z,\bar{z}) - 2Re\left\{(q^*)^T(0) f_\circ(z,\bar{z})q\right\},\ if\ \theta = 0 \end{array}$$

Then, we can rewrite (38) as

$$\dot{W} = AW + H(z,\bar{z},\theta), \tag{44}$$

where

$$H(z,\bar{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \ldots..$$

Besides,

$$\dot{W} = W_z\dot{Z} + W_{\bar{z}}\dot{\bar{Z}}. \tag{45}$$

Now by using (40) and (41) in above equation, we get

$$\dot{W} = i\tau_\circ\omega_\circ W_{20}(\theta)Z^2 - i\tau_\circ\omega_\circ W_{02}(\theta)\bar{Z}^2 + \ldots..$$

Again by (40) and (45), is given that

$$H(z,\bar{z},\theta) = (2i\tau_\circ\omega_\circ - A)W_{20}(\theta)\frac{z^2}{2} - AW_{11}(\theta)z\bar{z} - (2i\tau_\circ\omega_\circ + A)W_{02}(\theta)\frac{\bar{z}^2}{2} + \ldots..$$

By comparing the coefficients in above equation with those in equation (49), we obtain

$$\begin{aligned} &AW_{20}(\theta) = 2i\tau_\circ\omega_\circ W_{20}(\theta) - H_{20}(\theta), \\ &AW_{11}(\theta) + H_{11}(\theta) = 0, \\ &AW_{02}(\theta) = -2i\tau_\circ\omega_\circ W_{02}(\theta) - H_{02}(\theta). \end{aligned} \tag{46}$$

From (44) and (45), we have for $\theta \in [-1,0)$

$$\begin{aligned} &H(z,\bar{z},\theta) = -2Re\{\bar{K}^*(0)f_\circ(z,\bar{z})K(\theta)\} \\ &= -g(z,\bar{z})K(\theta) - \bar{g}(z,\bar{z})\bar{K}(\theta). \end{aligned}$$

Then,

$$\begin{aligned} H_{20}(\theta) &= -g_{20}K(\theta) - \bar{g}_{02}\bar{K}(\theta), \\ H_{11}(\theta) &= -g_{11}K(\theta) - \bar{g}_{11}\bar{K}(\theta), \\ H_{02}(\theta) &= -g_{02}K(\theta) - \bar{g}_{20}\bar{K}(\theta). \end{aligned}$$

Based on (46), we have

$$\begin{aligned} &\dot{W}_{20}(\theta) = 2i\tau_\circ\omega_\circ W_{20}(\theta) - H_{20}(\theta) \\ &= i\tau_\circ\omega_\circ W_{20}(\theta) + g_{20}K(\theta) + \bar{g}_{02}\bar{K}(\theta) \end{aligned} \tag{47}$$

$$\dot{W}_{11}(\theta) = -H_{11}(\theta) = g_{11}K(\theta) + \bar{g}_{11}\bar{K}(\theta) \tag{48}$$

with $\theta \in [-1,0)$, resolving for (47) and (48), then

$$W_{20}(\theta) = \frac{ig_{20}K(0)}{\tau_\circ\omega_\circ}e^{i\tau_\circ\omega_\circ\theta} + \frac{i\bar{g}_{02}\bar{K}(0)}{3\tau_\circ\omega_\circ}e^{-i\tau_\circ\omega_\circ\theta} + E_1e^{2i\tau_\circ\omega_\circ\theta},$$

$$W_{11}(\theta) = \frac{-ig_{11}K(0)}{\tau_\circ\omega_\circ}e^{i\tau_\circ\omega_\circ\theta} + \frac{i\bar{g}_{11}\bar{K}(0)}{\tau_\circ\omega_\circ}e^{-i\tau_\circ\omega_\circ\theta} + E_2,$$

where, $E_1 = (E_1^{(1)}, E_2^{(1)}, E_3^{(1)})^T$ and $E_2 = (E_1^{(2)}, E_2^{(2)}, E_3^{(2)})^T$ Can be deduced by $\theta = 0$. $W_{20}(\theta)$ and $W_{11}(\theta)$ are contentious on $[-1,0]$, if $\theta = 0$, then

$$W_{20}(\theta) = \frac{ig_{20}K(0)}{\tau_\circ\omega_\circ} + \frac{i\bar{g}_{02}\bar{K}(0)}{3\tau_\circ\omega_\circ} + E_1,$$

$$W_{11}(\theta) = \frac{-ig_{11}K(0)}{\tau_\circ\omega_\circ} + \frac{i\bar{g}_{11}\bar{K}(0)}{\tau_\circ\omega_\circ} + E_2.$$

Now, we know that

$$AW_{20}(0) = \int_{-1}^{0} d\eta(\xi)W_{20}(\theta),$$

$$AW_{11}(0) = \int_{-1}^{0} d\eta(\xi) W_{11}(\theta).$$

So we get

$$(2i\tau_\circ\omega_\circ I - A_1 - A_2 e^{-2i\tau_\circ\omega_\circ})E_1 = 2\tau_\circ \cdot \begin{bmatrix} -\beta_1 z_3 - \beta_2 z_4 e^{i\tau_\circ\omega_\circ} \\ q\beta_2 z_4 e^{i\tau_\circ\omega_\circ} \\ (1-q)\beta_2 z_4 e^{i\tau_\circ\omega_\circ} - \beta_3 z_2 z_3 \\ \beta_1 z_3 + \beta_3 z_2 z_3 \end{bmatrix}. \tag{49}$$

And

$$(A_1 + A_2)E_2 = \tau_\circ \begin{bmatrix} -\beta_1(z_3 + \bar{z}_3) - \beta_2(z_4 + \bar{z}_4) \\ q\beta_2(z_4 + \bar{z}_4) \\ (1-q)\beta_2(z_4 + \bar{z}_4) - \beta_3(z_2\bar{z}_3 + \bar{z}_2 z_3) \\ \beta_1(z_3 + \bar{z}_3) + \beta_3(z_2\bar{z}_3 + \bar{z}_2 z_3) \end{bmatrix}, \tag{50}$$

with I is identity matrix, thus

$$E_1 = 2\tau_\circ(2i\tau_\circ\omega_\circ I - A_1 - A_2 e^{-2i\tau_\circ\omega_\circ})^{-1} \cdot \begin{bmatrix} -\beta_1 z_3 - \beta_2 z_4 e^{i\tau_\circ\omega_\circ} \\ q\beta_2 z_4 e^{i\tau_\circ\omega_\circ} \\ (1-q)\beta_2 z_4 e^{i\tau_\circ\omega_\circ} - \beta_3 z_2 z_3 \\ \beta_1 z_3 + \beta_3 z_2 z_3 \end{bmatrix}, \tag{51}$$

$$E_2 = \tau_\circ(A_1 + A_2)^{-1} \cdot \begin{bmatrix} -\beta_1(z_3 + \bar{z}_3) - \beta_2(z_4 + \bar{z}_4) \\ q\beta_2(z_4 + \bar{z}_4) \\ (1-q)\beta_2(z_4 + \bar{z}_4) - \beta_3(z_2\bar{z}_3 + \bar{z}_2 z_3) \\ \beta_1(z_3 + \bar{z}_3) + \beta_3(z_2\bar{z}_3 + \bar{z}_2 z_3) \end{bmatrix}. \tag{52}$$

Consequently, from previous analysis, we can compute the following values

$$N_1(0) = \frac{i}{2\tau_\circ\omega_\circ}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = \frac{-Re(N_1(0))}{Re(\acute{\lambda}(\tau_\circ))},$$

$$\beta_2^* = 2Re(N_1(0)),$$

$$T_2 = \frac{-ImN_1(0) + \mu_2 Im\acute{\lambda}(\tau_\circ)}{\tau_\circ\omega_\circ}.$$

Clearly, the above values determine the quantities of bifurcating periodic solutions at $\tau = \tau_\circ$, i.e. μ_2 shows the direction of the Hopf bifurcation so for that $\mu_2 > 0$ $(\mu_2 < 0)$ then the Hopf bifurcation is supercritical (subcritical); Further β^* shows the stability of the bifurcating periodic solutions so that the periodic solutions are stable (un stable) when $\beta_2^* > 0$ $(\beta_2^* < 0)$; Finally, if $T_2 > 0$ $(T_2 < 0)$, we have the period of the bifurcating periodic solutions increases (decreases).

6. Numerical Results

6.1. Numerical simulation. Here, the analytical results we have obtained will be illustrated by numerical simulations by taking hypothetical values of the parameters in system (1), with different initial conditions in order to determine the effect of each of them on the dynamic behavior of the system (1). In the following data with initial point (0.7,0.9,0.6,0.5,0.5), we obtain that the trajectory of system (1) convergent towards stability of (DFE) point when any $\tau \geq 0$, as shown that in figure 1.

$$
\begin{gathered}
p = 0.87;\ A = 8;\ \beta_1 = 0.0010;\ \beta_2 = 0.05;\ \beta_3 = 0.005; \\
\rho = 0.05;\ \mu = 0.015; q = 0.6\ \nu = 0.3;\ d = 0.002;\ \gamma = 0.3;\ \theta = 0.2
\end{gathered} \tag{53}
$$

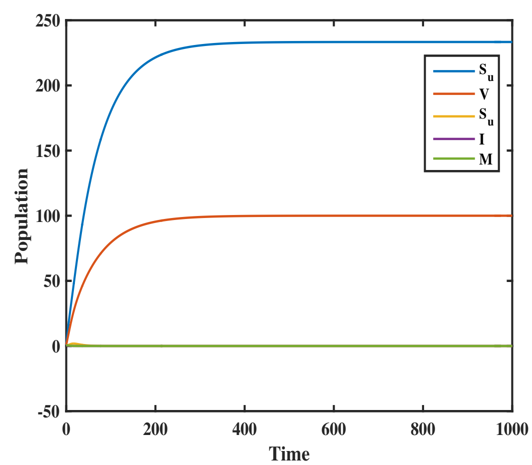

FIGURE 1. Stability of (DFE) point in system (1) when $\tau \geq 0$.

Obviously, Figure (1), shows that the solution of system (1) approaches asymptotically to the (DFE) point has asymptotically stable $E_\circ = (233, 99, 0, 0, 0)$.

Now, to study the stability of the (EE) point for system (1), we can take the same above data but $\beta_1 = 0.010$ and $0 \leq \tau \leq 4$ shown that in figure (2).

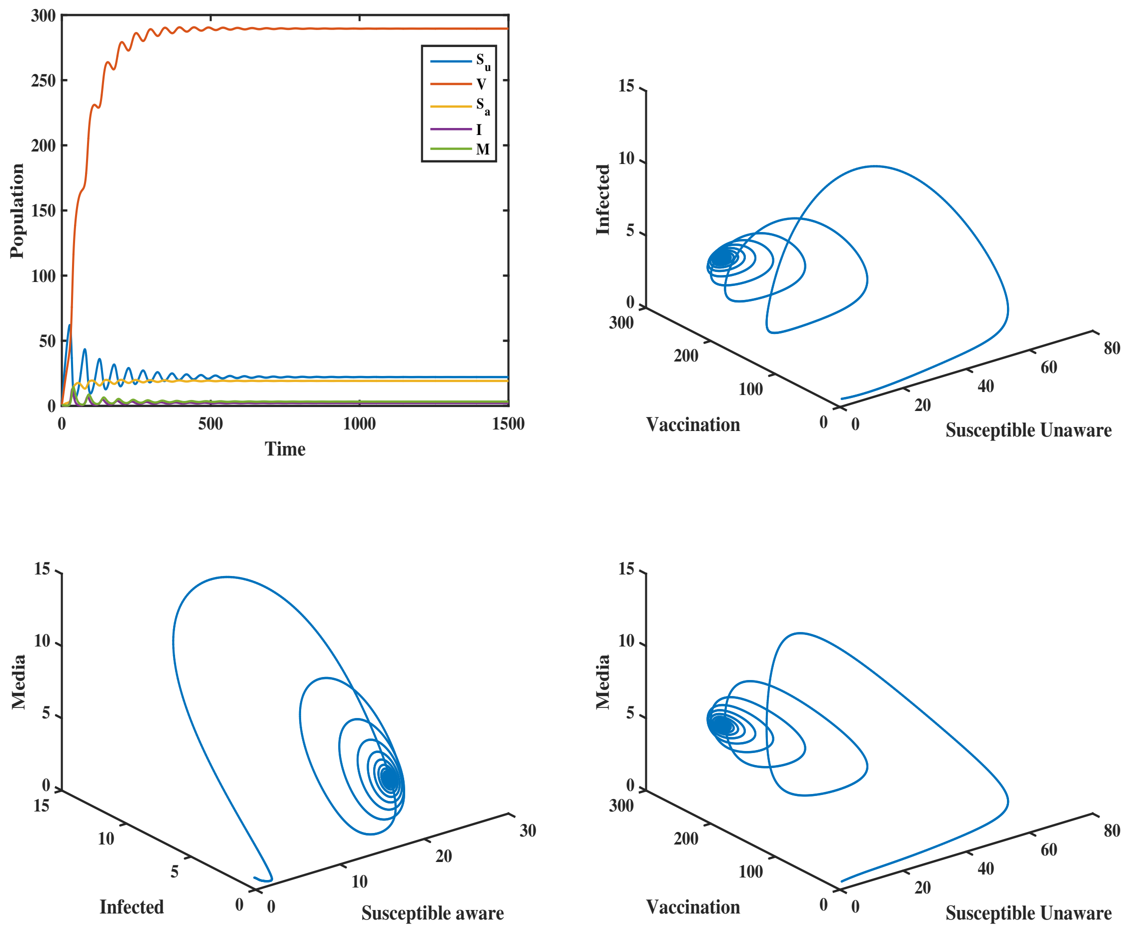

FIGURE 2. stability of (EE) point of system (1) when $0 \leq \tau \leq 4$.

Therefore, figure (2) as shown us, the endemic equilibrium point of system (1) is globally asymptotically stable and is identically to $E_1 = (22, 289, 19, 2, 3)$.

Now, we discuss the effect of delay on the dynamical behavior of system (1), hence we choose the data in equation (53) with $\beta_1 = 0.010$ and $\tau = \tau_\circ \approx 5$. Then, the (EE) point of system (1), become unstable and the solutions of system (1) has periodic as shown that in figure (3) and figure (4) with kepping data of figure (3) but we put $\tau = 15 > \tau_\circ \approx 5$.

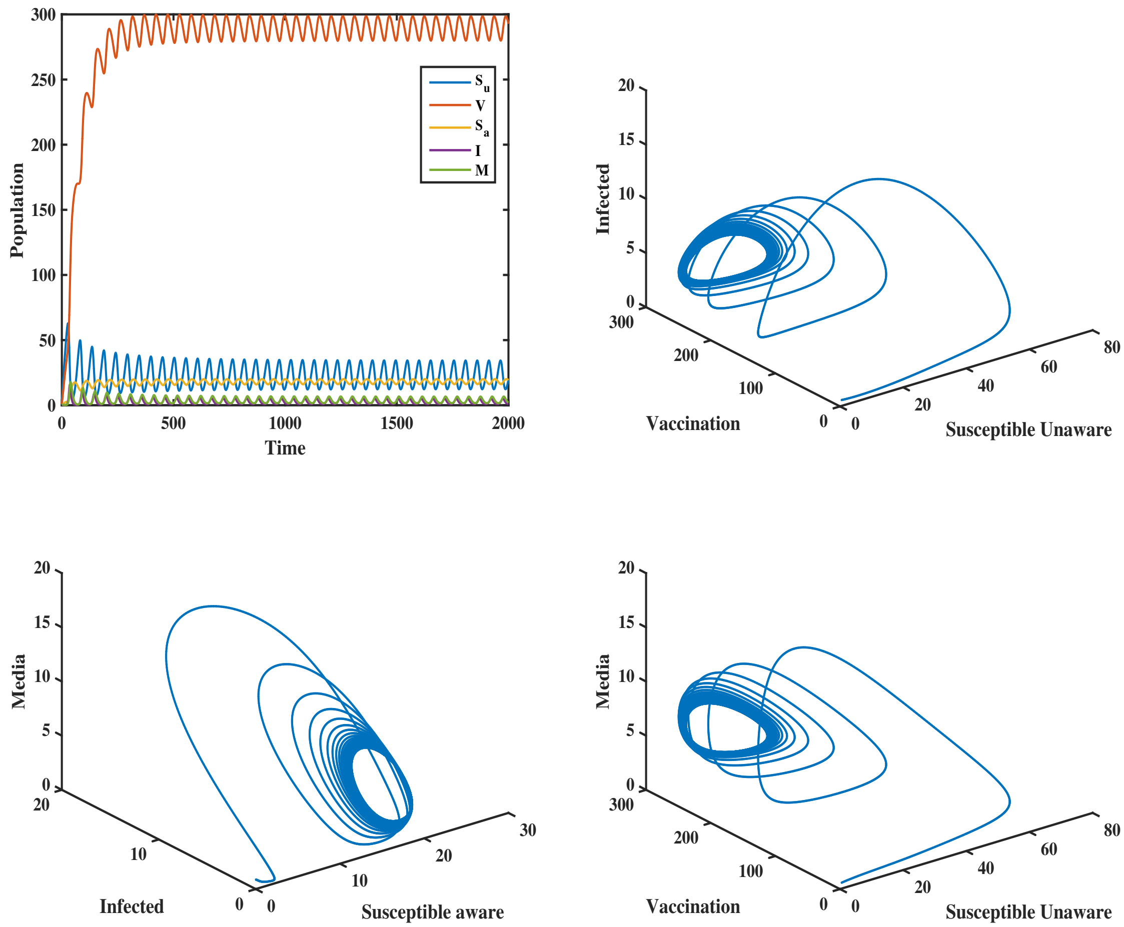

FIGURE 3. The (EE) point is loss stability for the data given by equation (53) with $\tau = \tau_\circ \approx 5$.

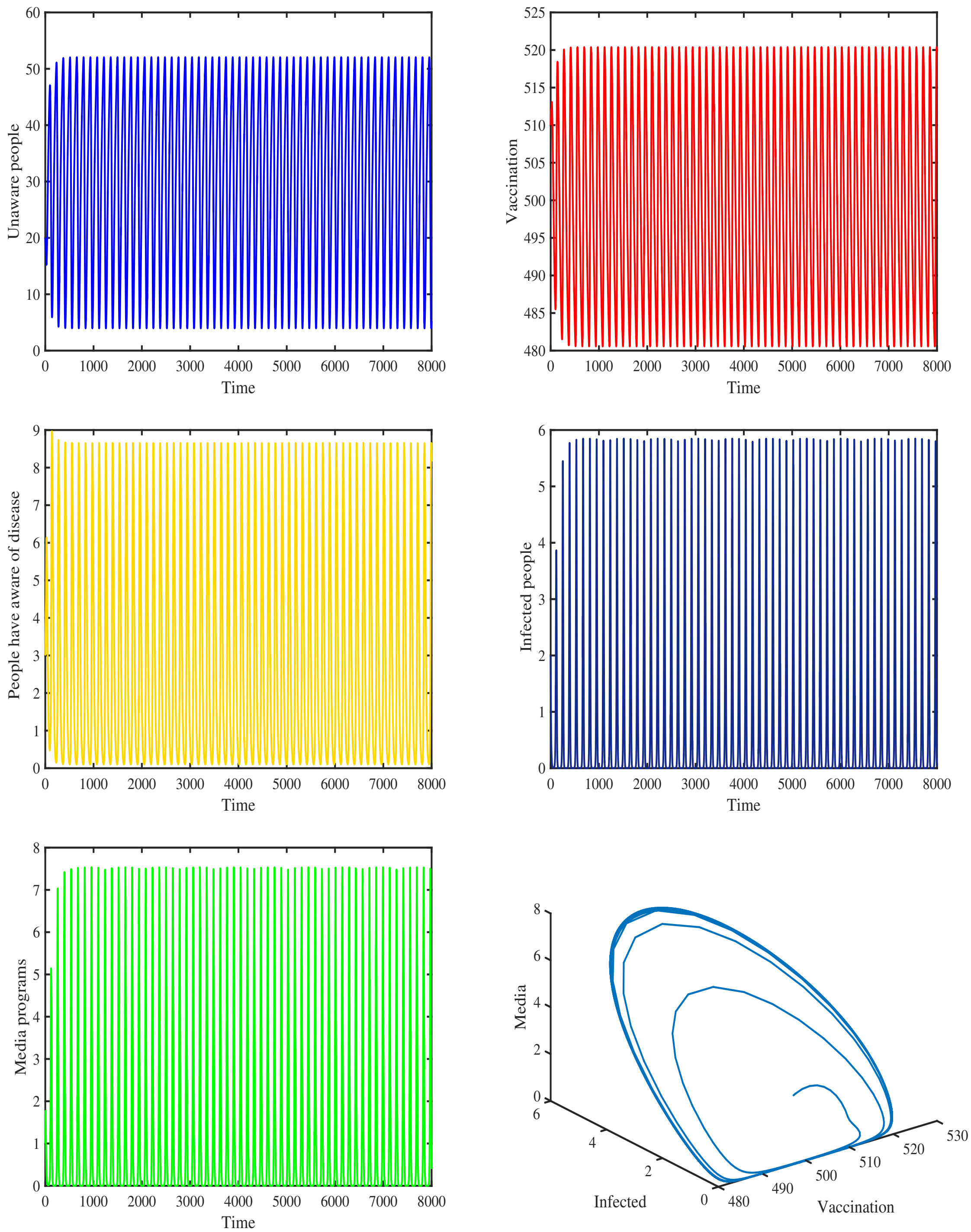

FIGURE 4. The (EE) point is loss stability for the data given by equation (53) with $\tau = 15 > \tau_{\circ} \approx 5$.

6.2. Sensitivity Analysis. The goal of discussing sensitivity analysis is to reveal a parametric effect of disease prevalence in epidemiological modeling. Our goal is to find out the parameters that control $\mathscr{R}_0$ and through which the dynamics behavior of the epidemiological model can be predicted. And all of this can be studied through the global sensitivity index of $\mathscr{R}_0$ and according to the parameter ϕ and it is denoted by $\Pi^{\phi}_{\mathscr{R}_0}$ and is defined by

$$\Pi^{\phi}_{\mathscr{R}_0} = \frac{\partial \mathscr{R}_0}{\partial \phi} \frac{\phi}{\mathscr{R}_0} \tag{54}$$

From the Eq (54), we note that as there is an increase or decrease in parameter rate ϕ by a certain percentage say m then the reproduction number $\mathscr{R}_0$ also increases or decreases by the same percentage m. Now, an increase of the value of any one of these parameters β_1 and A will increase the basic reproduction number. While, an increase of the value of other parameters p, d, ν and μ will decrease $\mathscr{R}_0$. Thus, we confirm that results and illustrate in the following figures (5) and (6)

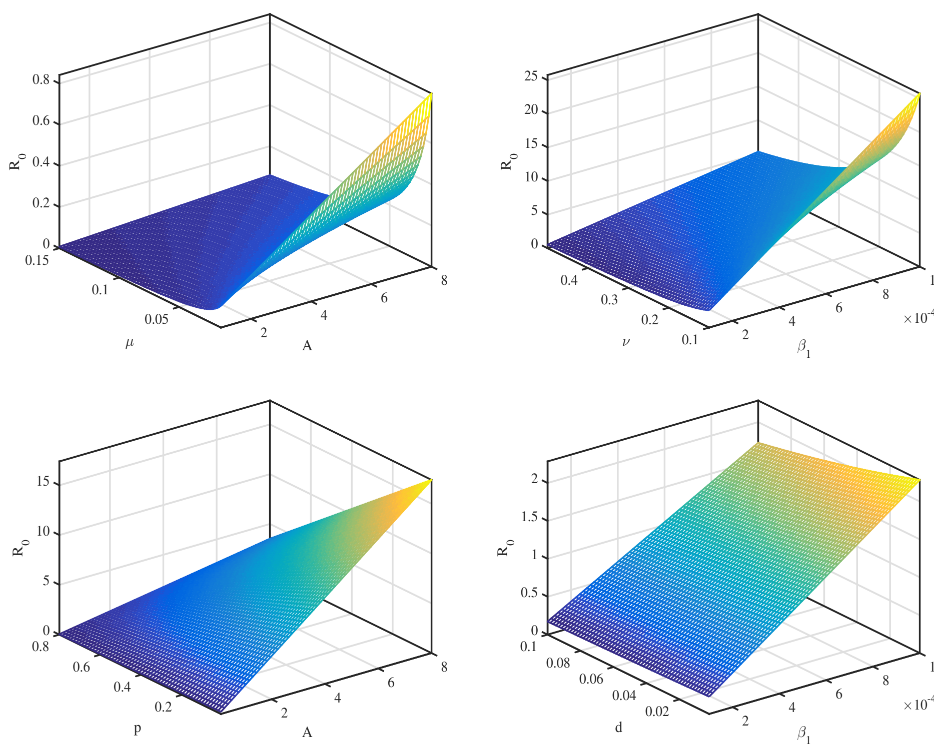

FIGURE 5. The relationship between $\mathscr{R}_0$ and several parameters.

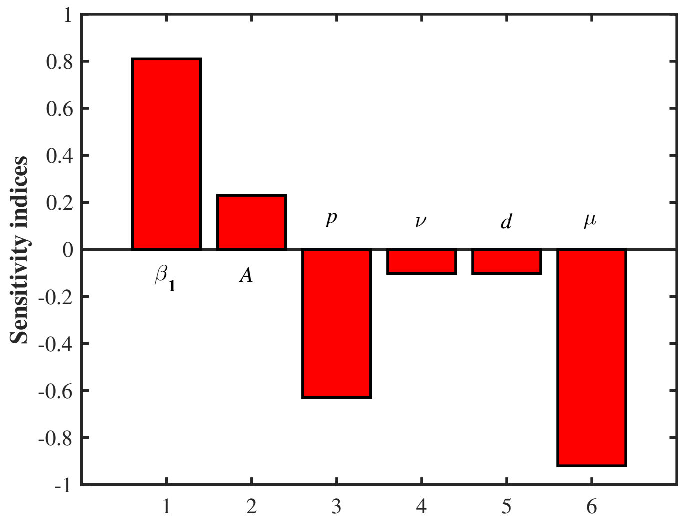

FIGURE 6. Sensitivity indices diagram for $\mathscr{R}_0$.

7. CONCLUSION

In this work, we investigated the dynamics of the epidemiological model with awareness and vaccination effect as well as we studied the effect of media programs time delay on the dynamic behavior of the epidemiological model. Besides determining the basic reproduction number and points of stability of the model. Then, a hopf bifurcation near the endemic point of proposed model was investigated, we also discussed the sensitive analysis to virtualize determined scenarios and choose a strategy that minimizes the spreading of the disease. The obtained numerical results are discussed and shown through graphs. For future work, we plan to modify the proposed model in a fractional order and presenting a results of the changes in the dynamic behavior of the infectious diseases.

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Not applicable.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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