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Commun. Math. Biol. Neurosci. 2023, 2023:61

https://doi.org/10.28919/cmbn/7994

ISSN: 2052-2541

# STABILITY ANALYSIS OF THE POPULATION MATRIX MODEL WITH TWO ITEROPAROUS SPECIES USING THE M-MATRIX

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**Abstract:** In this paper, we study the growth of two iteroparous species in the same community focusing on species with two age classes. It is modeled using the Leslie population projection matrix. The two age classes in view are age classes in units of time such as months and years. We assume the species is only capable of giving birth once per unit of time. We also assume that the growth of both species is influenced by density-dependent growth factors that only occur in the first age class and harvesting in the second age class. We consider two different models, one with the same and the other with different levels of intraspecific and interspecific competition. In both models, we analyse the existence conditions and local asymptotic stability of each equilibrium point. The local asymptotic stability is analysed using M-matrix theory. The inherent net reproductive number is derived and its relationship to

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Received April 18, 2023

the equilibrium point is explored. The results show that models with the same levels of intraspecific and interspecific competition do not have co-existence equilibrium points and vice versa. Then, the inherent net reproduction number and the levels of intraspecific and interspecific competition affect the existence and local asymptotic stability for each equilibrium point.

**Keywords:** Leslie matrix; iteroparous; age-structured model; competition; harvesting; stability; density-dependent; M-matrix.

**2020 AMS Subject Classification:** 39A60, 92D25.

## 1. INTRODUCTION

Different species may live in a community, which is a portion of an ecosystem. Species within a community can be divided into semelparous species and iteroparous species. Female individuals of semelparous species can only give birth once in the last age group, shortly before death. Cicadas [1]–[4], beetles [1], [5], [6], giant Australian cuttlefish [7]–[9], and salmons [10]–[13] are a few examples of semelparous species. An iteroparous species, on the other hand, has several reproductions throughout its entire lifespan. This species is abundant in a variety of environments.

In particular for species that are a source of food for humans, population dynamics must be studied in order to ascertain the state of the population in the future. Harvesting by humans to meet food demands is a factor that has an impact on growth in population [14], [15]. For this reason both researches those focus on the dynamics of the resources and those focus on the optimal harvesting of the resources are equally important [16]–[20]. In addition, several other factors can influence the number of species in that community. First, a species may be density-dependent because of the scarcity of natural resources like food and suitable habitat [21]–[23]. Second, competition between species is conceivable and may result from lacking natural resources [24]–[26].

For a very long time, population dynamics have been studied using mathematical models. In ecological studies on plants and animals, the population matrix model has grown in popularity recently [27], [28]. According to their characteristics, populations are categorized in the population matrix model. Age, developmental stage, body size (for instance, in small and big animals), and other traits might be considered as population characteristics. Leslie [29] first proposed an age-based discrete time population growth model in 1945. The Leslie matrix model is the name of this one. On sometimes, we are unable to determine the population's chronological age. According to their developmental stage, the population in this instance is divided into several categories [30]–[32]. The Lefkovitch matrix is a type of population matrix organized by developmental stage that was first presented in 1965 [32].

Research on population growth dynamics using the Leslie matrix model in the special situation of the multispecies scenario started in 1968. Pennycuick et al. [33] investigated the multispecies model using the Leslie matrix model and computer simulations. In that study, Pennycuick et al. [33] separated the instances into two categories of species interactions: predator-prey and competitive. The Leslie matrix model was then studied in 1980, with species divided into semelparous and iteroparous species. Travis et al.'s [34] stability requirements for the Leslie matrix model were created for two semelparous species that compete or are mutually exclusive. Kon [35] investigated the Leslie matrix model of two species, one of which has two age classes and the other of which has one. The next year, Kon [36] investigated how the coprime number of numbers affected the age classes of two semelparous species. When two numbers have the largest common factor of 1, they are called coprime. Later, Kon [37] expanded the earlier study to include an arbitrary number of semelparous species and an arbitrary number of age classes for each species. Furthermore, Hasibuan et al. [38] expanded on Kon [37]'s research by including the harvesting factor, although only for two semelparous species with two age classes. Next, Hasibuan et al. [39] established models of [38] for one semelparous species and one iteroparous species in the same community.

As an extension of the models investigated in [39], we study two iteroparous species, each with two age classes. It is possible to use the Leslie matrix model with two age classes for animals with life spans of two years, two months, two weeks, or two-time units. In this study, the population dynamics of the two iteroparous species are assumed to be affected by human harvesting and are density dependent between the two species. In both iteroparous species, harvesting occurs in the second age class. Then, in both iteroparous species, density dependence occurs only in the first age class. In addition, we also assumed that intraspecific and interspecific competition affected the growth of populations of both iteroparous species. Competition in multispecies can be divided into two competitions: interspecific competition and intraspecific competition. Interspecific competition occurs between species, while intraspecific competition occurs within the same species [40]. Furthermore, we divide this problem into two cases: iteroparous species with the same and different levels of intraspecific ($a > 0$) and interspecific ($b > 0$) competition. Hence, there are two models formed from the two problems. In both models, we derived the inherent net reproductive number which is often applied in research related to the Leslie matrix model [35]–[39], [41]. Next, we determine the equilibrium points of both models and their existence conditions. Finally, we analyze the asymptotic local stability of each equilibrium point of the two models using the M-matrix. Our aim is to investigate the impact of the level of intraspecific and interspecific competition along with inherent net reproductive number on the existence conditions and asymptotic local stability of each equilibrium point in both models.

## 2. MATERIALS AND METHODS

### 2.1. A Leslie Matrix Model of Two Iteroparous Species in a Community with Same Level of Intraspecific and Interspecific Competition.

In this section, we present a multispecies Leslie matrix model for two iteroparous species with the same level of intraspecific and interspecific competition in a community. It means that $a = b$. Because the species studied are iteroparous, both age classes of each species are assumed to be able to give birth. This problem is modelled with equations (1), and we refer to it as Model A.

$$\begin{cases} x_1(t+1) = \dfrac{f_{x_1}}{1+ax_1(t)+ay_1(t)}x_1(t) + f_{x_2}x_2(t) \\ x_2(t+1) = \dfrac{s_{x_1}\left(1-h_{x_2}\right)}{1+ax_1(t)+ay_1(t)}x_1(t) \\ y_1(t+1) = \dfrac{f_{y_1}}{1+ax_1(t)+ay_1(t)}y_1(t) + f_{y_2}y_2(t) \\ y_2(t+1) = \dfrac{s_{y_1}\left(1-h_{y_2}\right)}{1+ax_1(t)+ay_1(t)}y_1(t) \end{cases} \tag{1}$$

In Table 1, a description of the Model A parameters is given. The total population of age class $i$ of the species $x$ and $y$, respectively, is represented by $x_i(t)$ and $y_i(t)$ for $i = 1, 2$.

**Table 1.** Description of parameters in Model A.

| Parameter | Description |
|---|---|
| $f_{x_i} > 0$ | The birth rate at age $i$ for $i = 1,2$ of species $x$. |
| $f_{y_i} > 0$ | The birth rate at age $i$ for $i = 1,2$ of species $y$. |
| $0 < s_{x_1} \leq 1$ | The survival rate of the first age class of species $x$. |
| $0 < s_{y_1} \leq 1$ | The survival rate of the first age class of species $y$. |
| $0 < h_{x_2} \leq 1$ | The harvesting rate of the second age class of species $x$. |
| $0 < h_{y_2} \leq 1$ | The harvesting rate of the second age class of species $y$. |

The population density of the first age class at time $t+1$ is, to put it simply, the sum of the populations of the species $x$ born in the first and second age classes at time $t$, according to the first equation in Model A. In this case, density-dependent factors that happen in both species' first age classes affect natural births in the first age classes of both species. The second equation in Model A then states that the total surviving population of the first age class of species $x$ at time $t$ is equal to the population density of the second age class at time $t+1$. The density dependence between the first age classes of both species also influences the survival rate. Furthermore, Model A's third and fourth equations have the same meaning as its first and second equations, respectively.

## 2.2. A Leslie Matrix Model of Two Iteroparous Species in a Community with Different Levels of Intraspecific and Interspecific Competition

This section presents a multispecies Leslie matrix model for two iteroparous species with different levels of intraspecific and interspecific competition in a community, which is an extension of Model A. We present an extension of Model A in equations (2) and we refer to it as Model B.

$$\begin{cases} x_1(t+1) = \dfrac{f_{x_1}}{1+ax_1(t)+by_1(t)}x_1(t) + f_{x_2}x_2(t) \\ x_2(t+1) = \dfrac{s_{x_1}\left(1-h_{x_2}\right)}{1+ax_1(t)+by_1(t)}x_1(t) \\ y_1(t+1) = \dfrac{f_{y1}}{1+bx_1(t)+ay_1(t)}y_1(t) + f_{y2}y_2(t) \\ y_2(t+1) = \dfrac{s_{y1}\left(1-h_{y2}\right)}{1+bx_1(t)+ay_1(t)}y_1(t) \end{cases} \tag{2}$$

Parameters and the details regarding the explanation of Model B are almost the same as Model A. The difference between the two models is the competition that affects the birth rate and survival of the first age class in both species $x$ and $y$.

### 2.3. $M$-matrix and Asymptotically Local Stability Criterion Using the $M$-matrix

The asymptotic determination of the local stability of a discrete system or model can be seen through the absolute values of all the eigenvalues of the Jacobian matrix. Nevertheless, working with a system's eigenvalues from its Jacobian matrix to determine the asymptotic local stability is not easy. Therefore, we employ another method using asymptotic local stability introduced by Travis et al. [34]. The definition and theorem regarding the $M$-matrix can be seen below in Definition 1 and Theorem 1.

**Definition 1** [34]**:**

A square matrix $A$ of size $n$ is said to be an $M$-Matrix if it satisfies two conditions. First, element $a_{ij} \leq 0$ for $i \neq j$. Second, one of the following five conditions is met

i) All minor principals of matrix $A$ are positive.

ii) All real parts of the eigenvalues of matrix $A$ are positive.

iii) Matrix $A$ is a non-singular matrix and $A^{-1}$ is a positive matrix.

iv) There is a vector $v > 0$ so that $Av > 0$.

v) There is a vector $w > 0$ so that $A^T w > 0$.

**Theorem 1** [34]**:**

Assume matrix $J$ is as follows:

$$J = \begin{bmatrix} A_{k\times k} & B_{k\times l} \\ C_{l\times k} & D_{l\times l} \end{bmatrix}.$$

If $G = I - SJS^{-1}$ is an $M$-Matrix with

$$S = I_{(k+l)\times(k+l)} \quad \text{if } B_{k\times l} \text{ and } C_{l\times k} \geq 0$$
$$S = \begin{bmatrix} I_{k\times k} & O_{k\times l} \\ O_{l\times k} & I_{l\times l} \end{bmatrix} \quad \text{if } B_{k\times l} \text{ and } C_{l\times k} \leq 0'$$

where $I_{(k+l)\times(k+l)}$, $I_{k\times k}$, $I_{l\times l}$, $O_{k\times l}$, and $O_{l\times k}$ respectively are identity matrices of size $(k + l) \times (k + l)$, $k \times k$, and $l \times l$ and zero matrices of size $k \times l$ and $l \times k$, then the matrix $J$ has an absolute value of all eigenvalues less than one.

## 3. MAIN RESULTS

### 3.1. Inherent Net Reproductive Number from Model A and Model B

The inherent net reproductive number, which has been studied in research [35]–[39], [41], is one of the frequently applied essential aspects, particularly in the study of the Leslie matrix model. This quantity refers to the number of offspring expected per individual over a lifetime. There are two inherent net reproductive numbers, denoted by the letters $R_x$ and $R_y$ because our attention is on the situation of two species, species $x$ and $y$. The detailed step-by-step explanation of the inherent net reproductive number for species $x$ and $y$ can be found in [42].

The fertility matrix $F$ and transition matrix $T$ of Model A for species $x$, i.e.

$$F_x = \begin{bmatrix} \dfrac{f_{x1}}{1 + ax_1(t) + ay_1(t)} & f_{x2} \\ 0 & 0 \end{bmatrix}$$

and

$$T_x = \begin{bmatrix} 0 & 0 \\ \dfrac{s_{x1}(1 - h_{x2})}{1 + ax_1(t) + ay_1(t)} & 0 \end{bmatrix}.$$

Then, matrix $(I_2 - T_x)$ is

$$I_2 - T_x = \begin{bmatrix} 1 & 0 \\ -\dfrac{s_{x1}(1 - h_{x2})}{1 + ax_1(t) + ay_1(t)} & 1 \end{bmatrix}$$

and

$$F_x(I_2 - T_x)^{-1}(\mathbf{0}) = \begin{bmatrix} f_{x1} + f_{x2}s_{x1}(1 - h_{x2}) & f_{x2} \\ 0 & 0 \end{bmatrix}.$$

Then, the dominant eigenvalue of $F_x(I - T_x)^{-1}(\mathbf{0})$ is $f_{x1} + f_{x2}s_{x1}(1 - h_{x2})$. It is therefore known as the $R_x$, or the inherent net reproductive number of species $x$.

Second, the fertility matrix $F$ and the transition matrix $T$ of Model A for species $y$, i.e.

$$F_y = \begin{bmatrix} \dfrac{f_{y1}}{1 + ax_1(t) + ay_1(t)} & f_{y2} \\ 0 & 0 \end{bmatrix}$$

and

$$T_y = \begin{bmatrix} 0 & 0 \\ \dfrac{s_{y1}(1 - h_{y2})}{1 + ax_1(t) + ay_1(t)} & 0 \end{bmatrix}.$$

By using the same method as for spesies $x$, we obtained

$$F_y(I_2 - T_y)^{-1}(\mathbf{0}) = \begin{bmatrix} f_{y1} + f_{y2}s_{y1}(1 - h_{y2}) & f_{y2} \\ 0 & 0 \end{bmatrix}.$$

Then, the dominant eigenvalue of $F_y(I_2 - T_y)^{-1}(\mathbf{0})$ is $f_{y1} + f_{y2}s_{y1}(1 - h_{y2})$. It is therefore known as the $R_y$, or the inherent net reproductive number of species $y$.

Next, the inherent net reproductive number of Model B for both species is determined. The $F$ and $T$ matrices for each species, i.e.

$$F_x = \begin{bmatrix} \dfrac{f_{x1}}{1 + ax_1(t) + by_1(t)} & f_{x2} \\ 0 & 0 \end{bmatrix}, \quad T_x = \begin{bmatrix} 0 & 0 \\ \dfrac{s_{x1}(1 - h_{x2})}{1 + ax_1(t) + by_1(t)} & 0 \end{bmatrix},$$

$$F_y = \begin{bmatrix} \dfrac{f_{y1}}{1 + bx_1(t) + ay_1(t)} & f_{y2} \\ 0 & 0 \end{bmatrix},$$

and

$$T_y = \begin{bmatrix} 0 & 0 \\ \dfrac{s_{y1}(1 - h_{y2})}{1 + bx_1(t) + ay_1(t)} & 0 \end{bmatrix}.$$

Our results show that the dominant eigenvalues of $F_x(I_2 - T_x)^{-1}\,(0)$ and $F_x(I_2 - T_x)^{-1}(0)$ have the same values as the results in Model A.

### 3.2. The Equilibrium Points of Model A

The first step to find the equilibrium points is to construct an equilibrium model for Model A. the equilibrium model. The equilibrium model can be derived by expressing that the left-hand side of Model A depends on time $t$. In light of this, the equilibrium model of Model A is as follows:

$$\begin{cases} x_1(t) = \dfrac{f_{x_1}}{1 + ax_1(t) + ay_1(t)} x_1(t) + f_{x_2} x_2(t) \\ x_2(t) = \dfrac{s_{x_1}\left(1 - h_{x_2}\right)}{1 + ax_1(t) + ay_1(t)} x_1(t) \\ y_1(t) = \dfrac{f_{y_1}}{1 + ax_1(t) + ay_1(t)} y_1(t) + f_{y_2} y_2(t) \\ y_2(t) = \dfrac{s_{y_1}\left(1 - h_{y_2}\right)}{1 + ax_1(t) + ay_1(t)} y_1(t) \end{cases}, \qquad (3)$$

The next step is to find solutions from (3) and the solutions are:

i) The extinction equilibrium point for the species $x$ and $y$ is

$$E_0 = [x_1^*, x_2^*, y_1^*, y_2^*]^T = [0,0,0,0]^T.$$

ii) The equilibrium point with species $y$ extinct, i.e.

$$E_x = [x_1^*, x_2^*, y_1^*, y_2^*]^T = \left[\frac{R_x - 1}{a}, \frac{s_{x_1}(1 - h_{x2})(R_x - 1)}{aR_x}, 0,0\right]^T.$$

iii) The equilibrium point with species $x$ extinct, i.e.

$$E_y = [x_1^*, x_2^*, y_1^*, y_2^*]^T = \left[0,0, \frac{R_y - 1}{a}, \frac{s_{y_1}\left(1 - h_{y_2}\right)\left(R_y - 1\right)}{aR_y}\right]^T.$$

The thing that is often studied at the equilibrium point is to determine the existing condition at the equilibrium point. Only the $E_x$ and $E_y$ equilibrium points, according to model (3) solutions, do not have all of the element values equal to zero. The conditions for the existence of equilibrium points $E_x$ and $E_y$ of Model A are provided in Theorem 2 below.

**Theorem 2**

For Model A that

i) The equilibrium point $E_x$ exist if $R_x > 1$.

ii) The equilibrium point $E_y$ exist if $R_y > 1$.

**Proof**. In $E_x$ and $E_y$, it can be seen that the nonpositive generators are $R_x - 1$ and $R_y - 1$. Therefore, if $R_x > 1$, then $E_x$ will exist; if $R_y > 1$, then $E_y$ will exist.

■

### 3.3. Asymptotically Local Stability at Equilibrium Points of Model A

This section presents a theorem and an analysis of the asymptotic local stability for each equilibrium point of Model A. Following is Theorem 3, which details the asymptotically local stability of each equilibrium point of Model A.

**Theorem 3**

For Model A that

i) If $R_x < 1$ and $R_y < 1$, the equilibrium point $E_0$ is asymptotically stable locally.
ii) If $R_x > 1$ and $R_x > R_y$, the equilibrium point $E_x$ is asymptotically stable locally.
iii) If $R_y > 1$ and $R_y > R_x$, the equilibrium point $E_y$ is asymptotically stable locally.

**Proof**. The first thing to do in the local stability problem is to determine the Jacobian matrix of model. The Jacobian matrix of Model A is

$$J(E^*) = J\left(\begin{bmatrix} x_1^* \\ x_2^* \\ y_1^* \\ y_2^* \end{bmatrix}\right) = \begin{bmatrix} \dfrac{f_{x_1}(1+ay_1^*)}{(1+ax_1^*+ay_1^*)^2} & f_{x2} & -\dfrac{f_{x_1}x_1^*a}{(1+ax_1^*+ay_1^*)^2} & 0 \\ P_x(1+ay_1^*) & 0 & -P_x x_1^* a & 0 \\ -\dfrac{f_{y_1}x_1^*a}{(1+ax_1^*+ay_1^*)^2} & 0 & \dfrac{f_{y_1}(1+ax_1^*)}{(1+ax_1^*+ay_1^*)^2} & f_{y2} \\ -P_y y_1^* a & 0 & P_y(1+ax_1^*) & 0 \end{bmatrix} \tag{4}$$

with

$$P_x = \frac{s_{x_1}\left(1-h_{x_2}\right)}{(1+ax_1^*+ay_1^*)^2}$$

and

$$P_y = \frac{s_{y_1}\left(1-h_{y_2}\right)}{(1+ax_1^*+ay_1^*)^2}.$$

Referring to the existence of equilibrium points, it follows that$J(E^*)_{1i} \leq 0$, $J(E^*)_{2i} \leq 0$, $J(E^*)_{3j} \leq 0$, dan $J(E^*)_{4j} \leq 0$ with $i = 3, 4$ and $j = 1, 2$, so by Theorem 1 we obtain

$$S = \begin{bmatrix} 1 & 0 & -0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The Jacobian matrix for each equilibrium point of Model A is then determined by replacing each equilibrium point in (4) with the corresponding equilibrium point of Model A. Following is an analysis of the Jacobian Matrix for each equilibrium point of Model A and its stability using the $M$-matrix theory:

i) For $E_0$, the Jacobian matrix is

$$J(E_0) = \begin{bmatrix} f_{x_1} & f_{x_2} & 0 & 0 \\ s_{x_1}(1-h_{x_2}) & 0 & 0 & 0 \\ 0 & 0 & f_{y_1} & f_{y_2} \\ 0 & 0 & s_{y_1}(1-h_{y_2}) & 0 \end{bmatrix}.$$

After that, the matrix

$$G = \begin{bmatrix} 1-f_{x_1} & -f_{x_2} & 0 & 0 \\ -s_{x_1}(1-h_{x_2}) & 1 & 0 & 0 \\ 0 & 0 & 1-f_{y_1} & -f_{y_2} \\ 0 & 0 & -s_{y_1}(1-h_{y_2}) & 1 \end{bmatrix}.$$

It should be noted that all $g_{ij} \leq 0$ for $i \neq j$ meet Theorem 1's first condition. The next step is to establish the requirement that all of matrix $G$'s minor principles are positive. Based on the calculations obtained, the matrix $G$'s minor principles are

$$PM_1 = |g_{11}| = 1 - f_{x_1}, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = 1 - R_x,$$

$$PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = (f_{y_1} - 1)(1 - R_x),$$

and $$PM_4 = |G| = (1 - R_x)(1 - R_y).$$

Take note that if $R_x < 1$, $PM_2 > 0$. Because $0 < R_x < 1$ consequently $f_{x_1} < 1$ so that $PM_1 > 0$. Then, since $R_x < 1$, it follows that $PM_4 > 0$ in the case of $R_y < 1$. Furthermore, due to $R_y < 1$ and $R_x < 1$ it follows that $PM_3 > 0$. Therefore, $G$ is an $M$-matrix. Then, the equilibrium point $E_0$ is locally asymptotically stable if $R_x < 1$ and $R_y < 1$.

ii) For $E_x$, the Jacobian matrix is

$$J(E_x) = \begin{bmatrix} \frac{f_{x_1}}{R_x^2} & f_{x_2} & \frac{f_{x_1}(1-R_x)}{R_x^2} & 0 \\ \frac{s_{x_1}(1-h_{x_2})}{R_x^2} & 0 & \frac{s_{x_1}(1-h_{x_2})(1-R_x)}{R_x^2} & 0 \\ 0 & 0 & \frac{f_{y_1}}{R_x} & f_{y_2} \\ 0 & 0 & \frac{s_{y_1}(1-h_{y_2})}{R_x} & 0 \end{bmatrix}.$$

After that, the matrix

$$G = \begin{bmatrix} \frac{R_x^2 - f_{x_1}}{R_x^2} & -f_{x_2} & -\frac{f_{x_1}(R_x-1)}{R_x^2} & 0 \\ -\frac{s_{x_1}(1-h_{x_2})}{R_x^2} & 1 & \frac{s_{x_1}(1-h_{x_2})(1-R_x)}{R_x^2} & 0 \\ 0 & 0 & \frac{R_x - f_{y_1}}{R_x} & -f_{y_2} \\ 0 & 0 & -\frac{s_{y_1}(1-h_{y_2})}{R_x} & 1 \end{bmatrix}.$$

Let's note that, since $R_x > 1$, all values of $g_{ij} \leq 0$ for $i \neq j$ are satisfied, hence the first condition of being an $M$-matrix in Definition 1 is satisfied. The next step is to determine the conditions for all the minor principals of matrix $G$ to be positive. The minor principals of matrix $G$ are

$$PM_1 = \frac{R_x^2 - f_{x_1}}{R_x^2}, \qquad PM_2 = -\frac{(1-R_x)}{R_x}, \qquad PM_3 = -\frac{(1-R_x)(R_x - f_{y_1})}{R_x^2}$$

and

$$PM_4 = \frac{(1-R_x)(R_y - R_x)}{R_x^2}.$$

Note that $PM_2 > 0$ if $R_x > 1$. Because $R_x > 1$ consequently $PM_4 > 0$ if $R_x > R_y$. It is clear that $R_x^2 = \left(f_{x_1} s_{x_1}(1-h_{x_2})\right)^2 > f_{x_1}$ so that $PM_1 > 0$. Then, because $R_x > R_y > 0$ and $R_x > 1$, $R_x > f_{y_1}$ is satisfied. Therefore, $G$ is an $M$-matrix and equilibrium point $E_x$ is asymptotically stable locally if $R_x > 1$ and $R_x > R_y$.

iii) For $E_y$, the Jacobian matrix is

$$J(E_y) = \begin{bmatrix} \dfrac{f_{x_1}}{R_y} & f_{x_2} & 0 & 0 \\ \dfrac{s_{x_1}(1-h_{x_2})}{R_y} & 0 & 0 & 0 \\ -\dfrac{f_{y_1}(R_y-1)}{R_y^2} & 0 & \dfrac{f_{y_1}}{R_y^2} & f_{y_2} \\ \dfrac{s_{y_1}(1-h_{y_2})(1-R_y)}{R_y^2} & 0 & \dfrac{s_{y_1}(1-h_{y_2})}{R_y^2} & 0 \end{bmatrix}.$$

After that, the matrix

$$G = \begin{bmatrix} \dfrac{R_y - f_{x_1}}{R_y} & -f_{x_2} & 0 & 0 \\ -\dfrac{s_{x_1}(1-h_{x_2})}{R_y} & 1 & 0 & 0 \\ -\dfrac{f_{y_1}(R_y-1)}{R_y^2} & 0 & \dfrac{R_y^2 - f_{y_1}}{R_y^2} & -f_{y_2} \\ \dfrac{s_{x_1}(1-h_{y_2})(1-R_y)}{R_y^2} & 0 & -\dfrac{s_{y_1}(1-h_{y_2})}{R_y^2} & 1 \end{bmatrix}.$$

Note that, since $R_y > 1$ consequently all values of $g_{ij} \leq 0$ for $i \neq j$ are satisfied, the first condition is said to be $M$-matrix in Definition 1 is satisfied. The next step is to show that all the minor principles of $G$ are positive. Based on the calculations obtained,

$$PM_1 = \frac{R_y - f_{x_1}}{R_y}, \qquad PM_2 = \frac{(R_y - R_x)}{R_y}, \qquad PM_3 = \frac{(R_y - R_x)(R_y^2 - f_{y_1})}{R_y^2}$$

and

$$PM_4 = -\frac{(1-R_y)(R_y-R_x)}{R_y^2}.$$

Note that $PM_2 > 0$ if $R_y > R_x$. Because $R_y > 1$ and $R_y > R_x$ consequently $PM_4 > 0$. Then, since $0 < R_x < R_y$ it is clear that $R_y > f_{x_1}$ so that $PM_1 > 0$. $PM_3$ is satisfied to be positive since it is clear that $R_y^2 = \left(f_{y_2} s_{y_1}(1-h_{y_2}) + f_{y_1}\right)^2 > f_{y_1}$ and $R_y > R_x$. Therefore, $G$ is an $M$-Matrix, and the equilibrium point $E_y$ is asymptotically stable locally if $R_y > 1$ and $R_y > R_x$.

■

## 3.4. The Equilibrium Points of Model B

In this section, we present the equilibrium points of Model B and analyze their existing conditions. The equilibrium representation of Model B is

$$\begin{cases} x_1(t) = \dfrac{f_{x_1}}{1 + ax_1(t) + by_1(t)} x_1(t) + f_{x_2} x_2(t) \\ x_2(t) = \dfrac{s_{x_1}\left(1 - h_{x_2}\right)}{1 + ax_1(t) + by_1(t)} x_1(t) \\ y_1(t) = \dfrac{f_{y_1}}{1 + bx_1(t) + ay_1(t)} y_1(t) + f_{y_2} y_2(t) \\ y_2(t) = \dfrac{s_{y_1}\left(1 - h_{y_2}\right)}{1 + bx_1(t) + ay_1(t)} y_1(t) \end{cases}. \tag{5}$$

Then, the following four Model B equilibrium points:

$$E_0 = [0,0,0,0]^T, \qquad E_x = \left[\frac{R_x - 1}{a}, \frac{(R_x - 1)s_{x_1}\left(1 - h_{x_2}\right)}{aR_x}, 0,0\right]^T,$$

$$E_y = \left[0,0, \frac{R_y - 1}{a}, \frac{\left(R_y - 1\right)s_{y_1}\left(1 - h_{y_2}\right)}{aR_y}\right]^T,$$

and

$$E_{xy} = \left[\frac{A_x}{C}, \frac{s_{x_1}\left(1 - h_{x_2}\right)A_x}{CR_x}, \frac{A_y}{C}, \frac{s_{y_1}\left(1 - h_{y_2}\right)A_y}{CR_y}\right]^T$$

where

$$A_x = a(R_x - 1) - b\left(R_y - 1\right), A_y = a\left(R_y - 1\right) - b(R_x - 1), C = a^2 - b^2.$$

Finally, Theorem 4 presents the conditions needed for the $E_x$, $E_y$, and $E_{xy}$ equilibrium points of Model B to exist.

**Theorem 4**

For Model B that

i) The equilibrium point $E_x$ exist if $R_x > 1$.

ii) The equilibrium point $E_y$ exist if $R_y > 1$.

iii) The equilibrium point $E_{xy}$ exists if $a > b, A_x > 0$, and $A_y > 0$ or $a < b$, $A_x < 0$, and $A_y < 0$.

**Proof**. Similarly to the equilibrium points $E_x$ and $E_y$ in Model A, it can be seen that the nonpositive generators are $R_x - 1$ and $R_y - 1$. Hence, the same conclusion is obtained: $E_x$ exists if $R_x > 1$ and $E_y$ exists if $R_y > 1$. The remaining equilibrium point is $E_{xy}$. Consider that the equilibrium point $E_{xy}$ has the same denominator. Also note that the first element of $E_{xy}$ is included in the second element, and the third element of $E_{xy}$ is also included in the fourth element of $E_{xy}$, which must be guaranteed to be positive. Therefore, $E_{xy}$ is positive if $A_x$ and $A_y$ have the same sign as $C$. In other words, $E_{xy} > 0$ if $C > 0$ is $a > b$, $A_x > 0$, and $A_y > 0$ or $a < b$, $A_x < 0$, and $A_y < 0$.

In this model, we obtain a co-existence equilibrium point, namely an equilibrium point with both species existing, which shows the influence of competition. This condition should be expected in a community so that the two species can coexist in one community.

## 3.5. Asymptotically Local Stability at Equilibrium Points of Model B

The asymptotic local stability analysis is also performed on the equilibrium points of Model B. Theorem 5 below provides the conditions that the asymptotic local stability of the equilibrium points of Model B must satisfy.

**Theorem 5**

For Model B that

i) If $R_x < 1$ and $R_y < 1$, the equilibrium point $E_0$ is asymptotically stable locally.

ii) If $R_x > 1$ and $R_y < 1 + \frac{b}{a}(R_x - 1)$, the equilibrium point $E_x$ is asymptotically stable locally.

iii) If $R_y > 1$ and $R_x < 1 + \frac{b}{a}(R_y - 1)$, the equilibrium point $E_y$ is asymptotically stable locally.

iv) If $a > b$, $a(R_y - 1) > b(R_x - 1)$, $a(R_x - 1) > b(R_y - 1)$, $R_y^2 > R_x$, and $f_{x_1}\left(a(a - b) - b(bR_x - aR_y)\right) < (a^2 - b^2)R_x^2$, the equilibrium point $E_{xy}$ is asymptotically stable locally.

**Proof**. The Jacobian matrix of Model B is

$$J(E^*) = J\left(\begin{bmatrix} x_1^* \\ x_2^* \\ y_1^* \\ y_2^* \end{bmatrix}\right) = \begin{bmatrix} \dfrac{f_{x_1}(1+by_1^*)}{(1+ax_1^*+by_1^*)^2} & f_{x2} & -\dfrac{bf_{x_1}x_1^*}{(1+ax_1^*+by_1^*)^2} & 0 \\ Q_x(1+by_1^*) & 0 & -Q_x x_1^* b & 0 \\ -\dfrac{bf_{y_1}x_1^*}{(1+bx_1^*+ay_1^*)^2} & 0 & \dfrac{f_{y_1}(1+bx_1^*)}{(1+bx_1^*+ay_1^*)^2} & f_{y2} \\ -Q_y y_1^* b & 0 & Q_y(1+bx_1^*) & 0 \end{bmatrix} \tag{6}$$

with $Q_x = \frac{s_{x_1}(1-h_{x_2})}{(1+ax_1^*+by_1^*)^2}$

and $Q_y = \frac{s_{y_1}(1-h_{y_2})}{(1+bx_1^*+ay_1^*)^2}$.

Referring to the existence of equilibrium points of Model B, it follows that $J(E^*)_{1i} \leq 0$, $J(E^*)_{2i} \leq 0$, $J(E^*)_{3j} \leq 0$, and $J(E^*)_{4j} \leq 0$ with $i = 3,4$ and $j = 1,2$, so by Theorem 1 we choose

$$S = \begin{bmatrix} 1 & 0 & -0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The Jacobian matrix for each equilibrium point of Model A is then determined by replacing each equilibrium point in (6) with the corresponding equilibrium point of Model B. Following is an analysis of the Jacobian Matrix for each equilibrium point of Model A and its stability using the $M$-matrix theory:

i) For $E_0$, the Jacobian matrix is

$$J(E_0) = \begin{bmatrix} f_{x_1} & f_{x_2} & 0 & 0 \\ s_{x_1}(1-h_{x_2}) & 0 & 0 & 0 \\ 0 & 0 & f_{y_1} & f_{y_2} \\ 0 & 0 & s_{y_1}(1-h_{y_2}) & 0 \end{bmatrix}.$$

After that, the matrix

$$G = \begin{bmatrix} 1-f_{x_1} & -f_{x_2} & 0 & 0 \\ -s_{x_1}(1-h_{x_2}) & 1 & 0 & 0 \\ 0 & 0 & 1-f_{y_1} & -f_{y_2} \\ 0 & 0 & -s_{y_1}(1-h_{y_2}) & 1 \end{bmatrix}.$$

It should be noted that all $g_{ij} \leq 0$ for $i \neq j$ meet Theorem 1's first condition. The next step is to establish the requirement that all of matrix $G$'s minor principles are positive.

Based on the calculations obtained, the matrix $G$'s minor principles are

$$PM_1 = 1 - f_{x_1},\ PM_2 = 1 - R_x,\quad PM_3 = \left(f_{y_1} - 1\right)(1 - R_x),$$

and
$$PM_4 = (1 - R_x)\left(1 - R_y\right).$$

Take note that if $R_x < 1$, $PM_2 > 0$. Because $0 < R_x < 1$ consequently $f_{x_1} < 1$ so that $PM_1 > 0$. Then, since $R_x < 1$, it follows that $PM_4 > 0$ in the case of $R_y < 1$. Furthermore, due to $R_y < 1$ and $R_x < 1$ it follows that $PM_3 > 0$. Therefore, $G$ is an $M$-matrix. Then, the equilibrium point $E_0$ is locally asymptotically stable if $R_x < 1$ and $R_y < 1$.

ii) For $E_x$, the Jacobian matrix is

$$J(E_x) = \begin{bmatrix} \dfrac{f_{x_1}}{R_x^2} & f_{x_2} & -\dfrac{bf_{x_1}(R_x - 1)}{aR_x^2} & 0 \\ \dfrac{s_{x_1}\left(1 - h_{x_2}\right)}{R_x^2} & 0 & -\dfrac{s_{x_1}\left(1 - h_{x_2}\right)b(R_x - 1)}{aR_x^2} & 0 \\ 0 & 0 & \dfrac{af_{y_1}}{a + b(R_x - 1)} & f_{y_2} \\ 0 & 0 & \dfrac{as_{y_1}\left(1 - h_{y_2}\right)}{a + b(R_x - 1)} & 0 \end{bmatrix}.$$

After that, the matrix

$$G = \begin{bmatrix} \dfrac{R_x^2 - f_{x_1}}{R_x^2} & -f_{x_2} & -\dfrac{bf_{x_1}(R_x - 1)}{aR_x^2} & 0 \\ -\dfrac{s_{x_1}\left(1 - h_{x_2}\right)}{R_x^2} & 1 & -\dfrac{bs_{x_1}\left(1 - h_{x_2}\right)(R_x - 1)}{aR_x^2} & 0 \\ 0 & 0 & 1 - \dfrac{af_{y_1}}{a + b(R_x - 1)} & -f_{y_2} \\ 0 & 0 & -\dfrac{s_{y_1}\left(1 - h_{y_2}\right)}{a + b(R_x - 1)} & 1 \end{bmatrix}.$$

Let's note that, since $R_x > 1$, all values of $g_{ij} \leq 0$ for $i \neq j$ are satisfied, hence the first condition of being an $M$-matrix in Definition 1 is satisfied. The next step is to determine the conditions for all the minor principals of matrix $G$ to be positive. The minor principals of matrix $G$ are

$$PM_1 = \frac{R_x^2 - f_{x_1}}{R_x^2}, \qquad PM_2 = \frac{R_x - 1}{R_x},$$

$$PM_3 = -\frac{(R_x - 1)\left(a\left(f_{y_1} - 1\right) + b(1 - R_x)\right)}{R_x\left(a + b(R_x - 1)\right)}$$

and

$$PM_4 = \frac{(R_x - 1)\left(b(R_x - 1) - a\left(R_y - 1\right)\right)}{R_x\left(a + b(R_x - 1)\right)}.$$

Because $R_x > 1$ consequently $PM_2 > 0$. Then, it is clear that $R_x^2 = \left(f_{x_2}s_{x_1}(1 - h_{x_2}) + f_{x_1}\right)^2 > f_{x_1}$ so that $PM_1 > 0$. Because $R_x > 1$ consequently $PM_4 > 0$ if $a\left(R_y - 1\right) < b(R_x - 1)$. Furthermore, since $R_x > 1$ and $a\left(R_y - 1\right) < b(R_x - 1)$ it is obvious that $PM_3 > 0$. Therefore, $G$ is an $M$-Matriks and the equilibrium point $E_x$ is asymptotically stable locally if $R_x > 1$ and $a\left(R_y - 1\right) < b(R_x - 1)$ or $R_y < 1 + \frac{b}{a}(R_x - 1)$.

iii) For $E_y$, the Jacobian matrix is

$$J\left(E_y\right) = \begin{bmatrix} \frac{af_{x_1}}{a + b(R_y - 1)} & f_{x_2} & 0 & 0 \\ \frac{as_{x_1}\left(1 - h_{x_2}\right)}{a + b(R_y - 1)} & 0 & 0 & 0 \\ -\frac{bf_{y_1}(R_y - 1)}{aR_y^2} & 0 & \frac{f_{y_1}}{R_y^2} & f_{y_2} \\ -\frac{s_{y_1}\left(1 - h_{y_2}\right)b(R_y - 1)}{aR_y^2} & 0 & \frac{s_{y_1}\left(1 - h_{y_2}\right)}{R_y^2} & 0 \end{bmatrix}.$$

After that, the matrix

$$G = \begin{bmatrix} 1 - \frac{af_{x_1}}{a + b\left(R_y - 1\right)} & -f_{x_2} & 0 & 0 \\ -\frac{as_{x_1}\left(1 - h_{x_2}\right)}{a + b\left(R_y - 1\right)} & 1 & 0 & 0 \\ -\frac{bf_{y_1}(R_y - 1)}{aR_y^2} & 0 & \frac{R_y^2 - f_{y_1}}{R_y^2} & -f_{y_2} \\ -\frac{s_{y_1}\left(1 - h_{y_2}\right)b(R_y - 1)}{aR_y^2} & 0 & -\frac{s_{y_1}\left(1 - h_{y_2}\right)}{R_y^2} & 1 \end{bmatrix}.$$

Note that, since $R_y > 1$ consequently all values of $g_{ij} \leq 0$ for $i \neq j$ are satisfied, the first condition is said to be $M$-matrix in Definition 1 is satisfied. The next step is to show that all the minor principles of $G$ are positive. Based on the calculations obtained,

$$PM_1 = \frac{b\left(R_y - 1\right) - a\left(f_{x_1} - 1\right)}{a + b\left(R_y - 1\right)}, \qquad PM_2 = \frac{\left(b\left(R_y - 1\right) - a(R_x - 1)\right)}{a + b\left(R_y - 1\right)},$$

$$PM_3 = \frac{\left(b\left(R_y - 1\right) - a(R_x - 1)\right)\left(R_y^2 - f_{y_1}\right)}{\left(a + b\left(R_y - 1\right)\right)R_y^2}$$

and

$$PM_4 = \frac{\left(R_y - 1\right)\left(b\left(R_y - 1\right) - a(R_x - 1)\right)}{R_y\left(a + b\left(R_y - 1\right)\right)}.$$

Note that since $R_y > 1$ the result is $PM_2 > 0$ and $PM_4 > 0$ if $a(R_x - 1) < b(R_y - 1)$. Then, it is clear that $R_y^2 = \left(f_{y_2} s_{y_1}\left(1 - h_{y_2}\right) + f_{y_1}\right)^2 > f_{y_1}$ so that $PM_3 > 0$. Furthermore, since $R_y > 1$ and $a(R_x - 1) < b\left(R_y - 1\right)$ it is clear that $PM_1 > 0$. Therefore, $G$ is an $M$-Matrix and the equilibrium point $E_y$ is asymptotically stable locally if $R_y > 1$ and $a(R_x - 1) < b(R_y - 1)$ or $R_x < 1 + \frac{b}{a}(R_y - 1)$.

iv) For $E_{xy}$, the Jacobian matrix is

$$J\left(E_{xy}\right) = \begin{bmatrix} \frac{D_x f_{x_1}}{CR_x^2} & f_{x_2} & -\frac{A_x b f_{x_1}}{CR_x^2} & 0 \\ \frac{D_x s_{x_1}\left(1 - h_{x_2}\right)}{CR_x^2} & 0 & -\frac{b s_{x_1}\left(1 - h_{x_2}\right) A_x}{CR_x^2} & 0 \\ -\frac{A_y b f_{y_1}}{CR_y^2} & 0 & \frac{D_y f_{y_1}}{CR_y^2} & f_{y_2} \\ -\frac{s_{y_1}\left(1 - h_{y_2}\right) b A_y}{CR_y^2} & 0 & \frac{D_y s_{y_1}\left(1 - h_{y_2}\right)}{CR_y^2} & 0 \end{bmatrix}$$

with

$$D_x = a^2 - ab\left(1 - R_y\right) - b^2 R_x$$

and

$$D_y = a^2 - ab(1 - R_x) - b^2 R_y.$$

After that, the matrix

$$G=\begin{bmatrix} 1-\frac{D_x f_{x_1}}{CR_x^2} & -f_{x_2} & -\frac{b f_{x_1} A_x}{CR_x^2} & 0 \\ -\frac{D_x s_{x_1}\left(1-h_{x_2}\right)}{CR_x^2} & 1 & -\frac{b s_{x_1}\left(1-h_{x_2}\right) A_x}{CR_x^2} & 0 \\ -\frac{A_y b f_{y_1}}{CR_y^2} & 0 & 1-\frac{D_y f_{y_1}}{CR_y^2} & -f_{y_2} \\ -\frac{s_{y_1}\left(1-h_{y_2}\right) b A_y}{CR_y^2} & 0 & \frac{D_y s_{y_1}\left(1-h_{y_2}\right)}{CR_y^2} & 1 \end{bmatrix}.$$

Note that all values of $g_{ij} \leq 0$ for $i \neq j$ with conditions $A_x,\ A_y,\ C>0$ or $A_x,\ A_y,$ $C<0$, which are conditions for the existence of $E_{xy}$ so that the first condition is said to be $G$ as an $M$-Matrix fulfilled. The next step is to establish the requirement that all of matrix $G$'s minor principles are positive. Based on the calculations obtained, the matrix $G$'s minor principles are

$$PM_1 = 1-\frac{D_x f_{x_1}}{CR_x^2},\qquad PM_2=\frac{aA_x}{CR_x},\qquad PM_3=\frac{\left(a\left(R_y^2-f_{y_1}\right)-bf_{y_1}(R_x-1)\right)A_x}{CR_xR_y^2}$$

and

$$PM_4=\frac{A_xA_y}{CR_xR_y}.$$

Note that $PM_2>0$ and $PM_4>0$ if $A_x$, $A_y$, and $C>0$. Because $C>0$ which results in $a>b$ so that $PM_3>0$ if $\left(R_y^2-f_{y_1}\right)>f_{y_1}(R_x-1)$ or $R_y^2>R_x$ where it is clear that $R_y^2=\left(f_{y_2}s_{y_1}\left(1-h_{y_2}\right)+f_{y_1}\right)^2>f_{y_1}$. Then, $PM_1>0$ if $f_{x_1}D_x<CR_x^2$. Therefore, $G$ is an $M$-Matriks and equilibrium point $E_{xy}$ is asymptotically stable locally if $a>b$, $a\left(R_y-1\right)>b(R_x-1)$, $a(R_x-1)>b(R_y-1)$, $R_y^2>R_x$, and $f_{x_1}\left(a(a-b)-b\left(bR_x-aR_y\right)\right)<(a^2-b^2)R_x^2$. ■

## 4. NUMERICAL SOLUTIONS

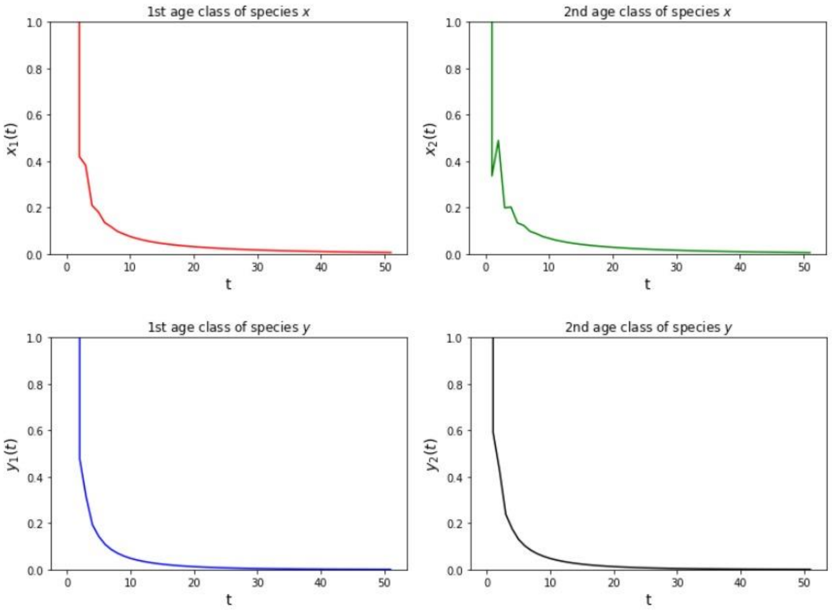
The numerical simulation of the analysis from the Results and Discussions section is covered in this section. Only Theorems 3 and 5 are numerically and graphically demonstrated by the numerical simulations provided in this section. As a result, the parameters described in this section are hypothetical parameter values. Based on the stability criteria of each equilibrium point in Theorems 3 and 5, Models A and B simulations will be shown in 3 and 4 cases, respectively. All of the parameters that we include in these simulations are measured in terms of time units, and the population total is determined per individual. In this simulation, it is considered that both Model A and Model B have $x_1(0) = 30{,}000$, $x_2(0) = 10{,}000$, $y_1(0) = 50{,}000$, and $y_2(0) = 10{,}000$ individuals.

The numerical simulation of Model A is separated into three cases based on the number of equilibrium points from Model A in Theorem 3. The intraspecific and interspecific competition levels are assumed to be $a = b = 1$. The values of other parameters from the numerical simulations of Model A for its three cases are presented in Table 2.

**Table 2.** Parameter values for each case of the numerical simulation of Model A.

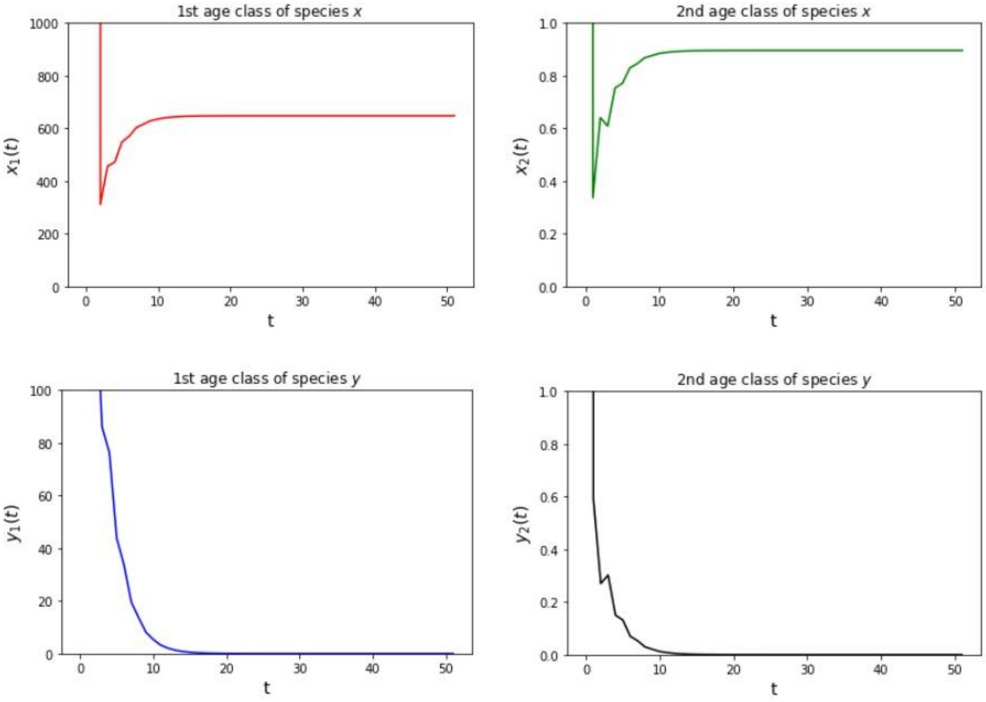
| **Case** | **Species $x$** | | | | **Species $y$** | | | | $R_x$ | $R_y$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $f_{x_1}$ | $s_{x_1}$ | $f_{x_2}$ | $h_{x_2}$ | $f_{y_1}$ | $s_{y_1}$ | $f_{y_2}$ | $h_{y_2}$ | | |
| I | 0.4 | 0.9 | 0.6 | 0.003 | 0.4 | 0.95 | 0.5 | 0.005 | 0.94 | 0.87 |
| II | 200 | 0.9 | 500 | 0.003 | 100 | 0.95 | 200 | 0.005 | 648.65 | 289.05 |
| III | 100 | 0.9 | 300 | 0.003 | 200 | 0.95 | 400 | 0.005 | 369.19 | 578.10 |

FIGURE 1. Population growth of each age classes in case I Model A.



Based on the parameters presented in Table 2, for the case I that $R_x = 0.94 < 1$ and $R_y = 0.87 < 1$. The simulation results from the case I in Table I are presented in Figure 1, where Figure 1 interprets that when $R_x < 1$ and $R_y < 1$, the system is asymptotically stable locally towards the equilibrium point $E_0$. In that sense, the populations of both species $x$ and $y$ are extinct. Furthermore, for case II in Table 2 that $R_x = 648.45$ and $R_y = 289.05$ where $R_x$ exceeds the threshold of one and $R_x > R_y$ indicates an asymptotically stable locally system towards $E_x = [647.65,0.90,0,0]^T$ shown in Figure 3. In the sense that the population $x$ exists because it has an inherent net reproductive number greater than $y$ and exceeds the threshold. In case III from Table 2, the population $y$ exists where the system is asymptotically stable locally towards the equilibrium point $E_y = [0,0,577.1,0.94]^T$, which is shown graphically in Figure 3. This is because $R_y = 369.19 > R_x = 578.10$ and $R_y > 1$. In a sense, species $y$ has an inherent net reproductive number greater than $x$ and exceeds the threshold.

FIGURE 2. Population growth of each age phase in case II Model A.



The numerical simulation of Model B is separated into four cases based on the number of equilibrium points from model B in Theorem 5. The values of each parameter except the level of competition from the numerical simulations of Model B for its four cases are provided in Table 3. Furthermore, the intraspecific and interspecific competition levels are assumed to be $a = 0.002$ and $b = 0.001$, respectively.

FIGURE 3. Population growth of each age phase in case III Model A.

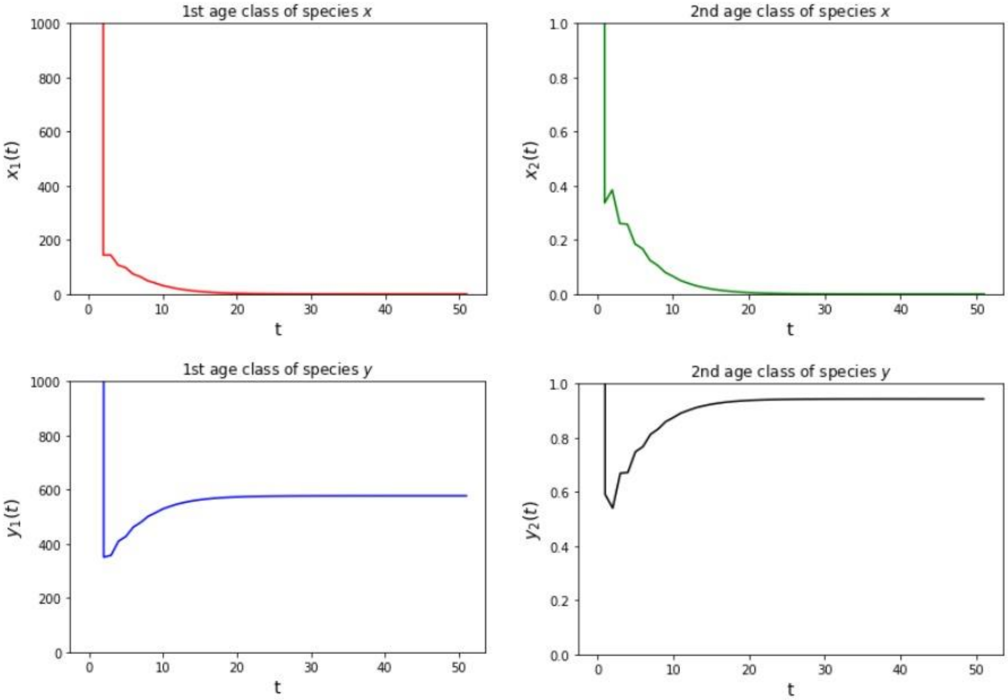


**Table 3.** Parameter values for each case of the numerical simulation of Model B.

| **Case** | **Species $x$** | | | | **Species $y$** | | | | $R_x$ | $R_y$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $f_{x_1}$ | $s_{x_1}$ | $f_{x_2}$ | $h_{x_2}$ | $f_{y_1}$ | $s_{y_1}$ | $f_{y_2}$ | $h_{y_2}$ | | |
| I | 0.25 | 1 | 0.7 | 0.003 | 0.3 | 0.98 | 0.6 | 0.005 | 0.95 | 0.88 |
| II | 500 | 1 | 500 | 0.003 | 120 | 0.98 | 300 | 0.005 | 998.5 | 412.53 |
| III | 100 | 1 | 250 | 0.003 | 600 | 0.98 | 600 | 0.005 | 349.25 | 1185.06 |
| IV | 500 | 1 | 500 | 0.003 | 600 | 0.98 | 600 | 0.005 | 998.5 | 1185.06 |

FIGURE 4. Population growth for each age class in case I Model B.

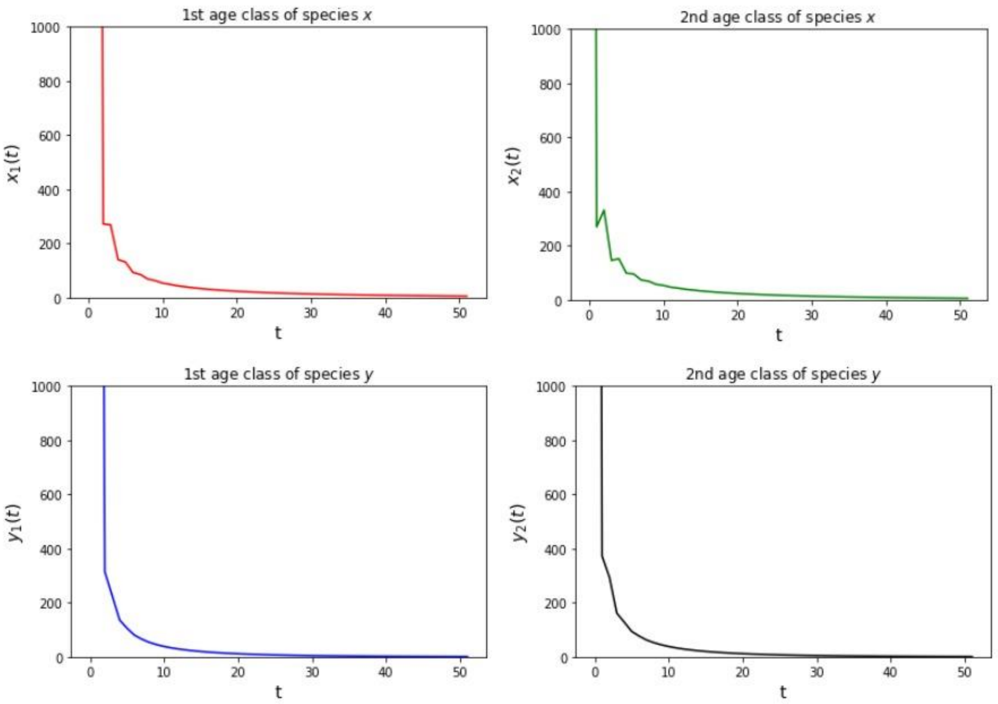


Figure 4-7 displays the Model B's numerical simulation results for the parameters in Table 3. Because the parameter values in case I from Table 3 satisfy the first condition of Theorem 5, Figure 4 shows that the system is asymptotically stable locally towards the equilibrium point where all species become extinct or $E_0$. The parameter values in case II Table 3 fulfil the second condition in Theorem 5, where the value of $R_x$ exceeds the threshold and fulfils the condition $a(R_y - 1) < b(R_x - 1)$. Consequently, the numerical simulation results in Figure 5 show that the system is asymptotically stable locally towards the equilibrium point $E_x = [498750, 498, 0,0]^T$. In a sense, the population that survives is only the population in species $x$ for both age classes. Then, Figure 6 shows that the system is asymptotically stable locally towards the equilibrium point $E_y = [0,0,592030,487]^T$. In a sense, the population that survives is only the population in species $y$ for both age classes. This is because the selected parameter values in case III Table 3 fulfil the third condition of Theorem 5 where $R_y$ exceeds the threshold and $a(R_x - 1) < b(R_y - 1)$. Next, Figure 7 shows the asymptotically stable system towards the co-existence equilibrium point, namely $E_{xy} = [270313,270,456873,376]^T$. Figure 7 shows that populations $x$ and $y$ can simultaneously survive in a community if the fourth condition of Theorem 5 is met with the simulation example selected in case IV Table 3.

FIGURE 5. Population growth for each age class in case II Model B.

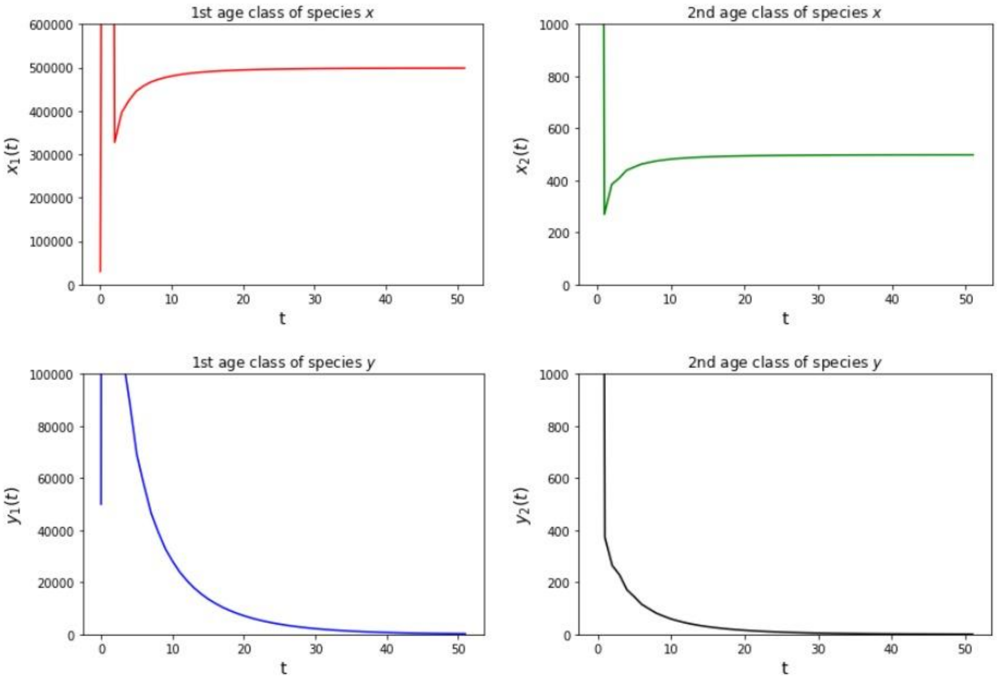


FIGURE 6. Population growth for each age class in case III Model B.

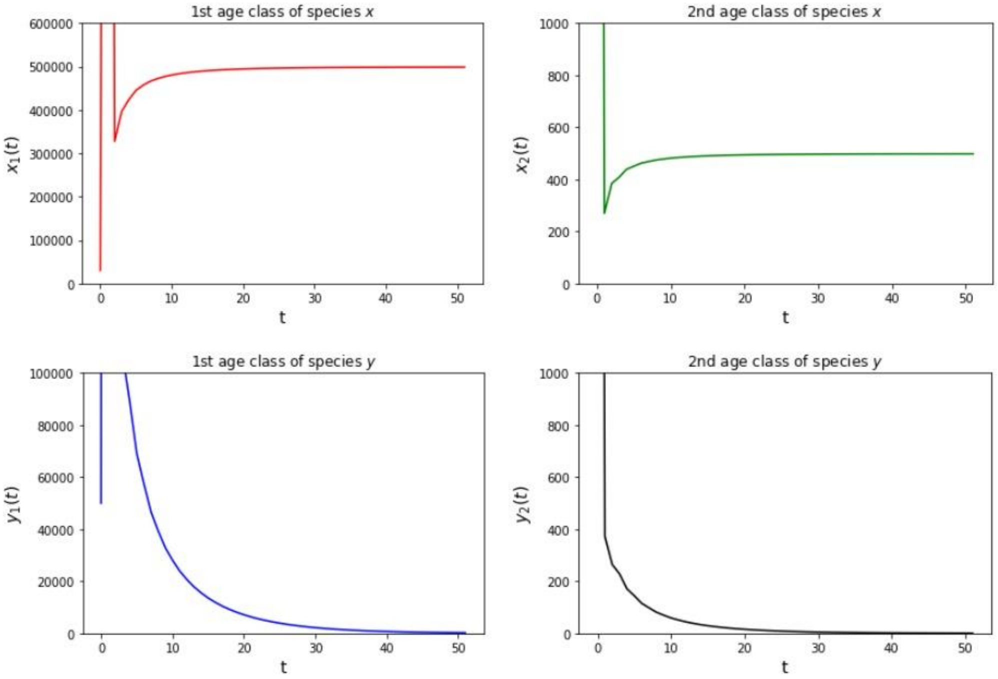
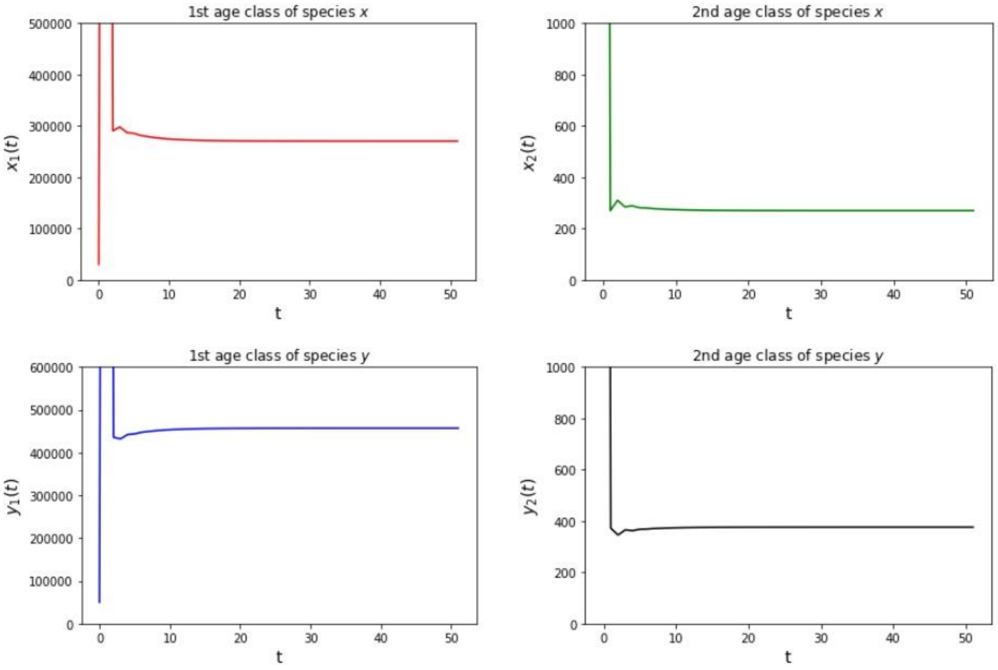


FIGURE 7. Population growth for each age class in case IV Model B.



## 5. CONCLUSION

The problem of growth dynamics of two iteroparous species with two age classes for each species is developed using two models in this work. The two models consist of models on the growth of species affected by same and different levels of intraspecific and interspecific competition. The two models are referred to as Model A and Model B, respectively. Density dependency and harvesting were taken into account in these models. These models were established using the Leslie Matrix model for multispecies. In this paper, the equilibrium points of both models were found, and the asymptotically local stability for each equilibrium point was also analyzed using M-matrix theory. There were three equilibrium points obtained from Model A where no co-existence equilibrium point was found. Unlike the case in Model B, there was an additional one type of equilibrium point, namely the co-existence equilibrium point. The existence and stability of each equilibrium point in models A and B were characterized by the inherent net reproductive number of each species, namely $R_x$ and $R_y$; for a species to exist and be locally asymptotically stable, its value must exceed a threshold of one. Conversely, if $R_x$ and

$R_y$ are smaller than one, both species would become extinction in the long term. However, this condition is not enough, so there are other conditions. The Model A showed that the equilibrium point with one species existing will be asymptotically locally stable if that species has a larger inherent net reproductive number. This demonstrates the existence of the competition exclusion principle. A species that is dominant no matter how small will also dominate in the long run over other species. The Model B showed that the equilibrium point with one species existing will be asymptotically stable if it satisfies the other conditions stated in Theorem 5. Then, the co-existence equilibrium point will be asymptotically stable if the degree of intraspecific competition is greater than interspecific competition and other conditions which are complex enough to be interpreted biologically.

The model presented and studied in this research is still open for development into a more realistic and in-depth model. The research conducted in this study is the basis to be used as a reference for us or other researchers to be able to develop a more general model as done by Kon [37], namely on an arbitrary number of iteroparous species with an arbitrary number of age classes. In the end, the more general model is more applicable to various species. Further developments that can be made from the generalization of the model include studying global stability, studying bifurcations in the model, and many more.

## ACKNOWLEDGMENTS

We thank the University of Padjadjaran who funded this research with grant number 2203/UN6.3.1/PT.00/2022 through the Padjadjaran Doctoral Program Scholarship scheme.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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