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A NEW DISCRETE-TIME EPIDEMIC MODEL DESCRIBING INFORMATION SPREAD AND ITS IMPACT ON THE AGREE-DISAGREE MODEL

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Abstract. In this study, we propose a new modified discrete-time epidemic model that characterizes the diffusion of information and its impact on the agreement/disagreement model. So the goal is to increase the amount of information to influence people's opinions. To do this, we proposed a control strategy based on the increase in the number of posts that influence people to agree with the subject studied (election, vaccine against COVID 19). The Pontryagin maximum principle is used to describe optimal control. Finally, numerical simulations are performed to verify the theoretical analysis using MATLAB.

Keywords: mathematical model; discrete-time systems; optimal control; information; Pontryagin maximum.

2020 AMS Subject Classification: 92C60.

1. Introduction

The internet presents an enormous network of growth which leads to the production of unsecured information, everyone could post and share anything online [1]. The anarchic nature of social media is desirable for maintaining open debates without control, this dilemma creates issues about the quality of information circulated in different kinds of network websites [2].

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Recently, the use of social media has become the first information source due to its increasing popularity among various user groups [3], and its relationship between emotions Besides information diffusion in a network setting, each data shared by a user generates sentimental content that affects other users to keep sharing and discussing the fact, emotionally charged messages tend to be powerful more than neutral ones [4].

However, the information spread through social networks not only affects the development of political processes and other spheres of society, but also is influenced by the economy, culture, education, etc. Recently, the whole world has seen a debate of agreeing and disagreeing about vaccination based on irresponsible anti-vaccination propaganda, due to fear of reactions to vaccination and complications after the procedure especially after the exchange of some negative experiences of vaccinated people, the information shared in mass media and social networks has a great influence on the formation of the population's attitude to vaccination [5, 6]. To change others mind, a variety of language tools are often used such as metaphors and words with an emotional connotation to touch people's hearts.

In the 21st century, people can talk about events, and express their attitudes to others on social networks. Today, Facebook, Twitter, Instagram, YouTube, Ticktock, and others are not just platforms for posting photos, but also public areas for discussion, expression, and opinion formation. Gathering like-minded people and organizing protests is now much easier. Thus, the internet has greatly simplified the propagation of information and increased the level of political participation of the population. Coordination of modern actions and mobilization of citizens takes place in social networks [7, 8]. These debates of agree and disagree are created and keep circulating in social media in which every part tries to convince and influence public opinion. For this reason, a mathematical model is being studied describing the evolution of agree and disagree opinions, Then the population will be devised into three compartments the first is Indifferent people or people who do not know about the subject or decide to not participate in the debate. The second category is agreed or Approved Individuals refers to people in agreement with the idea being studied. The third one concerns Disagree or Disapproved Individuals in disagreement with the idea being studied [9, 10]. The objective of this paper is to introduce a new compartment to *IAD* model [9, 11, 12], this compartment called Quantity of information

F_A which denotes the amount of information that supports the topic under examination, that exists on websites and social media, to illustrate the power of its information in changing the opinions of people to be in agreement [13].

The paper is structured as follows. In Section 2, we briefly describe our proposed model, and we give some basic properties of the model. In Section 3, we introduce the control problem and provide some results. The Existence and Characterization of Optimal Control Using Pontryagin's Maximum Principle. Numerical simulation on MATLAB supports the theoretical results in Section 4. Lastly, we conclude our work in Section 5.

2. Presentation of the Model

Many models consider that the propagation of information is analogous to an epidemic [14, 11]. Moreover, in the modeling of the transmission of information, the population is supposed to be divided into three compartments Ignored, Agreed, and Aisagreed similar to the SIR models [11, 15, 16].

This type of model can then be used to describe the impact of social media on the human population through their publication [11].

In our paper, we put ourselves in the position in front of a problem of public opinion that divided the population into three categories: Ignorant (I), Agree (A), Disagree (D), Example (elections, Corona vaccine), without losing generalities we suppose that we want to increase the number of people who agree, for this, we consider the model of information transmission *IAD* [11], where the compartment "Ignorant" is used to indicate the people who have no idea of the subject of the study or who are not interested in the subject, the compartment "Agree" is used to indicate that a person agrees with a studied subject. The compartment "Disagree" is used to indicate people do not agree with the topic studied.

In the innovation, we introduced a new compartment called Quantity of information F_A that denotes the amount of information that supports to be with the subject examined, that exists on the websites and social media.

The model resulting from these arguments is governed by the following system:

$$
\begin{cases}
I_{i+1} &= I_i - \alpha A_i I_i - \beta D_i I_i - \theta F_i^A I_i + \delta_1 A_i + \delta_2 D_i \\
A_{i+1} &= A_i + \alpha A_i I_i + \theta F_i^A I_i - \delta_1 A_i \\
D_{i+1} &= D_i + \beta D_i I_i - \delta_2 D_i \\
F_{i+1}^A &= F_i^A + P_A F_i^A + \sigma_1 F_i^A A_i - \mu F_i^A
\end{cases} \tag{1}
$$

Where $I_i > 0, A_i > 0, D_i > 0, F_i^A > 0$ for all i, and α is the proportion in which an ignorant person meets/contacts an agreeing person and also becomes agreed, (β) is the probability that an ignorant person will contact a person who disagrees and also become disagreed. The probability that an ignorant person will encounter the amount of information that influences him to agree is θ. Users can delete messages, videos, and images for any reason at a rate of μ. Agree and disagree individuals can change their opinions and become ignorant people with respective rates δ_1, and δ_2, respectively. The meaning of each parameter is given in the table (1).

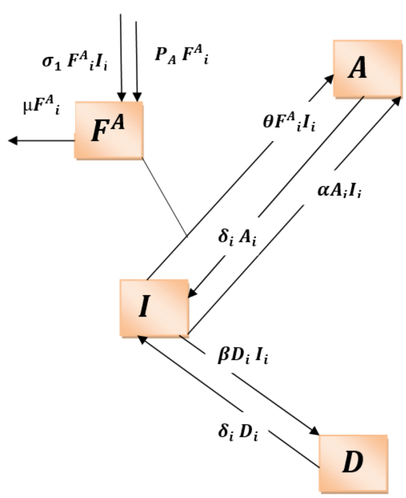

FIGURE 1. A compartment diagram of the transmittance dynamics of $IADF^A$.

TABLE 1. Model parameters meaning

Parameter	Description
α	the rate at which an ignorant person will contact someone who disagrees and who also disagrees
β	the rate at which an ignorant person will contact someone who disagrees and who also disagrees
δ_1	The rate at which a person agrees becomes ignorant
δ_2	The rate at which a person disagrees becomes ignorant
θ	The rate at which an ignorant person will encounter the amount of information that will influence them to agree.
σ_1	Factor of loss of interest of individuals in agreement
μ	Probability that the agreed information is deleted
P_A	A novel information (agree) posting rate

3. THE OPTIMAL CONTROL PROBLEM

We study the impact of information on people's opinions, for example, an influencer with thousands of fans who sells cosmetics, by publishing a video in which she shows how a product gives very satisfactory results. She thus changes the opinion of some girls on this product. Another example: during elections, a group of candidates chooses social media sites to persuade a large group of young people to vote through a series of publications. The World Health Organization and country governments have used the media to convince people to take the covid 19 vaccine. So our noted control strategy (u) is the new posts (advertisement, videos, images) which consist of increasing the amount of information to influence people's opinions to agree with the topic under study. We present the new control variable u as follows:

$$
\begin{cases}
I_{i+1} &= I_i - \alpha A_i I_i - \beta D_i I_i - \theta F_i^A I_i + \delta_1 A_i + \delta_2 D_i \\
A_{i+1} &= A_i + \alpha A_i I_i + \theta F_i^A I_i - \delta_1 A_i \\
D_{i+1} &= D_i + \beta D_i I_i - \delta_2 D_i \\
F_{i+1}^A &= F_i^A + P_A F_i^A + \sigma_1 F_i^A A_i - \mu F_i^A + u_i F_i^A
\end{cases} \tag{2}
$$

where $I_i > 0, A_i > 0, D_i > 0, F_i^A > 0$ for all i.

3.1. Objective functional.

The goal of the optimal control problem is to minimize the objective function given by

$$J(u) = (\alpha_D D_{\mathcal{N}} - \alpha_A A_{\mathcal{N}}) + \sum_{i=0}^{\mathcal{N}-1} \left(\alpha_D D_i - \alpha_A A_i + \frac{A}{2} (u_i)^2 \right) \tag{3}$$

where $A > 0$, $\alpha_D > 0$, $\alpha_A > 0$ are the constants of weights of the controls, the sharers and the withdrawn, separately, $u = (u_0, \cdots, u_{\mathcal{N}-1})$, and $\mathcal{N}$ is the final moment of our control strategize. The goal is to reduce the number of people who disagree and maximize the number of people who agree with an optimal cost. In other words, we are seeking the control u^* such that

$$J(u^*) = min\{J(u)/u \in \mathcal{U}\} \tag{4}$$

where $\mathcal{U}$ is the control defined by

$$\mathcal{U} = \{u/u_{min} \leq u_i \leq u_{max}, i = 0, \cdots, \mathcal{N} - 1\} \tag{5}$$

such that

$$0 < u_{min} < u_{max} < 1$$

3.2. Sufficient conditions.

Theorem 3.1. *There exists an optimal control $u^* \in \mathcal{U}$ such that*

$$J(u^*) = min\{J(u)/u \in \mathcal{U}\}$$

subject to the control system (2)-(4) *and initial conditions.*

Proof. Given that the parameters of the system are bounded and that there is a finite number of time steps, i.e. I, A, D, F^A and are uniformly bounded for all u in the control set $\mathcal{U}$, thus $J(u)$ is also bounded for all $u \in \mathcal{U}$. This implies that $\inf_{u \in \mathcal{U}} J(u)$ is finite, and there exists a sequence $u^n \in \mathcal{U}$ such that

$$\lim_{n \to +\infty} J(u^n) = \inf_{u \in \mathcal{U}} J(u)$$

and corresponding sequences of states I^n, A^n, D^n , F^{An}.

Since there is a finite number of uniformly bounded sequences, there exists $u^* \in \mathscr{U}$ and I^*, A^*, D^* F^{A*} such that, on a sequence,

$$u^n \to u^*$$

$$I^n \to I^*$$

$$A^n \to A^*$$

$$D^n \to D^*$$

$$F^A \to F^{A*}$$

Lastly, due to the finite dimension structure of the system (2)-(4) and the goal function $J(u)$, we obtain that (u^*) is an optimal control with corresponding states I^*, A^*, D^* , F^{A*}. Which completes the proof. ■

3.3. Necessary conditions. Using the Pontryagin maximum principle [17, 18, 19], we determine the necessary conditions for our optimal controls. For this goal, we set the Hamiltonian as

$$\begin{aligned}
\mathscr{H}(i) &= \alpha_D D_i - \alpha_A A_i + \frac{A}{2}(u_i)^2 \\
&+ \zeta_{1,i+1}\left[I_i - \alpha A_i I_i - \beta D_i I_i - \theta F_i^A I_i + \delta_1 A_i + \delta_2 D_i\right] \\
&+ \zeta_{2,i+1}\left[A_i + \alpha A_i I_i + \theta F_i^A I_i - \delta_1 A_i\right] + \zeta_{3,i+1}\left[D_i + \beta D_i I_i - \delta_2 D_i\right] \\
&+ \zeta_{4,i+1}\left[F_i^A + P_A F_i^A + \sigma_1 F_i^A A_i - \mu F_i^A + u_i F_i^A\right]
\end{aligned} \tag{6}$$

Theorem 3.2. *With optimal control u^* and solutions I^*, A^*, D^* , F^{A*} , there exists $\zeta_{k,i}$, $i = 0 \ldots \mathscr{N} - 1$, $k = 1,2,3,4$ the adjoint variables meeting the following equations:*

$$\Delta\zeta_{1,i} = -\left[\zeta_{1,i+1}\left(1 - \alpha A_i - \beta D_i - \theta F_i^A\right) + \zeta_{2,i+1}\left(\alpha A_i + \theta F_i^A\right) + \zeta_{3,i+1}\beta D_i\right] \tag{7}$$

$$\Delta\zeta_{2,i} = -\left[-\alpha_{\mathrm{A}} + \zeta_{1,i+1}\left(-\alpha I_i + \delta_1\right) + \zeta_{2,i+1}\left(1 + \alpha I_i - \delta_1\right) + \zeta_{4,i+1}\sigma_1 F_i^A\right] \tag{8}$$

$$\Delta\zeta_{3,i} = -\left[\alpha_{\mathrm{D}} + \zeta_{1,i+1}\left(-\beta I_i + \delta_2\right) + \zeta_{3,i+1}\left(1 + \beta I_i - \delta_2\right)\right] \tag{9}$$

$$\Delta\zeta_{4,i} = -\left[-\zeta_{1,i+1}\theta I_i + \zeta_{2,i+1}\theta I_i + \zeta_{4,i+1}(1 + P_A + \sigma_1 A_i - \mu + u_i)\right] \tag{10}$$

whit $\zeta_{1,\mathcal{N}} = 0 = \zeta_{2,\mathcal{N}} = -\alpha_A$, $\zeta_{3,\mathcal{N}} = \alpha_D$, $\zeta_{4,\mathcal{N}} = 0$ *are the transversality conditions.*
In addition, for $i = 0, 1, \cdots, \mathcal{N} - 1$ *we obtain the optimal control* u_i^* *as*

$$
\begin{aligned}
u_i^* &= min\left\{max\left\{u_{min}, \frac{-F_i^A\ \zeta_{4,i+1}}{A}\right\}, u_{max}\right\} \\
& \quad i = 0, ..., \mathcal{N} - 1
\end{aligned} \tag{11}
$$

Proof. If we use the discrete version of Pontryagin's maximum principle [20], we get the following adjoint equations:

$$
\begin{aligned}
\Delta\zeta_{1,i} &= \frac{-\partial H}{\partial I} \\
&= -\left[\zeta_{1,i+1}\left(1 - \alpha A_i - \beta D_i - \theta F_i^A\right) + \zeta_{2,i+1}\left(\alpha A_i + \theta F_i^A\right) + \zeta_{3,i+1}\beta D_i\right] \\
\Delta\zeta_{2,i} &= \frac{-\partial H}{\partial A} \\
&= -\left[-\alpha_{\mathrm{A}} + \zeta_{1,i+1}\left(-\alpha I_i + \delta_1\right) + \zeta_{2,i+1}\left(1 + \alpha I_i - \delta_1\right) + \zeta_{4,i+1}\sigma_1 F_i^A\right] \\
\Delta\zeta_{3,i} &= \frac{-\partial H}{\partial D} \\
&= -\left[\alpha_{\mathrm{D}} + \zeta_{1,i+1}\left(-\beta I_i + \delta_2\right) + \zeta_{3,i+1}\left(1 + \beta I_i - \delta_2\right)\right] \\
\Delta\zeta_{4,i} &= \frac{-\partial H}{\partial F_i^A} \\
&= -\left[-\zeta_{1,i+1}\theta I_i + \zeta_{2,i+1}\theta I_i + \zeta_{4,i+1}(1 + P_A + \sigma_1 A_i - \mu + u_i)\right]
\end{aligned}
$$

With $\zeta_{1,\mathcal{N}} = 0 = \zeta_{2,\mathcal{N}} = -\alpha_A$, $\zeta_{3,\mathcal{N}} = \alpha_D$, $\zeta_{4,\mathcal{N}} = 0$ [19].
For $i = 0, \cdots, \mathcal{N} - 1$, the optimal control (u^*) can be determined from the optimality conditions

$$\frac{\partial H}{\partial u} = Au_i + \zeta_{4,i+1}F^A = 0$$

we obtain the optimal control as follows

$$u_i = \frac{-\zeta_{4,i+1}F_i^A}{A}.$$

By the limits in $\mathscr{U}$ of the control in the definition (5) , it is easy to get u_i^* in the following form

$$
\begin{aligned}
u_i^* &= min\left\{max\left\{u_{min}, \frac{-F_i^A\ \zeta_{4,i+1}}{A}\right\}, u_{max}\right\} \\
& \quad i = 0, ..., \mathcal{N} - 1.
\end{aligned}
$$

This achieve the demonstration. ■

4. Discussion

We give numerical simulations for the aforementioned optimization problem in this part. We use several types of data to model our work when writing the program in MATLAB. With a discrete iterative method that converges after a sufficient test akin to the FBSM, the optimality systems are solved. The adjoint system is then solved backward in time due to the transversality conditions after the state's system has first been solved with the starting hypothesis forward in time. The state and co-state resources obtained in the previous steps are used to update our optimal control settings. Lastly, we carry out the previous procedures up until the desired tolerance is reached.

We put in the spot where we want to raise the number of people who agree given a phenomenon, social discussion, or social opinion, where there are individuals who agree and people who are uninformed or uninterested (example: the number of people who agree with the vaccination against COVID 19, or the number of people who support a political party). In order to do that, we suggested a control approach based on the volume of data that supports the examined discussion and is noted F^A to the mathematical model IAD proposed by BIDAH et al [9]. The goals of this method of control are to illustrate how information is spread and its impact on social opinion, as well as how to influence people's opinions.

Numerical simulations of our model with MATLAB using the parameters in the table have demonstrated the effectiveness of our control strategy.

TABLE 2. Parameters values using in the simulation

Parameter	value
α	0.07
β	0.5
δ_1	0.3
δ_2	0.1
θ	0.1
σ_1	0.09
μ	0.1
P_A	0.1

It can be seen from Figure (7) that the amount of information that influences people to agree with the subject under study is slightly increased, which justifies that the number of individuals disagreeing has become greater than the number of people agreeing after 40 days (see Figure (2)).

We are interested in this case, and without losing generality we want to increase the number of people who agree, for this reason, we have proposed the u control, which is the new posts (advertisement, videos, images) that consist in increasing the amount of information in order to influence people's opinions so that they agree with the studied subject.

Because of the additional information provided by the u control, we notice from figures 1, 2 and 3 that the number of F^A information is increased in a remarkable way. Moreover, from figures (3) and (4), we can see that the control program allowed us to indirectly reduce the number of people disagreeing while indirectly increasing the number of people agreeing. The above results and Figure (5) illustrate that there were more people who were ignored at the beginning and that over time, most of the ignored people became in agreement.

The results obtained showed the influence of the spread of information on social networks on the agreed-disagreed debate, which further argues our work, the article by Puri Neha et al. [21] which examined the current position of social media platforms in the spread of vaccine hesitancy and explored the next steps in how social media can be used to improve health literacy and foster public confidence in vaccination. In addition, the study made by the National Bureau of Economic Research [22], which illustrated how social media influence the results of elections in the United States by using the variation of the number of Twitter users, the results obtained indicate that Twitter has decreased the share of Republican votes in the presidential elections of 2016 and 2020.

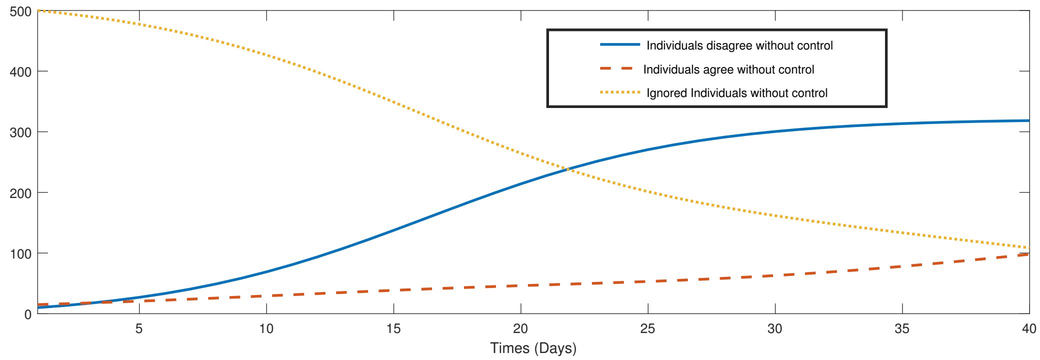

FIGURE 2. Individuals disagree, agree and ignored without control

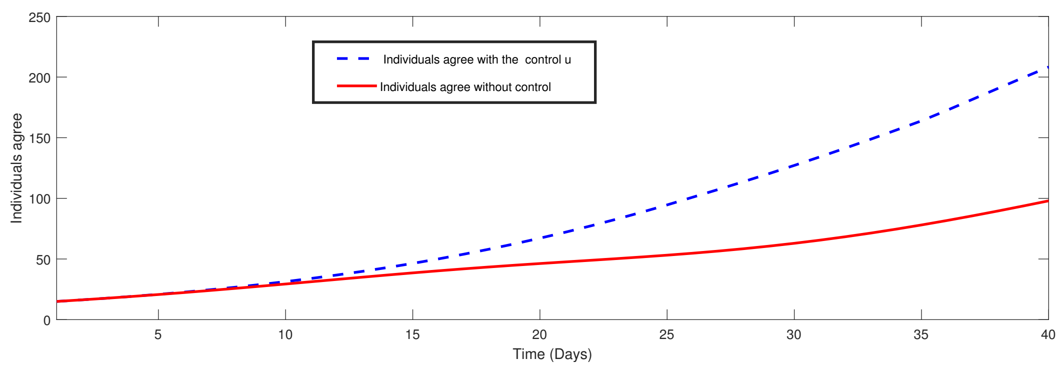

FIGURE 3. Individuals agree without and with the control u

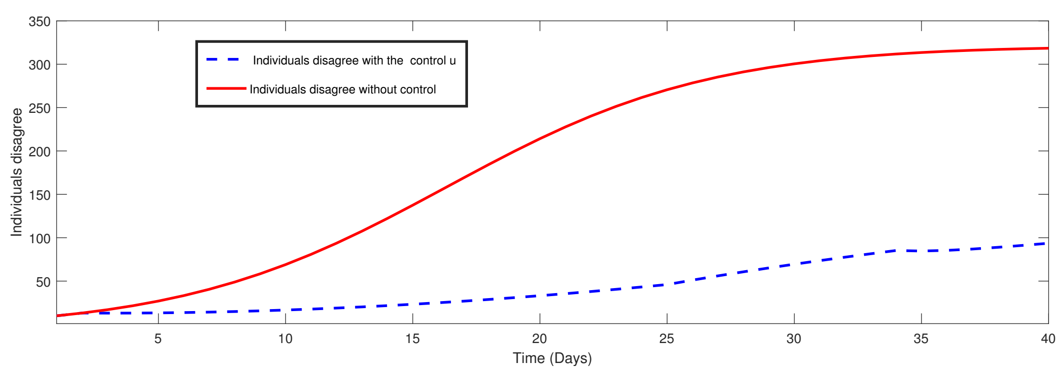

FIGURE 4. Individuals disagree without and with the control u

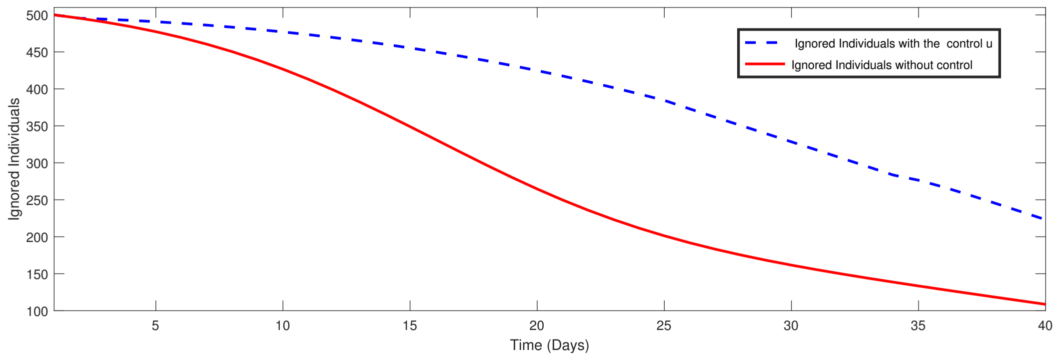

FIGURE 5. Ignored individuals without and with the control u

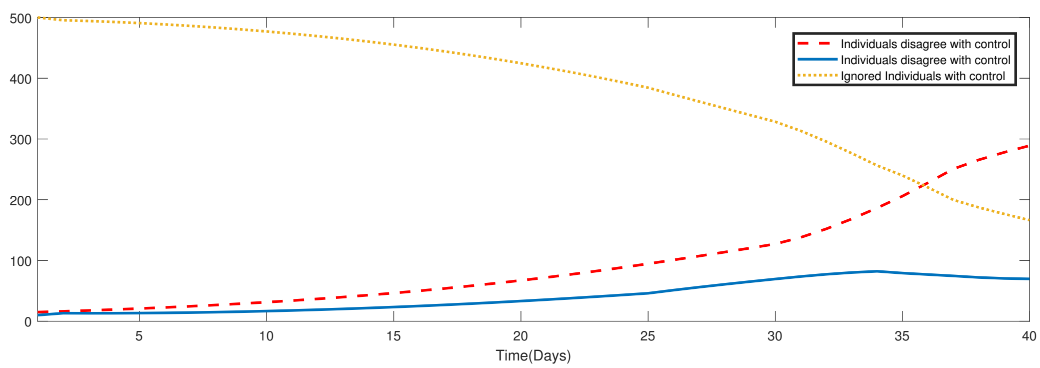

FIGURE 6. Individuals disagree, agree, and ignored with the control u

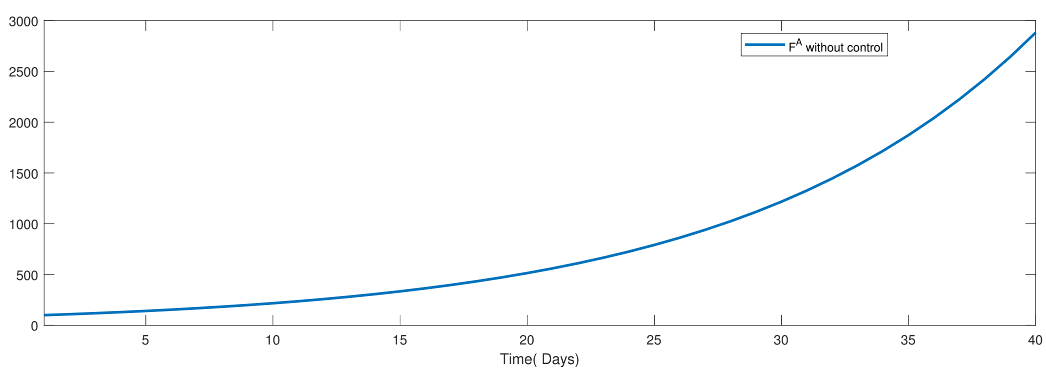

FIGURE 7. The amount of information that influences people to agree in the absence of control

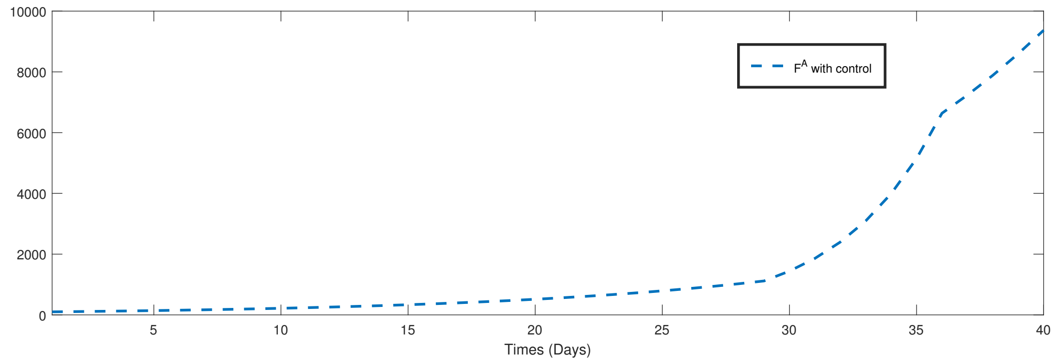

FIGURE 8. The amount of information that influences people to agree in the presence of the control u

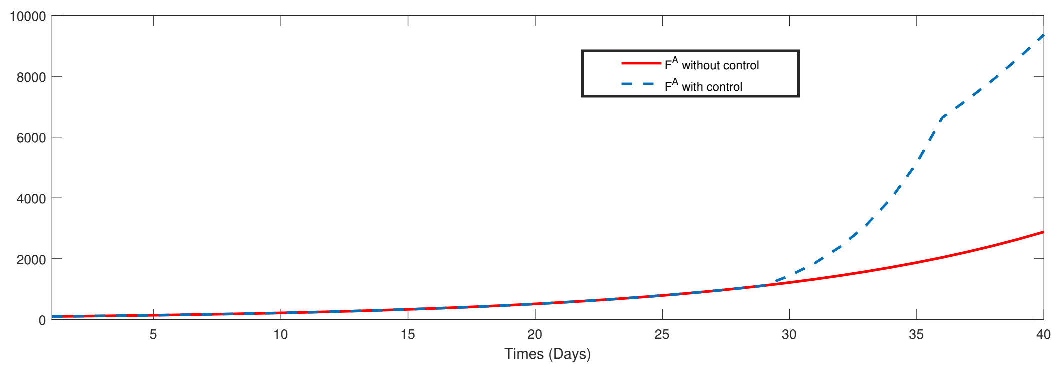

FIGURE 9. The amount of information that influences people to agree in the absence and the presence of the control u

5. CONCLUSION

In this work, a new simple discrete-time epidemic model describing the spread of information in certain types of online environments such as Facebook, WhatsApp, and Twitter is examined by adding a new compartment to the *IAD* model [10]. We additionally propose optimal control by increasing the amount of information via new publications in order to influence people's opinions on a topic under study (elections, the covid 19 vaccine debate). A discrete version of Pontryagin's maximum theorem was applied to define the necessary conditions and the description of our optimal controls. Finally, a simulation illustrates the effectiveness of our control strategy.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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