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THE DYNAMIC OF AN ECO-EPIDEMIOLOGICAL MODEL INVOLVING FEAR AND HUNTING COOPERATION

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Abstract: In the present paper, an eco-epidemiological model consisting of diseased prey consumed by a predator with fear cost, and hunting cooperation property is formulated and studied. It is assumed that the predator doesn't distinguish between the healthy prey and sick prey and hence it consumed both. The solution's properties such as existence, uniqueness, positivity, and bounded are discussed. The existence and stability conditions of all possible equilibrium points are studied. The persistence requirements of the proposed system are established. The bifurcation analysis near the non-hyperbolic equilibrium points is investigated. Numerically, some simulations are carried out to validate the main findings and obtain the critical values of the bifurcation parameters, if any. It is obtained that the existence of fear controls the disease outbreak and the system's persistence. While in the case of a rising hunting cooperation rate, the induced fear may control the outbreak of disease.

Keywords: eco-epidemiological model; fear; hunting cooperation; stability; bifurcation.

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1. INTRODUCTION

Ecology research has long focused on the interaction between predators and their prey, which is a very important aspect of the field. A significant and well-known topic of study in population dynamics and applied mathematical modeling is the prey-predator relationship. Such relationships are one of many types of inter-species interactions that are important in determining how complex ecological systems in our diverse planet behave [1]. According to studies, it has an impact on the ecosystem as a whole and not just the species of predator and prey that interact [2].

However, some recent theoretical and experimental research has challenged conventional wisdom. Studies have shown how crucial indirect effects (panic or fear) are in influencing both the dynamics of prey-predator relationships and the ecosystem as a whole. Although Cannon first proposed the concept of fear in 1915 [3], it is still a relatively new concept in the world of mathematical modeling. Prey individuals have been seen to alter their typical foraging activity in the presence of predator species as a result of the psychological stress of being captured and murdered by predators. In some ways, this helps the prey species at that specific time by boosting their chances of surviving, but in the long run, it could result in a significant loss. In addition to affecting their foraging habits, this perceived predation risk lowers both their birth rate and the likelihood that their children will survive more than typical adults. Several recent field trials and theoretical analyses back up the aforementioned assertions. According to certain paradoxical findings from studies [4] and the references therein, the influence of indirect fear may occasionally outweigh the effect of direct predation. Since direct predation is relatively simple to detect in nature, it is typically believed in traditional prey-predator models that predators only have an impact on prey populations by direct killing. However, the presence of a predator may drastically alter prey physiology and behavior to the point where it may have a greater impact on the prey population than direct predation [5-6]. Numerous mathematical models examined how fear affected the relationship between prey and predator, see for example [7-15]. Recently, a tri-trophic food web with a fear reaction for the base prey and a Lotka-Volterra functional response for predation by both a specialist predator and a superpredator was recently developed and studied by Fakhry et al.

[16]. They discovered a surprising result of the prey's fear of its expert predator, which is the potential extinction of the superpredator.

Even while epidemiology is a significant subject of research in and of itself, there has been a recent movement toward combining it with ecology to better understand how species interact in ecosystems under the influence of epidemiological causes. Because no species lives alone in nature but interacts with many other species directly or indirectly, studying the impact of disease in the context of interspecies interactions is more realistic than the one without it. As a result, it gave birth to a new branch of science called eco-epidemiology. This innovative approach is motivated by a curiosity to understand the impact of disease in prey-predator scenarios. The first to introduce eco-epidemiological modeling was Anderson and May [17]. In order to create a new essence of nature, scholars are becoming more and more interested in combining these two crucial fields of study. Eco-epidemiology is a new field of mathematical biology that addresses both ecological and epidemiological concerns. Eco-epidemiological systems, which are used to explain how illnesses interact with predators and prey in one population or both populations, must become crucial instruments in studying the transmission and management of infectious diseases. Therefore, several researchers examined ecological systems where the disease affects prey, predator, or both populations in eco-epidemiology systems [18–22]. On the other hand, others studies focused the eco-epidemiological systems in the existence of fear, see for example [23-25].

In the prey-predator concept, group hunting is also prevalent. Animals frequently engage in cooperative hunting, which helps predators survive by ensuring they have access to enough food [26]. The cooperative hunting strategy has been widely researched mathematically. Consequently, several researchers have recently included cooperative hunting strategies in their studies; see for example [25-27].

The analyses mentioned above inspired the development of a generic prey-predator model with fear cost, disease in the prey population, and hunting cooperation strategy. The prey population was split into two classes, susceptible prey, and diseased prey, with the former playing a substantial mathematical role. Predators are said to be unable to tell the difference between healthy and sick prey, therefore they both end up in their stomachs.

2. MODEL FORMULATION

In this section, an eco-epidemiological system incorporating a prey-predator with an infective disease in the prey population is proposed and studied. It is believed that there are two population classes that make up the entire population of prey: the susceptible prey class and the infected prey class, whose population densities are given by $S(t)$ and $I(t)$, respectively. While $Y(t)$ represents the predator population density. Therefore, to formulate the described system mathematically the following hypotheses are adopted.

1. It is assumed that the disease is spread among the prey population exclusively, that only the susceptible prey may reproduce, while the sick prey competes for the resource only, and that the disease is not genetically inherited.
2. It is assumed that the predator consumes both populations of the prey according to the Lotka-Volterra functional response. However, the prey population grows logistically in the absence of the predator.
3. It is thought that predation anxiety changes the foraging behavior of the prey population, which in turn reduces the risk of disease transmission among prey.
4. As the predator has a hunting cooperation capability, it will profit and successfully acquire prey. As a result, the predator population's attack rate, say $\alpha_1 > 0$, can be increased by the cooperation term to become $(\alpha_1 + \alpha_2 Y)$, where $\alpha_2 \geq 0$ describes the predator cooperation in hunting [26].

Accordingly, the dynamic of the described eco-epidemiological system can be represented using the following set of nonlinear first-order differential equations.

$$\begin{aligned}
\frac{dS}{dt} &= \frac{r}{1+\gamma_1 Y} S\left[1 - \frac{S+I}{k}\right] - \frac{\beta}{1+\gamma_2 Y} SI - (\alpha_1 + \alpha_2 Y)SY \\
\frac{dI}{dt} &= \frac{\beta}{1+\gamma_2 Y} SI - (\alpha_1 + \alpha_2 Y)IY - d_1 I \\
\frac{dY}{dt} &= (\alpha_1 + \alpha_2 Y)(c_1 S + c_2 I)Y - d_2 Y
\end{aligned} \quad , \qquad (1)$$

where $S(0) = S_0 \geq 0$, $I(0) = I_0 \geq 0$, and $Y(0) = Y_0 \geq 0$ represent the initial condition of the system (1), and all parameters are assumed nonnegative and can be described in Table 1.

Table 1: The parameters description

Parameters	Description
$r > 0$	The prey's intrinsic growth rate
$k > 0$	The environment-carrying capacity
$\gamma_1 \geq 0$	The level of fear that reduces the growth of the prey
$\gamma_2 \geq 0$	The level of fear that reduces the disease transmission
$\beta > 0$	The disease transmission rate
$\alpha_1 > 0$	The attack rate of the predator on the prey
$\alpha_2 > 0$	The predator cooperation in hunting
$d_1 > 0$	The death rates of the infected prey populations
$d_2 > 0$	The death rates of the predator populations
$c_1 \in (0,1]$	The conversion efficiency from susceptible prey biomass to predator biomass
$c_2 \in (0,1]$	The conversion efficiency from infected prey biomass to predator biomass

To non-dimensionalize the system (1), the following transformation is used.

$$rt = \bar{t},\ \frac{S}{k} = x_1,\ \frac{I}{k} = x_2,\ \frac{\alpha_2}{\alpha_1} Y = x_3.$$

Then, after dropping the bar, the system (1) reduces to the following form

$$\begin{aligned}
\frac{dx_1}{dt} &= x_1 \left[\frac{1-x_1-x_2}{1+w_1x_3} - \frac{w_2x_2}{1+w_3x_3} - w_4(1+x_3)x_3\right] = x_1 f_1(x_1,x_2,x_3),\\
\frac{dx_2}{dt} &= x_2 \left[\frac{w_2x_1}{1+w_3x_3} - w_4(1+x_3)x_3 - w_5\right] = x_2 f_2(x_1,x_2,x_3),\\
\frac{dx_3}{dt} &= x_3[w_6(1+x_3)(c_1x_1+c_2x_2) - w_7] = x_3 f_3(x_1,x_2,x_3),
\end{aligned} \tag{2}$$

where $w_1 = \gamma_1 \frac{\alpha_1}{\alpha_2}$, $w_2 = \frac{\beta k}{r}$, $w_3 = \gamma_2 \frac{\alpha_1}{\alpha_2}$, $w_4 = \frac{\alpha_1^2}{r\alpha_2}$, $w_5 = \frac{d_1}{r}$, $w_6 = \frac{\alpha_1 k}{r}$, $w_7 = \frac{d_2}{r}$.

It is clear from the system (2) that, the interaction functions $x_i f_i(x_1,x_2,x_3); i = 1,2,3$ in the right-hand side of the system (2), are continuous and have continuous partial derivatives on the domain $\mathbb{R}_+^3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. Hence, they are locally Lipschitz functions in $\mathbb{R}_+^3$. Consequently, due to the fundamental existence and uniqueness theorem, it is obtained that system (2) with any non-negative initial condition $x_1(0) \geq 0$, $x_2(0) \geq 0$, and $x_3(0) \geq 0$ there exists $T > 0$ so that the system (2) has a unique solution defined in $\mathbb{R}_+^3$.

3. PROPERTIES OF THE SOLUTION

This section treats the properties of the solution of system (2), such as positivity and bounded as presented in the next theorems.

Theorem 1: All system (2)'s solutions with the initial conditions belong to $int.\mathbb{R}_+^3$ are positively invariant.

Proof. From the first equation of the system (2), it is obtained:

$$\frac{dx_1}{x_1} = f_1(x_1, x_2, x_3)dt$$

Then integrating the above equation within the limit [0, t], gives that:

$$x_1(t) = x_1(0)e^{\int_0^t f_1(x_1(s), x_2(s), x_3(s))ds} > 0;\ \forall\, t$$

Similarly, the second and third equations, it is obtained

$$x_2(t) = x_2(0)e^{\int_0^t f_2(x_1(s), x_2(s), x_3(s))ds} > 0;\ \forall\, t$$

$$x_3(t) = x_3(0)e^{\int_0^t f_3(x_1(s), x_2(s), x_3(s))ds} > 0;\ \forall\, t$$

This completes the proof.

Theorem 2: All system (2)'s solutions with the initial conditions belonging to $\mathbb{R}_+^3$ are uniformly bounded

Proof. From system (2), it is easy to verify that

$$\frac{dx_1}{dt} \leq x_1(1 - x_1)$$

Then according to the lemma (2.2) (Chen, 2005), it is obtained that

$$x_1(t) \leq \left[1 + \left(\frac{1}{x_1(0)} - 1\right)e^{-t}\right]^{-1}$$

Hence for $t \to \infty$, it is obtained that $x_1(t) \leq 1$.

Let $= x_1 + x_2 + \frac{w_4}{w_6}x_3$, then using the fact that $c_i \in (0,1]; i = 1,2$, system (2) gives that:

$$\frac{dW}{dt} \leq x_1(1 - x_1) - w_5x_2 - \frac{w_4w_7}{w_6}x_3 \mp Mx_1,$$

where $M = min\left\{w_5, \frac{w_4 w_7}{w_6}\right\}$. Hence, simple manipulation yields

$$\frac{dW}{dt} + MW \leq \frac{(1+M)^2}{4}.$$

Then according to the lemma (2.1) [28], it is obtained that

$$W(t) \leq \frac{(1+M)^2}{4M}\left[1 + \left(\frac{4MW(0)}{(1+M)^2} - 1\right)e^{-Mt}\right]$$

Therefore, for $t \to \infty$, it is obtained that:

$$W(t) \leq \frac{(1+M)^2}{4M}.$$

That completes the proof.

4. EXISTENCE OF EQUILIBRIUM POINTS AND STABILITY ANALYSIS

The examination of each potential equilibrium point's stability is determined in this section. System (2) has the following equilibrium points (EPs):

The total extinction equilibrium point (TEEP) $p_1 = (0,0,0)$ always exists.

The axial equilibrium point (AEP) $p_2 = (1,0,0)$ always exists.

The predator-free equilibrium point (PFEP) $p_3 = (\bar{x}_1, \bar{x}_2, 0) = \left(\frac{w_5}{w_2}, \frac{w_2 - w_5}{w_2(1+w_2)}, 0\right)$, where

$$\bar{x}_1 = \frac{w_5}{w_2}, \bar{x}_2 = \frac{w_2 - w_5}{w_2(1+w_2)}, \tag{3}$$

exists provided that

$$w_5 < w_2. \tag{4}$$

The disease-free equilibrium point (DFEP) $p_4 = (\hat{x}_1, 0, \hat{x}_3)$, where

$$\hat{x}_1 = \frac{w_7}{c_1 w_6 (1+\hat{x}_3)}, \tag{5}$$

while $\hat{x}_3$ is a positive root of the following fourth-degree polynomial equation.

$$\begin{aligned} c_1 w_1 w_4 w_6 x_3^4 + c_1 w_4 w_6 (1 + 2w_1) x_3^3 + c_1 w_4 w_6 (2 + w_1) x_3^2 \\ + c_1 (w_4 - 1) w_6 x_3 - c_1 w_6 + w_7 = 0. \end{aligned} \tag{6}$$

Obviously, this equation has a unique positive root provided that

$$w_7 < c_1 w_6 \tag{7}$$

It may have two positive roots or zero positive roots provided that

$$\left.\begin{array}{r} w_4 < 1 \\ w_7 > c_1 w_6 \end{array}\right\}. \tag{8}$$

The positive equilibrium point (PEP) $p_5 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, where

$$\left.\begin{array}{r} \tilde{x}_1 = \frac{(1+w_3\tilde{x}_3)(w_5+w_4\tilde{x}_3+w_4\tilde{x}_3{}^2)}{w_2} \\ \tilde{x}_2 = \frac{w_2w_7-c_1w_6(1+\tilde{x}_3)(1+w_3\tilde{x}_3)(w_5+w_4\tilde{x}_3+w_4\tilde{x}_3{}^2)}{w_2c_2w_6(1+\tilde{x}_3)} \end{array}\right\}. \tag{9}$$

While $\tilde{x}_3$ represents a positive root of the following fifth-order polynomial equation.

$$A_0x_3^5 + A_1x_3^4 + A_2x_3^3 + A_3x_3^2 + A_4x_3 + A_5 = 0, \tag{10}$$

where

$$A_0 = w_3w_4w_6(c_2 - c_1)(w_1w_2 + w_3).$$

$$A_1 = w_4w_6(c_2 - c_1)[w_1w_2 + 2w_3 + w_2w_3 + 2w_1w_2w_3 + 2w_3^2].$$

$$\begin{array}{c} A_2 = (c_2 - c_1)w_4w_6[1 + w_2 + 2w_1w_2 + 4w_3 + 2w_2w_3 + w_1w_2w_3 + w_3^2] \\ +w_5w_6[(c_2 - c_1)w_3^2 - c_1w_1w_2w_3] \end{array}.$$

$$\begin{array}{c} A_3 = w_6[-(c_2 + c_1w_5)w_2w_3 + 2(c_2 - c_1)w_4 + 2(c_2 - c_1)w_2w_4 + (c_2 - c_1)w_1w_2w_4 \\ +2(c_2 - 2c_1)w_3w_4 + (c_2 - c_1)w_2w_3w_4 - c_1w_1w_2w_5(1 + w_3) \\ +2(c_2 - c_1)w_3w_5 + (c_2 - c_1)w_3^2w_5] \end{array}.$$

$$\begin{array}{c} A_4 = -c_2w_2w_6(1 + w_3)(c_2 - c_1)\, w_4w_6 + (c_2 - c_1)w_2w_4w_6 + (c_2 - c_1)w_5w_6 \\ -c_1w_2w_5w_6(1 + w_1) + 2(c_2 - c_1)\, w_3w_5w_6 - c_1w_2w_3w_5w_6 \\ +w_2w_7(w_2w_1 + w_3) \end{array}.$$

$$A_5 = -(c_2w_2 + c_1w_5)w_6 + (c_2 - c_1w_2)w_5w_6 + w_2w_7(1 + w_2).$$

Accordingly, due to the discarding rule of signs, equation (10) has at least one positive root provided that one set of the following sets of conditions occurs.

$$\left.\begin{array}{l} A_0 > 0\,,\ A_5 < 0 \\ A_0 < 0\,,\ A_5 > 0 \end{array}\right\}. \tag{11}$$

However, it has a unique positive root provided that one set of the following sets of conditions occurs.

$$\left.\begin{array}{c} A_0 > 0, A_1 > 0, A_2 > 0, A_3 > 0, A_4 > 0, A_5 < 0 \\ A_0 > 0, A_1 > 0, A_2 < 0, A_3 < 0, A_4 < 0, A_5 < 0 \\ A_0 > 0, A_1 > 0, A_2 > 0, A_4 < 0, A_5 < 0 \\ A_0 < 0, A_1 < 0, A_2 < 0, A_3 < 0, A_4 < 0, A_5 > 0 \\ A_0 < 0, A_1 < 0, A_2 > 0, A_3 > 0, A_4 > 0, A_5 > 0 \\ A_0 < 0, A_1 < 0, A_2 < 0, A_4 > 0, A_5 > 0 \\ A_0 = 0, A_1 = 0, A_2 < 0, A_3 < 0, A_5 > 0 \end{array}\right\}. \tag{12}$$

Keeping the existence of a unique positive root of equation (10) that denoted by $\tilde{x}_3$, the PEP will be exists in the interior of positive octant provided that.

$$c_1 w_6 (1+\tilde{x}_3)(1+w_3\tilde{x}_3)\left(w_5 + w_4\tilde{x}_3 + w_4\tilde{x}_3{}^2\right) < w_2 w_7. \tag{13}$$

The local stability analysis of the above equilibrium points can be determined using the following determined Jacobain matrix (JM)

$$J = \left(q_{ij}\right)_{3\times 3}, \tag{14}$$

where

$$q_{11} = -w_4 x_3(1+x_3) - \frac{x_1}{1+w_1x_3} + \frac{1-x_1-x_2}{1+w_1x_3} - \frac{w_2x_2}{1+w_3x_3}.$$

$$q_{12} = -x_1\left(\frac{1}{1+w_1x_3} + \frac{w_2}{1+w_3x_3}\right).$$

$$q_{13} = x_1\left(-w_4x_3 - w_4(1+x_3) - \frac{w_1(1-x_1-x_2)}{(1+w_1x_3)^2} + \frac{w_2w_3x_2}{(1+w_3x_3)^2}\right).$$

$$q_{21} = \frac{w_2x_2}{1+w_3x_3}.$$

$$q_{22} = -w_5 - w_4x_3(1+x_3) + \frac{w_2x_1}{1+w_3x_3}.$$

$$q_{23} = -x_2\left(w_4x_3 + w_4(1+x_3) + \frac{w_2w_3x_1}{(1+w_3x_3)^2}\right).$$

$$q_{31} = c_1w_6x_3(1+x_3).$$

$$q_{32} = c_2w_6x_3(1+x_3).$$

$$q_{33} = -w_7 + w_6(c_1x_1 + c_2x_2)x_3 + w_6(c_1x_1 + c_2x_2)(1+x_3).$$

Then at $p_1 = (0,0,0)$, the JM becomes

$$J(p_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -w_5 & 0 \\ 0 & 0 & -w_7 \end{pmatrix}. \tag{15}$$

Therefore, the eigenvalues of $J(p_1)$ are given by

$$\lambda_{11} = 1 > 0\,, \lambda_{12} = -w_5 < 0\,, \lambda_{13} = -w_7 < 0. \tag{16}$$

Hence, p_1 is a saddle point.

At $p_2 = (1,0,0)$, the JM becomes.

$$J(p_2) = \begin{pmatrix} -1 & -1-w_2 & -w_4 \\ 0 & w_2 - w_5 & 0 \\ 0 & 0 & c_1w_6 - w_7 \end{pmatrix}. \tag{17}$$

Therefore, the eigenvalues of $J(p_2)$ are given by

$$\lambda_{21} = -1 < 0\,, \lambda_{22} = w_2 - w_5, \lambda_{23} = c_1 w_6 - w_7. \tag{18}$$

Hence, p_2 is locally asymptotically stable (LAS) provided that the following two conditions are met.

$$\left.\begin{array}{c} w_2 < w_5 \\ c_1 w_6 < w_7 \end{array}\right\} \tag{19}$$

However, the point p_2 is a saddle point when at least one of these inequalities given is reflected. Finally, the AEP becomes a non-hyperbolic point if any one of these inequalities becomes equality.

At $p_3 = \left(\frac{w_5}{w_2}, \frac{w_2 - w_5}{w_2(1+w_2)}, 0\right)$ the JM can be written as:

$$J(p_3) = \begin{pmatrix} -\frac{w_5}{w_2} & -\frac{(1+w_2)w_5}{w_2} & \frac{w_5\left[-w_4 + \frac{w_3(w_2-w_5)}{1+w_2} - w_1\left(1 - \frac{w_2-w_5}{w_2(1+w_2)} - \frac{w_5}{w_2}\right)\right]}{w_2} \\ \frac{w_2-w_5}{1+w_2} & 0 & -\frac{(w_2-w_5)(w_4+w_3w_5)}{w_2(1+w_2)} \\ 0 & 0 & \left(\frac{c_2(w_2-w_5)}{w_2(1+w_2)} + \frac{c_1w_5}{w_2}\right)w_6 - w_7 \end{pmatrix}. \tag{20}$$

Hence, the characteristic equation of $J(p_3)$ can be written as:

$$\left[\left(\frac{c_2(w_2-w_5)}{w_2(1+w_2)} + \frac{c_1w_5}{w_2}\right)w_6 - w_7 - \lambda\right] + \left[\lambda^2 + \left(\frac{w_5}{w_2}\right)\lambda + \frac{(w_2-w_5)w_5}{w_2}\right] = 0. \tag{21}$$

Direct computation gives the following roots

$$\left.\begin{array}{l} \lambda_{31} = -\frac{w_5}{2w_2} + \frac{1}{2}\sqrt{\left(\frac{w_5}{w_2}\right)^2 - 4\frac{(w_2-w_5)w_5}{w_2}} \\ \lambda_{32} = -\frac{w_5}{2w_2} - \frac{1}{2}\sqrt{\left(\frac{w_5}{w_2}\right)^2 - 4\frac{(w_2-w_5)w_5}{w_2}} \\ \lambda_{33} = \left(\frac{c_2(w_2-w_5)}{w_2(1+w_2)} + \frac{c_1w_5}{w_2}\right)w_6 - w_7 \end{array}\right\}. \tag{22}$$

Hence, as the λ_{31} and λ_{32} have negative real parts, the point p_3 is LAS provided that

$$\left(\frac{c_2(w_2-w_5)}{w_2(1+w_2)} + \frac{c_1w_5}{w_2}\right)w_6 < w_7. \tag{23}$$

Otherwise, the PFEP will be saddle point if the condition (23) is reflected and becomes a non-hyperbolic point when the inequality of the condition (23) transfers to quality.

At $p_4 = (\hat{x}_1, 0, \hat{x}_3)$ the JM can be written as

$$J(p_4) = \left(b_{ij}\right)_{3\times3} \tag{24}$$

where

$$b_{11} = -\frac{\hat{x}_1}{1+w_1\hat{x}_3},\ b_{12} = -\hat{x}_1\left(\frac{1}{1+w_1\hat{x}_3} + \frac{w_2}{1+w_3\hat{x}_3}\right),$$

$$b_{13} = \hat{x}_1\left[-w_4\hat{x}_3 - w_4(1+\hat{x}_3) - \frac{w_1(1-\hat{x}_1)}{(1+w_1\hat{x}_3)^2}\right],$$

$$b_{22} = -w_5 - w_4\hat{x}_3(1+\hat{x}_3) + \frac{w_2\hat{x}_1}{1+w_3\hat{x}_3},$$

$$b_{31} = c_1w_6\hat{x}_3(1+\hat{x}_3),\ b_{32} = c_2w_6\hat{x}_3(1+\hat{x}_3),\ b_{33} = c_1w_6\hat{x}_1\hat{x}_3.$$

Hence, the characteristic equation of $J(p_4)$ can be written as:

$$[b_{22} - \lambda][\lambda^2 - (b_{11} + b_{33})\lambda + b_{11}b_{33} - b_{13}b_{31}] = 0. \quad (25)$$

Consequently, the eigenvalues of the $J(p_4)$ can be written as

$$\left.\begin{array}{l} \lambda_{41} = \frac{b_{11}+b_{33}}{2} + \frac{1}{2}\sqrt{(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})} \\ \lambda_{42} = -w_5 - w_4\hat{x}_3(1+\hat{x}_3) + \frac{w_2\hat{x}_1}{1+w_3\hat{x}_3} \\ \lambda_{43} = \frac{b_{11}+b_{33}}{2} - \frac{1}{2}\sqrt{(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})} \end{array}\right\}. \quad (26)$$

Direct computation shows that all the eigenvalues given by equation (26) have negative real parts if and only if the following two conditions are met.

$$c_1w_6\hat{x}_1\hat{x}_3 < \frac{\hat{x}_1}{1+w_1\hat{x}_3} < \left[w_4(1+2\hat{x}_3) + \frac{w_1(1-\hat{x}_1)}{(1+w_1\hat{x}_3)^2}\right](1+\hat{x}_3). \quad (27)$$

$$\frac{w_2\hat{x}_1}{1+w_3\hat{x}_3} < w_5 + w_4\hat{x}_3(1+\hat{x}_3). \quad (28)$$

However, violating any one of these two conditions makes the DFEP unstable.

Finally, at $p_5 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ the JM will be written as

$$J(p_5) = (a_{ij})_{3\times 3}, \quad (29)$$

where

$$a_{11} = -\frac{\tilde{x}_1}{1+w_1\tilde{x}_3},\ a_{12} = -\tilde{x}_1\left(\frac{1}{1+w_1\tilde{x}_3} + \frac{w_2}{1+w_3\tilde{x}_3}\right),$$

$$a_{13} = \tilde{x}_1\left[-w_4(1+2\tilde{x}_3) - \frac{w_1(1-\tilde{x}_1-\tilde{x}_2)}{(1+w_1\tilde{x}_3)^2} + \frac{w_2w_3\tilde{x}_2}{(1+w_3\tilde{x}_3)^2}\right],$$

$$a_{21} = \frac{w_2\tilde{x}_2}{1+w_3\tilde{x}_3},\ a_{22} = 0,\ a_{23} = -\tilde{x}_2\left[w_4(1+2\tilde{x}_3) + \frac{w_2w_3\tilde{x}_1}{(1+w_3\tilde{x}_3)^2}\right],$$

$$a_{31} = c_1w_6\tilde{x}_3(1+\tilde{x}_3),\ a_{32} = c_2w_6\tilde{x}_3(1+\tilde{x}_3),\ a_{33} = w_6(c_1\tilde{x}_1 + c_2\tilde{x}_2)\tilde{x}_3.$$

The characteristic equation of $J(p_5)$ can be written as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \quad (30)$$

where

$$A_1 = -(a_{11} + a_{33}),$$

$$A_2 = -a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32},$$

$$A_3 = -[a_{23}(a_{12}a_{31} - a_{11}a_{32}) + a_{21}(a_{13}a_{32} - a_{12}a_{33})],$$

$$\Delta = A_1A_2 - A_3 = -(a_{11} + a_{33})[a_{11}a_{33} - a_{13}a_{31}] + a_{12}(a_{11}a_{21} + a_{23}a_{31}) + a_{32}(a_{23}a_{33} + a_{13}a_{21}).$$

According to the Routh-Hurwitz criterion [1] the characteristic equation (30) has three eigenvalues with negative real parts if the following conditions are satisfied $A_1 > 0;\ A_3 > 0$, and $\Delta = A_1A_2 - A_3 > 0$. Therefore, the following theorem for local stability of the PEP is follows.

Theorem 3: The PEP of the system (2) is LAS if and only if the following set of conditions is met.

$$w_6(c_1\tilde{x}_1 + c_2\tilde{x}_2)\tilde{x}_3 < \frac{\tilde{x}_1}{1+w_1\tilde{x}_3}. \tag{31}$$

$$\frac{\tilde{x}_1}{1+w_1\tilde{x}_3}c_2 > \tilde{x}_1\left(\frac{1}{1+w_1\tilde{x}_3} + \frac{w_2}{1+w_3\tilde{x}_3}\right)c_1. \tag{32}$$

$$\left(\frac{1}{1+w_1\tilde{x}_3} + \frac{w_2}{1+w_3\tilde{x}_3}\right)(c_1\tilde{x}_1 + c_2\tilde{x}_2) < \left[w_4(1+2\tilde{x}_3) + \frac{w_1(1-\tilde{x}_1-\tilde{x}_2)}{(1+w_1\tilde{x}_3)^2} - \frac{w_2w_3\tilde{x}_2}{(1+w_3\tilde{x}_3)^2}\right]c_2(1+\tilde{x}_3). \tag{33}$$

$$(a_{11} + a_{33})a_{13}a_{31} + a_{12}(a_{11}a_{21} + a_{23}a_{31}) > (a_{11} + a_{33})a_{11}a_{33} - a_{32}(a_{23}a_{33} + a_{13}a_{21}). \tag{34}$$

Proof. Direct with the application of the Routh-Hurwitz criterion.

5. PERSISTENCE

This section studies an eco-epidemiological model's persistence and extinction property involving fear and hunting cooperation. The objective is to investigate the influence of fear and hunting cooperation within a diseased prey-predator system, on the persistence and extinction of system species. In order to determine the conditions that ensure the continuity, the dynamics at the boundary levels of the system must be understood.

It is clear that system (2) has two subsystems; the first subsystem can be representing in case of the absence of predator, and the second subsystem can be representing in the absence of disease from the system. Therefore, these two subsystems can be written in the following forms respectively.

The first subsystem is

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(1 - x_1 - x_2 - w_2 x_2) = x_1 g_{11}(x_1, x_2),\\ \frac{dx_2}{dt} &= x_2(w_2 x_1 - w_5) = x_2 g_{12}(x_1, x_2).\end{aligned} \tag{35}$$

The second subsystem is

$$\begin{aligned}\frac{dx_1}{dt} &= x_1\left(\frac{1-x_1}{1+w_1x_3} - w_4(1 + x_3)x_3\right) = x_1 g_{21}(x_1, x_3),\\ \frac{dx_3}{dt} &= x_3(w_6 c_1 x_1(1 + x_3) - w_7) = x_3 g_{22}(x_1, x_3).\end{aligned} \tag{36}$$

The first subsystem (35) has the following equilibrium points $p_{11} = (0,0)$, $p_{12} = (1,0)$, and $p_{13} = \left(\frac{w_5}{w_2}, \frac{w_2-w_5}{w_2(1+w_2)}\right)$, while the second subsystem has the equilibrium points $p_{21} = (0,0)$, $p_{22} = (1,0)$, and $p_{23} = (\hat{x}_1, \hat{x}_3)$, where $\hat{x}_1$ is given by equation (5) and $\hat{x}_3$ exists uniquely under the condition (7). Obviously, the equilibrium points of the above two subsystems coincide with the boundary equilibrium points of the system (2). Therefore, they have the same local stability conditions. Now, to investigate the possibilities of non-existence of periodic dynamics in the interior of positive quadrants corresponding to these two subsystems, Dulac-Bendixon criterion is applied.

Theorem 4: There are no periodic dynamics fall entirely:

1. In the interior of positive quadrant of x_1x_2 −plane.
2. In the interior of positive quadrant of x_1x_3 − plane, provided that the following condition is met.

$$w_6 c_1 > \frac{1}{x_3(1+w_1x_3)}. \tag{37}$$

Proof. (1) Consider the continuously differential function $D_1(x_1, x_2) = \frac{1}{x_1x_2}$ on a simple connected region of the interior of positive quadrant of x_1x_2 −plane. Then the expiration

$$\Delta = \frac{\partial(D_1 g_{11})}{\partial x_1} + \frac{\partial(D_1 g_{12})}{\partial x_2} = -\frac{1}{x_2} < 0.$$

It's clear that Δ has the same sign and does not equal to zero. Therefore, due to Dulac-Bendixon criterion the first subsystem (35) do not have periodic dynamic in the interior of positive quadrant of x_1x_2 −plane.

(2) Similarly, consider a continuously differential function $D_2(x_1, x_3) = \frac{1}{x_1 x_3}$ on the simple connected region of the interior of positive quadrant of x_1x_3 −plane. Then the expiration

$$\Delta = \frac{\partial(D_2 g_{21})}{\partial x_1} + \frac{\partial(D_2 g_{22})}{\partial x_3} = -\frac{1}{x_3(1+w_1 x_3)} + w_6 c_1$$

So using condition (37), subsystem (36) do not have periodic dynamic in the interior of positive quadrant of x_1x_3 −plane.

Theorem 5: Assume that condition (37) holds, then system (2) is uniformly persistent if the following conditions are met.

$$w_5 < w_2. \tag{38}$$

$$w_7 < c_1 w_6. \tag{39}$$

$$w_7 < w_6\left(c_1 \frac{w_5}{w_2} + c_2 \frac{w_2 - w_5}{w_2(1+w_2)}\right). \tag{40}$$

$$w_4 \hat{x}_3(1 + \hat{x}_3) + w_5 < \frac{w_2 \hat{x}_1}{1 + w_3 \hat{x}_3}. \tag{41}$$

Proof. Define the function $(x_1, x_2, x_3) = {x_1}^a {x_2}^b {x_3}^c$, where $a, b,$ and c are positive constants. Hence, $\rho(x_1, x_2, x_3)$ is a non-negative continuously differentiable function that satisfies $\rho(x_1, x_2, x_3) \to 0$ if any one of $x_i \to 0$. Moreover,

$$\rho'(x_1, x_2, x_3) = \frac{\partial\rho}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial\rho}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial\rho}{\partial x_3}\frac{dx_3}{dt}.$$

Then, it is obtained that

$$\varphi(x_1, x_2, x_3) = \frac{\rho'(x_1,x_2,x_3)}{\rho(x_1,x_2,x_3)} = af_1(x_1, x_2, x_3) + bf_2(x_1, x_2, x_3) + cf_3(x_1, x_2, x_3),$$

where $f_i;\ i = 1,2,3$ are given in system (2).

Consequently, due to the average Lyapunov function method, the proof will follows if and only if $\varphi(p_i) > 0$, for all boundary points p_i.

Now, we have

$$\varphi(p_1) = a - bw_5 - cw_7.$$

$$\varphi(p_2) = b(w_2 - w_5) + c(c_1 w_6 - w_7).$$

$$\varphi(p_3) = c\left[w_6\left(c_1 \frac{w_5}{w_2} + c_2 \frac{w_2 - w_5}{w_2(1+w_2)}\right) - w_7\right].$$

$$\varphi(p_4) = b\left[\frac{w_2 \hat{x}_1}{1 + w_3 \hat{x}_3} - w_4 \hat{x}_3(1 + \hat{x}_3) - w_5\right].$$

Obviously, $\varphi(p_1) > 0$ for suitable choice of positive constants $a, b,$ and c. However, $\varphi(p_2)$, $\varphi(p_3)$, and $\varphi(p_4)$ are positive under the conditions (38), (39), (40), and (41) respectively. Hence the proof is complete.

6. LOCAL BIFURCATION

The occurrence of local bifurcation is investigated in this section using the Sotomayor theorem [29]. Recall that a non-hyperbolic equilibrium point represents a necessary but not sufficient condition for a local bifurcation to occur. Therefore, in the following theorems, the bifurcation parameter is selected so that the equilibrium point becomes a non-hyperbolic point.

Now, rewrite the system (2) in the vector form as:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, \mu),\ \mathbf{X} = (x_1, x_2, x_3)^{\mathrm{T}},\ \mathbf{F} = (x_1 f_1(\mathbf{X}, \mu), x_2 f_2(\mathbf{X}, \mu), x_3 f_3(\mathbf{X}, \mu))^{\mathrm{T}}, \tag{42}$$

where $\mu \in \mathbb{R}$ represents a bifurcation parameter. Hence the second directional derivatives for (42) can be written as:

$$D^2\mathbf{F}(\boldsymbol{X}, \mu)(\mathbf{V}, \mathbf{V}) = [n_{i1}]_{3\times1}, \tag{43}$$

where $\mathbf{V} = (v_1, v_2, v_3)^T$ be any vector with

$$\begin{aligned} n_{11} = {} & -2v_3w_4(v_1 + v_3x_1) - 4v_1v_3w_4x_3 - \frac{2v_3^2w_1^2x_1(-1+x_1+x_2)}{(1+w_1x_3)^3} \\ & + \frac{2v_3w_1(v_2x_1+v_1(-1+2x_1+x_2))}{(1+w_1x_3)^2} - \frac{2v_1(v_1+v_2)}{1+w_1x_3} \\ & - \frac{2v_3^2w_2w_3^2x_1x_2}{(1+w_3x_3)^3} + \frac{2v_3w_2w_3(v_2x_1+v_1x_2)}{(1+w_3x_3)^2} - \frac{2v_1v_2w_2}{1+w_3x_3}, \end{aligned}$$

$$\begin{aligned} n_{21} = {} & -2v_3w_4(v_2 + v_3x_2) - 4v_2v_3w_4x_3 + \frac{2v_3^2w_2w_3^2x_1x_2}{(1+w_3x_3)^3} \\ & - \frac{2v_3w_2w_3(v_2x_1+v_1x_2)}{(1+w_3x_3)^2} + \frac{2v_1v_2w_2}{1+w_3x_3}, \end{aligned}$$

$$n_{31} = 2v_3w_6(c_1(v_3x_1 + v_1(1 + 2x_3)) + c_2(v_3x_2 + v_2(1 + 2x_3))).$$

However, the third directional derivatives for (42) can be written as:

$$D^3\mathbf{F}(\mathbf{X}, \mu)(\mathbf{V}, \mathbf{V}, \mathbf{V}) = [n_{i2}]_{3\times1}, \tag{44}$$

where

$$\begin{aligned} n_{12} = 6v_3 \Big[& -v_1v_3w_4 + \frac{v_3^2w_1^3x_1(-1+x_1+x_2)}{(1+w_1x_3)^4} \\ & - \frac{v_3w_1^2(v_2x_1+v_1(-1+2x_1+x_2))}{(1+w_1x_3)^3} + \frac{v_1(v_1+v_2)w_1}{(1+w_1x_3)^2} \\ & + \frac{v_3^2w_2w_3^3x_1x_2}{(1+w_3x_3)^4} - \frac{v_3w_2w_3^2(v_2x_1+v_1x_2)}{(1+w_3x_3)^3} + \frac{v_1v_2w_2w_3}{(1+w_3x_3)^2} \Big]. \end{aligned}$$

$$n_{22} = \frac{6v_3}{(1+w_3x_3)^4}[v_1w_2w_3(1+w_3x_3)(-v_3w_3x_2+v_2(1+w_3x_3)) \\ +v_3(v_3w_2w_3^3x_1x_2+v_2(1+w_3x_3)(-w_2w_3^2x_1+w_4(1+w_3x_3)^3))],$$

$$n_{32} = 6(c_1v_1+c_2v_2)v_3^2w_6.$$

Theorem 6: The system (2) undergoes a Transcritical bifurcation (TB) near AEP when the parameter w_2 passes through the value $w_2^* = w_5$ provided that $c_1w_6 < w_7$.

Proof. From the equation (17) with $w_2 = w_2^*$ the JM becomes

$$J_1^* = J(p_2, w_2^*) = \begin{pmatrix} -1 & -1-w_2^* & -w_4 \\ 0 & 0 & 0 \\ 0 & 0 & c_1w_6-w_7 \end{pmatrix}.$$

Therefore, the eigenvalues of J_1^* are given by

$$\lambda_{21}(w_2^*) = -1 < 0\,, \lambda_{22}(w_2^*) = 0, \lambda_{23}(w_2^*) = c_1w_6-w_7 < 0.$$

Thus AEP is a non-hyperbolic point at $w_2 = w_2^*$.

Let $\mathbf{V}_1 = (v_{11}, v_{12}, v_{13})^{\mathrm{T}}$ be the eigenvector conjugate with the eigenvalue $\lambda_{22}(w_2^*) = 0$. Thus, $J_1^*\mathbf{V}_1 = \mathbf{0}$, gives that $\mathbf{V}_1 = (-(1+w_2^*),1,0)^{\mathrm{T}}$.

Now, let $\mathbf{U}_1 = (u_{11}, u_{12}, u_{13})^{\mathrm{T}}$ represents the eigenvector conjugate with the eigenvalue $\lambda_{22}(w_2^*) = 0$, of the matrix ${J_1^*}^{\mathrm{T}}$. Thus, ${J_1^*}^{\mathrm{T}}\mathbf{U}_1 = \mathbf{0}$ gives that $\mathbf{U}_1 = (0,1,0)^T$.

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_2}\mathbf{F}(\mathbf{X}, w_2) = \begin{pmatrix} -\frac{x_1x_2}{1+w_3x_3} \\ \frac{x_1x_2}{1+w_3x_3} \\ 0 \end{pmatrix}; \Rightarrow \frac{\partial}{\partial w_2}\mathbf{F}(p_2, w_2^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, ${\mathbf{U}_1}^T\mathbf{F}_{w_2}(p_2, w_2^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_2}(\mathbf{X}, w_2) = \begin{bmatrix} -\frac{x_2}{1+w_3x_3} & -\frac{x_1}{1+w_3x_3} & \frac{w_3x_1x_2}{(1+w_3x_3)^2} \\ \frac{x_2}{1+w_3x_3} & \frac{x_1}{1+w_3x_3} & -\frac{w_3x_1x_2}{(1+w_3x_3)^2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow D\mathbf{F}_{w_2}(p_2, w_2^*)\mathbf{V}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore

$${\mathbf{U}_1}^{\mathrm{T}}D\mathbf{F}_{w_2}(p_2, w_2^*)\mathbf{V}_1 = 1 \neq 0.$$

Also, by using equation (43), it is obtained that

$$D^2\mathbf{F}(p_2,w_2^*)(\mathbf{V}_1,\mathbf{V}_1)=\begin{bmatrix}0\\-2(1+w_2^*)w_2^*\\c_2\end{bmatrix}\Rightarrow\ \mathbf{U}_1{}^{\mathrm{T}}D^2\mathbf{F}(p_2,w_2^*)(\mathbf{V}_1,\mathbf{V}_1)=-2(1+w_2^*)w_2^*\neq 0.$$

Hence a TB take place near AEP.

Theorem 7: The system (2) undergoes a TB near PFEP when the parameter w_7 passes through the value $w_7^*=\left(\frac{c_2(w_2-w_5)}{w_2(1+w_2)}+\frac{c_1w_5}{w_2}\right)w_6$ provided that the following condition holds

$$c_1\left(\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}\right)+c_2\left(\bar{x}_2+\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right)\neq 0, \tag{45}$$

where all the new symbols will be defined in the proof. Otherwise, pitchfork bifurcation (PB) takes place.

Proof. From the equation (17) with $w_7=w_7^*$ the JM becomes

$$J_2^*=J(p_3,w_7^*)=\begin{bmatrix}-\frac{w_5}{w_2} & -\frac{(1+w_2)w_5}{w_2} & \frac{w_5[-w_4(1+w_2)+(w_3-w_1)(w_2-w_5)]}{w_2(1+w_2)}\\ \frac{w_2-w_5}{1+w_2} & 0 & -\frac{(w_2-w_5)(w_4+w_3w_5)}{w_2(1+w_2)}\\ 0 & 0 & 0\end{bmatrix}=\left[\rho_{ij}\right]_{3\times 3}.$$

Therefore, the eigenvalues of J_2^* are given by

$$\left.\begin{aligned}\lambda_{31}(w_7^*)&=-\frac{w_5}{2w_2}+\frac{1}{2}\sqrt{\left(\frac{w_5}{w_2}\right)^2-4\frac{(w_2-w_5)w_5}{w_2}}\\ \lambda_{32}(w_7^*)&=-\frac{w_5}{2w_2}-\frac{1}{2}\sqrt{\left(\frac{w_5}{w_2}\right)^2-4\frac{(w_2-w_5)w_5}{w_2}}\\ \lambda_{33}(w_7^*)&=0\end{aligned}\right\}.$$

Thus PFEP is a non-hyperbolic point at $w_7=w_7^*$.

Let $\mathbf{V}_2=(v_{21},v_{22},v_{23})^{\mathrm{T}}$ be the eigenvector conjugate with the eigenvalue $\lambda_{33}(w_7^*)=0$. Thus, $J_2^*\mathbf{V}_2=\mathbf{0}$, gives that $\mathbf{V}_2=\left(-\frac{\rho_{23}}{\rho_{21}},\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}},1\right)^{\mathrm{T}}$.

Now, let $\mathbf{U}_2=(u_{21},u_{22},u_{23})^{\mathrm{T}}$ represents the eigenvector conjugate with the eigenvalue $\lambda_{33}(w_7^*)=0$, of the matrix $J_2^{*\mathrm{T}}$. Thus, $J_2^{*\mathrm{T}}\mathbf{U}_2=\mathbf{0}$ gives that $\mathbf{U}_2=(0,0,1)^T$.

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_7}\mathbf{F}(\mathbf{X},w_7)=\begin{pmatrix}0\\0\\-x_3\end{pmatrix};\ \Rightarrow\ \frac{\partial}{\partial w_7}\mathbf{F}(p_3,w_7^*)=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Therefore, $\mathbf{U}_2{}^{\mathrm{T}}\mathbf{F}_{w_7}(p_3,w_7^*)=0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_7}(\mathbf{X},w_7)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}\Rightarrow D\mathbf{F}_{w_7}(p_3,w_7^*)\mathbf{V}_2=\begin{bmatrix}0\\0\\-1\end{bmatrix}.$$

Therefore

$$\mathbf{U}_2{}^{\mathrm{T}}D\mathbf{F}_{w_7}(p_3,w_7^*)\mathbf{V}_2=-1\neq 0.$$

Also, by using equation (43), it is obtained that

$$D^2\mathbf{F}(p_3,w_7^*)(\mathbf{V}_2,\mathbf{V}_2)=[n_{i1}(p_3,w_7^*)],$$

where

$$\begin{aligned}n_{11}(p_3,w_7^*)&=-2w_4\left(-\frac{\rho_{23}}{\rho_{21}}+\bar{x}_1\right)-2w_1^2\bar{x}_1(-1+\bar{x}_1+\bar{x}_2)\\&+2w_1\left[\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}(-1+2\bar{x}_1+\bar{x}_2)\right]\\&+2\left(\frac{\rho_{23}}{\rho_{21}}\right)\left[-\frac{\rho_{23}}{\rho_{21}}+\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right]-2w_2w_3^2\bar{x}_1\bar{x}_2\\&+2w_2w_3\left[\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}\bar{x}_2\right]-2\left(-\frac{\rho_{23}}{\rho_{21}}\right)\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}w_2\end{aligned},$$

$$\begin{aligned}n_{21}(p_3,w_7^*)&=-2w_4\left(\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}+\bar{x}_2\right)+2w_2w_3^2\bar{x}_1\bar{x}_2\\&-2w_2w_3\left[\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}\bar{x}_2\right]\\&+2\left(-\frac{\rho_{23}}{\rho_{21}}\right)\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}w_2\end{aligned},$$

$$n_{31}(p_3,w_7^*)=2w_6\left[c_1\left(\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}\right)+c_2\left(\bar{x}_2+\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right)\right].$$

Then, when the condition (45) is met, it is obtained that

$$\mathbf{U}_2{}^{T}D^2\mathbf{F}(p_3,w_7^*)(\mathbf{V}_2,\mathbf{V}_2)=2w_6\left[c_1\left(\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}\right)+c_2\left(\bar{x}_2+\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right)\right]\neq 0.$$

Hence a TB take place near PFEP. Otherwise, by using equation (44), it is obtained that

$$D^3\mathbf{F}(p_3,w_7^*)(\mathbf{V}_2,\mathbf{V}_2,\mathbf{V}_2)=[n_{i2}(p_3,w_7^*)],$$

where

$$\begin{aligned}n_{12}(p_3,w_7^*)&=6\left[\frac{\rho_{23}}{\rho_{21}}w_4+w_1^3\bar{x}_1(-1+\bar{x}_1+\bar{x}_2)\right.\\&-w_1^2\left(\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}(-1+2\bar{x}_1+\bar{x}_2)\right)\\&-\frac{\rho_{23}}{\rho_{21}}\left(-\frac{\rho_{23}}{\rho_{21}}+\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right)w_1+w_2w_3^3\bar{x}_1\bar{x}_2\\&\left.-w_2w_3^2\left(\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\bar{x}_1-\frac{\rho_{23}}{\rho_{21}}\bar{x}_2\right)-\frac{\rho_{23}}{\rho_{21}}\frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}w_2w_3\right].\end{aligned}$$

$$n_{22}(p_3, w_7^*) = 6\left[-\frac{\rho_{23}}{\rho_{21}} w_2 w_3 \left(-w_3 \bar{x}_2 + \frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right) + w_2 w_3^3 \bar{x}_1 \bar{x}_2 + \frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\left(-w_2 w_3^2 \bar{x}_1 + w_4\right)\right].$$

$$n_{32}(p_3, w_7^*) = 6\left[-c_1 \frac{\rho_{23}}{\rho_{21}} + c_2 \frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right] w_6.$$

Accordingly, since it is assumed $c_1\left(\bar{x}_1 - \frac{\rho_{23}}{\rho_{21}}\right) + c_2\left(\bar{x}_2 + \frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right) = 0$, it is obtained

$$\mathbf{U}_2{}^T D^3 \mathbf{F}(p_3, w_7^*)(\mathbf{V}_2, \mathbf{V}_2, \mathbf{V}_2) = 6\left[-c_1 \frac{\rho_{23}}{\rho_{21}} + c_2 \frac{\rho_{11}\rho_{23}-\rho_{13}\rho_{21}}{\rho_{12}\rho_{21}}\right] w_6 \neq 0.$$

Therefore, PB takes place near PFEP, and the proof is complete.

Theorem 8: Assume that condition (27) is met, then system (2) undergoes a TB near DFEP when the parameter w_5 passes through the value $w_5^* = -w_4 \hat{x}_3(1 + \hat{x}_3) + \frac{w_2 \hat{x}_1}{1+w_3 \hat{x}_3}$ provided that the following condition holds

$$-2\sigma_2 w_4 - 4\sigma_2 w_4 \hat{x}_3 - \frac{2\sigma_2 w_2 w_3 \hat{x}_1}{(1+w_3\hat{x}_3)^2} + \frac{2\sigma_1 w_2}{1+w_3\hat{x}_3} \neq 0, \tag{46}$$

where all the new symbols will be defined in the proof. Otherwise, pitchfork bifurcation (PB) takes place provided that the following condition holds.

$$\sigma_1 w_2 w_3 (1 + w_3\hat{x}_3)^2 - w_2 w_3^2 \sigma_2 (1 + w_3\hat{x}_3)\hat{x}_1 + w_4 \sigma_2 (1 + w_3\hat{x}_3)^4 \neq 0. \tag{47}$$

Proof. From the equation (24) with $w_5 = w_5^*$ the JM becomes

$$J_3^* = J(p_4, w_5^*) = \left(b_{ij}^*\right)_{3\times 3},$$

where $b_{ij}^* = b_{ij}$, for all $i, j = 1,2,3$ with $b_{22}^* = 0$.

Therefore, the eigenvalues of J_3^* are given by

$$\left.\begin{array}{c} \lambda_{41}^* = \frac{b_{11}+b_{33}}{2} + \frac{1}{2}\sqrt{(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})} \\ \lambda_{42}^* = 0 \\ \lambda_{43}^* = \frac{b_{11}+b_{33}}{2} - \frac{1}{2}\sqrt{(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})} \end{array}\right\}.$$

Thus DFEP is a non-hyperbolic point at $w_5 = w_5^*$, with two other eigenvalues λ_{41}^* and λ_{43}^* have negative real parts due to condition (27).

Let $\mathbf{V}_3 = (v_{31}, v_{32}, v_{33})^{\mathrm{T}}$ be the eigenvector conjugate with the eigenvalue $\lambda_{42}^* = 0$. Thus, $J_3^* \mathbf{V}_3 = \mathbf{0}$, gives that $\mathbf{V}_3 = \left(\frac{b_{13}b_{32}-b_{12}b_{33}}{b_{11}b_{33}-b_{13}b_{31}}, 1, \frac{b_{12}b_{31}-b_{11}b_{32}}{b_{11}b_{33}-b_{13}b_{31}}\right)^{\mathrm{T}} = (\sigma_1, 1, \sigma_2)^{\mathrm{T}}$.

Now, let $\mathbf{U}_3 = (u_{31}, u_{32}, u_{33})^{\mathrm{T}}$ represents the eigenvector conjugate with the eigenvalue $\lambda_{42}^* =$

0, of the matrix $J_3^{*\mathrm{T}}$. Thus, $J_3^{*\mathrm{T}}\mathbf{U}_3 = \mathbf{0}$ gives that $\mathbf{U}_3 = (0,1,0)^T$.

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_5}\mathbf{F}(\mathbf{X}, w_5) = \begin{pmatrix} 0 \\ -x_2 \\ 0 \end{pmatrix}; \Rightarrow \frac{\partial}{\partial w_5}\mathbf{F}(p_4, w_5^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $\mathbf{U}_3{}^{\mathrm{T}}\mathbf{F}_{w_5}(p_4, w_5^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_5}(\mathbf{X}, w_5) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow D\mathbf{F}_{w_5}(p_4, w_5^*)\mathbf{V}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{U}_3{}^{\mathrm{T}} D\mathbf{F}_{w_5}(p_4, w_5^*)\mathbf{V}_3 = -1 \neq 0.$$

Also, by using equation (43), it is obtained that

$$D^2\mathbf{F}(p_4, w_5^*)(\mathbf{V}_3, \mathbf{V}_3) = [n_{i1}(p_4, w_5^*)],$$

where

$$n_{11}(p_4, w_5^*) = -2\sigma_1\sigma_2 w_4 - 2{\sigma_2}^2 w_4\hat{x}_1 - 4\sigma_1\sigma_2 w_4\hat{x}_3 - \frac{2{\sigma_2}^2 w_1^2\hat{x}_1(-1+\hat{x}_1)}{(1+w_1\hat{x}_3)^3} + \frac{2\sigma_2 w_1\hat{x}_1 + 2\sigma_1\sigma_2 w_1(-1+2\hat{x}_1)}{(1+w_1\hat{x}_3)^2} - \frac{2{\sigma_1}^2 + 2\sigma_1}{1+w_1\hat{x}_3} + \frac{2\sigma_2 w_2 w_3\hat{x}_1}{(1+w_3\hat{x}_3)^2} - \frac{2\sigma_1 w_2}{1+w_3\hat{x}_3}.$$

$$n_{21}(p_4, w_5^*) = -2\sigma_2 w_4 - 4\sigma_2 w_4\hat{x}_3 - \frac{2\sigma_2 w_2 w_3\hat{x}_1}{(1+w_3\hat{x}_3)^2} + \frac{2\sigma_1 w_2}{1+w_3\hat{x}_3}.$$

$$n_{31}(p_4, w_5^*) = 2c_1{\sigma_2}^2 w_6\hat{x}_1 + 2c_1\sigma_1\sigma_2 w_6(1+2\hat{x}_3) + 2c_2\sigma_2 w_6(1+2\hat{x}_3).$$

Then, when the condition (46) is met, it is obtained that

$$\mathbf{U}_3{}^T D^2\mathbf{F}(p_4, w_5^*)(\mathbf{V}_3, \mathbf{V}_3) = -2\sigma_2 w_4 - 4\sigma_2 w_4\hat{x}_3 - \frac{2\sigma_2 w_2 w_3\hat{x}_1}{(1+w_3\hat{x}_3)^2} + \frac{2\sigma_1 w_2}{1+w_3\hat{x}_3} \neq 0.$$

Hence a TB take place near DFEP. Otherwise, by using equation (44), it is obtained that

$$D^3\mathbf{F}(p_4, w_5^*)(\mathbf{V}_3, \mathbf{V}_3, \mathbf{V}_3) = [n_{i2}(p_4, w_5^*)],$$

where

$$n_{12}(p_4, w_5^*) = 6\sigma_2\Big[-\sigma_1\sigma_2 w_4 + \frac{{\sigma_2}^2 w_1^3\hat{x}_1(-1+\hat{x}_1)}{(1+w_1\hat{x}_3)^4} - \frac{\sigma_2 w_1^2(\hat{x}_1 + \sigma_1(-1+2\hat{x}_1))}{(1+w_1\hat{x}_3)^3} + \frac{\sigma_1(\sigma_1+1)w_1}{(1+w_1\hat{x}_3)^2} - \frac{\sigma_2 w_2 w_3^2\hat{x}_1}{(1+w_3\hat{x}_3)^3} + \frac{\sigma_1 w_2 w_3}{(1+w_3\hat{x}_3)^2}\Big]$$

$$n_{22}(p_4, w_5^*) = \frac{6\sigma_2}{(1+w_3\hat{x}_3)^4}[\sigma_1 w_2 w_3(1+w_3\hat{x}_3)^2 - w_2 w_3^2\sigma_2(1+w_3\hat{x}_3)\hat{x}_1 + w_4\sigma_2(1+w_3\hat{x}_3)^4].$$

$n_{32}(p_4, w_5^*) = 6(c_1\sigma_1 + c_2){\sigma_2}^2 w_6.$

Accordingly, due to condition (47), it is obtained

$$\mathbf{U}_3{}^T D^3 \mathbf{F}(p_4, w_5^*)(\mathbf{V}_3, \mathbf{V}_3, \mathbf{V}_3) = n_{22}(p_4, w_5^*) \neq 0.$$

Therefore, PB takes place near PFEP, and the proof is complete.

Theorem 9: Assume that condition (31) is met along with the following condition

$$\frac{(c_1\tilde{x}_1 + c_2\tilde{x}_2)}{1 + w_1\tilde{x}_3} < \left[w_4^*(1 + 2\tilde{x}_3) + \frac{w_1(1 - \tilde{x}_1 - \tilde{x}_2)}{(1 + w_1\tilde{x}_3)^2} - \frac{w_2 w_3 \tilde{x}_2}{(1 + w_3\tilde{x}_3)^2}\right] c_1(1 + \tilde{x}_3). \quad (48)$$

Then system (2) undergoes a saddle-node bifurcation (SNB) near PEP when the parameter w_4 passes through the value w_4^*, provided that the following conditions hold

$$-\sigma_5(1 + \tilde{x}_3)\tilde{x}_1\tilde{x}_3 - \sigma_6(1 + \tilde{x}_3)\tilde{x}_2\tilde{x}_3 \neq 0, \quad (49)$$

$$\sigma_5 n_{11}(p_5, w_4^*) + \sigma_6 n_{21}(p_5, w_4^*) + n_{31}(p_5, w_4^*) \neq 0, \quad (50)$$

where

$$w_4^* = \left[\frac{-\tilde{x}_1 \frac{w_1(1 - \tilde{x}_1 - \tilde{x}_2)}{(1 + w_1\tilde{x}_3)^2} a_{21}a_{32} + \frac{w_2 w_3 \tilde{x}_1 \tilde{x}_2}{(1 + w_3\tilde{x}_3)^2}[a_{21}a_{32} - a_{12}a_{31} + a_{11}a_{32}] - a_{12}a_{21}a_{33}}{(1 + 2\tilde{x}_3)[(a_{12}a_{31} - a_{11}a_{32})\tilde{x}_2 + \tilde{x}_1 a_{21}a_{32}]}\right].$$

Proof. From the equation (24) with $w_4 = w_4^*$ the JM becomes

$$J_4^* = J(p_5, w_4^*) = \left(a_{ij}^*\right)_{3\times 3},$$

where $a_{ij}^* = a_{ij}(w_4^*)$, for all $i, j = 1,2,3$.

Therefore, it is easy to verify that the coefficient $A_3 = 0$ at $w_4 = w_4^*$ in equation (30). Hence the characteristic equation (30) becomes

$$(\lambda^2 + A_1^*\lambda + A_2^*)\lambda = 0,$$

where $A_1^* = A_1(w_4^*)$ and $A_2^* = A_2(w_4^*)$ with A_1 and A_2 as given in equation (30). Clearly, $A_1^* > 0$ under the condition (31a) and $A_2^* > 0$ under the condition (48). Therefore, due to Routh-Hurwitz criterion, the above obtained characteristic equation has two eigenvalues with negative real parts and third zero eigenvalues:

$$\left.\begin{aligned} \lambda_{51}^* &= -\frac{A_1^*}{2} + \frac{1}{2}\sqrt{{A_1^*}^2 - 4A_2^*} \\ \lambda_{52}^* &= -\frac{A_1^*}{2} + \frac{1}{2}\sqrt{{A_1^*}^2 - 4A_2^*} \\ \lambda_{53}^* &= 0 \end{aligned}\right\}.$$

Thus PEP is a non-hyperbolic point at $w_4 = w_4^*$.

Let $\mathbf{V}_4 = (v_{41}, v_{42}, v_{43})^{\mathrm{T}}$ be the eigenvector conjugate with the eigenvalue $\lambda_{53}^* = 0$. Thus, $J_4^*\mathbf{V}_4 = \mathbf{0}$, gives that $\mathbf{V}_4 = \left(-\frac{a_{23}^*}{a_{21}^*}, \frac{a_{11}^* a_{23}^* - a_{13}^* a_{21}^*}{a_{12}^* a_{21}^*}, 1\right)^{\mathrm{T}} = (\sigma_3, \sigma_4, 1)^{\mathrm{T}}$.

Now, let $\mathbf{U}_4 = (u_{41}, u_{42}, u_{43})^{\mathrm{T}}$ represents the eigenvector conjugate with the eigenvalue $\lambda_{53}^* = 0$, of the matrix $J_4^{*\mathrm{T}}$. Thus, $J_4^{*\mathrm{T}}\mathbf{U}_4 = \mathbf{0}$ gives that $\mathbf{U}_4 = \left(-\frac{a_{32}^*}{a_{12}^*}, \frac{a_{11}^* a_{32}^* - a_{12}^* a_{31}^*}{a_{12}^* a_{21}^*}, 1\right)^{\mathrm{T}} = (\sigma_5, \sigma_6, 1)^{\mathrm{T}}$.

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_4}\mathbf{F}(\mathbf{X}, w_4) = \begin{pmatrix} -(1 + x_3)x_1x_3 \\ -(1 + x_3)x_2x_3 \\ 0 \end{pmatrix}; \Rightarrow \frac{\partial}{\partial w_4}\mathbf{F}(p_5, w_4^*) = \begin{pmatrix} -(1 + \tilde{x}_3)\tilde{x}_1\tilde{x}_3 \\ -(1 + \tilde{x}_3)\tilde{x}_2\tilde{x}_3 \\ 0 \end{pmatrix}$$

Therefore due to condition (49) the following is obtained

$$\mathbf{U}_4{}^{\mathrm{T}}\mathbf{F}_{w_4}(p_5, w_4^*) = -\sigma_5(1 + \tilde{x}_3)\tilde{x}_1\tilde{x}_3 - \sigma_6(1 + \tilde{x}_3)\tilde{x}_2\tilde{x}_3 \neq 0$$

Also, by using equation (43), it is obtained that

$$D^2\mathbf{F}(p_5, w_4^*)(\mathbf{V}_4, \mathbf{V}_4) = [n_{i1}(p_5, w_4^*)],$$

where

$$\begin{aligned} n_{11}(p_5, w_4^*) = {} & -2w_4^*(\sigma_3 + \tilde{x}_1) - 4\sigma_3 w_4^*\tilde{x}_3 - \frac{2w_1^2\tilde{x}_1(-1+\tilde{x}_1+\tilde{x}_2)}{(1+w_1\tilde{x}_3)^3} \\ & + \frac{2w_1(\sigma_4\tilde{x}_1+\sigma_3(-1+2\tilde{x}_1+\tilde{x}_2))}{(1+w_1\tilde{x}_3)^2} - \frac{2\sigma_3(\sigma_3+\sigma_4)}{1+w_1\tilde{x}_3} \\ & - \frac{2w_2w_3^2\tilde{x}_1\tilde{x}_2}{(1+w_3\tilde{x}_3)^3} + \frac{2w_2w_3(\sigma_4\tilde{x}_1+\sigma_3\tilde{x}_2)}{(1+w_3\tilde{x}_3)^2} - \frac{2\sigma_3\sigma_4 w_2}{1+w_3\tilde{x}_3} \end{aligned},$$

$$\begin{aligned} n_{21}(p_5, w_4^*) = {} & -2w_4^*(\sigma_4 + \tilde{x}_2) - 4\sigma_4 w_4^*\tilde{x}_3 + \frac{2w_2w_3^2\tilde{x}_1\tilde{x}_2}{(1+w_3\tilde{x}_3)^3} \\ & - \frac{2w_2w_3(\sigma_4\tilde{x}_1+\sigma_3\tilde{x}_2)}{(1+w_3\tilde{x}_3)^2} + \frac{2\sigma_3\sigma_4 w_2}{1+w_3\tilde{x}_3} \end{aligned},$$

$$n_{31}(p_5, w_4^*) = 2w_6(c_1[\tilde{x}_1 + \sigma_3(1 + 2\tilde{x}_3)] + c_2[\tilde{x}_2 + \sigma_4(1 + 2\tilde{x}_3)]).$$

Thus, due to condition (50) the following is obtained.

$$\mathbf{U}_4{}^{T}D^2\mathbf{F}(p_5, w_4^*)(\mathbf{V}_4, \mathbf{V}_4) = \sigma_5 n_{11}(p_5, w_4^*) + \sigma_6 n_{21}(p_5, w_4^*) + n_{31}(p_5, w_4^*) \neq 0.$$

Therefore, SNB takes place as the $w_4 = w_4^*$.

7. Numerical Simulation

Numerical simulations were carried out using MATLAB with a set of non-dimensionalized parameters as listed in the following equation to support our analytical results and comprehend the impact of changing parameter values.

$$\begin{gathered} w_1 = 0.2, w_2 = 0.6, w_3 = 0.2, w_4 = 0.75, w_5 = 0.1 \\ w_6 = 0.5, w_7 = 0.1, c_1 = 0.2, c_2 = 0.4. \end{gathered} \tag{51}$$

It is observed that, for the dataset given by equation (51), system (2) starting from different initial values approach asymptotically to the PEP as shown in figure (1). In all the following figures, the star represents the attracting equilibrium point, and the magenta color is used for expressing the trajectory of the system (2). In contrast, the blue, green, and red colors are used to describe the trajectory of x_1, x_2, and x_3 respectively in the time series.

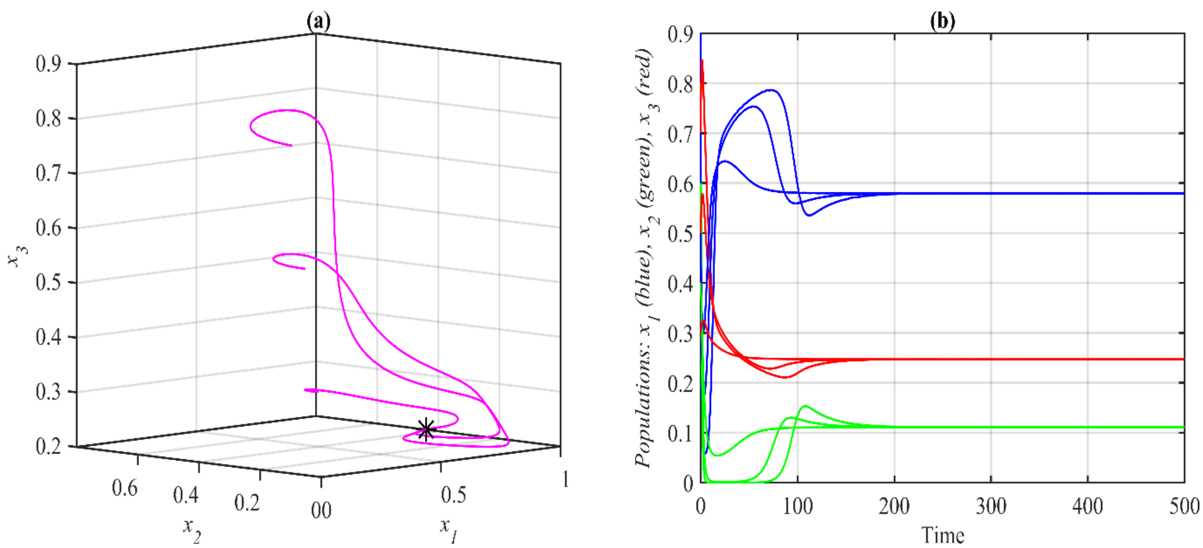


Fig. 1: The dynamics of system (2) utilizing the dataset (51) approach asymptotically to $p_5 = (0.57,0.11,0.24)$. (a) 3D phase portrait. (b) Time series.

Note that, figure (1) represents the existence of a unique PEP that is asymptotic stable point. In contrast, figures (2) and (3) describe the influence of the fear rates w_1, and w_3 on the dynamics of the system (3).

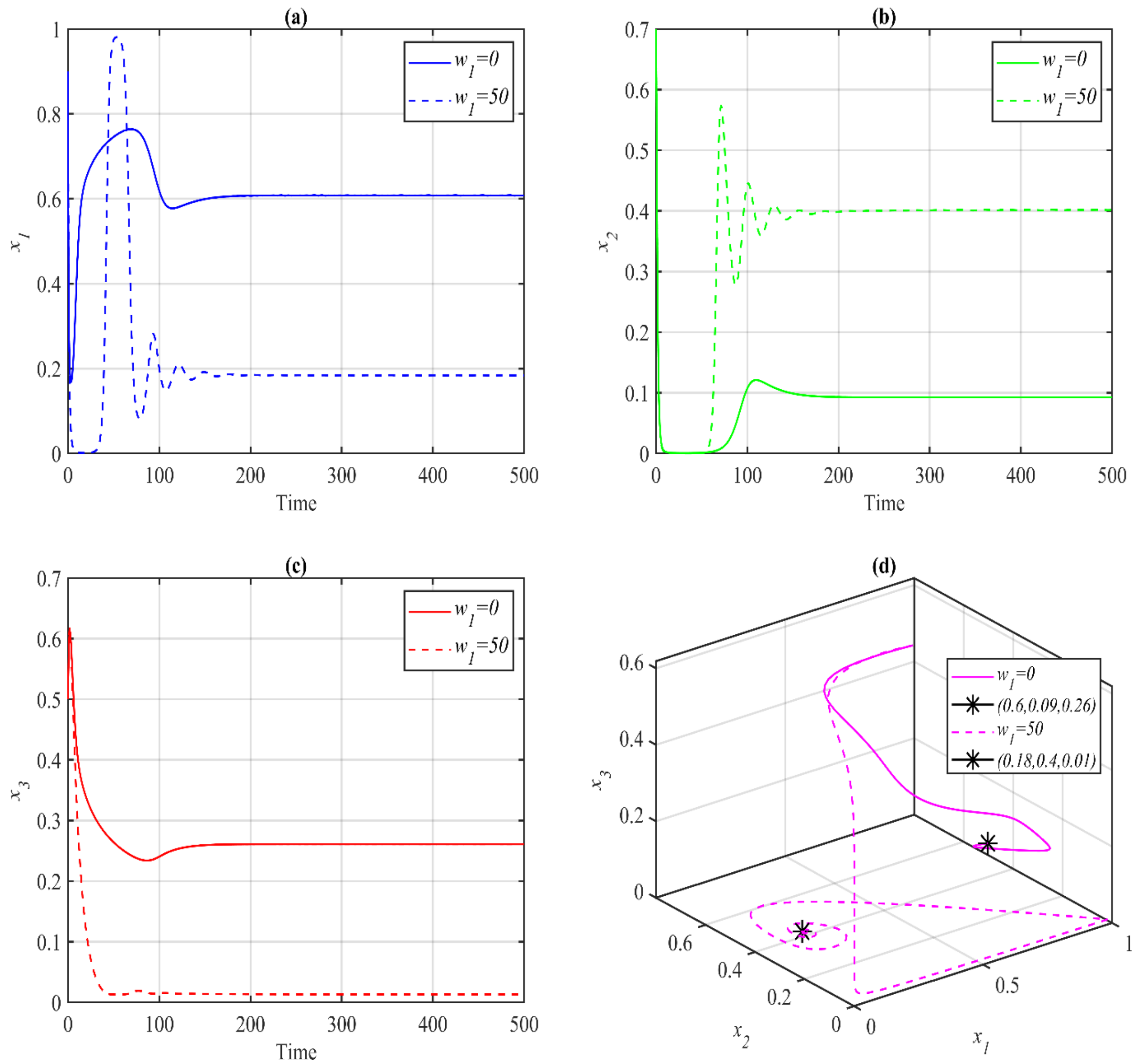


Fig. 2: The dynamics of system (2) utilizing the dataset (51) with two different values of $w_1 = 0, 50$. (a) The trajectory of system (2). (b) Time series.

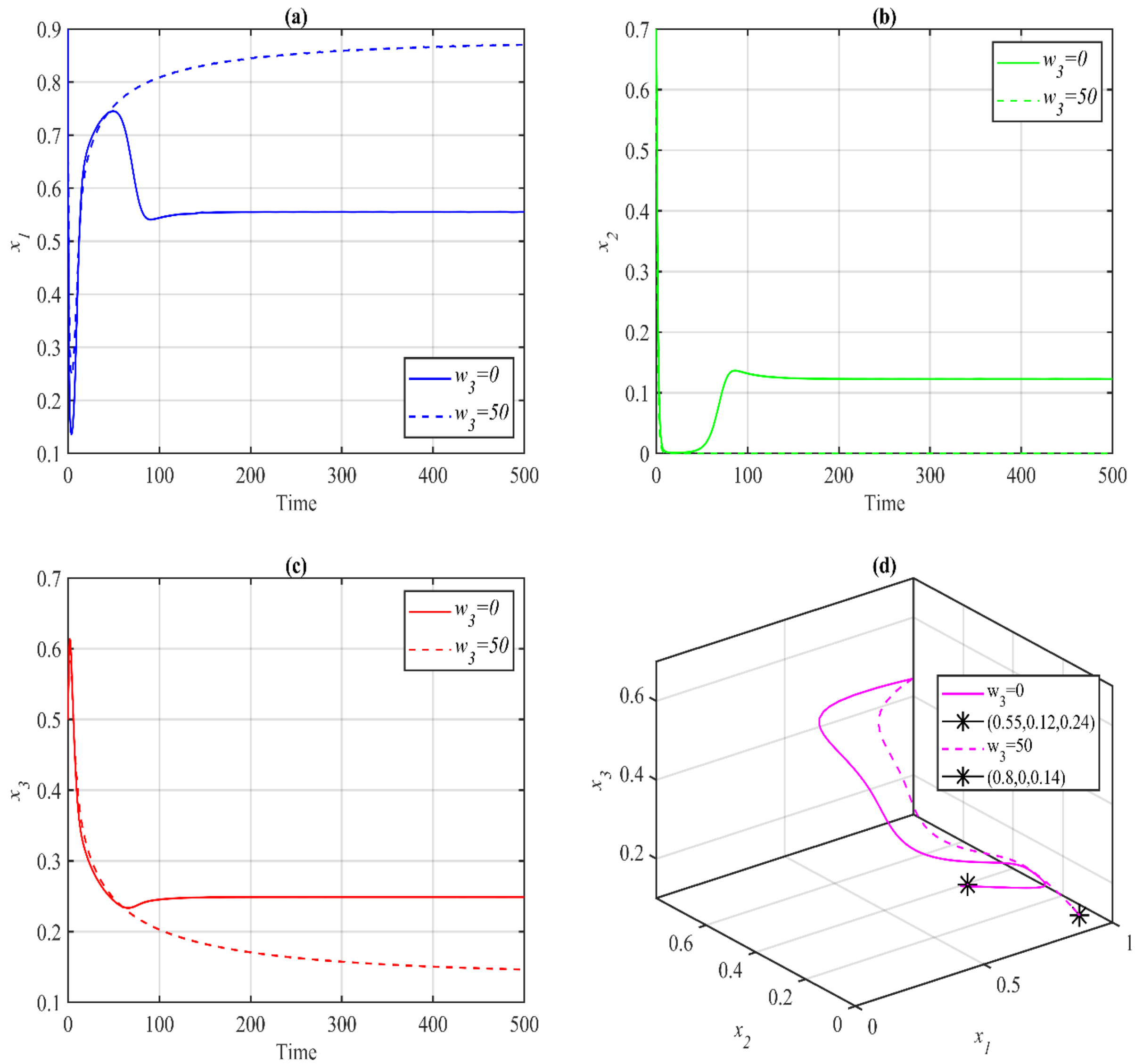


Fig. 3: The dynamics of system (2) utilizing the dataset (51) with two different values of $w_3 = 0, 50$. (a) The trajectory of system (2). (b) Time series.

According to figure (2), as w_1 increases, the population density of the species x_2 approaches zero. In contrast, figure (3) shows the approaching of the population density species x_3 to zero when the parameter w_3 increases. Now the influence of altering w_2 is explored through figure (4).

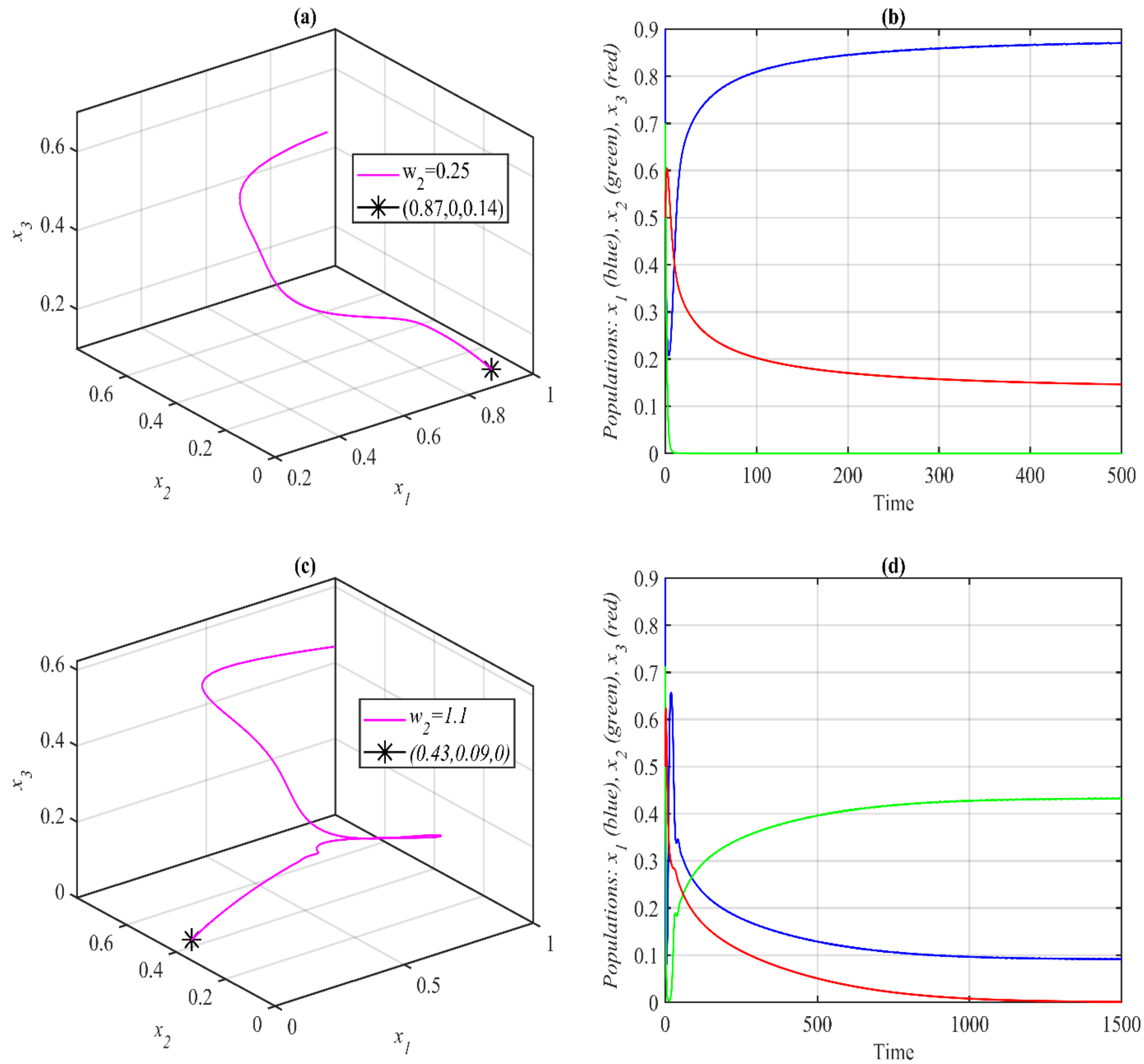


Fig. 4: The dynamics of system (2) utilizing the dataset (51) with two different values of w_2. (a) - (b) The trajectory of system (2) for $w_2 = 0.25$, and their time series. (c) - (d) The trajectory of system (2) for $w_2 = 1.1$, and their time series.

Clearly, increasing the value w_2 leads to PFEP, while decreasing it leads to DFEP. The altering of w_4 is investigated through figure (5) below.

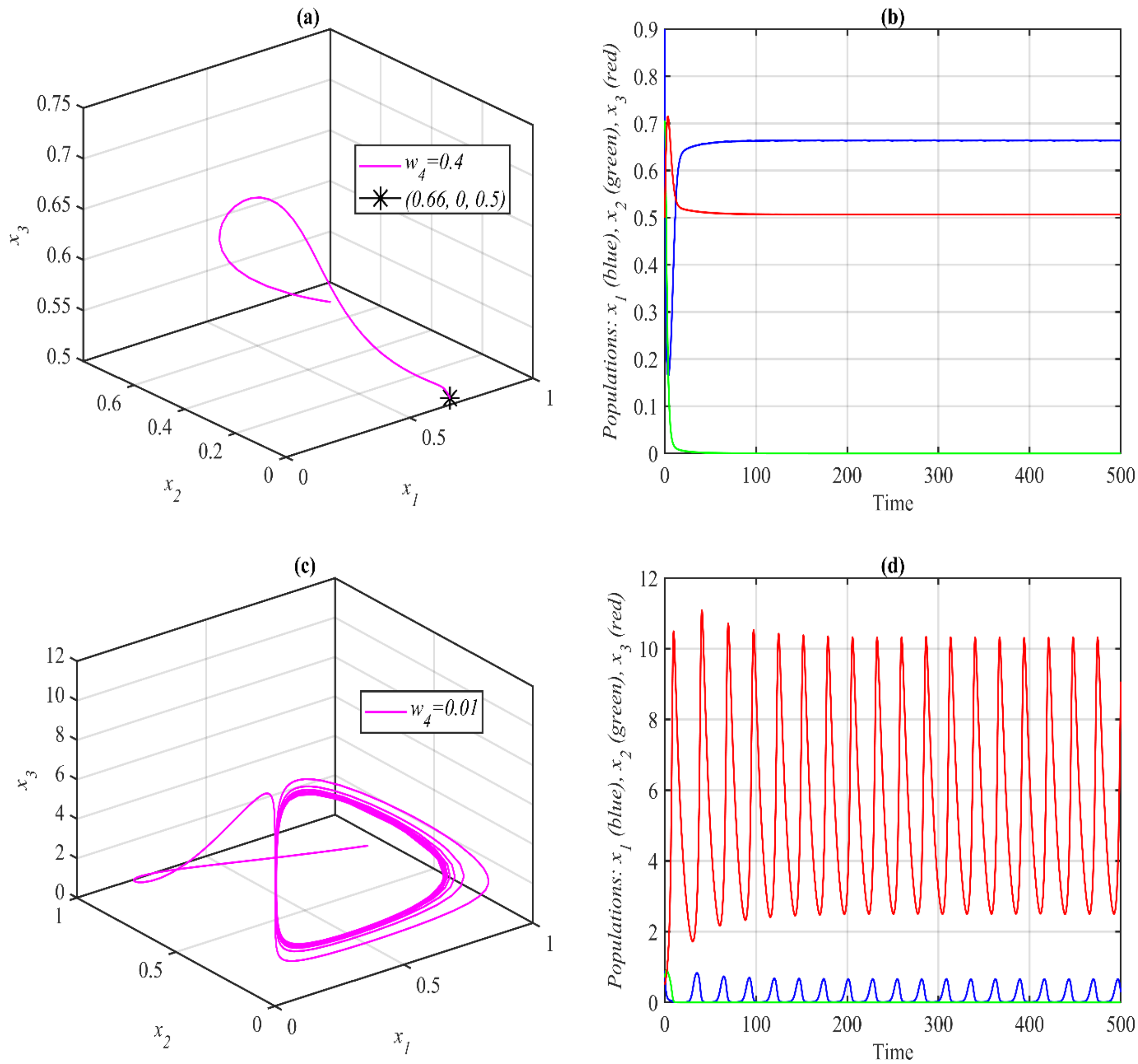


Fig. 5: The dynamics of system (2) utilizing the dataset (51) with two different values of w_4. (a) - (b) The trajectory of system (2) for $w_4 = 0.4$, and their time series. (c) - (d) The trajectory of system (2) for $w_4 = 0.01$, and their time series.

Clearly, decreasing the value w_4 leads to DFEP, while decreasing it further leads to periodic dynamics in the x_1x_2 −plane. The altering of w_5, and w_6 are investigated through figures (5) – (6) respectively.

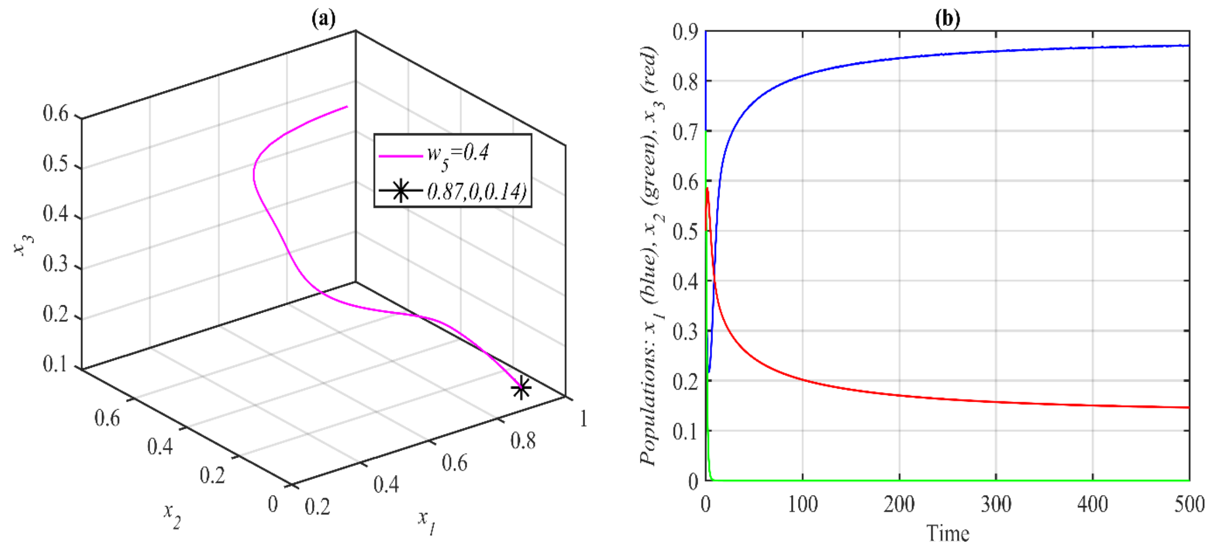
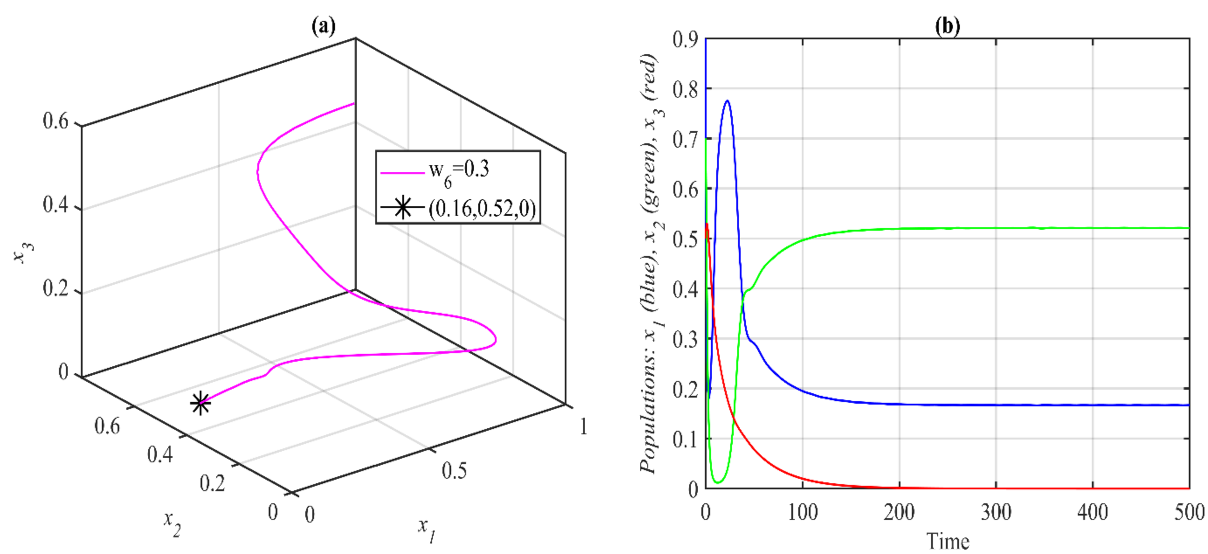


Fig. 6: The dynamic of system (2) utilizing the dataset (51) with $w_5 = 0.4$ approaches asymptotically to $p_4 = (0.87, 0, 0.14)$. (a) The trajectory of system (2). (b) Time series.



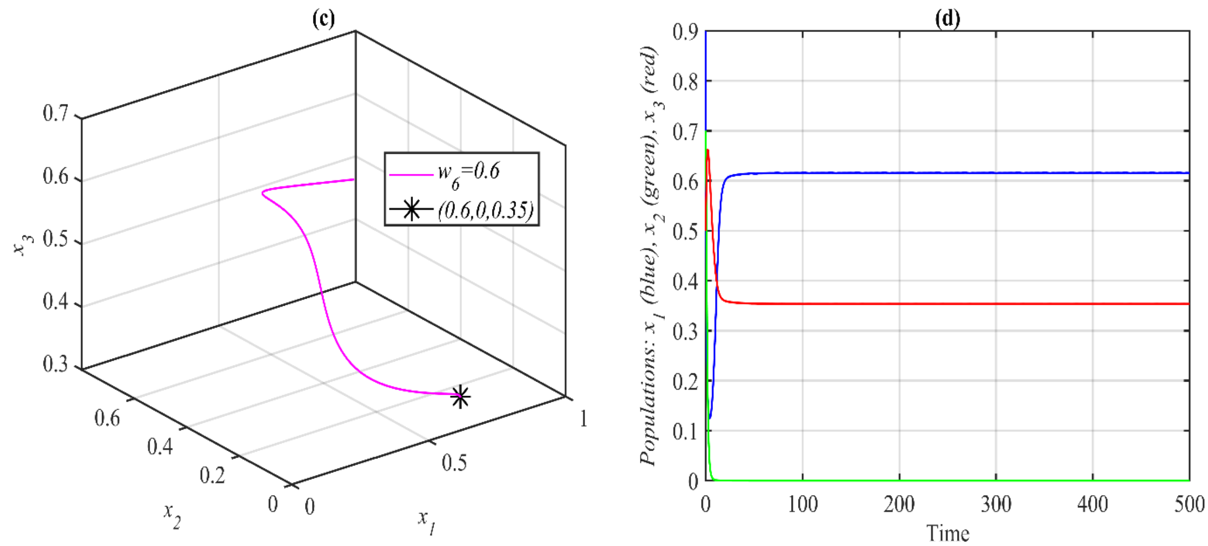


Fig. 7: The dynamic of system (2) utilizing the dataset (51) with different values of w_6 approaches asymptotically to $p_4 = (0.87, 0, 0.14)$. (a) – (b) The trajectory of system (2) for $w_6 = 0.3$, and their time series. (c) – (d) The trajectory of system (2) for $w_6 = 0.6$, and their time series.

According to figure (6), as w_5 increases, the population density of the species x_2 approaches zero. On the other hand, figure (7) shows the approaching of the population density of the species x_3 to zero when the parameter w_6 decreases, while the approaching of the population density of the species x_2 approaching to zero as w_6 increases.

It is observed that, the parameters w_7, and c_1 have similar influence as that for w_2, and w_5 respectively on the dynamics of the system (2). Finally, the influence of the parameter c_2 can be detected fron the figure (8).

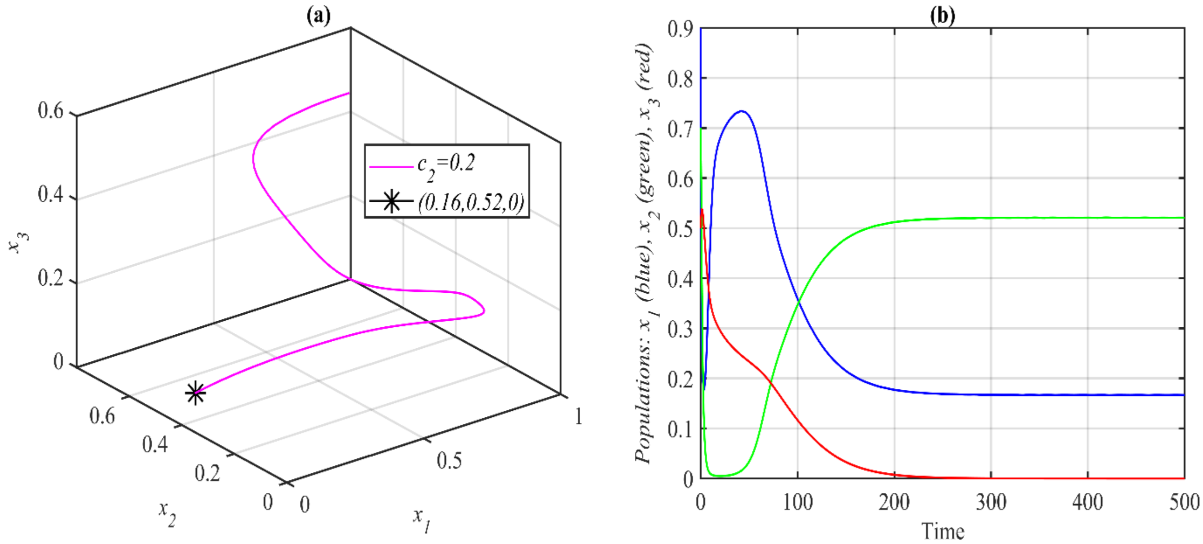


Fig. 8: The dynamic of the system (2) utilizing the dataset (51) with $c_2 = 0.2$ approaches asymptotically to $p_3 = (0.16, 0.52, 0)$. (a) The trajectory of system (2). (b) Time series.

Obviously from figure (8), as c_2 decreases, the population density of the species x_3 approaches zero.

8. Discussion and Conclusions

This paper suggested and researched the use of an eco-epidemiological system with a prey-predator and an infectious disease in the prey population. Investigated were the effects of predation-related fear and the predator's hunting cooperation. All the properties of the solution of the system (2) were studied. The local stability of all the biologically feasible equilibrium points was investigated along with their existing requirements. The persistence conditions of the system were established. The local bifurcation near the non-hyperbolic points was studied. Finally, all the analytical findings were confirmed using the numerical simulation depending on the hypothetical dataset (51), and the obtained results are summarized as follows.

Increasing the fear rate that reduces the growth of the prey (or fear rate that reduces the disease transmission) above a vital value leads to extinction in the population of the predator (infected prey) and hence the system losses the persistence. Decreasing the infection rate (or predator death rate) below a specific value or increasing the infection rate (or predator death rate) above a vital value causes extinction in the infected prey or predator population, respectively. This indicates the occurrence of a transcritical bifurcation. Reducing the attack rate ($w_4 = \frac{\alpha_1^2}{r\alpha_2}$), which represents a ratio of squared attack rate to predator cooperation in hunting rate, causes extinction in infected prey species and the system approaches to DFEP, while reducing this value further leads to losing the stability of the DFEP and the solution approaches asymptotically to periodic dynamics. Noting that, reducing the parameter w_4 is equivalent to rising the hunting cooperation rate. Moreover, it is observed that rising the value of infected prey's death rate (or the conversion efficiency from susceptible prey biomass to predator biomass) causes extinction of infected prey species and the system approaches asymptotically DFEP. Decreasing the infection rate ($w_6 = \frac{\alpha_1 k}{r}$), which represents the ratio of the product of attack rate and carrying capacity to birth rate), below a specific value, or increasing it above a vital value causes extinction in the predator population or infected prey, respectively. Finally, decreasing the conversion efficiency from infected prey biomass to predator biomass below a specific value causes the extinction of a predator species.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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