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GLOBAL STABILITY OF A FRACTIONAL-ORDER ECO-EPIDEMIOLOGICAL MODEL WITH INFECTED PREDATOR: THEORETICAL ANALYSIS

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Abstract. A theoretical knowledge of the global stability of an eco-epidemiological model is not only important in itself but is also important in understanding the results of numerical simulations. In this paper the global stability of a fractional-order eco-epidemiological model with infected predator and harvesting is investigated using the Lyapunov function.

Keywords: fractional-order model; global stability; eco-epidemiological.

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1. INTRODUCTION

Mathematical models of the relationship between predator and prey in the presence of infectious diseases, which play an important role in the dynamics, are called eco-epidemiological

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models. There have been various studies of such models with disease being present in the constituent populations. These studies include [1]-[9].

Harvesting can influence the dynamics of eco-epidemiological models. In recent years the demand for greater resources has resulted in over-exploitation. Therefore there is a need for a sustainable strategy to protect ecosystems [10].

Mathematical models incorporating fractional differential equations have attracted much attention in recent years. Such models are believed to be more suitable for models that depend on past history [11, 12]. Further, such models are more realistic and less prone to errors [13]. Studies on fractional-order eco-epidemiological models include [14]-[21]. Ghosh et al. [1] studied a fractional-order eco-epidemiological model incorporating fear, treatment, and hunting cooperation effects to explore the memory effect in an ecological system through Caputo-type fractional-order derivative. In the work by Mukherjee [21], the author investigated a fractional-order predator–prey system with fear effect. Moustafa et al. [14] described the dynamical behavior of fractional-order Rosenzweig-MacArthur model allowing for a prey refuge. The influence of an infectious disease on a prey-predator model equipped with a fractional-order derivative is studied in [3]. The effect of fractional-order derivative on a prey–predator model with infection and harvesting is discussed by Moustafa et al. [16]. However, these papers did not deal with fractional-order eco-epidemiological model with infected predator and harvesting as such.

This paper is a theoretical study of the global stability of a fractional-order eco-epidemiological model with infected predator and harvesting. There has, so far as we are aware, been no theoretical studies of such a model.

2. Model Description

This paper investigates the global dynamic properties of a generalisation of the integer-order eco-epidemiological model introduced in [22]. The Caputo fractional derivative of order q $({}^{c}D^{q})$ is introduced and harvesting (H) is included. This generalised(fractional) model can be written as:

$$ {}^{c}D^{q}x(t) = rx\left(1-\frac{x}{k}\right) - \frac{c_1xy}{a+x} - \frac{c_2xz}{a+x},\ x(0)=x_0, $$

$$
\begin{aligned}
{}^{c}D^{q}y(t) &= \frac{m_1xy}{a+x} - \frac{\lambda yz}{b+y} - d_1y,\ y(0)=y_0,\\
{}^{c}D^{q}z(t) &= \frac{m_2xz}{a+x} + \frac{\lambda yz}{b+y} - d_2z - Hz,\ z(0)=z_0,
\end{aligned} \tag{1}
$$

where $q \in (0,1)$. The population is divided into: prey population density (x), susceptible predator population density (y) and infected predator population density (z).

3. Equilibrium Points and Global Stability

So as to evaluate the equilibrium points of model (1), let

$$ {}^{c}D^{q}x(t)=0,\ {}^{c}D^{q}y(t)=0 \text{ and } {}^{c}D^{q}z(t)=0. $$

Thus, the equilibrium points of model (1) are as follows:

(1) $E_0=(0,0,0)$, which always exists.
(2) $E_1=(k,0,0)$, which always exists.
(3) $E_2=\left(\frac{ad_1}{m_1-d_1}, \frac{rd_1(k+a)}{c_1k(m_1-d_1)}(\Re_{01}-1)(a+x_2), 0\right)$, which exists if $\Re_{01}>1$ and $m_1>d_1$, where, $x_2=\frac{ad_1}{m_1-d_1}$.
(4) $E_3=\left(\frac{a\zeta}{m_2-\zeta}, 0, \frac{r\zeta(k+a)}{c_2k(m_2-\zeta)}(\Re_{02}-1)(a+x_3)\right)$, which exists if $\Re_{02}>1$ and $m_2>\zeta$, where, $x_3=\frac{a\zeta}{m_2-\zeta}$.
(5) $E_4=(x_4,y_4,z_4)$ where

$$ y_4=\frac{b(\zeta(a+x_4)-m_2x_4)}{m_2x_4+(\lambda-\zeta)(a+x_4)},\ z_4=\frac{b(m_1x_4-d_1(a+x_4))}{m_2x_4+(\lambda-\zeta)(a+x_4)}, $$

and that x_4 needs to be a positive root of the following cubic polynomial:

$$ {x_4}^3+v_1{x_4}^2+v_2x_4+v_3=0, \tag{2} $$

where

$$
\begin{aligned}
v_1 &= a-k+\frac{a(\lambda-\zeta)}{\lambda-\zeta+m_2},\\
v_2 &= \frac{a(a-2k)r\lambda+\zeta(bkc_1-a(a-2k)r)-bkc_2(d_1-m_1)-k(ar+bc_1)m_2}{r(\lambda-\zeta+m_2)},\\
v_3 &= \frac{ak(ar(\zeta-\lambda)+b(c_1\zeta-c_2d_1))}{r(\lambda-\zeta+m_2)}.
\end{aligned}
$$

In accordance with Theorem 3.4 in [23], the analytical conditions about the existence of the equilibrium point E_4 can be illustrated in Table 1, Table 2 and Table 3.

TABLE 1. $\Theta > 0$.

Conditions	Equilibria of model (1)
$v_1 < 0, v_2 \in \mathbb{R}, v_3 > 0$	Two distinct positive equilibria
$v_1 \geq 0, v_2 < 0, v_3 > 0$	Two distinct positive equilibria
$v_1 \geq 0, v_2 \in \mathbb{R}, v_3 < 0$	One positive equilibrium
$v_1 < 0, v_2 > 0, v_3 < 0$	Three distinct positive equilibria
$v_1 < 0, v_2 \leq 0, v_3 < 0$	One positive equilibrium

TABLE 2. $\Theta = 0$.

Conditions	Equilibria of model (1)
$v_1 < 0, v_2 \in \mathbb{R}, v_3 > 0$	Two same positive equilibria
$v_1 \geq 0, v_2 < 0, v_3 > 0$	Two same positive equilibria
$v_1 \geq 0, v_2 \in \mathbb{R}, v_3 < 0$	One positive equilibrium
$v_1 < 0, v_2 > 0, v_3 < 0$	Two same positive equilibria
$v_1 < 0, v_2 \leq 0, v_3 < 0$	One positive equilibrium

TABLE 3. $\Theta < 0$.

Conditions	Equilibria of model (1)
For any value of v_1, $v_2 \in \mathbb{R}, v_3 \geq 0$	No positive equilibrium exists
For any value of v_1, $v_2 \in \mathbb{R}, v_3 < 0$	One positive equilibrium.

with

$$\Theta = 18v_1v_2v_3 - 4v_1^3v_3 + v_1^2v_2^2 - 4v_2^3 - 27v_2^2.$$

The following theorems investigate the global stability of the equilibrium points E_1, E_2, E_3 and E_4.

Theorem 1. *The equilibrium point E_1 of model* (1) *is globally asymptotically stable if* $d_1 > \min\left\{\frac{m_1k}{a+k}, \frac{c_1k}{a}\right\}$ *and* $\zeta > \min\left\{\frac{c_2k}{a}, \frac{m_2k}{a+k} + \lambda\right\}$.

Proof. The following positive definite Lyapunov function can be considered:

$$V = x - k - k\ln\left(\frac{x}{k}\right) + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2 + y + z.$$

Calculating the q-order derivative of V along the solution of model (1) and using Lemma 3.1 in [24],

$$\begin{aligned}
{}^cD^qV \leq & \left(\frac{x-k}{x}\right){}^cD^qx + y\,{}^cD^qy + z\,{}^cD^qz + {}^cD^qy + {}^cD^qz \\
=&(x-k)\left(r\left(1-\frac{x}{k}\right) - \frac{c_1y}{a+x} - \frac{c_2z}{a+x}\right) \\
&+\left(\frac{m_1x}{a+x} - \frac{\lambda z}{b+y} - d_1\right)y^2 + \left(\frac{m_2x}{a+x} + \frac{\lambda y}{b+y} - \zeta\right)z^2 \\
&+\frac{m_1xy}{a+x} + \frac{m_2xz}{a+x} - d_1y - \zeta z \\
=&-\frac{r}{k}(x-k)^2 - \frac{c_1xy}{a+x} - \frac{c_2xz}{a+x} + \frac{c_1ky}{a+x} + \frac{c_2kz}{a+x} \\
&+\left(\frac{m_1x}{a+x} - \frac{\lambda z}{b+y} - d_1\right)y^2 + \left(\frac{m_2x}{a+x} + \frac{\lambda y}{b+y} - \zeta\right)z^2 \\
&+\frac{m_1xy}{a+x} + \frac{m_2xz}{a+x} - d_1y - \zeta z \\
\leq&-\frac{r}{k}(x-k)^2 + \left(\frac{m_1k}{a+k} - d_1\right)y^2 + \left(\frac{m_2k}{a+k} + \lambda - \zeta\right)z^2 \\
&+\left(\frac{c_1k}{a} - d_1\right)y + \left(\frac{c_2k}{a} - \zeta\right)z.
\end{aligned}$$

Thus, ${}^cD^qV \leq 0$ if $d_1 > \min\left\{\frac{m_1k}{a+k}, \frac{c_1k}{a}\right\}$ and $\zeta > \min\left\{\frac{c_2k}{a}, \frac{m_2k}{a+k} + \lambda\right\}$. By Lemma 4.6 in [25], it is proof that the equilibrium point E_1 is globally asymptotically stable. □

Theorem 2. *The equilibrium point E_2 of model* (1) *is globally asymptotically stable if $y_2 < \frac{r(a+x_2)a}{c_1k}$ and $\frac{\lambda y_2}{b} + \frac{m_2k}{a+k} + \frac{Lc_2x_2}{a} < \zeta$.*

Proof. The following positive definite Lyapunov function is considered.

$$V = L\left(x - x_2 - x_2\ln\left(\frac{x}{x_2}\right)\right) + y - y_2 - y_2\ln\left(\frac{y}{y_2}\right) + z.$$

By calculating the q-order derivative of V along the solution of model (1) and using Lemma 3.1 in [24],

$$
\begin{aligned}
{}^cD^qV \leq & L(x-x_2)\left(r-\frac{rx}{k}-\frac{c_1y}{a+x}-\frac{c_2z}{a+x}\right)\\
& +(y-y_2)\left(\frac{m_1x}{a+x}-\frac{\lambda z}{b+y}-d_1\right)+\frac{m_2xz}{a+x}+\frac{\lambda yz}{b+y}-\zeta z\\
\leq & L(x-x_2)\left(\frac{rx_2}{k}+\frac{c_1y_2}{a+x_2}-\frac{rx}{k}-\frac{c_1y}{a+x}-\frac{c_2z}{a+x}\right)\\
& +(y-y_2)\left(\frac{m_1x}{a+x}-\frac{\lambda z}{b+y}-\frac{m_1x_2}{a+x_2}\right)+\frac{m_2xz}{a+x}+\frac{\lambda yz}{b+y}-\zeta z\\
\leq & L\left(\frac{c_1y_2}{(a+x)(a+x_2)}-\frac{r}{k}\right)(x-x_2)^2+\frac{(x-x_2)(y-y_2)}{(a+x)(a+x_2)}(am_1-aLc_1-Lc_1x_2)\\
& +\frac{\lambda y_2z}{b+y}+\frac{m_2xz}{a+x}-\frac{Lc_2zx}{a+x}+\frac{Lc_2x_2z}{a+x}-\zeta z\\
\leq & L\left(\frac{c_1y_2}{a(a+x_2)}-\frac{r}{k}\right)(x-x_2)^2+\frac{(x-x_2)(y-y_2)}{(a+x)(a+x_2)}(am_1-aLc_1-Lc_1x_2)\\
& +z\left(\frac{\lambda y_2}{b}+\frac{m_2k}{a+k}+\frac{Lc_2x_2}{a}-\zeta\right).
\end{aligned}
$$

Suppose $L=\frac{am_1}{c_1(a+x_2)}$. Thus, ${}^cD^q\leq 0$ when $y_2<\frac{r(a+x_2)a}{c_1k}$ and $\frac{\lambda y_2}{b}+\frac{m_2k}{a+k}+\frac{Lc_2x_2}{a}<\zeta$. Hence the theorem is proved. □

Theorem 3. *The equilibrium point E_3 of model* (1) *is globally asymptotically stable if $z_3<\frac{r(a+x_3)a}{c_2k}$ and $\frac{m_1k}{a+k}+\frac{Mc_1x_3}{a}-\frac{\lambda z_3}{b}<d_1$.*

Proof. It can be used the following positive definite Lyapunov function.

$$V=M\left(x-x_3-x_3\ln\left(\frac{x}{x_3}\right)\right)+y+z-z_3-z_3\ln\left(\frac{z}{z_3}\right).$$

Computing the time derivative of V along the solution of model (1) and utilizing Lemma 3.1 in [24],

$$
\begin{aligned}
{}^cD^qV \leq & M(x-x_3)\left(r-\frac{rx}{k}-\frac{c_1y}{a+x}-\frac{c_2z}{a+x}\right)\\
& +\frac{m_1xy}{a+x}-\frac{\lambda yz}{b+y}-d_1y+(z-z_3)\left(\frac{m_2x}{a+x}+\frac{\lambda y}{b+y}-\zeta\right)\\
\leq & M(x-x_3)\left(\frac{rx_3}{k}-\frac{rx}{k}+\frac{c_2z_3}{a+x_3}-\frac{c_1y}{a+x}-\frac{c_2z}{a+x}\right)\\
& +\frac{m_1xy}{a+x}-\frac{\lambda yz}{b+y}-d_1y+(z-z_3)\left(\frac{m_2x}{a+x}-\frac{m_2x_3}{a+x_3}+\frac{\lambda y}{b+y}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq M\left(\frac{c_2 z_3}{(a+x)(a+x_3)}-\frac{r}{k}\right)(x-x_3)^2+\frac{(x-x_3)(z-z_3)}{(a+x)(a+x_3)}\left(am_2-aMc_2-Mc_2x_3\right)\\
&\quad-\frac{\lambda y z_3}{b+y}+\frac{m_1 xy}{a+x}-\frac{Mc_1 xy}{a+x}+\frac{Mc_1 x_3 y}{a+x}-d_1 y\\
&\leq M\left(\frac{c_2 z_3}{a(a+x_3)}-\frac{r}{k}\right)(x-x_3)^2+\frac{(x-x_3)(z-z_3)}{(a+x)(a+x_3)}\left(am_2-aMc_2-Mc_2x_3\right)\\
&\quad+y\left(\frac{m_1 k}{a+k}+\frac{Mc_1 x_3}{a}-\frac{\lambda z_3}{b}-d_1\right).
\end{aligned}
$$

Suppose $M=\frac{am_2}{c_2(a+x_3)}$. Thus, ${}^cD^q\leq 0$ when $z_3<\frac{r(a+x_3)a}{c_2 k}$ and $\frac{m_1 k}{a+k}+\frac{Mc_1x_3}{a}-\frac{\lambda z_3}{b}<d_1$. Hence the theorem is proved. □

Theorem 4. *The equilibrium point E_4 of model* (1) *is globally asymptotically stable if* $\xi\nu^2+\lambda\nu^2 z_4+\xi y_4 z_4+\lambda y_4^2 z_4-\xi\rho z_4-\lambda\rho y_4 z_4-\xi y_4\rho-\lambda\rho y_4 z_4<0$ *and* $r>k(c_1y_4+c_2z_4)$.

Proof. The following positive definite Lyapunov function can be used.

$$
V=N_1\left(x-x_4-x_4\ln\left(\frac{x}{x_4}\right)\right)+y-y_4-y_4\ln\left(\frac{y}{y_4}\right)+N_2\left(z-z_4-z_4\ln\left(\frac{z}{z_4}\right)\right).
$$

Computing the time derivative of V along the solution of model (1) and utilizing using Lemma 3.1 in [24],

$$
\begin{aligned}
{}^cD^qV\leq\;& N_1(x-x_4)\left(r-\frac{rx}{k}-\frac{c_1y}{a+x}-\frac{c_2z}{a+x}\right)+(y-y_4)\left(\frac{m_1x}{a+x}-\frac{\lambda z}{b+y}-d_1\right)\\
&+N_2(z-z_4)\left(\frac{m_2x}{a+x}+\frac{\lambda y}{b+y}-\zeta\right)\\
\leq\;& N_1(x-x_4)\left(\frac{rx_4}{k}-\frac{rx}{k}+\frac{c_1y_4}{a+x_4}-\frac{c_1y}{a+x}+\frac{c_2z_4}{a+x_4}-\frac{c_2z}{a+x}\right)\\
&+(y-y_4)\left(\frac{m_1x}{a+x}-\frac{m_1x_4}{a+x_4}+\frac{\lambda z_4}{b+y_4}-\frac{\lambda z}{b+y}\right)\\
&+N_2(z-z_4)\left(\frac{m_2x}{a+x}-\frac{m_2x_4}{a+x_4}+\frac{\lambda y}{b+y}-\frac{\lambda y_4}{b+y_4}\right)\\
\leq\;& N_1\left(\frac{c_1y_4+c_2z_4}{(a+x)(a+x_4)}-\frac{r}{k}\right)(x-x_4)^2+(am_1-N_1c_1a-N_1c_1x_4)\frac{(x-x_4)(y-y_4)}{(a+x)(a+x_4)}\\
&+(N_2m_2a-N_1c_2a-N_1c_2x_4)\frac{(x-x_4)(z-z_4)}{(a+x)(a+x_4)}-\frac{b\lambda(y-y_4)(z-z_4)}{(b+y)(b+y_4)}\\
&+\frac{\lambda z_4(y-y_4)^2}{(b+y)(b+y_4)}-\frac{\lambda y_4(y-y_4)(z-z_4)}{(b+y)(b+y_4)}+\frac{N_2\lambda b(y-y_4)(z-z_4)}{(b+y)(b+y_4)}
\end{aligned}
$$

$$
\begin{aligned}
\leq & N_1\left(c_1y_4+c_2z_4-\frac{r}{k}\right)(x-x_4)^2+(am_1-N_1c_1a-N_1c_1x_4)\frac{(x-x_4)(y-y_4)}{(a+x)(a+x_4)}\\
&+(N_2m_2a-N_1c_2a-N_1c_2x_4)\frac{(x-x_4)(z-z_4)}{(a+x)(a+x_4)}\\
&+(N_2\lambda b-\lambda b-\lambda y_4)\frac{(y-y_4)(z-z_4)}{(b+y)(b+y_4)}+\frac{\lambda z_4(y-y_4)^2}{(b+y)(b+y_4)}\\
\leq & N_1\left(c_1y_4+c_2z_4-\frac{r}{k}\right)(x-x_4)^2+(am_1-N_1c_1a-N_1c_1x_4)\frac{(x-x_4)(y-y_4)}{(a+x)(a+x_4)}\\
&+(N_2m_2a-N_1c_2a-N_1c_2x_4)\frac{(x-x_4)(z-z_4)}{(a+x)(a+x_4)}\\
&+\frac{\xi yz+\lambda y^2z_4+\xi y_4z_4+\lambda y_4^2z_4-\xi yz_4-\lambda yy_4z_4-\xi y_4z-\lambda yy_4z_4}{(b+y)(b+y_4)}\\
\leq & N_1\left(c_1y_4+c_2z_4-\frac{r}{k}\right)(x-x_4)^2+(am_1-N_1c_1a-N_1c_1x_4)\frac{(x-x_4)(y-y_4)}{(a+x)(a+x_4)}\\
&+(N_2m_2a-N_1c_2a-N_1c_2x_4)\frac{(x-x_4)(z-z_4)}{(a+x)(a+x_4)}\\
&+\frac{\xi\nu^2+\lambda\nu^2z_4+\xi y_4z_4+\lambda y_4^2z_4-\xi\rho z_4-\lambda\rho y_4z_4-\xi y_4\rho-\lambda\rho y_4z_4}{(b+y)(b+y_4)},
\end{aligned}
$$

where $\xi = N_2\lambda b-\lambda b-\lambda y_4$. Suppose $N_1 = \frac{am_1}{c_1(a+x_4)}$, $N_2 = \frac{m_1c_2}{m_2c_1}$, and $\nu < z$, $y < \rho$. Thus, ${}^cD^qV(x,y,z) \leq 0$, when $\xi\nu^2+\lambda\nu^2z_4+\xi y_4z_4+\lambda y_4^2z_4-\xi\rho z_4-\lambda\rho y_4z_4-\xi y_4\rho-\lambda\rho y_4z_4 < 0$ and $r > k(c_1y_4+c_2z_4)$. By Lemma 4.6 in [25], it is proof that E_4 is globally asymptotically stable.

□

Now, the proof of the existence of transcritical bifurcation around the equilibrium point $E_1(k,0,0)$ is given by using Sotomayor's theorem.

Theorem 5 (Transcritical bifurcation around E_1)**.** *The fractional-order model* (1) *undergoes a transcritical bifurcation with respect to the bifurcation parameter H around $E_1(k,0,0)$ when $H = H_{tr1} = \frac{m_2k}{a+k}-d_2$ and keeping $\Re_{01} < 1$, while no saddle-node bifurcation can occur.*

Proof. The Jacobian matrix for the model (1) around E_1 when $H = H_{tr1}$ is as follows:

$$
J(E_1) = \begin{pmatrix} -r & -\frac{c_1k}{a+k} & -\frac{c_2k}{a+k}\\ 0 & d_1(\Re_{01}-1) & 0\\ 0 & 0 & 0 \end{pmatrix}. \tag{3}
$$

A straightforward computation indicates that the Jacobian matrix (3) has zero eigenvalue μ_3. Here, $\mu_1 = -r < 0$ and $\mu_2 = d_1(\Re_{01} - 1) <$ when $\Re_{01} < 1$. Let $V = (v_1,\ v_2,\ v_3)^T = (\frac{-c_2 k v_3}{r(a+k)},\ 0,\ v_3)^T$ and $W = (\tau_1,\ \tau_2,\ \tau_3)^T = (0,\ 0,\ \tau_3)^T$ be the two eigenvectors corresponding to the zero eigenvalue of the matrices $J(E_1)$ and $(J(E_1))^T$, respectively. Where v_3 and τ_3 are any non zero real numbers. Therefore,

$$W^T(F_H(E_1, H_{tr1})) = 0,$$

$$W^T(DF_H(E_1, H_{tr1})V) = -v_3\tau_3 \neq 0,$$

$$W^T(D^2F(E_1, H_{tr1})(V,V)) = \left(\frac{2b^2\lambda v_2 v_3}{b^3} + \frac{2am_2 v_1 v_3}{(a+k)^2}\right)\tau_3 \neq 0.$$

By Sotomayor's Theorem for local bifurcation [26], the fractional-order model (1) has a transcritical bifurcation around E_1 when $H = H_{tr1} = \frac{m_2 k}{a+k} - d_2$ as H passes through the value H_{tr1}, while no saddle-node bifurcation can occur. □

4. Conclusion

In this paper, a fractional-order eco-epidemiological model with infected predator and harvesting has been formulated and analyzed. The equilibrium points were identified and their global properties were investigated. The existence of transcritical bifurcation was shown using Sotomayor's theorem. The threshold parameters ($\Re_{01}$ and $\Re_{02}$) were used to determine the existence conditions of the equilibrium points.

Conflict of Interests

The authors declare that there is no conflict of interests.

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