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THE DYNAMICS OF A STAGE-STRUCTURE PREY-PREDATOR MODEL WITH HUNTING COOPERATION AND ANTI-PREDATOR BEHAVIOR

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Abstract: The mathematical construction of an ecological model with a prey-predator relationship was done. It presumed that the prey consisted of a stage structure of juveniles and adults. While the adult prey species had the power to fight off the predator, the predator, and juvenile prey worked together to hunt them. Additionally, the effect of the harvest was considered on the prey. All the solution's properties were discussed. All potential equilibrium points' local stability was tested. The prerequisites for persistence were established. Global stability was investigated using Lyapunov methods. It was found that the system underwent a saddle-node bifurcation near the coexistence equilibrium point while exhibiting a transcritical bifurcation near the vanishing and predator-free equilibrium points. The analytical results are then validated using a numerical approach. It is discovered that the cooperative hunting rate and conversion rate persistently affect the system. In contrast, the anti-predator rate leads to the extinction of the predator.

Keywords: prey-predator; stage-structure; stability; anti-predator; stability; persistence; bifurcation.

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1. INTRODUCTION

Ecology is the study of the dispersion, resources, connections, and interactions of organisms with their surroundings. Ecological research aims to investigate every factor that influences how

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individual organisms interact with one another and their surroundings to produce findings supporting the ecosystem's continued sustainability. An important area of study in ecological systems has centered on the prey-predator relationship [1]. Lotka and Volterra propose and examine the earliest model of the prey-predator interaction, which includes two first-order, nonlinear differential equations. Following that, multiple attempts have been made to expand the groundbreaking work, see [2-8].

The rate at which prey is consumed by the predator population or the predator's functional response is a key element in the dynamics of prey-predator interaction. It aids in more accurate prey-predator dynamics analysis. Predation rates vary depending on several factors including age category, corpulence, environment characteristics, interference, and cooperation among members of a specific species. A linear function called the Holling type I functional response consumes the prey if it is weak, tiny, juvenile, or readily available. Functional responses come in various categories: prey-dependent, prey-predator-dependent, and ratio-dependent. In the literature, numerous investigations of prey-predator systems with various categories have been conducted, see [5-6, 9-13]. Most biological species depend on individuals' age or stage of development to determine their survival and rate of reproduction. Adult and juvenile stages of a species' life cycle can be distinguished from one another. For adult versus juvenile prey, a predator's nature is entirely different. While adult prey has greater potential to escape than juvenile prey, the predator has a strong attraction to juvenile prey. Therefore, it makes sense to incorporate the impact of a species' past life to get more accurate results. Many academics analyze stage-structured models in an effort to overcome the drawbacks of traditional Lotka-Volterra models [14-18]. Even while biologists categorize animals as either predators or prey, the ecological function of an individual is frequently unclear. There are numerous instances of predators and prey switching roles, where an adult prey attacks juvenile, weak predators, see [19] and the references therein. This suggests that young prey can grow up, escape from predators, and subsequently become a threat to weak predators. Anti-predator adaptations are biological defenses created by evolution to aid prey creatures in their ongoing conflict with predators. For every stage of this conflict, adaptations have developed throughout the animal kingdom. Recently, some researchers have proposed and studied prey-predator models with anti-predator properties; see for example [20-21].

On the other hand, some ecosystem predators engage in cooperative behavior when hunting and frightening their prey. During hunting, wolves participate as a potential keystone species, and they indirectly impact their prey [22]. Numerous studies examined the function of hunting cooperation in predator-prey systems [23–24]. To our knowledge, no research has been done on the combined impact of hunting cooperation, stage structure, and harvest in a prey-predator system. The current study aims to simultaneously examine the effects of cooperation and stage structure in a harvested prey-predator scenario.

2. Mathematical Model Construction

In this section, an ecological model consisting of a prey-predator system has been constructed mathematically. Different biological factors are included in this system according to the following assumptions.

1. The prey population is a stage structure species consisting of the juvenile prey population and the adult prey population, which are denoted to their population densities at time T by $X_1(T)$ and $X_2(T)$ so that the total prey population density is $X_1(T) + X_2(T)$. On the other hand, $X_3(T)$ is represented the predator population density at time T.
2. The prey population grows logistically in the absence of the predation process, while the predator decays exponentially in the absence of their prey.
3. The adult prey has anti-predator defensive property so that it can kill the attacked predator. However, the predator cooperates in hunting the juvenile prey according to the Lotka-Volterra type of functional response.
4. An external force harvests the prey population only.

According to the above assumptions, the dynamics of the described prey-predator system can be simulated mathematically using the following set of non-linear first-order differential equations.

$$\begin{aligned} \frac{dX_1}{dT} &= rX_2\left(1 - \frac{X_1}{K}\right) - aX_1 - (\rho + \alpha X_3)X_1X_3 - h_1X_1, \\ \frac{dX_2}{dT} &= aX_1 - d_1X_2 - h_2X_2, \\ \frac{dX_3}{dT} &= e(\rho + \alpha X_3)X_1X_3 - d_2X_3 - bX_2X_3, \end{aligned} \tag{1}$$

where $X_1(0) \geq 0, X_2(0) \geq 0$, and $X_3(0) \geq 0$ are the initial values of the populations. Moreover, all the system parameters are positive and described in Table 1.

Table 1. Description of Parameters

Parameter	Description
r	The birth rate of the juvenile prey
K	Carrying capacity of the prey environment.
a	The grown-up rate of juvenile prey to adult prey
ρ	The attack rate of the predator to the juvenile prey.
α	The hunting cooperative rate of a predator on the juvenile prey.
h_1	The harvesting rate of the juvenile prey.
h_2	The harvesting rate of adult prey.
d_1	The death rate of adult prey
d_2	The death rate of predator
e	The conversion rate of juvenile prey biomass to predator biomass.
b	The anti-predator rate by the adult prey

To simplify the study of a system (1) and reduce the number of its parameters, the following non-dimensional system is examined instead of the corresponding system (1).

$$\begin{aligned}
\frac{dy_1}{dt} &= y_2(1-y_1) - w_1y_1 - (1+w_2y_3)y_1y_3 - w_3y_1, \\
\frac{dy_2}{dt} &= w_1y_1 - w_4y_2 - w_5y_2, \\
\frac{dy_3}{dt} &= w_6(1+w_2y_3)y_1y_3 - w_7y_3 - w_8y_2y_3,
\end{aligned} \tag{2}$$

with the initial values $y_1(0) \geq 0, y_2(0) \geq 0$, and $y_3(0) \geq 0$, where the dimensionless variables and parameters are given by:

$$y_1 = \frac{X_1}{K},\ y_2 = \frac{X_2}{K},\ y_3 = \frac{X_3}{K},\ t = rT.$$

$$w_1 = \frac{a}{r},\ w_2 = \frac{r\alpha}{\rho^2},\ w_3 = \frac{h_1}{r},\ w_4 = \frac{d_1}{r},\ w_5 = \frac{h_2}{r},\ w_6 = \frac{e\rho K}{r},\ w_7 = \frac{d_2}{r},\ w_8 = \frac{bK}{r}.$$

It is obvious, the interaction functions on the right-hand side of the system (2), are continuous and have continuous partial derivatives on the domain $\mathbb{R}_+^3 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\}$. Then, they are locally Lipschitz functions in $\mathbb{R}_+^3$. Accordingly, there exists $T > 0$ so that the fundamental existence and uniqueness theorem guarantee that system (2) with any given non-negative initial condition has a unique solution defined in $\mathbb{R}_+^3$.

Now it is easy to prove that, the solution of system (2) has the following properties.

Theorem 1: All system (2) solutions with positive initial conditions are positively invariant.

Proof. From the first equation of the system (2), it is obtained:

$$y_1(t) = y_1(0)e^{\int_0^t\left[\frac{y_2(s)(1-y_1(s))}{y_1(s)} - w_1 - (1+w_2y_3(s))y_3(s) - w_3\right]ds} > 0;\ \forall\, t$$

Similarly, the second and third equations, it is obtained

$$y_2(t) = y_2(0)e^{\int_0^t\left[\frac{w_1y_1(s)}{y_2(s)} - w_4 - w_5\right]ds} > 0;\ \forall\, t$$

$$y_3(t) = y_3(0)e^{\int_0^t[w_6(1+w_2y_3(s))y_1(s) - w_7 - w_8y_2(s)]ds} > 0;\ \forall\, t$$

This completes the proof.

Theorem 2: All system (2) solutions with nonnegative initial conditions are uniformly bounded.

Proof. Assume that $N_1(t) = y_1 + y_2$ is the total prey density, then from the system (2), it is easy to verify that

$$\frac{dN_1}{dt} = y_2(1-y_1) - (1+w_2y_3)y_1y_3 - w_3y_1 - (w_4+w_5)y_2$$

Since $N_1(t) = y_1 + y_2$ represents the same species that is growing logistically in the absence of the predator, its density will be bounded by the environment carrying capacity. Hence, it is obtained that

$$\frac{dN_1}{dt} \le N_1(1-N_1) - m_1N_1 \le \frac{1}{4} - m_1N_1,$$

where $m_1 = \min\{w_3, (w_4+w_5)\}$. Thus according to the lemma (2.1) [25], it is obtained that

$$N_1(t) \le \frac{1}{4m_1}[1 + (4m_1N_1(0) - 1)e^{-m_1t}]$$

Therefore, for $t \to \infty$, it is obtained that:

$$N_1(t) \le \frac{1}{4m_1}$$

Let $N_2 = y_1 + y_2 + \frac{y_3}{w_6}$, then system (2) gives that:

$$\begin{aligned}\frac{dN_2}{dt} &= y_2(1-y_1) - w_3y_1 - (w_4+w_5)y_2 - \frac{w_7}{w_6}y_3 - \frac{w_8}{w_6}y_2y_3 \\ &\le 2y_2 - w_3y_1 - (1+w_4+w_5)y_2 - \frac{w_7}{w_6}y_3 \le 2y_2 - m_2N_2\end{aligned},$$

where $m_2 = min\left\{w_3, (1+w_4+w_5), \frac{w_7}{w_6}\right\}$. Hence, using the bound of $N_1(t) = y_1 + y_2 \le \frac{1}{4m_1}$, it is obtained:

$$\frac{dN_2}{dt} + m_2N_2 \le \frac{1}{2m_1}.$$

Then according to the lemma (2.1) [25], it is obtained that

$$N_2(t) \leq \frac{1}{2m_1m_2}[1 + (2m_1m_2N_2(0) - 1)e^{-m_2t}]$$

Therefore, for $t \to \infty$, it is obtained that:

$$N_2(t) \leq \frac{1}{2m_1m_2}.$$

That completes the proof.

3. STABILITY ANALYSIS

In this section, the existence and stability analysis of all possible equilibrium points are investigated. System (2) has at most three nonnegative equilibrium points given by:

The vanishing equilibrium point is represented by $e_0 = (0,0,0)$ always exists.

The predator-free equilibrium point is represented by $e_1 = (\bar{y}_1, \bar{y}_2, 0)$ where

$$\bar{y}_1 = \frac{w_1 - (w_1 + w_3)(w_4 + w_5)}{w_1},\ \bar{y}_2 = \frac{w_1\bar{y}_1}{w_4 + w_5}. \tag{3}$$

Clearly, e_1 exists if and only if the following condition holds.

$$(w_1 + w_3)(w_4 + w_5) < w_1. \tag{4}$$

The coexistence equilibrium point is represented by $e_2 = (y_1^*, y_2^*, y_3^*)$ where

$$y_1^* = \frac{(w_4 + w_5)w_7}{w_6(w_4 + w_5) - w_1w_8 + w_2(w_4 + w_5)w_6y_3^*},\ y_2^* = \frac{w_1y_1^*}{w_4 + w_5}\ . \tag{5}$$

While y_3^* is a positive root of the following third-order polynomial equation.

$$B_1y_3^3 + B_2y_3^2 + B_3y_3 + B_4 = 0, \tag{6}$$

where

$$B_1 = -w_2^2w_6(w_4 + w_5)^2 < 0.$$
$$B_2 = -2w_2w_6(w_4 + w_5)^2 + w_1w_2w_8(w_4 + w_5).$$
$$B_3 = w_1(w_4 + w_5)[w_2w_6 + w_8] - w_6(w_4 + w_5)^2[1 + w_1w_2 + w_2w_3].$$
$$B_4 = w_1(w_4 + w_5)[w_6 + w_3w_8 - w_7] - w_6(w_4 + w_5)^2[w_1 + w_3] \\ -w_1^2w_8[1 + w_4 + w_5]$$

Obviously, the point e_2 exists uniquely in the interior of $\mathbb{R}_+^3 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\}$ provided that the following conditions are met

$$w_1w_8 + w_2(w_4 + w_5)w_6y_3^* < w_6(w_4 + w_5). \tag{7}$$

$$\left.\begin{array}{l} w_6(w_4 + w_5)^2[w_1 + w_3] + w_1^2w_8[1 + w_4 + w_5] \\ \qquad < w_1(w_4 + w_5)[w_6 + w_3w_8 - w_7] \end{array}\right\}. \tag{8}$$

The Jacobian matrix of the system (2) at the point (y_1, y_2, y_3) can be written as:

$$J = [k_{ij}]_{\mathbf{3\times3}}, \tag{9}$$

where

$$k_{11} = -w_1 - w_3 - y_2 - y_3(1 + w_2 y_3),\ k_{12} = 1 - y_1,\ k_{13} = -y_1(1 + 2w_2 y_3),$$
$$k_{21} = w_1,\ k_{22} = -(w_4 + w_5),\ k_{23} = 0,\ k_{31} = w_6 y_3(1 + w_2 y_3),\ k_{32} = -w_8 y_3,$$
$$k_{33} = -w_7 - w_8 y_2 + w_6 y_1(1 + 2w_2 y_3).$$

Accordingly, the Jacobian matrix at the point e_0 can be written as:

$$J(e_0) = \begin{bmatrix} -(w_1 + w_3) & 1 & 0 \\ w_1 & -(w_4 + w_5) & 0 \\ 0 & 0 & -w_7 \end{bmatrix}. \tag{10}$$

The characteristic equation of $J(e_0)$ can be written as

$$[\lambda^2 + (w_1 + w_3 + w_4 + w_5)\lambda + (w_1 + w_3)(w_4 + w_5) - w_1](-w_7 - \lambda) = 0. \tag{11}$$

Direct computation shows that the eigenvalues of $J(e_0)$ are given by:

$$\begin{gathered} \lambda_{01}, \lambda_{02} = \frac{-(w_1+w_3+w_4+w_5)}{2} \pm \frac{\sqrt{(w_1+w_3+w_4+w_5)^2 - 4[(w_1+w_3)(w_4+w_5) - w_1]}}{2}. \\ \lambda_{03} = -w_7 \end{gathered} \tag{12}$$

Obviously, the Jacobian matrix $J(e_0)$ has three negative real parts eigenvalues if and only if the following condition is met.

$$w_1 < (w_1 + w_3)(w_4 + w_5). \tag{13}$$

Therefore, the point e_0 is locally asymptotically stable. However, if condition (4) holds (that is the predator-free equilibrium point exists) then the vanishing point becomes a saddle point.

The Jacobian matrix at the point e_1 can be written as:

$$J(e_1) = \begin{bmatrix} -w_1 - w_3 - \frac{\Gamma}{w_4+w_5} & 1 - \frac{\Gamma}{w_1} & -\frac{\Gamma}{w_1} \\ w_1 & -(w_4 + w_5) & 0 \\ 0 & 0 & \frac{\Gamma w_6}{w_1} - w_7 - \frac{\Gamma w_8}{w_4+w_5} \end{bmatrix} = [a_{ij}], \tag{14}$$

with $\Gamma = w_1 - (w_1 + w_3)(w_4 + w_5) > 0$ due to existence condition. The characteristic equation of $J(e_1)$ can be written as

$$[\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}](a_{33} - \lambda) = 0. \tag{15}$$

Direct computation shows that the eigenvalues of $J(e_1)$ are given by:

$$\lambda_{11}, \lambda_{12} = \frac{(a_{11}+a_{22})}{2} \pm \frac{\sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2},\ \lambda_{13} = a_{33}. \tag{16}$$

Obviously, the above eigenvalues have negative real parts provided that the following condition is met.

$$1 < \frac{\Gamma}{w_1} < \frac{1}{w_6}\left(w_7 + \frac{\Gamma w_8}{w_4+w_5}\right). \tag{17}$$

Therefore, condition (17) guarantees the local stability of the predator-free equilibrium point.

Now, the Jacobian matrix of the system (2) at the coexistence equilibrium point is computed by:

$$J(e_2) = \begin{bmatrix} -w_1 - w_3 - y_2^* - y_3^*(1 + w_2 y_3^*) & 1 - y_1^* & -y_1^*(1 + 2w_2 y_3^*) \\ w_1 & -w_4 - w_5 & 0 \\ w_6 y_3^*(1 + w_2 y_3^*) & -w_8 y_3^* & w_2 w_6 y_1^* y_3^* \end{bmatrix} = [b_{ij}]. \quad (18)$$

Then the characteristic equation of $J(e_2)$ can be written by:

$$\lambda^3 + B_1\lambda^2 + B_2 \lambda + B_3 = 0, \quad (19)$$

where

$$B_1 = -(b_{11} + b_{22} + b_{33}),$$
$$B_2 = b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} - b_{13}b_{31} + b_{22}b_{33},$$
$$B_3 = -b_{22}[b_{11}b_{33} - b_{13}b_{31}] - b_{21}[b_{13}b_{32} - b_{12}b_{33}],$$

with

$$\Delta = B_1B_2 - B_3 = -(b_{11} + b_{22})[b_{11}b_{22} - b_{12}b_{21}] - (b_{11} + b_{33})[b_{11}b_{33} - b_{13}b_{31}] - b_{22}b_{33}(2b_{11} + b_{22} + b_{33}) + b_{13}b_{21}b_{32}$$

Theorem 3. The coexistence equilibrium point e_2 is locally asymptotically stable provided that the following set of sufficient conditions are satisfied.

$$y_1^* < 1. \quad (20)$$

$$w_2w_6y_1^*y_3^* < w_1 + w_3 + y_2^* + y_3^*(1 + w_2y_3^*) < \frac{(1+2w_2y_3^*)(1+w_2y_3^*)}{w_2}. \quad (21)$$

$$\frac{w_8y_1^*y_3^*(1+2w_2y_3^*)}{w_2w_6y_1^*y_3^*} < (1 - y_1^*) < \frac{(w_1+w_3+y_2^*+y_3^*(1+w_2y_3^*))(w_4+w_5)}{w_1}. \quad (22)$$

$$b_{22}b_{33}(2b_{11} + b_{22} + b_{33}) < b_{13}b_{21}b_{32}. \quad (23)$$

Proof. According to the Routh-Hurwitz criterion, the proof follows if the following requirements $B_1 > 0$, $B_3 > 0$, and $\Delta = B_1B_2 - B_3 > 0$ are satisfied. Straightforward computation shows that, conditions (20), (21), and (22) guarantee that $B_1 > 0$, $B_3 > 0$. However, the conditions (20), (21), and (22) in addition to (23) guarantee that $\Delta = B_1B_2 - B_3 > 0$. Hence the proof is done.

4. PERSISTENCE AND GLOBAL STABILITY

In order to discuss the persistence the possibility of the existence of periodic dynamics in the boundary planes is discussed using a Bendixson criterion [26] that provides a sufficient condition for the non-existence of periodic solutions within simply connected domains in the phase plane. Since system (2) has only one subsystem falling in the y_1y_2 −plane and is defined by:

$$\begin{aligned} \frac{dy_1}{dt} &= y_2(1 - y_1) - w_1y_1 - w_3y_1 = g_1(y_1, y_2). \\ \frac{dy_2}{dt} &= w_1y_1 - w_4y_2 - w_5y_2 = g_1(y_1, y_2). \end{aligned} \quad (24)$$

Accordingly, there are no periodic dynamics in the $y_1 y_2$ −plane if and only if the following expression is not identically zero over any subdomain of $D \subseteq \mathbb{R}^2$ and does not change the sign over D

$$dif\ \mathbf{g} = \frac{\partial g_1(y_1,y_2)}{\partial y_1} + \frac{\partial g_2(y_1,y_2)}{\partial y_2}.$$

Direct computation on the system (24) shows that:

$$dif\ \mathbf{g} = -y_2 - w_1 - w_3 - w_4 - w_5.$$

Clearly, $dif\ \mathbf{g}$ is not identically zero and does not change the sign over any subdomain of $D \subseteq \mathbb{R}^2_+$. Thus the only possible attractor in the boundary planes of the system (2) is the equilibrium point E_1.

In the following theorem, the following Butler-McGhee lemma, which is stated in Freedman and Waltman [27], is used in the proof. Let $\Omega(x)$ stand for the omega limit set of an orbit, and let $\gamma(x)$ stand for the orbit via a point x. The stable and unstable manifolds of q are denoted by $\mathrm{M}_s(q)$ and $\mathrm{M}_u(q)$, respectively, if q is a hyperbolic equilibrium.

Lemma A1. Let q be an isolated hyperbolic equilibrium in the omega limit set $\Omega(x)$ of an orbit $\gamma(x)$. Then either $\Omega(x) = q$ or there exist points $p_1,\ p_2$ in $\Omega(x)$ with $p_1 \in \mathrm{M}_s(q)$ and $p_2 \in \mathrm{M}_u(q)$.

Theorem 4. The system (2) persists if the following conditions are met

$$(w_1 + w_3)(w_4 + w_5) < w_1. \tag{25}$$

$$w_7 + \frac{\Gamma w_8}{w_4 + w_5} < \frac{\Gamma w_6}{w_1}. \tag{26}$$

Proof. Assume that z belongs to the interior of $\mathbb{R}^3_+$ and $\gamma(z)$ represents the orbit through z and suppose $\Omega(z)$ be the omega limit set of the $\gamma(z)$. Recall that $\Omega(z)$ is bounded, due to the boundedness of the system (2).

To show that $e_0 \notin \Omega(z)$, consider the contrary. Since e_0 is a saddle point due to condition (25), e_0 cannot be the only point in $\Omega(z)$, and hence by Butler-McGhee lemma there is at least one other point w such that $w \in \mathrm{M}_s(e_0) \cap \Omega(z)$.

Now, since $\mathrm{M}_s(e_0)$ is the positive y_1- axis (or y_2- axis) and the entire orbit $\gamma(w)$ is contained in $\Omega(z)$. Then the positive specific axis (that containing w) is contained in $\Omega(z)$ that contradicting its boundedness.

Similarly, the proof of $e_1 \notin \Omega(z)$ under the condition (26) can be proved. Hence, as the system (2) has no periodic dynamics in the boundary plans the system (2) persists. ■

In the following, the global stability of the equilibrium points of system (2) is investigated utilizing the Lyapunov function.

Theorem 5. The vanishing equilibrium point of system (2) is globally stable provided that the following condition is satisfied.

$$1 < w_4 + w_5. \tag{28}$$

Proof. Consider the following real-valued function $L_0 = y_1 + y_2 + \frac{y_3}{w_6}$. Clearly, $L_0(0,0,0) = 0$, and $L_0(y_1, y_2, y_3) > 0$ for all $(y_1, y_2, y_3) \in \mathbb{R}^3_+$ and $(y_1, y_2, y_3) \neq (0,0,0)$. Now, differentiate L_0 with respect to t gives:

$$\begin{aligned}\frac{dL_0}{dt} &= y_2(1 - y_1) - w_3 y_1 - w_4 y_2 - w_5 y_2 - \frac{w_7}{w_6} y_3 - \frac{w_8}{w_6} y_2 y_3 \\ &\leq -w_3 y_1 + (1 - w_4 - w_5) y_2 - \frac{w_7}{w_6} y_3\end{aligned}$$

Clearly, condition (28) guarantees the negative definite of $\frac{dL_0}{dt}$. Hence E_0 is a global asymptotically stable in the $\mathbb{R}^3_+$.

Theorem 6. The predator-free equilibrium point is a globally asymptotically stable provided that the following conditions are satisfied

$$(1 + w_2 y_3)\bar{y}_1 < \frac{w_7}{w_6}. \tag{29}$$

$$k_{12}^2 < 4 k_{11} k_{22}, \tag{30}$$

where all the new symbols are given in the proof.

Proof. Consider the following real-valued function $L_1 = \left(y_1 - \bar{y}_1 - \bar{y}_1 \ln\left(\frac{y_1}{\bar{y}_1}\right)\right) + \frac{(y_2 - \bar{y}_2)^2}{2} + \frac{y_3}{w_6}$. Clearly, $L_1(\bar{y}_1, \bar{y}_2, 0) = 0$, and $L_1(y_1, y_2, y_3) > 0$ for all $(y_1, y_2, y_3) \in \mathbb{R}^3_+$ and $(y_1, y_2, y_3) \neq (\bar{y}_1, \bar{y}_2, 0)$. Now, differentiate L_1 with respect to t gives:

$$\begin{aligned}\frac{dL_1}{dt} &= -k_{11}(y_1 - \bar{y}_1)^2 + k_{12}(y_1 - \bar{y}_1)(y_2 - \bar{y}_2) - k_{22}(y_2 - \bar{y}_2)^2 \\ &\quad + (1 + w_2 y_3)\bar{y}_1 y_3 - \frac{w_7}{w_6} y_3 - \frac{w_8}{w_6} y_2 y_3,\end{aligned}$$

where $k_{11} = \frac{[(w_1 + w_3) + y_2]}{y_1}$, $k_{12} = \left[w_1 + \frac{(1 - \bar{y}_2)}{y_1}\right]$, and $k_{22} = (w_4 + w_5)$.

Therefore, using the above-given conditions gives that

$$\frac{dL_1}{dt} \leq -\left[\sqrt{k_{11}}(y_1 - \bar{y}_1) - \sqrt{k_{22}}(y_2 - \bar{y}_2)\right]^2 - \left[\frac{w_7}{w_6} - (1 + w_2 y_3)\bar{y}_1\right] y_3.$$

Clearly, $\frac{dL_1}{dt}$ is a negative definite and hence the predator-free equilibrium point is an asymptotically stable point. Since, the Lyapunov function L_1 is a radially unbounded function in the $\mathbb{R}^3_+$, hence it's a globally asymptotically stable point.

Theorem 7. The coexistence equilibrium point e_2 has a basin of attraction that satisfies the following conditions.

$$\left(w_1 + \frac{(1-y_1)}{y_1}\right)^2 < (w_4 + w_5)\frac{y_2^*}{y_1^* y_1}. \tag{31}$$

$$(w_2 y_3)^2 < \frac{y_2^*}{y_1^* y_1} \ . \tag{32}$$

$$\left(\frac{w_8}{w_6}\right)^2 < (w_4 + w_5). \tag{33}$$

$$(1 + w_2 y_1)(y_3 - y_3^*)^2 < M, \tag{34}$$

where

$$M = \frac{1}{2}\left[\sqrt{\frac{y_2^*}{y_1^* y_1}}\,(y_1 - y_1^*) - \sqrt{(w_4 + w_5)}(y_2 - y_2^*)\right]^2 + \frac{1}{2}\left[\sqrt{\frac{y_2^*}{y_1^* y_1}}\,(y_1 - y_1^*) - (y_3 - y_3^*)\right]^2 + \frac{1}{2}\left[\sqrt{(w_4 + w_5)}(y_2 - y_2^*) - (y_3 - y_3^*)\right]^2.$$

Proof. Consider the following real-valued function $L_2 = \left(y_1 - y_1^* - y_1^* \ln\left(\frac{y_1}{y_1^*}\right)\right) + \frac{(y_2 - y_2^*)^2}{2} + \frac{1}{w_6}\left(y_3 - y_3^* - y_3^* \ln\left(\frac{y_3}{y_3^*}\right)\right)$. Clearly, $L_2(y_1^*, y_2^*, y_3^*) = 0$, and $L_1(y_1, y_2, y_3) > 0$ for all $(y_1, y_2, y_3) \in \mathbb{R}_+^3$ and $(y_1, y_2, y_3) \neq (y_1^*, y_2^*, y_3^*)$. Now, differentiate L_2 with respect to t gives after some algebraic manipulations:

$$\begin{aligned} \frac{dL_2}{dt} = &-\frac{y_2^*}{y_1^* y_1}(y_1 - y_1^*)^2 + \left(w_1 + \frac{(1-y_1)}{y_1}\right)(y_1 - y_1^*)(y_2 - y_2^*) \\ &-(w_4 + w_5)(y_2 - y_2^*)^2 - w_2 y_3 (y_1 - y_1^*)(y_3 - y_3^*) - (y_3 - y_3^*)^2 \\ &-\frac{w_8}{w_6}(y_2 - y_2^*)(y_3 - y_3^*) + (1 + w_2 y_1)(y_3 - y_3^*)^2 \end{aligned}.$$

Therefore, by using the conditions (31)-(33), it is obtained that

$$\begin{aligned} \frac{dL_2}{dt} < &-\frac{1}{2}\left[\sqrt{\frac{y_2^*}{y_1^* y_1}}\,(y_1 - y_1^*) - \sqrt{(w_4 + w_5)}(y_2 - y_2^*)\right]^2 \\ &-\frac{1}{2}\left[\sqrt{\frac{y_2^*}{y_1^* y_1}}\,(y_1 - y_1^*) - (y_3 - y_3^*)\right]^2 + (1 + w_2 y_1)(y_3 - y_3^*)^2 \\ &-\frac{1}{2}\left[\sqrt{(w_4 + w_5)}(y_2 - y_2^*) - (y_3 - y_3^*)\right]^2 \end{aligned}.$$

Clearly, $\frac{dL_2}{dt}$ is a negative definite under the condition (34) and hence the coexistence equilibrium point is asymptotically stable point and has a basin of attraction satisfies the given conditions.

5. Bifurcation Analysis

This section examines the potential that altering a parameter could lead to a change in quality.

Write the system (2) in the vector norm shown below.

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}, \theta),\ \ \mathbf{Y} = (y_1, y_2, y_3),\ \ \theta \in \mathbb{R}_+. \tag{35}$$

Here $\mathbf{F}$ is the interaction functions vector of system (2). The second and third directional derivative of $\mathbf{F}(\mathbf{Y}, \theta)$ can be written:

$$D^2\mathbf{F}(\mathbf{Y}, \theta) = \begin{pmatrix} -2(v_3^2 w_2 y_1 + v_1[v_2 + v_3(1 + 2w_2 y_3)]) \\ 0 \\ 2v_3(-v_2 w_8 + w_6[v_3 w_2 y_1 + v_1(1 + 2w_2 y_3)]) \end{pmatrix}, \tag{36}$$

and

$$D^3\mathbf{F}(\mathbf{Y}, \theta) = \begin{pmatrix} -6v_1 v_3^2 w_2 \\ 0 \\ 6v_1 v_3^2 w_2 w_6 \end{pmatrix}, \tag{37}$$

with $\mathbf{V} = (v_1, v_2, v_3)$ be any vector.

Theorem 8: Assume that the parameter w_1 fulfills $w_1 = (w_1 + w_3)(w_4 + w_5) \equiv (w_1^*)$, then at the vanishing equilibrium point, the system (2) undergoes a transcritical bifurcation.

Proof. For $w_1^* = (w_1^* + w_3)(w_4 + w_5)$ the Jacobian matrix becomes

$$J_0 = J(e_0, w_1^*) = \begin{bmatrix} -(w_1^* + w_3) & 1 & 0 \\ w_1^* & -(w_4 + w_5) & 0 \\ 0 & 0 & -w_7 \end{bmatrix}.$$

Note that, J_0 has the eigenvalues $\lambda_{01} = 0$, $\lambda_{02} = -(w_1^* + w_3 + w_4 + w_5)$, and $\lambda_{03} = -w_7$. Therefore, the eigenvectors of J_0 and $J_0{}^{\mathrm{T}}$ corresponding $\lambda_{01} = 0$ can be determined as $\mathbf{V}_0 = (v_{01}, v_{02}, v_{03})^{\mathrm{T}}$ and $\mathbf{U}_0 = (u_{01}, u_{02}, u_{03})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_0 = \left(\frac{1}{(w_1^* + w_3)}, 1, 0\right)^{\mathrm{T}},\ \ \mathbf{U}_0 = \left(\frac{w_1^*}{(w_1^* + w_3)}, 1, 0\right)^{\mathrm{T}}.$$

In addition, equation (36) yields the result that

$$\mathbf{F}_{w_1} = \begin{pmatrix} -y_1 \\ y_1 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_{w_1}(e_0, w_1^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{U}_0{}^{\mathrm{T}}\mathbf{F}_{w_1}(e_0, w_1^*) = 0$$

$$D\mathbf{F}_{w_1}(e_0, w_1^*).\mathbf{V}_0 = \begin{pmatrix} -\frac{1}{(w_1^* + w_3)} \\ \frac{1}{(w_1^* + w_3)} \\ 0 \end{pmatrix} \Rightarrow \mathbf{U}_0{}^{\mathrm{T}}\left[D\mathbf{F}_{w_1}(e_0, w_1^*).\mathbf{V}_0\right] = \frac{w_3}{(w_1^* + w_3)^2}$$

$$D^2\mathbf{F}(e_0, w_1^*)(\mathbf{V}_0, \mathbf{V}_0) = \begin{pmatrix} -\frac{2}{(w_1^* + w_3)} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{U}_0{}^{\mathrm{T}}[D^2\mathbf{F}(e_0, w_1^*)(\mathbf{V}_0, \mathbf{V}_0)] = -2\frac{w_1^*}{(w_1^* + w_3)^2}$$

The Sotomayor theorem [26] causes the system (2) to face a transcritical bifurcation at the equilibrium point e_0 as the parameter w_1 swings through w_1^*.

Theorem 9: Assume that the parameter w_7 fulfills $w_7 = \frac{\Gamma w_6}{w_1} - \frac{\Gamma w_8}{w_4+w_5} \equiv (w_7^*)$, then at the predator-free equilibrium point, the system (2) undergoes a transcritical bifurcation provided that

$$1 < \frac{\Gamma}{w_1}, \tag{38}$$

$$-\delta_2 w_8 + w_2 w_6 \bar{y}_1 + w_6 \delta_1 \neq 0, \tag{39}$$

where $\Gamma = w_1 - (w_1 + w_3)(w_4 + w_5)$ that is positive due to existence condition. Furthermore, if we reflect condition (39) the system (2) undergoes a pitchfork bifurcation.

Proof. For $w_7^* = \frac{\Gamma w_6}{w_1} - \frac{\Gamma w_8}{w_4+w_5}$ the Jacobian matrix becomes

$$J_1 = J(e_1, w_7^*) = \begin{bmatrix} -w_1 - w_3 - \frac{\Gamma}{w_4+w_5} & 1 - \frac{\Gamma}{w_1} & -\frac{\Gamma}{w_1} \\ w_1 & -(w_4 + w_5) & 0 \\ 0 & 0 & 0 \end{bmatrix} = [a_{ij}].$$

Note that, J_1 has two eigenvalues that are given in equation (16) with negative real parts due to condition (37), and $\lambda_{13}(w_7^*) = 0$. Thus, the eigenvectors of J_1 and ${J_1}^{\mathrm{T}}$ corresponding $\lambda_{13}(w_7^*) = 0$ can be determined as $\mathbf{V}_1 = (v_{11}, v_{12}, v_{13})^{\mathrm{T}}$ and $\mathbf{U}_1 = (u_{11}, u_{12}, u_{13})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_1 = \left(-\frac{a_{13}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \frac{a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, 1\right)^{\mathrm{T}} = (\delta_1, \delta_2, 1)^{\mathrm{T}},\ \mathbf{U}_1 = (0,0,1)^{\mathrm{T}}.$$

Obviously, δ_1 and δ_2 are negative. In addition, equation (36) yields the result that

$$\mathbf{F}_{w_7} = \begin{pmatrix} 0 \\ 0 \\ -y_3 \end{pmatrix} \Rightarrow \mathbf{F}_{w_7}(e_1, w_7^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow {\mathbf{U}_1}^{\mathrm{T}} \mathbf{F}_{w_7}(e_1, w_7^*) = 0$$

$$D\mathbf{F}_{w_7}(e_1, w_7^*).\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \Rightarrow {\mathbf{U}_1}^{\mathrm{T}}\left[D\mathbf{F}_{w_7}(e_1, w_7^*).\mathbf{V}_1\right] = -1$$

$$D^2\mathbf{F}(e_1, w_7^*)(\mathbf{V}_1, \mathbf{V}_1) = \begin{pmatrix} -2(w_2\bar{y}_1 + \delta_1\delta_2 + \delta_1) \\ 0 \\ 2(-\delta_2 w_8 + w_2 w_6 \bar{y}_1 + w_6\delta_1) \end{pmatrix}$$

Therefore,

$${\mathbf{U}_1}^{\mathrm{T}}[D^2\mathbf{F}(e_1, w_7^*)(\mathbf{V}_1, \mathbf{V}_1)] = 2(-\delta_2 w_8 + w_2 w_6 \bar{y}_1 + w_6 \delta_1)$$

Therefore condition (39) guarantees that system (2) faces a transcritical bifurcation at the equilibrium point e_1 as the parameter w_7 swings through w_7^*.

On the other hand, if we reflect condition (39), then using equation (37) gives that

$$D^3\mathbf{F}(e_1, w_7^*)(\mathbf{V}_1, \mathbf{V}_1) = \begin{pmatrix} -6\delta_1 w_2 \\ 0 \\ 6\delta_1 w_2 w_6 \end{pmatrix}.$$

Hence, it is obtained that $\mathbf{U}_1{}^{\mathrm{T}}[D^3\mathbf{F}(e_1,w_7^*)(\mathbf{V}_1,\mathbf{V}_1)] = 6\delta_1 w_2 w_6 < 0$.

Therefore, pitchfork bifurcation takes place and the proof is done.

Theorem 10: Assume that conditions (20) and (21) are met. Then if the parameter w_5 fulfills $w_5 = \frac{b_{21}[b_{13}b_{32}-b_{12}b_{33}]}{[b_{11}b_{33}-b_{13}b_{31}]} - w_4 \equiv (w_5^*)$, at the coexistence equilibrium point, the system (2) undergoes a saddle-node bifurcation provided that

$$(1-y_1^*) < \min\left\{\frac{(w_1+w_3+y_2^*+y_3^*(1+w_2y_3^*)-w_2w_6y_1^*y_3^*)(w_4+w_5^*)}{w_1}, \frac{w_8y_1^*y_3^*(1+2w_2y_3^*)}{w_2w_6y_1^*y_3^*}\right\}. \tag{40}$$

$$2(-\delta_3\delta_4w_8 + w_2w_6y_1^* + \delta_3w_6(1+2w_2y_3^*)) \neq 2\delta_5\big(w_2y_1^* + \delta_3\delta_4 + \delta_3(1+2w_2y_3^*)\big). \tag{41}$$

Proof. For $w_5^* = \frac{b_{21}[b_{13}b_{32}-b_{12}b_{33}]}{[b_{11}b_{33}-b_{13}b_{31}]} - w_4$ the Jacobian matrix becomes

$$J_2 = J(e_2,w_5^*) = \begin{bmatrix} -w_1-w_3-y_2^*-y_3^*(1+w_2y_3^*) & 1-y_1^* & -y_1^*(1+2w_2y_3^*) \\ w_1 & -w_4-w_5^* & 0 \\ w_6y_3^*(1+w_2y_3^*) & -w_8y_3^* & w_2w_6y_1^*y_3^* \end{bmatrix} = [b_{ij}].$$

Here, $b_{22} = -w_4 - w_5^* \equiv (b_{22}^*)$. Note that, J_2 has two eigenvalues that are given in equation (16) with negative real parts due to conditions (20), (21), and (40); while the third eigenvalue is given by $\lambda_{23}(w_5^*) = 0$. Thus, the eigenvectors of J_2 and $J_2{}^{\mathrm{T}}$ corresponding $\lambda_{23}(w_5^*) = 0$ can be determined as $\mathbf{V}_2 = (v_{21},v_{22},v_{23})^{\mathrm{T}}$ and $\mathbf{U}_2 = (u_{21},u_{22},u_{23})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_2 = \left(-\frac{b_{13}b_{22}^*}{b_{11}b_{22}^*-b_{12}b_{21}}, \frac{b_{13}b_{21}}{b_{11}b_{22}^*-b_{12}b_{21}}, 1\right)^{\mathrm{T}} = (\delta_3,\delta_4,1)^{\mathrm{T}},$$

$$\mathbf{U}_2 = \left(\frac{b_{21}b_{32}-b_{22}b_{31}}{b_{11}b_{22}^*-b_{12}b_{21}}, -\frac{b_{11}b_{32}-b_{12}b_{31}}{b_{11}b_{22}^*-b_{12}b_{21}}, 1\right)^{\mathrm{T}} = (\delta_5,\delta_6,1)^{\mathrm{T}}.$$

Obviously, δ_3 and δ_4 are negative. In addition, equation (36) yields the result that

$$\mathbf{F}_{w_5} = \begin{pmatrix} 0 \\ -y_2 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_{w_5}(e_2,w_5^*) = \begin{pmatrix} 0 \\ -y_2^* \\ 0 \end{pmatrix} \Rightarrow \mathbf{U}_2{}^{\mathrm{T}}\mathbf{F}_{w_5}(e_2,w_5^*) = -\delta_6 y_2^*$$

$$D^2\mathbf{F}(e_2,w_5^*)(\mathbf{V}_2,\mathbf{V}_2) = \begin{pmatrix} -2(w_2y_1^* + \delta_3\delta_4 + \delta_3(1+2w_2y_3^*)) \\ 0 \\ 2(-\delta_3\delta_4w_8 + w_2w_6y_1^* + \delta_3w_6(1+2w_2y_3^*)) \end{pmatrix}$$

Therefore,

$$\begin{aligned}\mathbf{U}_2{}^{\mathrm{T}}[D^2\mathbf{F}(e_2,w_5^*)(\mathbf{V}_2,\mathbf{V}_2)] = {}& -2\delta_5\big(w_2y_1^* + \delta_3\delta_4 + \delta_3(1+2w_2y_3^*)\big) \\ & +2(-\delta_3\delta_4w_8 + w_2w_6y_1^* + \delta_3w_6(1+2w_2y_3^*)).\end{aligned}$$

Therefore condition (41) guarantees that system (2) faces a saddle-node bifurcation at the equilibrium point e_2 as the parameter w_5 swings through w_5^*.

6. NUMERICAL SIMULATION

To verify our theoretical findings and comprehend the impact of changing the parameter values on the system's dynamics, we run some numerical simulations of the system (2). The following fictitious parameters are used for the simulation that follows.

$$w_1 = 0.3, w_2 = 0.5, w_3 = 0.2, w_4 = 0.1, w_5 = 0.2, w_6 = 0.5, w_7 = 0.1, w_8 = 0.2. \quad (42)$$

It is observed that the system (2) has a globally asymptotically stable coexistence equilibrium point $e_2 = (0.28, 0.28, 0.19)$ using the data set (42) as shown in figure (1) below.

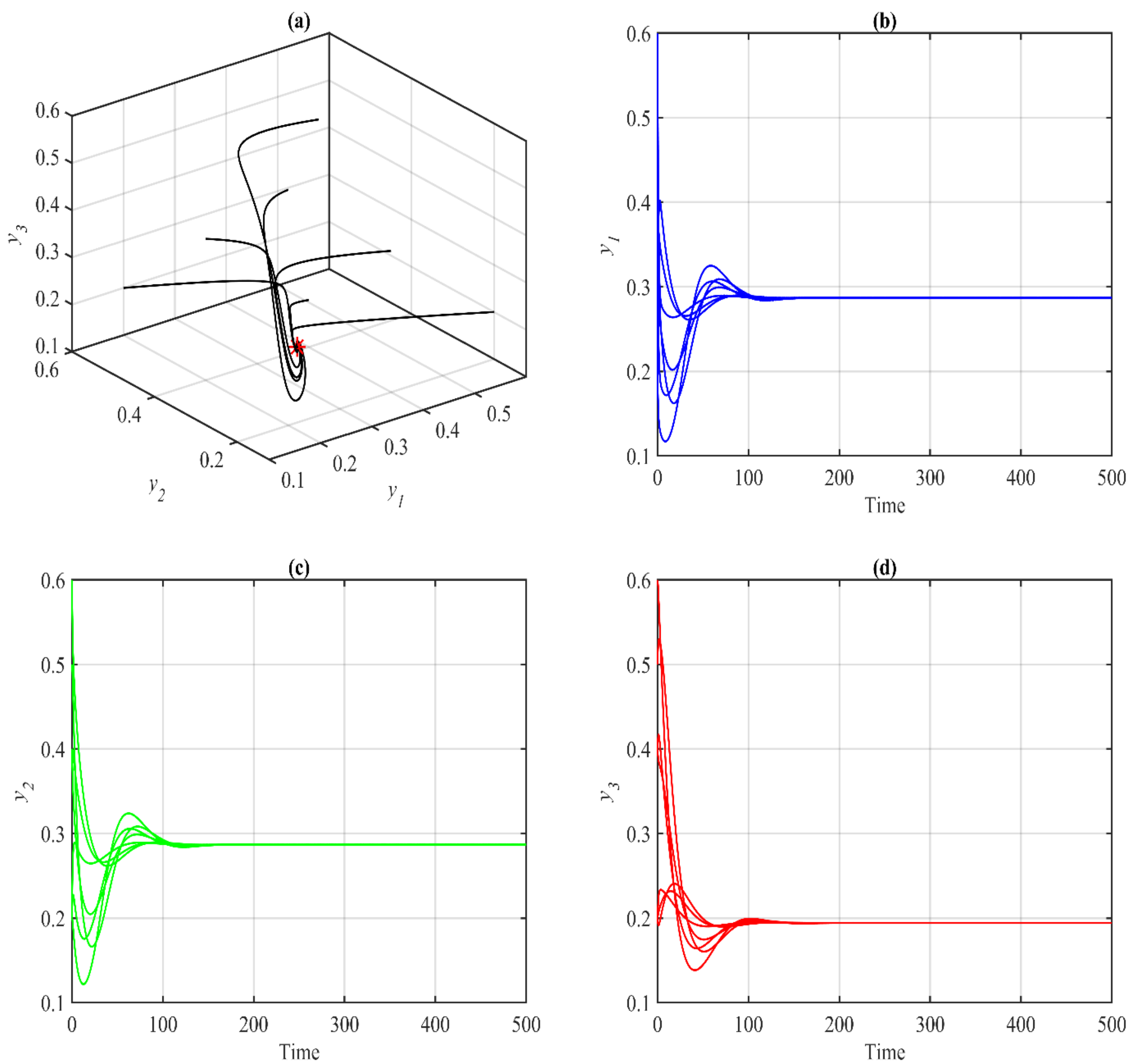

Fig. 1: The trajectories of system (2) using the set of data (42) and starting from multi initial points. (a) 3D phase plot converges to $e_2 = (0.28, 0.28, 0.19)$. (b) Trajectories of y_1 versus time. (c) Trajectories of y_2 versus time. (d) Trajectories of y_3 versus time.

Now, the influence of varying the value of the w_1 is studied. It is noted that for the ranges $(0,0.085]$, $(0.085,0.135]$, $(0.135,0.5)$, and $w_1 \geq 0.5$ the solution approaches asymptotically to e_0, e_1, e_2, and e_1 respectively, see figure (2) for the selected values.

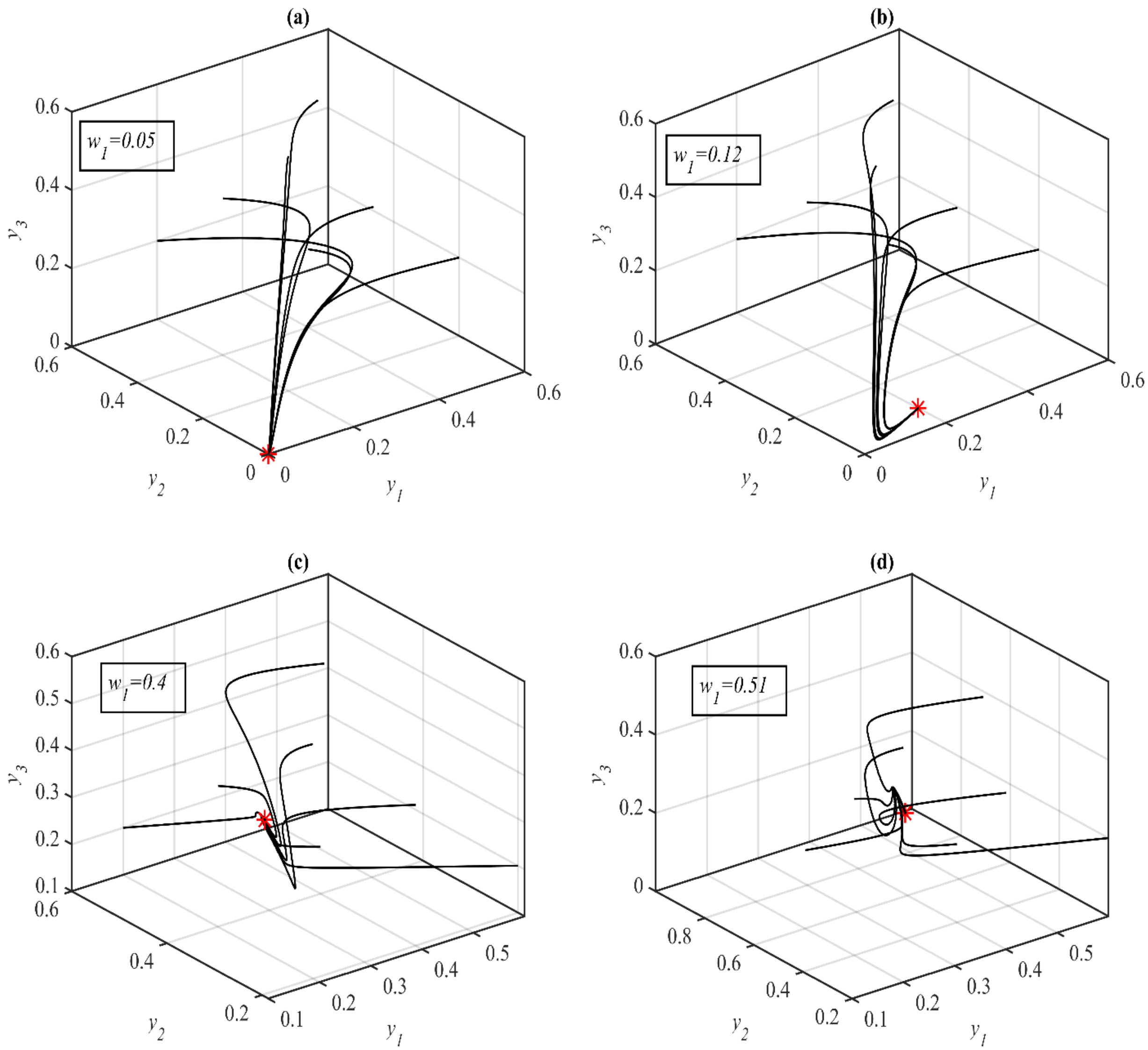

Fig. 2: The trajectories of system (2) using the set of data (42) with different values of w_1 and starting from multi initial points. (a) 3D phase plot converges to e_0 when $w_1 = 0.05$. (b) 3D phase plot converges to $e_1 = (0.2,0.08,0)$ when $w_1 = 0.12$. (c) 3D phase plot converges to $e_2 = (0.33,0.45,0.25)$ when $w_1 = 0.4$. (d) 3D phase plot converges to $e_1 = (0.58,0.99,0)$ when $w_1 = 0.51$.

Moreover, figure (3) explains the influence of w_2 on the system's (2) dynamics at a selected values.

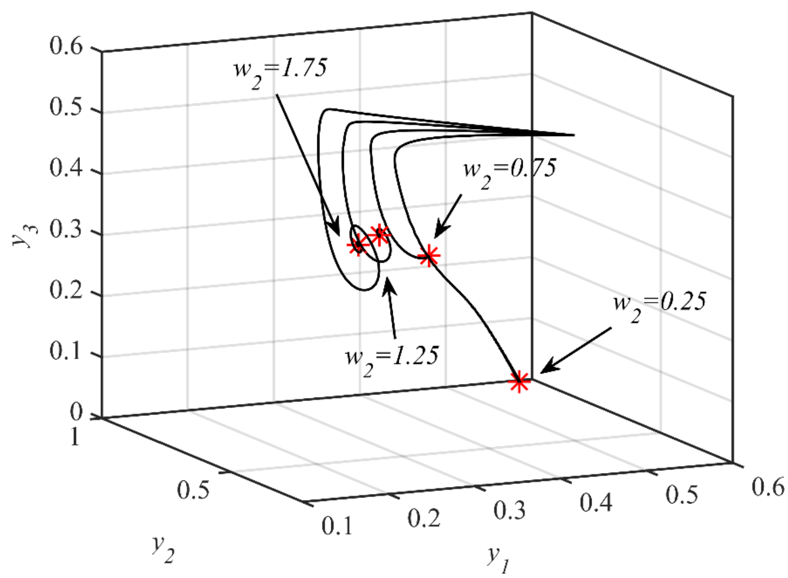

Fig. 3: The trajectories of system (2) using the set of data (42) with $w_1 = 0.51$ and different values of w_2. The system's solution approaches $e_1 = (0.58,0.99,0)$, $e_2 = (0.34,0.56,0.3)$, $e_2 = (0.25,0.44,0.38)$, and $e_2 = (0.19,0.33,0.38)$ for $w_2 = 0.25$, $w_2 = 0.75$, $w_2 = 1.25$, and $w_2 = 1.75$ respectively.

According to figure (3) as the value of w_2 increases, the predator population increases while that of prey (susceptible and infected) decreases. Figure (4) shows the influence of varying the value of w_3 at some selected values while the rest of parameters fixed at those values given in equation (42).

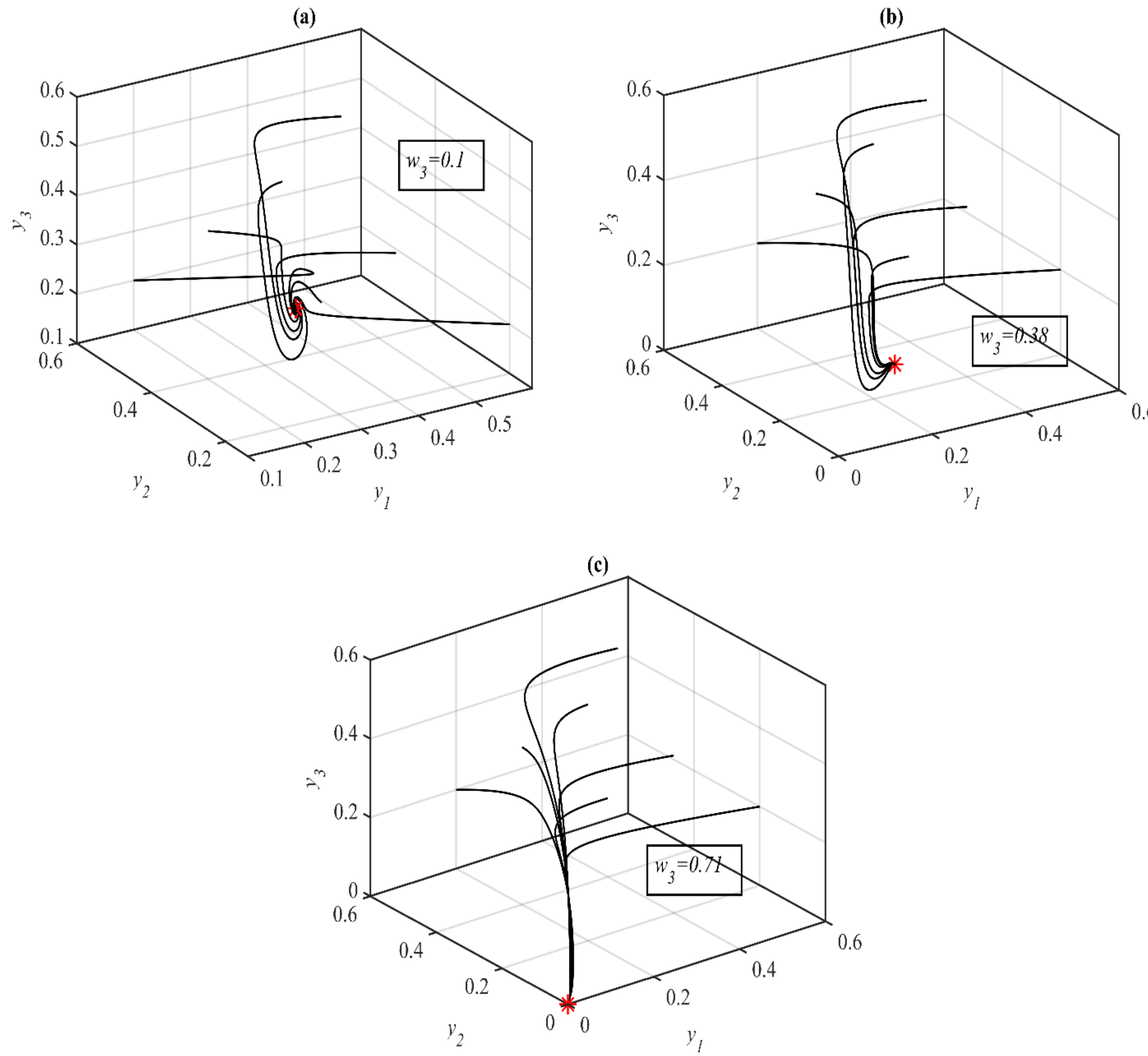

Fig. 4: The trajectories of system (2) using the set of data (42) with different values of w_3 and starting from multi initial points. (a) 3D phase plot converges to $e_2 = (0.26,0.26,0.28)$ when $w_3 = 0.1$. (b) 3D phase plot converges to $e_1 = (0.32,0.32,0)$ when $w_3 = 0.38$. (c) 3D phase plot converges to e_0 when $w_3 = 0.71$.

The system's solution is seen to approach e_2, e_1, and e_0 for w_3 that belongs to $(0,0.37)$, $[0.37,0.7]$, and $(0.7,1]$, respectively. When w_4 grows, the influence on the system's behavior (2) is similar to what is seen when w_3 is varied. The effects of changing the parameters w_5, w_6, and w_7, respectively, at chosen values are now described in figures (5), (6), and (7).

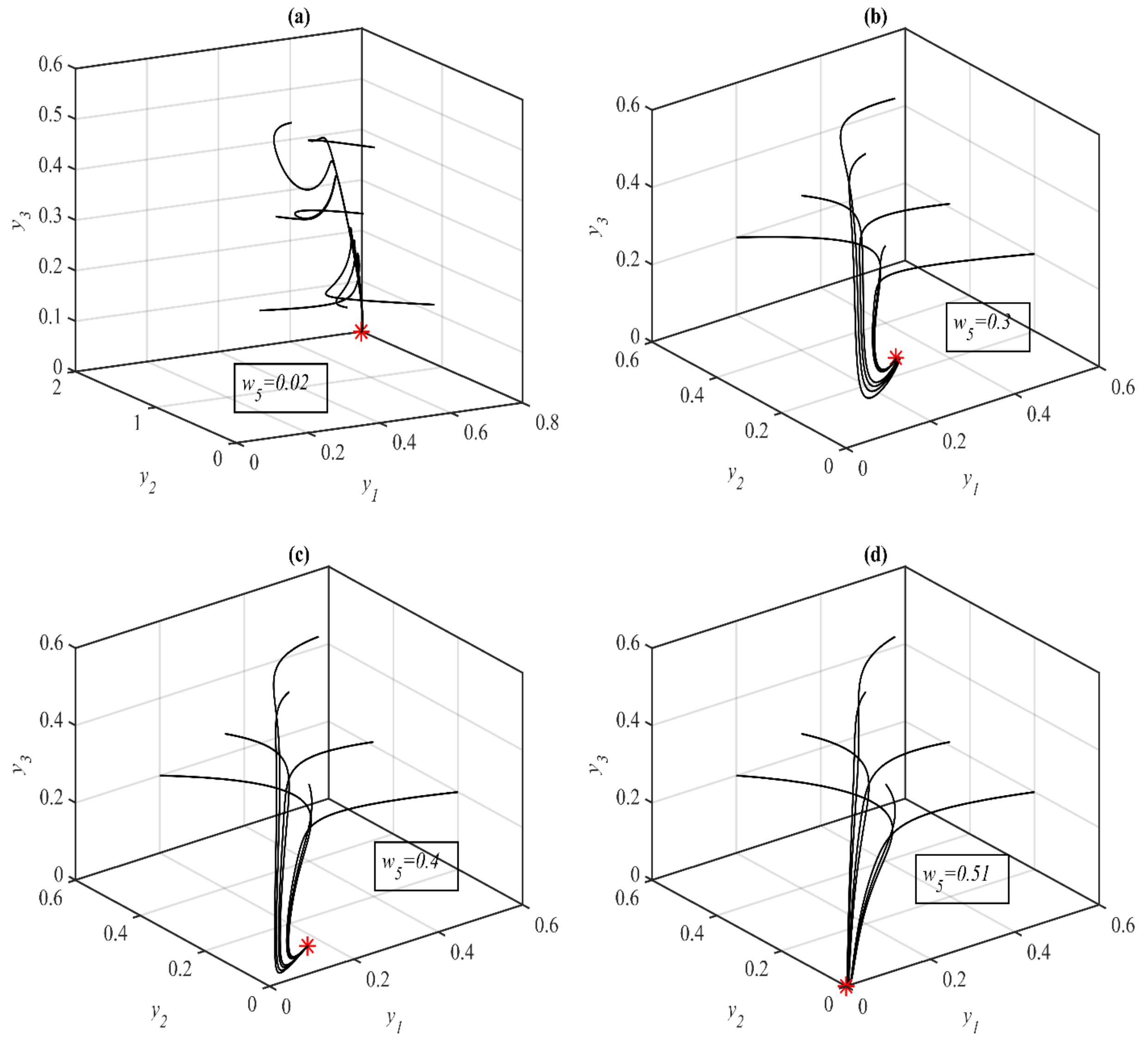

Fig. 5: The trajectories of system (2) using the set of data (42) with different values of w_5 and starting from multi initial points. (a) 3D phase plot converges to $e_1 = (0.8,0.2,0)$ when $w_5 = 0.02$. (b) 3D phase plot converges to $e_2 = (0.27,0.2,0.04)$ when $w_5 = 0.3$. (c) 3D phase plot converges to $e_1 = (0.16,0.1,0)$ when $w_5 = 0.4$. (d) 3D phase plot converges to e_0 when $w_5 = 0.51$.

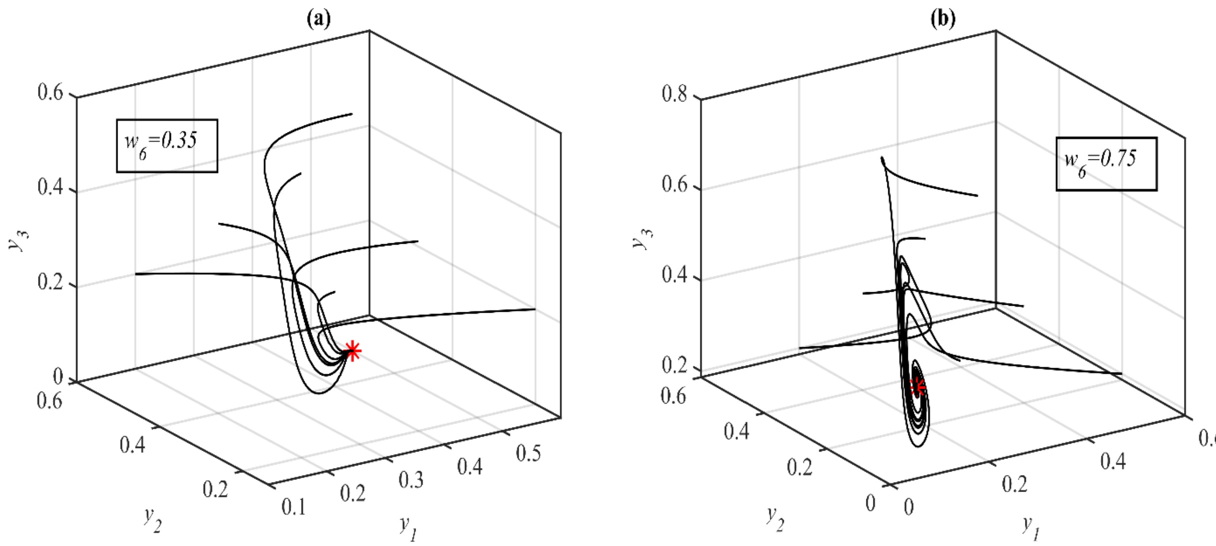

Fig. 6: The trajectories of system (2) using the set of data (42) with different values of w_6 and starting from multi initial points. (a) 3D phase plot converges to $e_1 = (0.5,0.5,0)$ when $w_6 = 0.35$. (b) 3D phase plot converges to $e_2 = (0.15,0.15,0.3)$ when $w_6 = 0.12$.

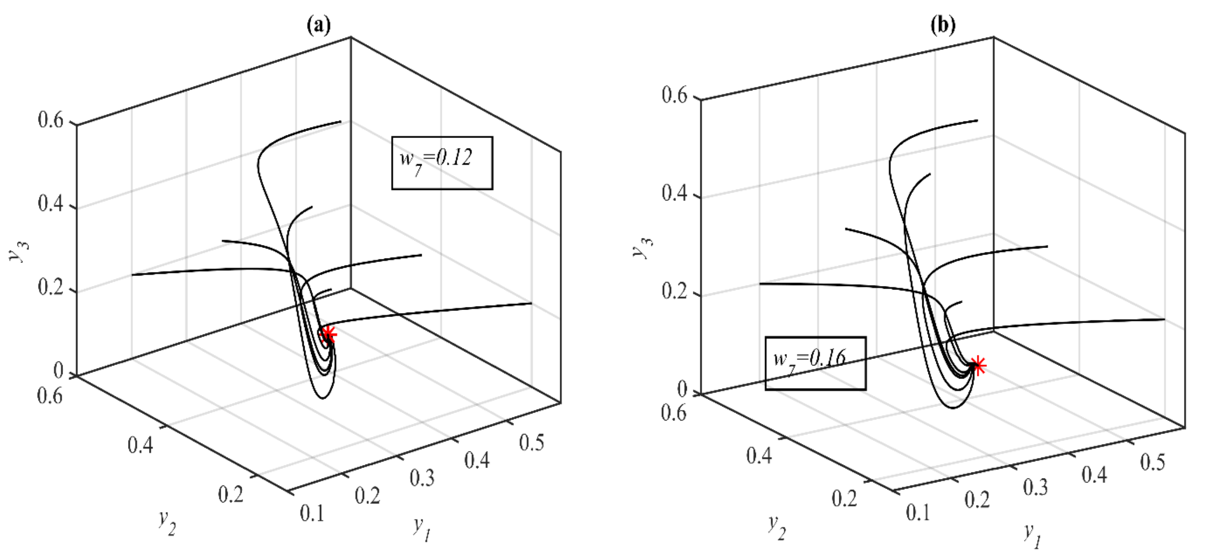

Fig. 7: The trajectories of system (2) using the set of data (42) with different values of w_7 and starting from multi initial points. (a) 3D phase plot converges to $e_2 = (0.36,0.36,0.13)$ when $w_7 = 0.12$. (b) 3D phase plot converges to $e_1 = (0.5,0.5,0)$ when $w_7 = 0.16$.

It is found that, for w_5 in the ranges $(0,0.04]$, $(0.04,0.33]$, $(0.33,0.5]$, and $w_5 > 0.5$, the solution of system (2) converges e_1, e_2, e_1, and e_0 respectively, however for w_6 in the ranges $(0,0.39]$, and $(0.39,1]$, the solution of system (2) converges e_1, and e_2, respectively, while for w_7 in the ranges $(0,0.2]$, and $(0.2,1]$, the solution of system (2) converges e_2, and e_1 respectively. The impact on the system's behavior (2) as w_8 increases is comparable to that observed when w_7 is altered.

7. Conclusions

In this study, a prey-predator system has been constructed mathematically. The prey is thought to be a species with a stage structure that includes juveniles and adults. While the adult prey species possesses antipredator abilities against the predator, the predator cooperates in pursuing juvenile prey. Additionally, the prey was believed to be under the influence of harvest. System (2) was shown to have three nonnegative equilibrium points. The system's persistence as well as local and global analyses of stability were investigated. The Sotomayor theorem is used to describe local bifurcation. It is determined that system (2) has saddle-node and transcritical bifurcation as its two types of bifurcation. Finally, using a fictitious set of parameter values, the following findings are derived numerically. The persistence and stability of the system at the positive equilibrium point are positively influenced by the cooperative hunting rate and the conversion rate of the hunted prey biomass to predator biomass. All other system parameters, on the other hand, have a negative impact on the system's persistence and stability at the positive equilibrium point.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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